Querying Big Graphs within Bounded Resources

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ABSTRACT
This paper studies the problem of querying graphs within bounded resources. Given a query $Q$, a graph $G$ and a small ratio $\alpha$, it aims to answer $Q$ in $G$ by accessing only a fraction $G_Q$ of $G$ of size $|G_Q| \leq \alpha |G|$. The need for this is evident when $G$ is big while our available resources are limited, as indicated by $\alpha$. We propose resource-bounded query answering via a dynamic scheme that reduces big $G$ to $G_Q$. We investigate when we can find the exact answers $Q(G)$ from $G_Q$, and if $G_Q$ cannot accommodate enough information, how accurate the approximate answers $Q(G_Q)$ are. To verify the effectiveness of the approach, we study two types of queries. One consists of pattern queries that have data locality, such as subgraph isomorphism and strong simulation. The other is the class of reachability queries, without data locality. We show that it is hard to get resource-bounded algorithms with 100% accuracy: NP-hard for pattern queries, and non-existing for reachability when $\alpha \neq 1$. Despite these, we develop resource-bounded algorithms for answering these queries. Using real-life and synthetic data, we experimentally evaluate the performance of the algorithms. We find that they scale well for both types of queries, and our approximate answers are accurate, even 100% for small $\alpha$.

Categories and Subject Descriptors
H.2.4 [Database Management]: Systems—Query processing

Keywords
bounded resource; graph querying; pattern matching

1. INTRODUCTION
Real-life graphs introduce challenges to query answering. (1) Such graphs are typically big. For instance, Facebook has 1 billion nodes and 140 billion links in its social graph\textsuperscript{1}, and a Web-scale graph is easily of PB size [17]. (2) Queries are routinely posed on these graphs, such as graph pattern queries [20, 33] and reachability queries [36]. Such queries are expensive. For a graph $G = (V, E)$ and a query $Q$, it takes $O(|V| + |E|)$ time when $Q$ is to test whether one node can reach another in $G$, $O(|Q||V|(|V| + |E|))$ time to find matches of $Q$ in $G$ when $Q$ is a graph pattern and matching is defined by strong simulation [20], and worse yet, it is NP-hard even to decide whether there exists a match of $Q$ in $G$ by subgraph isomorphism. It is often cost-prohibitive to find exact answers to these queries in big graphs.

Can we still answer such queries $Q$ in a big graph $G$ when we have limited resources, e.g., time and space? This question motivates us to study resource-bounded query answering. Given a small ratio $\alpha \in (0, 1)$ and $Q$ posed on $G$, we extract a fraction $G_Q$ of $G$ such that $|G_Q| \leq \alpha |G|$, and compute approximate answers $Q(G_Q)$. Here $\alpha$ is called a resource ratio and is determined by our available resources.

The idea behind resource-bounded query answering is to make big data “small”. While we cannot lower the complexity of computing $Q(G)$, we reduce the cost by using small $G_Q$ instead of $G$, and hence, make it feasible to answer expensive queries in big graphs. The need for this is evident: real-life searches require fast response (e.g., in less than 1 second [6]) with e.g., limited memory [18] and energy [15]. Computation of exact answers $Q(G)$ by accessing the entire $G$ is often beyond reach in these settings. We may have to settle with approximate answers, which often suffice in, e.g., updating ads based on trends in social networks [2] and mining patterns in social graphs [9, 16]. Moreover, for graph pattern matching by subgraph isomorphism, it is often necessary to adopt inexact query answers anyway.

Obviously the smaller the resource ratio $\alpha$ is, the less space and time it takes to compute $Q(G_Q)$; but as a price, the lower the accuracy of the approximate answers $Q(G_Q)$ is. Nonetheless, we show that even with small $\alpha$, we can often find answers with quite good quality, even exact answers (Section 6). Indeed, a typical Facebook Graph Search query\textsuperscript{2} can be answered by using nodes that are within 3 hops of a designated node in $G$, a small fraction of its entire social graph [34]. This is also the case for a range of personalized social search queries [1, 8]. However, note that simply extracting local information alone may not suffice: there could be more than $10^9$ nodes within 3 hops of a node [1].

Example 1: A fraction of a social network $G$ is shown in Fig. 1. There are three social groups in $G$: a hiking group (HG), a city cycling club (CC), and a separate group of

\textsuperscript{1}http://newsroom.fb.com/

\textsuperscript{2}https://www.facebook.com/about/graphsearch
cycling lovers (CL). A user Michael issues a query: “find me cycling lovers (CL) who know both my friends in LA cycling club (CC), and my friends in the hiking group (HG)”. Such a query is common in, e.g., Facebook Graph Search. It can be represented as graph pattern $Q$ shown in Fig. 1.

By strong simulation [20], $Q(G)$ returns two matches $cl_{n-1}$ and $cl_n$. To find them, there is no need to search the entire $G$. It suffices to consider a subgraph $G_Q$ of $G$ consisting of only those nodes within 2 hops of Michael. In fact, a small $G_Q$ with 7 nodes will do here. Resource-bounded query answering aims to identify such a small subgraph $G_Q$ of $G$ within resource ratio $\alpha$ such that $Q(G_Q)$ can still give us accurate or even exact answers. Note that if the resource ratio $\alpha$ allows us to visit, say, at most 16 nodes and edges in $G_Q$, it is nontrivial to identify a sensible $G_Q$ from (possibly thousands of) nodes within 2 hops of Michael.

The query $Q$ is localized: one can find matches locally, by searching only those nodes within $d_Q$ hops of a designated node Michael, where $d_Q$ is decided by $Q$. Thus to answer $Q$ we do not have to consider nodes far from Michael.

More challenging are non-localized queries. For example, Michael also asks whether he can reach a sport star Eric. We do not have to consider nodes far from Michael.

This example shows that resource-bounded query answering is feasible in practice, and it helps us query big data. To make practical use of it, however, several questions have to be answered. Given a ratio $\alpha$, how can we identify a small $G_Q$ that is within the bound w.r.t. $\alpha$, and gives us accurate or approximate answers to $Q$ in $G$? Can we get accuracy guarantees in $G_Q$, even 100% accurate? Does this approach work on both localized queries and non-localized queries? These questions are challenging even for localized queries.

**Contribution.** This work is a step towards effective approaches to answering queries within bounded resources.

1. We formalize resource-bounded query answering via a dynamic reduction scheme (Section 3). Given a graph $G$, a query $Q$ and a small $\alpha \in (0, 1)$, we propose to first reduce $G$ to $G_Q$ such that $|G_Q| \leq \alpha |G|$, by accessing a bounded amount of the data in $G$. We then compute $Q(G_Q)$ as approximate answers to $Q$ in $G$. We define query answer accuracy to evaluate the quality of approximate answers $Q(G_Q)$.

As a proof of concept, we study resource-bounded query answering for both localized and non-localized queries.

2. We develop resource-bounded algorithms for graph pattern matching in terms of strong simulation and subgraph isomorphism (Section 4). We show that for these localized queries, resource-bounded query answering is already non-trivial. It is NP-hard to decide, given $G$, $Q$ and $\alpha$, whether there exists a subgraph $G_Q$ of $G$ such that $|G_Q| \leq \alpha |G|$ and $Q(G) = Q(G_Q)$, i.e., whether it is possible to find exact answers from a small $G_Q$. Despite this, we develop a reduction strategy that identifies $G_Q$ by fetching nodes based on their dynamically maintained weights, guided by query $Q$. We show that our algorithms visit a bounded amount of data in $G$, and possess certain accuracy guarantees.

3. We extend the study to reachability queries (Section 5). We show that for these non-localized queries, there exist no algorithms that, given $G$, nodes $s$ and $t$ in $G$, and $\alpha < 1$, decide whether $s$ reaches $t$ with 100% accuracy by visiting no more than $\alpha |G|$ nodes. While reachability can be tested in linear time, it has to visit an unbounded number of nodes in $G$ or store an index larger than $G$. To this end, we provide an algorithm for answering reachability queries based on a hierarchical landmark index. Using the index, the algorithm drills-down or rolls-up in the search, visits at most $\alpha |G|$ nodes or edges, and guarantees 100% true positives.

4. We experimentally evaluate the effectiveness of the approach using real-life and synthetic graphs (Section 6). We find that our algorithms are (a) efficient: they are 5.5, 6.25 and 5.7 times faster than traditional algorithms for strong simulation, subgraph isomorphism and reachability, respectively, even after they are improved by employing our own optimization; (b) accurate: they often achieve 100% accuracy by accessing only 0.0015% (resp. 0.05%) of $G$. We show that for these non-localized queries, there exist no algorithms that, given $G$, nodes $s$ and $t$ in $G$, and $\alpha < 1$, decide whether $s$ reaches $t$ with 100% accuracy by visiting no more than $\alpha |G|$ nodes. While reachability can be tested in linear time, it has to visit an unbounded number of nodes in $G$ or store an index larger than $G$. To this end, we provide an algorithm for answering reachability queries based on a hierarchical landmark index. Using the index, the algorithm drills-down or rolls-up in the search, visits at most $\alpha |G|$ nodes or edges, and guarantees 100% true positives.

