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HIGGS BUNDLES ON WEIGHTED PROJECTIVE LINES AND LOOP CRYSTALS

GUILLAUME POUCHIN

ABSTRACT. We consider the space of nilpotent Higgs bundles on a weighted projective line, as a global analog of the nilpotent cone. We show that it is pure, compute its dimension, and define geometric correspondences between irreducible components. We prove it by constructing a loop analog of a cristal, in the spirit of Kashiwara and Saito, for some corresponding loop Kac-Moody algebra.

1. INTRODUCTION

The link between Kac-Moody algebras and geometry of quiver representation is known since the work of Ringel ([Ri]) and Lusztig ([L2]), and has led to the theory of canonical bases of quantum algebras. More recently, Kapranov and Bauman-Kassel ([BK]) have considered the Hall algebra of the category of coherent sheaves on the projective line and showed that some composition subalgebra is isomorphic to some positive part of the affine quantum algebra of sl_2 . Schiffmann ([Sc3]) generalized this situation to any weighted projective line as defined by Geigle and Lenzing in [GL], and showed that the Hall algebras obtained are affine versions of some quantum algebras. He used this setting to define canonical bases for some of these algebras. The category of coherent sheaves on weighted projective lines, or equivalently the category of bundles on \mathbb{P}^1 with a parabolic structure, also appears in the study of the Deligne-Simpson problem, see [CB1].

In the context of quivers, Lusztig considered a geometric construction of (enveloping) Kac-moody algebras via constructible functions on some nilpotent part Λ_Q of the cotangent bundle of the stack of representations of a quiver Q . He then defined semicanonical bases using that geometry, whose elements are indexed by irreducible components of Λ_Q . Kashiwara and Saito ([KS]) later considered natural correspondences between irreducible components to provide a geometric construction of the crystal associated to the semicanonical bases.

The aim of this article is to develop Kashiwara and Saito's ideas in the context of curves. We define the stack of nilpotent Higgs bundles on the stack $\underline{\Lambda}_{\mathbb{X}}$ of coherent sheaves on a weighted projective line \mathbb{X} , which was already considered in the case of the projective line in [La2] (see also [Gi], [Fa], [BD] for more general cases). We prove the following theorem:

Theorem 1.1. *For any α in the Grothendieck group $K^+(\text{Coh}_{\mathbb{X}})$, the stack $\underline{\Lambda}_{\mathbb{X}}^{\alpha}$ is pure of dimension $-\langle \alpha, \alpha \rangle$.*

Here the bracket $\langle \cdot, \cdot \rangle$ denotes the Euler form on the Grothendieck group.

We prove this by first constructing a stratification of this stack and then defining nice correspondences between strata which give rise to natural operators acting on the set irreducible components $\text{Irr}(\underline{\Lambda}_{\mathbb{X}})$, indexed by indecomposable rigid coherent sheaves on the curve. These operators have the same properties as the classical crystal operators:

Theorem 1.2. *For any $\alpha \in K^+(\text{Coh}_{\mathbb{X}})$ of the category $\text{Coh}_{\mathbb{X}}$ and any indecomposable rigid coherent sheaf $\mathcal{I} \in \underline{\text{Coh}}_{\mathbb{X}}$, there are operators:*

$$\begin{aligned} e_{\mathcal{I}} &: \text{Irr}(\underline{\Lambda}_{\mathbb{X}}^{\alpha}) \rightarrow \text{Irr}(\underline{\Lambda}_{\mathbb{X}}^{\alpha+[\mathcal{I}]}) \\ f_{\mathcal{I}} &: \text{Irr}(\underline{\Lambda}_{\mathbb{X}}^{\alpha}) \rightarrow \text{Irr}(\underline{\Lambda}_{\mathbb{X}}^{\alpha-[\mathcal{I}]} \cup \{0\}), \end{aligned}$$

and functions

$$wt : \text{Irr}(\underline{\Lambda}_{\mathbb{X}}) \rightarrow K^+(\underline{\text{Coh}}_{\mathbb{X}})$$

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and

$$\epsilon_{\mathcal{I}}, \phi_{\mathcal{I}} : \underline{\Delta}_{\mathbb{X}} \rightarrow \mathbb{Z}$$

with the following properties:

- (1) $wt(e_{\mathcal{I}}(Z)) = wt(Z) + \alpha_{\mathcal{I}}$ and $wt(f_{\mathcal{I}}(Z)) = wt(Z) - \alpha_{\mathcal{I}}$ if $f_{\mathcal{I}}(Z) \neq 0$,
- (2) $\epsilon_{\mathcal{I}}(e_{\mathcal{I}}(Z)) = \epsilon_{\mathcal{I}}(Z) + 1$ and $\phi_{\mathcal{I}}(e_{\mathcal{I}}(Z)) = \phi_{\mathcal{I}}(Z) - 1$,
- (3) $\epsilon_{\mathcal{I}}(f_{\mathcal{I}}(Z)) = \epsilon_{\mathcal{I}}(Z) - 1$ (resp. $\phi_{\mathcal{I}}(f_{\mathcal{I}}(Z)) = \phi_{\mathcal{I}}(Z) + 1$) if $\epsilon_{\mathcal{I}}(Z) \neq 0$, otherwise $f_{\mathcal{I}}(Z) = 0$,
- (4) $Z' = f_{\mathcal{I}}(Z)$ if and only if $e_{\mathcal{I}}(Z') = Z$.

Moreover for any two irreducible components Z and Z' in $\text{Irr}(\underline{\Delta}_{\mathbb{X}})$, there is a (finite) sequence of operators u_j of the form $e_{\mathcal{I}_j}$ or $f_{\mathcal{I}_j}$, for \mathcal{I}_j indecomposable rigid coherent sheaves, such that

$$Z = Z_0 \xrightarrow{u_1} Z_1 \xrightarrow{u_2} Z_2 \xrightarrow{u_3} \dots \xrightarrow{u_n} Z_n = Z'.$$

We define the *loop crystal* to be this new combinatorial data, as it is an analog of the crystal structure in the case of quivers constructed in [KS]. This is a first step towards a theory of crystals for loop Kac-Moody algebras, which, as in the usual Kac-Moody algebras case, is expected to carry very useful information about the representation theory of these algebras. The last property stated means that this loop crystal is connected. We will investigate the algebra of constructible functions on the space $\underline{\Delta}_{\mathbb{X}}$ in a forthcoming paper. It is worth noting that our methods should apply to the quiver case, and give operators indexed by positive roots, instead of just simple roots.

The organization of the paper is the following: in the first part we introduce the definitions and notation of the space of coherent sheaves on a weighted projective line. In the second part we explain the algebraic stack structure on the space of Higgs bundles on these curves. The third part is devoted to the first properties of this space, giving a decomposition of irreducible components into their locally free part and their torsion part, then describing the irreducible components on the torsion part using results of Lusztig. The fourth part is the main part, where we introduce a stratification of the space, then define correspondences between irreducible components labelled by indecomposable rigid vector bundles. As a consequence, we prove that the space $\underline{\Delta}_{\mathbb{X}}$ is Lagrangian. We also give some properties of the combinatorial data obtained, which we call *loop crystal*. Finally, we prove in the last part that the loop crystal is connected as a colored graph.

2. COHERENT SHEAVES ON WEIGHTED PROJECTIVE LINES

In this section we recall the definitions of weighted projectives lines and their categories of coherent sheaves.

2.1. Weighted projective lines. In this section we recall the definition of weighted projective lines, as introduced in [GL], as well as the properties we will use. We keep the notations of [Sc3]. Set $\mathbf{p} = (p_1, p_2, \dots, p_n)$ a tuple of n positive integers. We will consider that $n \geq 3$, adding some 1's if needed.

We define the following:

$$L(\mathbf{p}) = \oplus_{i=1}^n \mathbb{Z}\bar{x}_i / J,$$

to be the quotient of the free abelian group generated by elements \bar{x}_i , $i = 1 \dots n$, by the subgroup generated by the elements

$$p_i \bar{x}_i - p_1 \bar{x}_1$$

for every $i \geq 2$. Define $\bar{c} = p_1 \bar{x}_1 = p_i \bar{x}_i$. As $L(\mathbf{p})/\mathbb{Z}\bar{c} \simeq \oplus_i \mathbb{Z}/p_i \mathbb{Z}$, $L(\mathbf{p})$ is an abelian group of rank 1.