We contend that resource-bounded query answering is capable of finding accurate answers by accessing a small fraction of big graphs, and is promising for evaluating both localized and non-localized queries in real life. This also suggests how we can strike a balance between the resources needed and the accuracy of approximate answers computed.

**Related work.** We categorize the related work as follows.

*Indexing and compression.* There are typically two ways to reduce the search space: indexing and compression.

1. Graph indexing [10, 13, 26, 35] provides precomputed global information of $G$ to evaluate queries, with additional storage costs. For instance, given $G = (V, E)$ for reachability queries, a reachability matrix takes $O(|V|^2)$ space to store [36]. A 2-hop index takes $O(|V||E|^{1.5})$ space to store and $O(|E|^{1.2})$ time to query. These are not very practical when $G$ is big. Labeling-based methods for reachability queries are studied in [35] with reduced index size by pruning landmark and path labeling. In contrast, this work uses small indices to support dynamic reduction, while striking a balance between the amount of data accessed (bounded by a given small ratio) and the accuracy of query results.

2. Graph compression [4, 14] constructs a summary of $G$. To answer $Q$, however, it often needs decompression, sometimes restoring the entire $G$ [4]. In a similar sense, graph summarization gives sketches of $G$ [23, 32]. Query preserving compression [12] allows us to process $Q$ without decomposition. It compresses $G$ into a graph $G_r$ (5% and 43% of $|G|$ for reachability and graph simulation, respectively).
Distributed systems. Distributed systems, e.g., Pregel [21] and GraphLab [19], evaluate queries on vertices of a graph in parallel with multiple processors. In contrast, this work studies query evaluation with limited resources and a single processor. This said, the techniques of this work can be readily adapted to the distributed settings.

Budgeted search. Related is also prior work on finding error-bounded answers, as early as (weighted) $A^*$ [25], which was recently extended as optimistic search [30]. The prior work focuses on predicting how good a partial answer (as in a search tree) approximates the optimal solution, but the cost of finding such answers is not the major concern. Bounded-cost search was recently proposed [29,31] for planning, with the prior work being extended as optimistic search [30]. The prior work studies query evaluation with limited resources and a single processor. We denote by $N_r(v)$ the set of all nodes in $G$ within $r$ hops of $v$. For a node $v$ and a non-negative integer $r$, the $r$-neighborhood $G_r(v)$ of $v$ is the subgraph of $G$ induced by $N_r(v)$. (5) We say that $v$ is a parent of $v'$, or equivalently, $v'$ is a child of $v$, if $(v, v')$ is an edge in $E$.

We study two types of graph queries, given as follows.

Graph pattern queries. We study graph patterns for personalized social search [7,8]. A graph pattern is a graph $Q = (V_Q, E_Q, f)$, where (1) $V_Q$ and $E_Q$ are the set of query nodes and (directed) edges, respectively; (2) for each node $u$, $f(u)$ specifies a node label; and (3) $u_p$ and $u_o$ represent the personalized node and output node of $Q$, respectively.

In a data graph $G$, the personalized node $u_p$ has a unique match $v_p$, with $f(u_p) = L(v_p)$, often denoting the person who issues the query $Q$. The output node $u_o$ indicates the search intent of $Q$, and the label $f(u_o)$ specifies search constraints [7]. For instance, for the graph pattern $Q$ over graph $G$ of Fig. 1, node Michael is its personalized node, and has a unique match Michael in $G$. Node CL is the output node, indicating that the query is to find and return cycling lovers who satisfy the constraints of the pattern.

We consider two semantics for matching a graph pattern $Q = (V_p, E_p, f, u_p, u_o)$ to a data graph $G$.

Subgraph queries. A match of $Q$ in $G$ via subgraph isomorphism is a subgraph $G'$ of $G$ that is isomorphic to $Q$, i.e., there exists a bijection function $h$ from $V_{G'}$ to the set of nodes of $G'$ such that (1) for each node $u \in V_{G'}$, $f(u) = L(v)$; (2) $(u, u')$ is an edge in $Q$ if and only if $(h(u), h(u'))$ is an edge in $G'$; and (3) $h(u_p) = v_p$, i.e., $u_p$ matches the unique $v_p$.

The answer to $Q$ in $G$, denoted by $Q(G)$, is the set of nodes $h(u_o)$ that match the output node $u_o$ of $Q$ in $G'$, for all matches $G'$ of $Q$ in $G$. We refer to $Q$ as a subgraph query.

Simulation queries. For matching by strong simulation [20], a match of pattern $Q$ in $G$ is defined on the $d_Q$-neighborhood $G_{d_Q}(v_0) = (V_{d_Q}, E_{d_Q}, L_{d_Q})$ of nodes $v_0$ in $G$, where $d_Q$ is the diameter of $Q$. In this setting, we say that $G$ matches $Q$ if there exists a binary relation $R_{d_Q} \subseteq V_{d_Q} \times V_{d_Q}$ such that

- $(u_p, v_p) \in R_{d_Q}$, i.e., the match of $u_p$ is fixed to be $v_p$;
- for each node $u \in V_{d_Q}$, there exists a node $v \in V_{d_Q}$ such that $(u, v) \in R_{d_Q}$, referred to a match of $u$; and
- for each pair $(u, v) \in R_{d_Q}$, $f(u) = L_{d_Q}(v)$ and further, (a) for each edge $(u, u')$ in $E_{d_Q}$, there exists an edge $(v, v') \in E_{d_Q}$, such that $(u, v') \in R_{d_Q}$; and
  (b) for each edge $(u', v) \in E_{d_Q}$, there exists an edge $(v', v) \in E_{d_Q}$, such that $(u', v') \in R_{d_Q}$.

Conditions (a) and (b) above ensure that the match preserves the children and parent relationships, respectively.

2. PRELIMINARY

In this section we present localized queries and non-localized queries. We first review several basic notations.

Data graphs. We define a data graph as a node-labeled, directed graph $G = (V, E, L)$, where (1) $V$ is a finite set of data nodes; (2) $E \subseteq V \times V$ is a set of edges, in which $(v, v')$ denotes an edge from node $v$ to $v'$; and (3) for each node $v$ in $V$, $L(v)$ is the label of $v$. The label $L(v)$ may indicate e.g., the content of a page [3] or node attributes [27].

We use two types of subgraphs $G_s = (V_s, E_s, L_s)$ of $G$.

- Graph $G_s$ is a subgraph of $G$ if $V_s \subseteq V$, $E_s \subseteq E$, and $E_s$ (resp. $L_s$) is the restriction of $E$ (resp. $L$) on the nodes in $V_s$; i.e., for each edge $e = (v, v') \in E_s$, $v \in V_s$ and $v' \in V_s$; and for each $v \in V_s$, $L_s(v) = L(v)$.
- Graph $G_s$ is a subgraph of $G$ induced by $V_s$ if it is a subgraph of $G_s$ and for all nodes $v, v' \in V_s$, edge $(v, v') \in E_s$ if and only if $(v, v') \in E$: i.e., $E_s$ includes all the edge of $E$ that are defined on the nodes in $V_s$.

We will also use the following notations. (1) The size of a graph $G$, denoted as $|G|$, is the total number of the nodes and edges of $G$. We also use $|V|$ to denote the number of nodes in $G$; similarly for $|E|$. (2) The diameter of $G$ is the length of the longest shortest path between any two nodes in $G$. (3) We say that a node $v'$ is within $r$ hops of $v$ if there exists a path of at most $r$ edges from $v$ to $v'$ or from $v'$ to $v$. We denote by $N_r(v)$ the set of all nodes in $G$ within $r$ hops of $v$. (4) For a node $v$ and a non-negative integer $r$, the $r$-neighborhood $G_r(v)$ of $v$ is the subgraph of $G$ induced by $N_r(v)$.

For Web-scale graphs of PB size, however, this technique alone does not suffice. In contrast to compression that uses the same $G_c$ to answer all queries posed on $G$, we fetch a bounded $G_Q$ given each query $Q$ with information for answering the particular $Q$. This said, the technique of [12] can be seamlessly combined with ours as a preprocessing step.
The match relation $R$ of $Q$ in $G$ is defined as the union of $R_{v_0}$ for all nodes $v_0$ in $G$. For any $Q$ and $G$, it is known that there exists a unique, maximum match relation $R_{Q}$ via strong simulation [20]. We define the answer $Q(G)$ to $Q$ in $G$ to be the set of matches of the output node $u_0$, i.e., $Q(G) = \{v \mid (u_0, v) \in R_{Q}\}$. We refer to $Q$ as a simulation query.

For instance, for $Q$ and $G$ depicted in Fig. 1, $G$ matches $Q$ via strong simulation, in which the output node $CL$ has two matches $c_{l-1}$ and $c_{l}$, and $Q(G)$ is the set $\{c_{l-1}, c_{l}\}$.

Localized queries. A class of graph queries $Q$ is said to have data locality, referred to as localized queries, if for any graph $G$ and any node $v$ in $G$, one can decide whether $v$ is in $Q(G)$ locally, by inspecting only those nodes of $G$ that are within $d_Q$ hops of $v$, where $d_Q$ is determined only by $|Q|$. Otherwise, the class of queries is called non-localized.

Both subgraph and simulation queries are localized. To compute $Q(G)$, we only need to visit those nodes within $d_Q$ hops of $v_G$, where $d_Q$ is the diameter of $Q$, $d_Q \leq |Q|$, and $v_G$ is the match of the personalized node $u_G$ of $Q$. That is, we only need to consider the $d_Q$-neighborhood $G_{d_Q}(v_G)$ of $v_G$ in $G$. However, $G_{d_Q}(v_G)$ may be large [1].

Reachability queries. As an example of non-localized queries, we consider reachability queries. Given $G$ and query $Q$ as a pair of nodes $(v_p, v_o)$ in $G$, it returns true if and only if $v_p$ can reach $v_o$ in $G$, i.e., there is a path from $v_p$ to $v_o$.

For example, Example 1 gives an reachability query, to test whether Michael can reach Eric via social links.

Reachability queries are non-local: to compute $Q(G)$, we often have to visit nodes that reach $v_p$ or $v_o$, with a path of unbound length, even all the nodes in $G$ in the worst case.

The notations of this paper are summarized in Table 1.

### Table 1: Notations: graphs and queries

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Notations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_r(v)$</td>
<td>node set within $r$ hops of $v$</td>
</tr>
<tr>
<td>$G_r(v)$</td>
<td>$r$-neighborhood graph of $v$</td>
</tr>
<tr>
<td>$Q$</td>
<td>graph pattern</td>
</tr>
<tr>
<td>$(v_p, E_p, f_p, u_p, v_o, u_o)$</td>
<td>personalized (resp. output) node in $Q$</td>
</tr>
<tr>
<td>$G_{d_Q}(v_p)$</td>
<td>the diameter of $Q$ that matches $v_p$</td>
</tr>
<tr>
<td>$d_Q$</td>
<td>the diameter of $Q$ as an undirected graph</td>
</tr>
<tr>
<td>$d$</td>
<td>resource ratio such that $</td>
</tr>
<tr>
<td>$\eta$</td>
<td>accuracy ratio: accuracy$(Q, G)$ $\geq \eta$</td>
</tr>
<tr>
<td>$f$</td>
<td>the number of nodes in $Q_{d_q}(v_p)$ sharing the same label and a common parent or child</td>
</tr>
</tbody>
</table>

### 3. RESOURCE-BOUND QUERYING

To process a query $Q$ on a big graph $G$ while our resources are limited, we propose to identify and fetch a fraction $Q(G)$ of $G$ within a given bound on its size, and compute approximate answers $Q(G)$ with accuracy guarantees.

Accuracy of query answers. We define an accuracy measure for pattern queries, and then revise it for reachability.

Graph patterns. The exact answer to a pattern $Q$ in $G$ is a set $Q(G)$ of matches. Suppose that an algorithm $A$ computes a set $Y$ of approximate answers to $Q$ in $G$. We define the precision and recall of $Y$ for $(Q, G)$ in the standard way:

$$\text{precision}(Q, G, Y) = \frac{|Y \cap Q(G)|}{|Y|}, \text{ recall}(Q, G, Y) = \frac{|Y \cap Q(G)|}{|Q(G)|}.$$  

That is, $\text{precision}$ is the ratio of the number of correct matches in $Y$ to the total number of matches in $Y$, while $\text{recall}$ is the ratio of the number of correct matches in $Y$ to the total number of matches in $Q(G)$. Based on these, we define the accuracy of $Y$ for $(Q, D)$ as the usual $F$-measure:

$$\text{accuracy}(Q, G, Y) = \frac{\text{precision}(Q, G, Y) \cdot \text{recall}(Q, G, Y)}{\text{precision}(Q, G, Y) + \text{recall}(Q, G, Y)}.$$  

The larger accuracy $Q(G, Y)$ is, the more accurate $Y$ is.

When both $Q(G)$ and $Y$ are $\emptyset$, i.e., no match exists, we treat $\text{accuracy}(Q, G, Y)$ as 1; we consider $\text{precision}$ only if $Q(G)$ is $\emptyset$ but $Y$ is not, and recall only if $Y$ is $\emptyset$ but $Q(G)$ is not.

Reachability queries. When $Q$ is a reachability query, $Q(G)$ is a single truth value. Given a set $Q$ of reachability queries, we denote by $Q(G)$ the set of exact answers for all queries $Q$ in $G$, and by $Y$ the set of truth values computed by algorithms $A$ for $Q \in Q$. Then we define $\text{precision}(Q, G, Y)$, $\text{recall}(Q, G, Y)$ and $\text{accuracy}(Q, G, Y)$ in the same way as above. Here $\text{precision}(Q, G, Y)$ is the ratio of the number of true positives and true negatives to the total number of answers returned by $A$, which also include false positives and false negatives; similarly for $\text{recall}(Q, G, Y)$.

Resource-bounded query answering. We now present resource-bounded algorithms. Let $\alpha \in (0, 1)$ be a resource ratio, and $L$ be a class of queries (subgraph or simulation).

Given a graph $G$ and a query $Q$ in $L$, an algorithm $A$ for $L$ queries with resource-bound $\alpha$ does the following:

- fetches a fraction $G_Q$ of $G$ such that $|G_Q| \leq \alpha|G|$, by visiting at most $\alpha \times c \times |G|$ amount of data in $G$; and
- computes $Q(G_Q)$ as approximate answers, where $c$ is a coefficient such that $\alpha \times c \times 1 < 1$.

We say that $A$ has accuracy guarantee $\eta$ for $L$ if for all graphs $G$ and all queries $Q \in L$, $\text{accuracy}(G, Q, Q(G_Q)) \geq \eta$.

Note that the accuracy ratio $\eta$ is in the range $[0, 1]$. When $\eta = 1$, algorithm $A$ finds exact answers for all graphs $G$ and queries $Q$ i.e., with 100% accuracy.

Similarly such algorithms are defined for reachability.

As illustrated in Fig. 2, algorithm $A$ consists of two steps.

1. **Dynamic reduction.** Given a query $Q$, it reduces a possibly big $G$ to a small $G_Q$ within the bound. In contrast to graph indexing, compression and summarization that build the same structure for all queries (see Section 1), dynamic reduction finds $G_Q$ with only information needed for an input query $Q$, and hence, allows higher accuracy. One can use any techniques for dynamic reduction, including those for data synopses such as sampling and sketching, as long as the process visits a bounded amount of data in $G$. The reduction process may use some auxiliary information (e.g., indices) collected by offline preprocessing that is conducted once-for-all, to help us answer all queries posed on $G$.

2. **Approximate query answering.** Algorithm $A$ computes $Q(G_Q)$ by accessing $\alpha |G| \times c$ amount of data rather than the entire $G$. When $\alpha = 0.0015$ and $|G|$ is 1PB, e.g., $G_Q$ is of GB size and $\text{accuracy}(G, Q, Q(G_Q))$ is high (see Section 6).

**Example 2:** Recall $Q$ and $G$ from Fig. 1. Set resource ratio $\alpha = 1.6\%$, and $c = 1$. Suppose that $n = 96$ and $n = 900$, i.e., there are 1000 nodes within 2 hops of node Michael. Then a resource-bounded algorithm $A$ is allowed to visit at most 16 nodes and edges in $G$. Ideally, $A$ visits Michael, $cc_{1}$, $cc_{3}$, $cl_{-1}$, $cl_{4}$, and $hm$, and finds $G_Q$ to be the subgraph induced by the nodes (with 14 nodes and edges). If so, $A$ can find $Q(G_Q) = \{cl_{-1}, cl_{4}\}$ and $\text{accuracy}(G, Q, Q(G_Q)) = 100\%$.