We also set $L^+(\mathbf{p}) = \{l_1 \bar{x}_1 + \dots + l_n \bar{x}_n \in L(\mathbf{p}) \mid l_i \geq 0\}$ and define $\bar{x} \geq \bar{y}$ if $\bar{x} - \bar{y} \in L^+(\mathbf{p})$. Define $S(\mathbf{p}) = \mathbb{C}[X_1, \dots, X_n]$ equipped with the $L(\mathbf{p})$ -graduation given by $\deg(X_i) = \bar{x}_i$. Chose n distinct points λ_i in $\mathbb{P}^1(\mathbb{C})$, with $\lambda_1 = 0$, $\lambda_2 = \infty$ and $\lambda_3 = 1$. Consider the $L(\mathbf{p})$ -homogeneous ideal I_λ generated by polynomials $X_i^{p_i} - (X_1^{p_1} - \lambda_i X_2^{p_2})$ for $i \geq 3$. We write $S(\mathbf{p}, \lambda) = S(\mathbf{p})/I_\lambda$, still graded by $L(\mathbf{p})$.

Define the weighted projective line:

$$\mathbb{X} = \mathbb{X}_{\mathbf{p}, \lambda} := \text{Specgr} S(\mathbf{p}, \lambda),$$

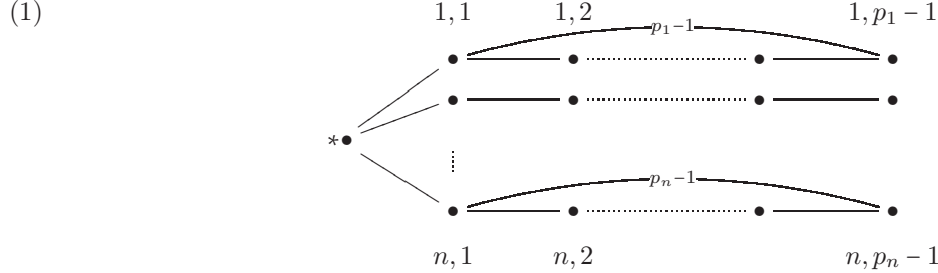
together with its Zariski topology. The closed points in \mathbb{X} are of two kinds:

- the ordinary points, corresponding to the ideal generated by $F(x_1^{p_1}, x_2^{p_2})$, where F is a prime homogeneous polynomial,
- the exceptional points $\lambda_1, \dots, \lambda_n$, corresponding to the ideals generated by x_i .

We set $p = \text{lcm}(p_1, \dots, p_n)$, and define the degree of a closed point by $\deg(x) = p$ if x is ordinary and $\deg(\lambda_i) = \frac{p}{p_i}$. We define the *genus* of \mathbb{X} to be

$$g_{\mathbb{X}} := 1 + \frac{1}{2} \left((n-2)p - \sum_{i=1}^n \frac{p}{p_i} \right).$$

We associate to this data the star-shaped Dynkin diagram with n branches of lengths $p_i - 1$:



As in the case of the projective line, we have a basis of the topology given by open subsets D_f , where f is any homogeneous element of $S(\underline{p}, \lambda)$, and:

$$D_f = \{\mathfrak{p} \in \mathbb{X} \mid f \in \mathfrak{p}\}$$

The structure sheaf $\mathcal{O}_{\mathbb{X}}$ of \mathbb{X} is defined as the sheaf associated to the presheaf, $D_f \mapsto S(\underline{p}, \lambda)_f$, where $S(\underline{p}, \lambda)_f$ is the usual localization. We write $\mathcal{O}_{\mathbb{X}}\text{-Mod}$ for the category of $\mathcal{O}_{\mathbb{X}}$ -modules on \mathbb{X} . There is also a description via Serre's theorem:

Theorem 2.1. (Serre) (see [GL]) *The category of $\mathcal{O}_{\mathbb{X}}$ is equivalent to the category of $L(\underline{p}, \lambda)$ -graded modules over $S(\underline{p}, \lambda)$ modulo finite length modules:*

$$\mathcal{O}_{\mathbb{X}} \simeq S(\underline{p}, \lambda) - \text{grmod} / S(\underline{p}, \lambda) - \text{grmod}_0,$$

where $S(\underline{p}, \lambda) - \text{grmod}_0$ are the $S(\underline{p}, \lambda)$ -modules of finite length. Thanks to this theorem, we can shift elements in this category by elements of $L(\underline{p}, \lambda)$. For any element $\vec{x} \in L(\underline{p}, \lambda)$, we denote by $\mathcal{O}_{\mathbb{X}}(\vec{x})$ the $\mathcal{O}_{\mathbb{X}}$ -module $\mathcal{O}_{\mathbb{X}}$ shifted by \vec{x} . Then for any $\mathcal{O}_{\mathbb{X}}$ -module \mathcal{M} , we denote by $\mathcal{M}(\vec{x})$ the tensor product $\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{X}}} \mathcal{O}_{\mathbb{X}}(\vec{x})$.

2.2. Coherent sheaves. The category $\text{Coh}_{\mathbb{X}}$ of coherent sheaves on \mathbb{X} is the category whose objects are $\mathcal{M} \in \mathcal{O}_{\mathbb{X}}\text{-Mod}$ with the following property: there exists an open covering $\{U_i\}$ of \mathbb{X} such that on each U_i there is an exact sequence of \mathcal{O}_{U_i} -modules

$$\bigoplus_{s=1}^N \mathcal{O}_{\mathbb{X}}(\vec{z}_s)|_{U_i} \rightarrow \bigoplus_{t=1}^M \mathcal{O}_{\mathbb{X}}(\vec{y}_t)|_{U_i} \rightarrow \mathcal{M}|_{U_i} \rightarrow 0$$

This category shares many properties with the category of coherent sheaves on \mathbb{P}^1 . We list the ones we will use later. We set $\vec{\omega} = (n-2)\vec{c} - \sum_{i=1}^n \vec{x}_i$.

Proposition 2.1. [GL] *We have the following properties:*

- (1) *The category $\underline{\text{Coh}}_{\mathbb{X}}$ is abelian and hereditary, i.e. for any $M, N \in \underline{\text{Coh}}_{\mathbb{X}}$, we have $\text{Ext}^i(M, N) = 0$ for $i > 1$,*
- (2) *For any $M, N \in \underline{\text{Coh}}_{\mathbb{X}}$, the spaces $\text{Hom}(M, N)$ and $\text{Ext}^1(M, N)$ are finite dimensional,*
- (3) *(Serre duality) For any $\mathcal{F}, \mathcal{G} \in \underline{\text{Coh}}_{\mathbb{X}}$ there is a canonical isomorphism:*

$$\text{Ext}^1(\mathcal{F}, \mathcal{G}) \simeq \text{Hom}(\mathcal{G}, \mathcal{F}(\vec{\omega}))^*$$

We denote by ξ the point of \mathbb{X} corresponding to the ideal (0) . The localisation $\mathcal{O}_{\mathbb{X}, \xi}$ is Morita equivalent to the field $\mathbb{C}(\frac{X_2^{p_2}}{X_1^{p_1}})$. For a coherent sheaf \mathcal{F} , the fiber $\mathcal{F}_{\xi} = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{X}}} \mathcal{O}_{\mathbb{X}, \xi}$ is a vector space over this field, and the dimension of this vector space is called the *rank* of the sheaf \mathcal{F} , denoted by $\text{rk}(\mathcal{F})$.

Sheaves of rank 0 are called *torsion sheaves*. For such a sheaf \mathcal{F} there are finitely many points $x \in \mathbb{X}$ where the fiber \mathcal{F}_x is non zero: the set of such points is called the support of the sheaf \mathcal{F} , denoted by $\text{supp}(\mathcal{F})$. Any torsion sheaf splits in a direct sum of sheaves with disjoint supports.

The category of sheaves supported at an ordinary point x is equivalent to the category of nilpotent representations of the loop quiver. We denote by $\mathcal{O}_x^{(d)}$ the unique indecomposable sheaf of degree $d\delta$, and any sheaf supported on x is isomorphic to a direct sum of sheaves of this type.