**Remarks.** The bound $\alpha |G|$ is essentially bounded by $c$, time, space and energy [6, 15, 18]. Disk-based algorithms
4.1 Resource-Bounded Strong Simulation

We start with a resource-bounded algorithm for simulation queries, denoted by \(\text{RBSim}\). Given a simulation query \(Q\), a graph \(G\) and a resource ratio \(\rho\), \(\text{RBSim}\) finds a subgraph \(G'\) of \(G\) such that \(|G'| \leq \rho|G|\), by visiting a bounded amount of data in \(G\). Moreover, for small \(\rho\), we can have 100% accuracy. Indeed, in practice \(\rho\), \(d\) and \(f\) are all quite small: (1) on average \(d\) is around 100 in Facebook [34]; this bound also applies to \(f\); (2) \(d\) and \(f\) are smaller than \(|Q|\), and \(|Q|\) is small in personalized social search [8] and ego network analysis [22]. In our experimental study using real-life graphs, while \(|G'_{d}(v_p)|\) is up to 0.01% of \(|G|\) with \(d\) up to 483, we find that we consistently get 100% accuracy even when \(\rho\) is 0.0015%, which is on average 3% of the theoretical bound given in Theorem 3(b), where \(|G'|\) is up to 19% of the size \(|G'_{d}(v_p)|\) (see Section 6).

We next prove Theorem 3 for simulation queries first (Section 4.1), and then adapt it to subgraph queries (Section 4.2). We focus on dynamic reduction to find \(G'\); after that, we simply use existing algorithms for strong simulation [20] and subgraph isomorphism [11] to compute \(Q(G')\).

4.4 ANSVERING LOCALIZED QUERIES

We now study resource-bounded algorithms for answering simulation and subgraph queries. This is nontrivial: Theorem 1 tells us that it is intractable to decide whether there exists a subgraph \(G'\) within a bound that preserves \(Q(G)\).

Despite this, we develop resource-bounded algorithms for graph pattern queries that still have certain performance guarantees. The main result of the section is as follows.

**Theorem 3:** There exist resource-bounded algorithms for simulation and subgraph queries such that given any resource ratio \(\rho < 1\), graph \(G\) and query \(Q\),
(a) they find a subgraph \(G'\) of \(G\) with \(|G'| \leq \rho|G|\), by visiting at most \(d\rho + \alpha|G|\) nodes and edges in \(G\), in \(O(d\rho|Q||G|)\) time; and
(b) \(Q(G')\) has 100% accuracy when \(\rho \geq \frac{2((d+1)^{d+1})}{(d+1)|G|}\).

Here \(d\) is the maximum degree of nodes in \(G_{d}(v_p)\) (corresponding to parameter \(c\) in resource-bounded query answering), \(d\) is the diameter of \(Q\) when \(Q\) is treated as an undirected graph, \(f\) is the number of distinct labels in \(Q\), and \(f\) is the maximum number of the nodes in \(G_{d}(v_p)\) that have the same label and a common parent or child.

Theorem 3 tells us that we can effectively find small \(G'\) by accessing a bounded amount of data in \(G\). Moreover, for small \(\rho\), we can have 100% accuracy. Indeed, in practice \(\rho\), \(d\) and \(f\) are all quite small: (1) on average \(d\) is around 100 in Facebook [34]; this bound also applies to \(f\); (2) \(d\) and \(f\) are smaller than \(|Q|\), and \(|Q|\) is small in personalized social search [8] and ego network analysis [22]. In our experimental study using real-life graphs, while \(|G_{d}(v_p)|\) is up to 0.01% of \(|G|\) with \(d\) up to 483, we find that we consistently get 100% accuracy even when \(\rho\) is 0.0015%, which is on average 3% of the theoretical bound given in Theorem 3(b), where \(|G'|\) is up to 19% of the size \(|G'_{d}(v_p)|\) (see Section 6).

We next prove Theorem 3 for simulation queries first (Section 4.1), and then adapt it to subgraph queries (Section 4.2). We focus on dynamic reduction to find \(G'\); after that, we simply use existing algorithms for strong simulation [20] and subgraph isomorphism [11] to compute \(Q(G')\).
ity of its 1-neighborhood $N_1(v)$ (or simply $N(v)$), consisting of the parents and children of $v$; and (b) a set $S_t$ of pairs $(\ell, q)$, where $\ell$ is a distinct label from $N(v)$, and $q$ is the number of occurrences of $\ell$ in $N(v)$. These can be found by a linear traversal of $G$ in an on-line-for-all offline preprocessing.

**Example 3:** Consider graph $G$ of Fig. 1. Let $|G| = 1000$, where $m = 96$ and $n = 900$. An offline preprocessing step computes, for each node $v$, a set $S_t = \{(\ell, q) : (\ell, q) \in V(G) \times \mathbb{R} \}$, and a set $S_c$ for each node $v$, consisting of heuristics for the estimated cost, where $\ell$ is a distinct label from $N(v)$, and $q$ is the number of occurrences of $\ell$ in $N(v)$. These can be found by a linear traversal of $G$ in an on-line-for-all offline preprocessing.

Given $a$, $b$, and $c$, the traversal is guided by pattern $P$, a graph $G$ with $|V(G)| = n$, and a resource ratio $\alpha$. The traversal starts with an empty stack $S$, and a node $v$. It terminates when the stack $S$ is empty. The traversal generates a ranked list of nodes that are candidates for query node $u$. The list is returned (line 7). If the stack $S$ is empty and no nodes are added to $G$, the query node $u$ is not found.

**Algorithm RBSim**

*Input:* A query node $u$, a graph $G$, a resource ratio $\alpha$.

*Output:* Approximate answers $Q(G|u)$.

1. $G_Q := \text{Search}(G, u, \alpha)$;
2. $Q(G) := \text{Match}(Q(G|u))$;
3. return $Q(G)$;

**Procedure Search**

*Input:* $Q$, $G$, and $\alpha$.

*Output:* Subgraph $G_Q$.

1. initialize graph $G_Q := \emptyset$; $b := 2$;
2. Stack $S$. push $(u, v)$; terminate := false; changed := false;
3. while terminate $\neq$ true do
4. pair $(u, v)$ := $S$. pop();
5. add $v$ to $G_Q$ if it is not already in $G_Q$;
6. update terminate; changed := true if $v$ is new to $G_Q$;
7. if terminate then return $G_Q$;
8. for each unvisited edge $(u, u')$ or $(u', u)$ do
9. ranked list $S_p$. push $(u', v, G, G|u, S)$;
10. for each $v' \in S_p$ do $S$. push $(u', v')$;
11. if changed and $S = \emptyset$ then
12. $b := b + 1$; $S$. push $(u_p, v_p)$; changed := false;
13. if not changed and $S = \emptyset$ then terminate := true;
14. return $G_Q$;

*Figure 3: Algorithm RBSim*

$G_0$ by including new nodes and edges, which are descendants or ancestors of $v_0$ in its $d_q$-neighborhood $G_{d_q}(v_0)$ (lines 3–13). It uses two flags to control the traversal (line 2): (a) terminate becomes true if either $|G| = \alpha|G|$, or no nodes within $d_q$ hops of $v_0$ can be added to $G_Q$, and (b) changed becomes false if for a given selection bound $b$, there are new nodes added to $G_Q$. More specifically, the traversal is guided by pattern $Q$. If a new node $v$ is added to $G_Q$ as a candidate for query node $u$ (line 5), initially $v_p$, we set changed := true (line 6). If now $G_Q$ reaches the bound $\alpha|G|$, then $G_Q$ is returned (line 7). Otherwise, it inspects both children and parents $u'$ of query node $u$ (line 8). For each such $u'$, it calls procedure Pick (line 9) to select a ranked list $S_p$ of best new candidates $v'$ for $u'$ from the neighborhood $N(u') \in G$, where $|S_p|$ is bounded by $b$. Each pair $(u', v')$ is then pushed onto the stack $S$, with the best candidate $v'$ at the top of $S$ (line 10).

If at this stage, stack $S$ is empty (i.e., no new insertions) but changed is true (i.e., there are still match candidates that are not yet in $G_Q$), we increase $b$ and start the search from $(u_p, v_p)$ again to find them (line 12). If $S$ is empty and changed is false, no more nodes can be added to $G_Q$, and we set terminate := true (line 13). The process proceeds until terminate becomes true, and then $G_Q$ is returned (line 14).

**Procedure Pick**

Given a node $u'$ in $Q$ and a node $v$ in $G$, Pick (omitted) finds a list $S_p$ of “top-ranked” nodes $v'$ in $N(v)$ that are not yet in the stack $S$. To do this, Pick keeps a max-heap to store $N(v)$, with the estimated cost as the sorting key. For nodes $v'$ in the max-heap that are not added to $G_Q$ yet (indicated by a dynamically maintained Boolean flag), it first checks whether the guarded condition $C(v', u') = true$, and then updates $C(v', u')$ and $p(v', u')$ by checking the neighborhoods $N(v')$ in $G_Q$ and $N(u')$ in $G$. It returns $S_p$ with the top-$b$ ones with the maximum weights in the max-heap that satisfy the guarded condition, where $b$ is the selection bound. Note that $S_p$ is possibly empty if no new candidates exist (e.g., all nodes in $N(v)$ are added to $G_Q$).
Example 4: Consider \( Q \) and \( G \) of Fig. 1. When \( \alpha = 1.6\% \) (and \( c = 1 \)), procedure Search finds a subgraph \( G_Q \) of \( G \) (Fig. 4) with size no more than 14, and visits no more than 16 nodes and edges as follows. (1) It first pushes the pair (Michael, Michael) onto stack \( S \) (line 2), and adds Michael to \( G_Q \) (line 5). (2) It then checks edge (Michael, CC) in \( Q \), and invokes procedure Pick to find top 2 candidates for CC. In this case, Pick returns \( cc_1 \) and \( cc_3 \) (to be explained soon). Hence, Search pushes (CC, \( cc_1 \)) and (CC, \( cc_3 \)) onto \( S \) (line 10), and inserts \( cc_3 \), the current top of \( S \), into \( G_Q \) (line 5). (3) In the same way, it processes query edge (CC, HG) (CC, CL) and then a “backward” edge (HG, CL). It includes in \( G_Q \) three new nodes \( cl_1 \), \( cl_{n-1} \) and \( hg_{m_1} \) along with edges between them. It next traces back to query edge (Michael, HG). As the parent of \( hg_{m_1} \) is already in \( G_Q \), no new node is added. (4) Moving up the stack \( S \), Search backtracks to \( cl_{n-1} \) and \( cc_1 \). As their neighborhoods are all in \( G_Q \), Search finally pops up the first pair (Michael, Michael), which makes \( S \) empty. At this moment \( G_Q \) already reaches its size bound 14. Hence, Search sets terminate true, and returns \( G_Q \). RBSim then invokes Match [20] to compute two matches \( cl_1 \) and \( cl_{n-1} \) from \( G_Q \), for the output node CL in \( Q \). In the entire process, 16 nodes and edges are visited. Note that RBSim visits each query edge once (line 8).

We now show how procedure Pick works. When Pick is invoked by Search for edge (Michael, CC), Pick rules out the node \( cc_2 \) since it does not satisfy the guarded condition, i.e., it has no CL child as required by the query node CC. For the two remaining nodes \( cc_1 \) and \( cc_3 \), it looks up the auxiliary structure \( S_1 \), and finds that (a) both have a cost 1, since query node CC requires a CL child of them to be in \( G_Q \), and its parent Michael already has a candidate Michael in the current \( G_Q \); and (b) \( p(cc_1, CC) = 3 \), indicating that there are 3 possible matches in \( N(cc_1) \), while \( p(cc_3, CC) = 2 \). Pick returns list \( Sp = [cc_1, cc_3] \) when the bound \( b = 2 \).

For edge (HG, CL) in \( Q \), Pick traces back to \( hg_{m_1} \), a parent of \( cl_1 \) and \( cl_{n-1} \). It updates the cost of \( hg_{m_1} \) from 1 to 0, as it already has a child \( cl_1 \) and parent Michael in \( G_Q \), while \( p(hg_{m_1}, HG) = 4 \). Pick finds node \( hg_{m_1} \) from the max-heap. Note that all the other HG nodes have cost 1, but do not get into \( G_Q \) as they have no CL child.

Performance analysis. We now prove Theorem 3 by analyzing algorithm RBSim. (1) RBSim extracts a subgraph \( G_Q \) with \( |G_Q| \leq \alpha |G| \), guaranteed by the termination condition. (2) For each newly added node \( v \) to \( G_Q \), procedure Search inspects at most 1-hop of \( v \) in \( G \), by calling Pick. Hence Search visits at most \( d_G \ast \alpha |G| \) nodes or edges, where \( d_G \) is the maximum node degree in \( G \). (3) For time complexity, note that Search executes the while loop (lines 3-13) at most \( \alpha |G| \) times. This is because (a) at least one new node is added to \( G_Q \) in each loop, and (b) \( |G_Q| \leq \alpha |G| \). For each node \( \varepsilon \) and query node \( \varepsilon' \), it checks the guarded conditions in \( O(d_G) \) time, and maintains the max-heap in \( \log |d_G| \) time. As there are in total \( |V_Q| \) query nodes, it takes at most \( O(d_G|Q| \log |G_Q|) \) time. These verify Theorem 3(a).

We next prove Theorem 3(b) by induction on the diameter \( d \) of \( Q \) (when \( G \) is treated as an undirected graph). The case when \( d = 1 \) is trivial. When \( d = 2 \), RBSim finds \( G_Q \), in the worst case, a two-level “tree” rooted at \( v_p \), with size at most \( 1 + 2 \ast l + f \leq 2((l+1)^2 - 1) \). Now assume Theorem 3(b) holds when \( d = k \). That is, RBSim identifies \( G_Q \) as a \( k \)-level, \( l \) -fary “tree”, which contains all possible matches for \( Q \) in \( G \).

For \( d = k + 1 \), RBSim only needs to explore at most \( l \ast f \) children for each leaf in \( G_Q \) to include any new matches at level \( k + 1 \). The new \( G_Q \) hence has size at most \( 2((l+1)^{k+1} - 1) \).

Hence, when \( \alpha \geq 2((l+1)^{k+1} - 1) \), \( Q(G_Q) = Q(G) \), i.e., RBSim finds \( G_Q \) of size \( \alpha |G| \) and guarantees 100% accuracy.

Putting these together, Theorem 3 follows.

4.2 Resource-Bounded Subgraph Queries

We now outline a resource-bounded algorithm for subgraph queries, denoted by RBSub. It revisits RBSim as follows: (1) we enrich the guarded condition and cost estimation for isomorphism test, and (2) after \( G_Q \) is found, we use a subgraph isomorphism algorithm [11] to compute \( Q(G_Q) \).

More specifically, we use the same termination condition as in RBSim, but revise the guarded condition \( C(v, u) \) for RBSub as follows: \( C(v, u) \) is true if and only if for every query node \( u' \in N(u) \) in \( Q \) with degree \( d_{u'} \), there exists a distinct \( v' \in N(v) \) in \( G \) with the same label and degree \( d_{v'} \) \( \geq d_{u'} \). That is, \( C(v, u) \) imposes additional degree constraints for subgraph isomorphism. Accordingly, (a) for a node \( v \) in \( G \) and a node \( u \) in \( Q \), RBSub defines estimated costs \( c(v, u) \) and potential \( p(v, u) \) by using the revised guarded condition \( C(v, u) \); and (b) procedure Pick in RBSub favors candidates for query nodes with larger degree and lower costs.

One can verify Theorem 3 for subgraph queries along the same lines as the proof above for simulation queries.

5. NON-LOCALIZED QUERYING

We next study resource-bounded query answering for reachability. Despite of Theorem 2, we develop a resource-bounded algorithm that guarantees 100% true positives.

Theorem 4: There exists a resource-bounded algorithm such that given any resource ratio \( \alpha \in (0, 1) \), data graph \( G \) and reachability query \( Q \), it

(a) visits at most \( \alpha |G| \) amount of data, by using an index of size \( O(\alpha |G|) \);

(b) takes \( O(\alpha |G|) \) time to approximately answer \( Q(G) \);

(c) and it returns true only if \( Q(G) \) is true.

The algorithm visits at most \( \alpha |G| \) nodes and edges (hence parameter \( c = 1 \)). It never returns “false positive”. We find from our experimental study that the algorithm constantly achieves 100% accuracy even when \( \alpha = 0.05\% \). Moreover, its \( \text{prec} \) and \( \text{recall} \) get higher over larger and denser \( G \).

We give a constructive proof for Theorem 4 by providing the algorithm. It requires an once-for-all preprocessing that compress \( G \) and constructs a hierarchical indexing.

Preprocessing. An once-for-all preprocessing first reduces a (possibly cyclic) \( G \) to a directed acyclic graph (DAG) \( G_{DACG} \) by using the compression method of [12]. It is reachability preserving, i.e., for all reachability queries \( Q \) posed on \( G \), \( Q(G) = Q(G_{DACG}) \). The reason for this step is twofold: (a)
Figure 5: A hierarchical landmark index

$G_{DAG}$ is much smaller than $G$ while retaining reachability; and (b) hierarchical indexing (to be given below) is more effective on DAGs. Below we simply refer to the DAG as $G$.

Hierarchical indexing. An hierarchical index $I$ is then constructed using landmarks [13]. Given a pair of nodes $(v_1, v_2)$ in $G$ such that $v_1$ reaches $v_2$, we say that a node $v$ in $G$ is a landmark for $(v_1, v_2)$ and covers $(v_1, v_2)$ if it is on a path from $v_1$ to $v_2$. We create $I$ of size $\alpha(G)$, again once for all $Q$ on $G$.

Query answering. Given a query $Q = (v_p, v_o)$ and $I$, we check whether $v_p$ reaches $v_o$ by searching $I$ instead of $G$, to find whether there is a landmark in $I$ that covers $(v_p, v_o)$.

Below we focus on hierarchical index (Section 5.1), and the resource-bounded reachability algorithm (Section 5.2).