The category of sheaves supported at an exceptional point λ_i has p_i simples denoted by $S_j^{(i)}$. The indecomposable sheaves are of the form $S_j^{(i)}(l)$, for l a positive integer; $S_j^{(i)}(l)$ is uniquely determined by its head $S_j^{(i)}$ and its length l .

A sheaf \mathcal{F} such that there exists a covering $\{U_i\}$ with $\mathcal{F}|_{U_i}$ isomorphic to some $\bigoplus_{i=1}^N \mathcal{O}_{\mathbb{X}}(\vec{y}_i)|_{U_i}$ is called a *vector bundle*.

Each coherent sheaf \mathcal{F} has a (non-canonical) splitting $\mathcal{F} \simeq \mathcal{F}^{\text{vec}} \oplus \mathcal{F}^{\text{tor}}$, where \mathcal{F}^{vec} is a vector bundle and \mathcal{F}^{tor} is a torsion sheaf (see [GL]).

We denote by $K(\mathbb{X}) = K(\text{Coh}_{\mathbb{X}})$ the Grothendieck group of $\text{Coh}_{\mathbb{X}}$, and by $K^+(\text{Coh}_{\mathbb{X}})$ its positive semi-group, whose elements α are such that there exists a coherent sheaf \mathcal{F} with $[\mathcal{F}] = \alpha$ (where $[\mathcal{F}]$ denotes the class of \mathcal{F} in $K(\text{Coh}_{\mathbb{X}})$). The Grothendieck group $K(\text{Coh}_{\mathbb{X}})$ is equipped with the Euler form: if $\alpha, \beta \in K^+(\text{Coh}_{\mathbb{X}})$ then

$$\langle \alpha, \beta \rangle = \dim \text{Hom}(\mathcal{F}, \mathcal{G}) - \dim \text{Ext}^1(\mathcal{F}, \mathcal{G})$$

for any coherent sheaves \mathcal{F}, \mathcal{G} of respective classes α and β .

2.3. Loop Kac-Moody algebra. The graph (1) has an associated (symmetric) Cartan matrix, and an associated Kac-Moody algebra \mathfrak{g} (see [K]). We denote by Δ the root system of \mathfrak{g} , by Q its root lattice, and by $\alpha_{i,j}$, $1 \leq j < p_i$, α_* the simple roots corresponding to the vertices of the graph (1). Denote also by $\mathcal{L}\mathfrak{g}$ the loop Kac-Moody algebra, defined as in [Sc3] or [En], which is an extension of the algebra $\mathfrak{g}[t, t^{-1}]$. Its root system $\widehat{\Delta}$ and its root lattice \widehat{Q} are then:

$$\widehat{\Delta} = (\Delta + \mathbb{Z}\delta) \cup \mathbb{Z}^*\delta \text{ and } \widehat{Q} = Q + \mathbb{Z}\delta.$$

The positive part of the root lattice is:

$$\widehat{Q}^+ = \{k\alpha_* + \sum_{i,j} k_{i,j}\alpha_{(i,j)} \mid k \in \mathbb{N} - \{0\}, k_{i,j} \in \mathbb{Z}\} \cup \{\sum_{i,j} k_{i,j}\alpha_{i,j} \mid k_{i,j} \geq 0\}$$

We have the following, see for instance [Sc1] lemma 2.1 or [Sc3] 5.1:

Lemma 2.1. *There is a canonical isomorphism $h : K(\mathbb{X}) \simeq \widehat{Q}$, which maps $K^+(\mathbb{X})$ to the positive root lattice \widehat{Q}^+ . The explicit isomorphism is given by:*

$$\begin{array}{lll} h : & K(\mathbb{X}) & \rightarrow & \widehat{Q} \\ & [S_j^{(i)}] & \mapsto & \alpha_{i,j} \\ & [S_j^{(i)}(l)] & \mapsto & \sum_{k=0}^{l-1} \alpha_{i,j-k} \\ & \mathcal{O}_x & \mapsto & \delta & (\text{ordinary point}) \\ & \mathcal{O}_{\mathbb{X}} & \mapsto & \alpha_* \end{array}$$