5.1 Hierarchical Landmark Index

One might want to find a minimum set $L_m$ of landmarks and take the subgraph induced by $L_m$, as $G_Q$, such that every pair of connected nodes in $G$ is covered by a landmark in $G_Q$. Given a query $(v_p, v_o)$, we could then test whether $v_p$ reaches $v_o$ by checking whether there is a landmark in $G_Q$ covering $(v_p, v_o)$. However, such a $G_Q$ may exceed $\alpha(G)$; moreover, the problem of finding a minimum $L_m$ is intractable [13].

In light of this, given $G$ and $\alpha$, we build a hierarchical landmark index $I$ of size $\alpha(G)$ to cover as many connected node pairs in $G$ as possible. Below we use $a$ to denote $\lfloor \frac{a}{2} \rfloor$.

Index structure. The index $I$ is a set of rooted trees (a forest), consisting of $\alpha(G)$ landmarks of $G$ in total. Each tree has a depth of at most $\log_2 |G| + 1$, i.e., it has at most $\log_2 |G| + 1$ levels with all its leaves at level 1. More specifically, (1) each node in $I$ is a landmark of $G$, and (2) there is an edge $(v_1, v_2)$ in $I$ if and only if either $v_1$ can reach $v_2$ or $v_2$ can reach $v_1$. Intuitively, $I$ organizes a set of landmarks into several "levels": (a) the leaves cover connected node pairs in $G$, and (b) those at level $i > 1$ specify the reachability among the landmarks at the lower levels.

Auxiliary information. We also maintain the following for reachability checking. (1) For each landmark $v$ in $I$, we use $v.cs$ to store its cover size, i.e., how many connected node pairs in $G$ are covered by $v$. (2) For each edge $(v_1, v_2)$ in $I$, we define a label $v_{2}.c$: it is $< v_1, v_1, i + 1 >$ (resp. $< v_1, i + 1 >$) if $v_1$ can reach $v_2$ (resp. $v_2$ can reach $v_1$), where $v_1$ is at level $i + 1$ and $v_2$ at level $i$. (3) For each node $v$ that is in $G$ but not in $I$, we define a set $v.E$ of triples such that for each leaf $v'$ in $I$, $< v, v', 1 >$ (resp. $< v, v', 1 >$) is in $v.E$ if $v$ can reach $v'$ (resp. $v'$ can reach $v$) by following a path that contains no landmark in $I$. Note that $|v.E| \leq \frac{\alpha(G)}{2}$ (4) We also define $v.d$ (resp. $v.r$), the degree (resp. the topological rank) of $v$ in $G$ (recall that $G$ is a DAG). Here $v.r$ is defined as follows: (a) $v.r = 0$ if $v$ has no child in $G$; (b) otherwise, $v.r = v'.r + 1$, where $v'$ is the child of $v$ with the largest rank. (4) For each landmark $v$ in $I$, we define its topological range $v.R$

Procedure RBlIndex
Input: A graph $G=(V, E, L)$ and $\alpha$.
Output: A hierarchical landmark index $I$.
1. $I := \emptyset$;
2. greedily select $\lfloor \frac{\alpha(G)}{2} \rfloor$ landmarks from $V$ as $L_M_1$;
3. construct landmark graph $G_1$ from $L_M_1$ and $G$;
4. update $I$ with $L_M_1$;
5. for $l$ from 2 to $\lfloor \log_2 |G| \rfloor + 1$ do /* $a = \lfloor \frac{a}{2} \rfloor$*/
6. greedily select $\lfloor \frac{\alpha(G_{l-1})}{2} \rfloor$ landmarks $L_M_{l-1}$ from $G_{l-1}$;
7. expand $I$ with $L_M_l$ /*move nodes in $L_M_l$ up and add edges*/
8. encode $(L_M_l, L_M_{l-1}, G_{l-1})$;
9. construct landmark graph $G_l$ from $L_M_l$ and $G_{l-1}$;
10. for each $v \in V \setminus L_M_l$ do
11. assign labels $v.E$ in terms of the leaves of $I$;
12. return $I$;

Figure 6: Procedure RBlIndex
$= [r_1, r_2]$, where $r_1$ (resp. $r_2$) is the smallest (resp. largest) rank of the landmarks in the subtree rooted at $v$ in $I$. The range is simply $[v.r, v.r]$ if $v$ is a leaf in $I$.

One may verify that $I$ guarantees the following.

Lemma 5: For any nodes $v$ and $v'$ in $G$, (1) $v$ can reach $v'$ if there exist landmarks $v_1, v_2, v_3$ in $I$ such that $v$ reaches $v_1$, $v_2$ reaches $v'$, $v_1, v.e < v_2$, $v_3, i >$ and $v_2, v.e < 0, v_3, i >$; (2) for any $v_4$ in $I$ with $v_3, v.R = [r_1, r_2]$, if $r_2 \leq v'.r$ or $r_1 \geq v.r$, then no node in the subtree rooted at $v_4$ covers $(v, v')$.

Example 5: Consider a DAG $G$ shown in Fig. 5, with $|G| = 128$. Let $\alpha = 0.25$ and $c = 1$. We show an index $I$ with 16 nodes (each corresponds to a landmark in $G$) and 15 edges in Fig. 5. Observe the following. (1) Cover size $c_h = 56$, i.e., $c_h$ covers 56 connected node pairs in $G$. (2) $E(c_h, c_1)$ in $I$ with the label $< 0, c_h, 2 >$ of $c_0$ indicates that $c_0$ reaches $c_h$. (3) The label set of Michael (which is not in $I$) is $[0, c_1, 1]$, which indicates that Michael can reach $c_1$ in $I$ by only accessing the nodes "outside" $I$. One may further verify that Michael reaches Eric in $G$ by using $I$.

Algorithm. We now present an algorithm, denoted by RBlIndex (Fig. 6), that builds a hierarchical index $I$ of size $\alpha(G)$. It selects a set $L_M_1$ of $\lfloor \frac{\alpha(G)}{2} \rfloor$ landmarks of $G$ (line 2), constructs a landmark graph $G_1$ such that $L_M_1$ is its node set, and there exists an edge $(v_1, v_2)$ in $G_1$ if and only if $v_1$ can reach $v_2$ in $G$ (line 3). The nodes of $L_M_1$ are added to (initially empty) $I$ as its level-1 nodes (leaves; line 4).

RBlIndex then expands $I$ "bottom-up" (lines 5-9). For each $l \in [2, \lfloor \log_2 |G| + 1 \rfloor]$, it does the following. (1) It selects a set $L_M_l$ of $\lfloor \frac{\alpha(G_{l-1})}{2} \rfloor$ landmarks from $G_{l-1}$ following a greedy strategy (to be described later; line 6). (2) It then moves the landmarks in $L_M_l$ up one level in $I$ to be its nodes at level $l + 1$. For each landmark $v$ in $L_M_l$, it adds an edge from $v$ to a node $v'$ at level $l + 1$ of $I$ if $v$ can reach $v'$ or can be reached from $v'$ in $G_{l-1}$ (line 7). Accordingly, it assigns a label to $v'$, and updates the topological range of $v$ (line 8), by invoking procedure Encode (omitted). We then build the landmark graph $G_l$ from $L_M_l$ and $G_{l-1}$ (line 9), such that $L_M_l$ is the node set of $G_l$, and $G_l$ has an edge $(v_1, v_2)$ if and only if $v_1$ can reach $v_2$ in $G_{l-1}$.

The process above repeats until only a single landmark remains (i.e., at level $\log_2 |G|$; line 5). RBlIndex then assigns labels to the rest of nodes in $G$ as described earlier (lines 10-11). After this, it returns the index $I$ (line 12).

Landmark selection. For each $G_l$, we want to select a set $L_M_l$ of $\lfloor \frac{\alpha(G_{l-1})}{2} \rfloor$ landmarks to cover a maximum number of node
Procedure RBReach
Input: A reachability query \( Q = (v_p, v_o) \), and index \( I \) of size \( \alpha|G| \).
Output: Approximate answers to \( Q(G) \).

1. terminate := false; answer := false;
2. active set \( v_p.Active := \{ (1, v, l) \in v_p.E | v \in \{ v_p \} \} \);
3. active set \( v_o.Active := \{ (0, v, l) \in v_o.E | v \in \{ v_o \} \} \);
4. update terminate and answer;
5. if answer = true then return answer;
6. while terminate = false do
7. \( v_o.Active := \text{PickLM}(v_o.Active, I) \);
8. \( v_p.Active := \text{PickLM}(v_p.Active, I) \);
9. update terminate and answer;
10. if answer = true then return answer;
11. if no node can be added to \( v_p.Active \) and \( v_o.Active \) then
12. terminate := true;
13. return answer.

Figure 7: Procedure RBReach

Drill down or roll up. To decide whether to roll up or drill down at a landmark \( v \) of \( I \), RBReach dynamically maintains the following. (1) \( \text{Boolean guarded condition} \) \( C(v, v_p, v_o) \), indicating whether \( v \) can possibly reach \( v_o \) via \( v_p \). We define \( C(v, v_p, v_o) = \text{true} \) if and only if for the topological range \( v.R = [r_1, r_2], r_2 > v_o.R \) and \( r_1 < v_p.R \). We filter the entire subtree rooted at \( v \) if \( C(v, v_p, v_o) = \text{false} \) (see Lemma 5(2)). (2) \( \text{Cost} \) \( c(v) \), defined as the size of the subtree rooted at \( v \) in \( I \), excluding the total size of the subtrees rooted at its children that are already visited in \( I \). The larger \( c(v) \) is, the more landmarks need to be inspected. (3) \( \text{Potential} \) \( p(v) \), which is the cover size \( v.cS \) subtracted by the sum of the cover sizes of its children that have been visited. The higher \( p(v) \) is, the more likely that \( v \) connects to \( v_p \) or \( v_o \). We define the weight \( w(v) \) of \( v \) to be \( \frac{p(v)}{\text{size}(v)} \) if \( C(v, v_p, v_o) = \text{true} \), and \( w(v) = -\infty \) otherwise. At landmark \( v \) of \( I \), we roll up to its parent \( v' \) if \( w(v') \) is the maximum, and drill down to a child \( v'' \) if \( w(v'') \) is the largest, if the edge \( (v', v'') \) or \( (v, v'') \) is not already visited, respectively.

Algorithm. We now present RBReach. It uses two Boolean flags to control the search: \( \text{answer} = \text{true} \) if it finds that \( v_p \) reaches \( v_o \), and terminate = \text{true} if all the landmarks in \( I \) have been visited. Initially, both are false (line 1). RBReach keeps track of the landmarks that \( v_p \) can reach and those that can reach \( v_o \), in sets \( v_p.Active \) and \( v_o.Active \), respectively, initially extracted from \( v_p.E \) and \( v_o.E \) (lines 2 and 3). To update terminate, it also maintains a set consisting of landmarks that are already visited (not shown).

After \( v_p.Active \) and \( v_o.Active \) are initialized, RBReach checks whether \( v_p \) can already be decided to reach \( v_o \); if so, it returns true (lines 4-9). Otherwise, it iteratively expands \( v_p.Active \) and \( v_o.Active \) by including landmarks in \( I \) that are reachable from the nodes in \( v_p.Active \) (line 7) and that can reach the nodes in \( v_o.Active \) (line 8), respectively. This is done by procedure PickLM (not shown), which rolls up or drills down \( I \) following the strategy described above. When new landmarks are added, termination and answer are updated accordingly (line 9). We set \( \text{answer} = \text{true} \) if there exists a landmark in both \( v_p.Active \) and \( v_o.Active \), i.e., the condition of Lemma 5(1) is satisfied. If so, it returns true (line 10). If the set visited includes all the nodes in \( I \), termination is set true (lines 11-12), and false is returned (line 13). To efficiently decide whether \( v_p.Active \) and \( v_o.Active \) share a node, RBReach stores a flag with a value \( v_p^* \) or \( v_o^* \) at each node to indicate if it is already in \( v_p.Active \) or \( v_o.Active \). This allows us to check Lemma 5(1) with little extra time.

Example 7: Given the index \( I \) of Fig. 5, RBReach checks whether \( v_p \) can reach Eric as follows. (1) It starts with \( \text{Michael.Active} = \{ c56 \} \) and \( \text{Eric.Active} = \{ c1046 \} \). (2) As termination and answer are false, RBReach calls PickLM, which rolls up to \( c10 \) from \( c56 \). We add \( c10 \) to Michael.Active. Similarly, \( c10 \) is added to Eric.Active. (3) PickLM finds that \( c10 \) has weight \( w(c10) = \frac{46}{10} = 5.1 \) (after visiting \( c38 \) and \( c9 \), the cost \( c(c10) \) is now \( 16 - 8 = 8 \), and its potential is updated to 56 - 10 = 46, with \( p(c10) = 34 - 26 = 8 \), and \( p(c9) = 30 - 28 = 2 \)). In contrast, \( w(c38) = \frac{3}{2} = 1.5 \), and the guarded condition of \( c_{10-1} \) is false since Eric has a topological rank 2 but the range of \( c_{10-1} \) is \([0, 0]\). Hence, it decides to roll up to \( c10 \) from \( c38 \) rather than to drill down. (4) For the same reason, it rolls up to \( c10 \) from \( c9 \). Now \( c10 \) is in both Michael.Active and Eric.Active, and true is returned.
Analysis. To show Theorem 4, observe the following. (1) RBReach visits at most \(\alpha|G|\) amount of data. In the worst case, it visits the entire \(G\). As shown in Section 5.1, \(|I| \leq \alpha|G| - 1\). (2) It answers \(Q\) in \(O(\alpha|G|)\) time, since it visits each edge in \(I\) at most twice. Moreover, RBReach only needs to check the flag of each newly added landmark to test the condition of Lemma 5(1), as remarked earlier. As the edge number is no larger than \(\frac{\alpha|G|}{2} - 1\), the total time is hence in \(O(\alpha|G|)\). (3) RBReach returns true only when there exists a landmark \(v\) in both \(v_p,\text{Active}\) and \(v_o,\text{Active}\). By Lemma 5(1), \(Q(G)\) is true. Hence, it guarantees 100% true positives.

6. EXPERIMENTAL STUDY

Using real-life and synthetic data, we conducted two sets of experiments to evaluate the accuracy, efficiency and scalability of our resource-bounded algorithms.

Experimental setting. We used two real-life datasets: (a) Youtube\(^3\), a video sharing network with 1,609,969 nodes (videos) and 4,509,826 edges (recommendations); and (b) Yahoo\(^4\), a snapshot of Yahoo Web graph with 3,009,022 nodes (Web pages) and 14,979,447 edges (links). We also designed a generator to produce synthetic graphs \(G = (V, E, L)\), controlled by the numbers of nodes \(|V|\) and edges \(|E|\), for \(L\) from a set \(\Sigma\) of 15 labels.

Query generator. We generated patterns controlled by the number \(|V_p|\) of query nodes and the number \(|E_p|\) of query edges. For patterns on real-life graphs, their labels were drawn from those datasets, and for synthetic graphs, they came from the alphabet \(\Sigma\). We randomly selected a personalized node and an output node for each query. For reachability tests, we randomly sampled a set of ordered node pairs from a data graph, each pair representing a query.

Algorithms. We implemented the following, all in Java: (a) RBSim (Section 4.1); (b) MatchOpt, an optimized version of the strong simulation algorithm [20], which only checks subgraphs within \(d_G\) hops of \(v_p\) for query \(Q\) (\(d_G\) is the diameter of \(G\), and \(v_p\) is the match of the personalized node of \(Q\)); (c) RBSub (Section 4.2); (d) VF2Opt, the subgraph isomorphism algorithm of [11] optimized like MatchOpt; (e) RBReach (Section 5); (f) BFS that tests reachability by breadth-first search, and BFsOpt, which compresses a graph first [12] and then runs BFS on the compressed graph; and (g) the reachability algorithm LM of [13] using landmark vectors. Note that VF2Opt, MatchOpt and BFS use our optimization.

Evaluation. We tested the impact of graph size \(|G|\), query (set) size \(|Q|\) and resource bound \(\alpha\) (with \(c = 1\)) on (a) running time, and (b) accuracy. We adopted the accuracy measures given in Section 3 for pattern and reachability queries.

All the experiments were run on a machine powered by an Intel Core(TM) i7-3520M 2.90GHz CPU with 8GB of memory, using 64 bit Windows 7. Each experiment was run 5 times and the average was stored here.

Experimental results. We next report our findings.

Exp-1: Graph patterns. The first set of experiments evaluated the accuracy, efficiency and scalability of (a) RBSim versus MatchOpt; and (b) RBSub versus VF2Opt. We report the results for simulation and subgraph queries together, as they were tested in the same setting.

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>Youtube</th>
<th>Yahoo</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.1</td>
<td>1.6</td>
</tr>
<tr>
<td>RBSim</td>
<td>7%</td>
<td>12%</td>
</tr>
<tr>
<td>RBSub</td>
<td>8%</td>
<td>15%</td>
</tr>
</tbody>
</table>

Table 2: The ratio of \(\alpha|G|\) to \(|G_{d_G}(v_p)|\) \((\alpha \times 10^{-5})\)

Varying \(\alpha\). We first evaluated the impact of \(\alpha\) using real-life graphs. Fixing \(|Q| = (4, 8)\) (i.e., \(|V_p| = 4\) and \(|E_p| = 8\)), we varied \(\alpha\) from 0.0011% to 0.002%, in 0.0001% increments.

(1) Efficiency. We report the response time of the four algorithms in Figures 8(a) and 8(b) on Youtube and Yahoo, respectively. The results tell us the following: on average, (a) for simulation, RBSim takes only 24.4% and 18.8% of the running time of MatchOpt on Youtube and Yahoo, respectively; (b) for subgraph queries, RBSub takes 16.7% and 14.4% of the time of VF2Opt on these two graphs; (c) the larger \(\alpha\) is, the longer RBSim and RBSub take, but only slightly, since \(|G_Q| = \alpha|G|\) gets larger when \(\alpha\) increases; and (d) RBSim and RBSub are efficient: they took 2 and 5 seconds on Yahoo, respectively, even when \(\alpha = 0.002\%\).

Moreover, our algorithms visit only a small part of the \(d_G\)-neighborhood \(G_{d_G}(v_p)\) of \(v_p\) (see Table 2 for examples). On average, RBSim visits from 7% to 19% of \(|G_{d_G}(v_p)|\) on Youtube, and from 8% to 21% on Yahoo, when \(\alpha\) ranges from 0.0011% to 0.002%; for RBSub, it is from 7% to 21% on Youtube and from 8% to 24% on Yahoo. This is why RBSim and RBSub outperform MatchOpt and VF2Opt, respectively.

These confirm that resource-bounded query answering indeed gives us the efficiency we need on real-life graphs.

(2) Accuracy. In the same setting, we report the corresponding accuracy results in Figures 8(c) and 8(d), respectively. The results tell us the following: on average, (a) both RBSim and RBSub achieve high accuracy even when \(\alpha\) is small. For example, the accuracy of RBSim ranges from 87% to 100% on Youtube, and 89% to 100% on Yahoo. (b) Better still, when \(\alpha \geq 0.0015\%), both RBSim and RBSub constantly get 100% accuracy. (c) When RBSim and RBSub achieve 100% accuracy, \(|G_Q|\) is on average only 3% of the space bound induced by the theoretical minimum \(\alpha\) given in Theorem 3(b), and it is between 17% and 19% of the size of the \(d_G\)-neighborhood of \(v_p\), respectively. (d) The larger \(\alpha\) is, the higher the accuracy is, as expected, since \(G_Q\) can accommodate more information when \(\alpha\) increases. These justify the effectiveness of resource-bounded query answering in practice.

Varying \(|Q|\). We also evaluated the impact of \(|Q|\). Fixing \(\alpha\) as 0.01%, we varied \(|Q|\) from (4, 8) to (8, 16).

(1) Efficiency. We report the efficiency of the algorithms on Youtube and Yahoo in Figures 8(e) and 8(f), respectively, which tell us the following. (a) The larger \(|Q|\) is, the longer all these algorithms take. For RBSim, it takes \(O(d_G|Q|G_Q))\) time to find \(G_Q = (V_G, E_G)\) (Section 4.1), and \(O(|Q||V_G|(|V_G| + |E_G|))\) time to find matches in \(G_Q\) (Section 1). Hence the larger \(Q\) is, the longer it takes; similarly for RBSub. Nonetheless, RBSim and RBSub are less sensitive to \(|Q|\) than MatchOpt and VF2Opt. (b) On average, RBSim and RBSub take 14.9% and 16.9% of running time of MatchOpt and VF2Opt, respectively. The improvement by our algorithms becomes more substantial for larger queries.

(2) Accuracy. Figures 8(g) and 8(h) report the accuracy results: (a) the larger \(|Q|\) is, the lower the accuracy is for
RBSim and RBSub. This is because bounded resources allow us to access at most $\alpha|G|$ amount of data regardless of $|Q|$. Nonetheless, the accuracy is above 86% for RBSim and above 80% for RBSub. Moreover, they achieve 100% for $Q$ as large as $(5, 10)$. In practice, $Q$ is typically small.

**Varying $|G|$: Efficiency and accuracy.** Fixing $|Q| = (4, 8)$ and $\alpha = 0.003\%$, we varied the node number $|V|$ of synthetic graphs from 2M to 10M, and set $|E| = 2|V|$. As shown in Fig. 8(i), (a) on average RBSim takes only 14.6% of the running time of Match and RBSub takes 13.8% of the time of VF2Opt. (b) Both RBSim and RBSub scale well with $|G|$, and are much less sensitive to the change of $|G|$.

As shown in Fig. 8(j), (a) in all cases, the accuracy is above 97% for RBSim and 94% for RBSub, and mostly 100%; and (b) the larger $|V|$ is, the more accurate the algorithms are, due to the locality of pattern queries and our search strategy.

**Exp-2: Reachability queries.** This set of experiment evaluated the performance of our algorithm RBRreach compared to BFS, BFSOpt and LM. We generated a set of 100 reachability queries, and report the average below. Following [13], we sampled $4 \times \log |V|$ landmarks for LM.

Varying $\alpha$: Efficiency and accuracy. Varying $\alpha$ from 0.01% to 0.1%, we report the response time of the algorithms on Youtube and Yahoo in Figures 8(k) and 8(l), respectively.

The results show the following. (a) RBRreach substantially outperforms BFS and BFSOpt in efficiency. It takes on average 1.6% and 17.4% of the running time of BFS and BFSOpt, respectively. (b) When $\alpha$ increases, the running time of RBRreach gets longer, as expected; but it is not very sensitive to $\alpha$. (c) RBRreach performs better than LM on Youtube. On Yahoo, LM does better when $\alpha > 0.07\%$. Nonetheless, as shown in Fig. 8(n), RBRreach achieves 100% accuracy on Youtube when $\alpha \leq 0.04\%$, when RBRreach is faster than LM.

Moreover, RBRreach is accurate. As shown in Figures 8(m) and 8(n), (a) in all cases, the accuracy is at least 96%, and is in general higher over denser graph Yahoo. (b) Moreover, when $\alpha \geq 0.05\%$, it is constantly 100% accurate! These verify that resource-bounded query answering is both efficient and accurate for non-localized reachability queries. The accuracy of LM, on the other hand, is from 69%-74%.

Varying $|V|$: Efficiency and accuracy. We varied $|V|$ of synthetic $G$ from 2M to 10M (where $|E| = 2|V|$), and set $\alpha$ as 0.02% and 0.01%. Figure 8(o) tells us that RBRreach scales well with $|G|$. (a) It is 58.8 and 5.2 times faster than BFS and BFSOpt, respectively. (b) It outperforms LM when $|V| \leq 5M$ for $\alpha = 0.02\%$; and is faster in all cases when $\alpha$ is small enough (e.g., 0.01\%), while the running time of LM is less sensitive to $|G|$ than RBRreach. The accuracy of RBRreach is above 97\% (resp. 94\%) for $\alpha = 0.02\%$ (resp. 0.01\%) in all.
cases (Fig. 8(p)). It increases with larger $G$, as the index $I$ covers slightly more node pairs (with $|I| \leq \alpha(G)$). In contrast, LM performs worse with larger $|V|$ as the number of landmarks sampled does not significantly increases.

**Summary.** We find the following. For patterns, (1) RBSim and RBS\textsubscript{Sub} are efficient: they are 5.5 times and 6.25 times faster than Match\textsubscript{opt} and VF2\textsubscript{opt}, respectively, on real-life graphs; (2) they are accurate: when $\alpha$ is as small as 0.0015%, both achieve 100% accuracy; and (3) they scale well with $|G|$, without much performance degradation when $G$ grows. The same holds on reachability queries: (4) RBReach is 62.5 and 5.7 times faster than BFS and BFS\textsubscript{opt} on average, respectively, on real-life graphs; while its efficiency is comparable to that of LM, it is more accurate: 96%-100% vs. 69%-74%; (5) it gives us mostly exact answers when $\alpha \geq 0.05\%$; and (6) it scales well with $|G|$: when $|G|$ increases, so does its accuracy, without much penalty in efficiency. Finally, the tunable performance (controlled by $\alpha$) of RBReach is more flexible than LM in balancing resource usage and accuracy.

**7. CONCLUSION.** We have proposed to query real-life graphs by resource-bounded query answering. We have studied its associated fundamental problems. We have also developed resource-bounded algorithms for answering localized (subgraph, simulation) and non-localized (reachability) queries. We have verified analytically and experimentally that these algorithms are able to efficiently find accurate approximate answers, even exact answers, with resource ratio $\alpha$ as small as 0.0015% for pattern queries, and 0.05% for reachability.

The study of resource-bounded query answering is still in its infancy. One topic is to explore resource-bounded algorithms for graph patterns without a personalized node. Another problem is to find, given a resource ratio $\alpha$, the maximum accuracy ratio $\eta$ that such algorithms can guarantee.

**Acknowledgment.** Fan, Wang and Wu are supported in part by 973 Programs 2011CB305032 and 2012CB316200, NSFC 61133002, Guangdong Innovative Research Team Program 2011ID0005, Shenzhen Peacock Program 11051000030834361, EPSRC EP/J015377/1 and NSHF III 1302212.

**8. REFERENCES**