Introduce the following set:

$$\Pi = \{\alpha_* + \sum_{i=1}^n \sum_{k=1}^{j_i} \alpha_{i,k} \mid 0 \leq j_i < p_i\}$$

Associated to an element t in Π , written $t = \alpha_* + \sum_{i=1}^n \sum_{k=1}^{j_i} \alpha_{i,k}$, we define the following vector bundle:

$$\mathcal{L}_t := \mathcal{O}_{\mathbb{X}}(\sum_{i=1}^n j_i \vec{x}_i).$$

3. THE STACK OF HIGGS BUNDLES ON \mathbb{X}

In this section we recall definitions and fix notation for the stack of Higgs bundles on a weighted projective line.

3.1. The algebraic stack of coherent sheaves. First we recall the stack structure on $\text{Coh}_{\mathbb{X}}$. We have a decomposition

$$\underline{\text{Coh}}_{\mathbb{X}} = \bigsqcup_{\alpha \in K^+(\text{Coh}_{\mathbb{X}})} \underline{\text{Coh}}_{\mathbb{X}}^{\alpha},$$

where $\underline{\text{Coh}}_{\mathbb{X}}^{\alpha}$ is the stack classifying isomorphism classes of coherent sheaves of class α . This stack has a local presentation as follows (see [Gr],[Le] or [Sc1]).

Let $\alpha \in K^+(\text{Coh}_{\mathbb{X}})$ and $\mathcal{E} \in \underline{\text{Coh}}_{\mathbb{X}}$. Define the following functor from the category of schemes over \mathbb{C} to the category of sets: for Σ a scheme over \mathbb{C} ,

$\underline{\text{Hilb}}_{\mathcal{E},\alpha}(\Sigma) = \{\phi_{\Sigma} : \mathcal{E} \boxtimes \mathcal{O}_{\Sigma} \twoheadrightarrow \mathcal{F} \mid \mathcal{F} \text{ is a coherent } \Sigma\text{-flat sheaf, } \mathcal{F}_{\sigma} \text{ is of class } \alpha \text{ for all closed point } \sigma \in \Sigma\} / \sim$ where two such maps are equivalent if they have same kernels. This functor is representable by a projective scheme $\text{Hilb}_{\mathcal{E},\alpha}$ (see [Gr] when $\mathbb{X} = \mathbb{P}^1$, and [Sc1] in the general case of a weighted projective line).

Fix an integer $n \in \mathbb{Z}$ and define for $t \in \Pi$:

$$d_t(n, \alpha) = \langle [\mathcal{L}_t(n\bar{c})], \alpha \rangle \text{ and } \mathcal{E}_n^{\alpha} = \bigoplus_{t \in \Pi} \mathbb{C}^{d_t(n, \alpha)} \otimes \mathcal{L}_t(n\bar{c}) \text{ if all } d_t(n, \alpha) \geq 0.$$

A map ϕ_{Σ} induces for each closed point $\sigma \in \Sigma$ linear maps $\phi_{t*,\sigma} : \mathbb{C}^{d_t(n, \alpha)} \rightarrow \text{Hom}(\mathcal{L}_t(n\bar{c}), \mathcal{F}_{\sigma})$. Let us consider the subfunctor defined by:

$$\Sigma \mapsto \{(\phi_{\Sigma} : \mathcal{E}_n^{\alpha} \boxtimes \mathcal{O}_{\Sigma} \twoheadrightarrow \mathcal{F}) \in \underline{\text{Hilb}}_{\mathcal{E}_n^{\alpha}, \alpha}(\Sigma) \mid \forall \sigma \in \Sigma, \forall t \in \Pi, \phi_{t*,\sigma} : \mathbb{C}^{d_t(n, \alpha)} \simeq \text{Hom}(\mathcal{L}_t(n\bar{c}), \mathcal{F}_{\sigma})\} / \sim.$$

This subfunctor is representable by a smooth open quasiprojective subscheme Q_n^{α} of $\text{Hilb}_{\mathcal{E}_n^{\alpha}, \alpha}$ (see [Le]).

The group $G_n^{\alpha} = \prod_{t \in \Pi} \text{Aut}(\mathcal{L}_t(n\bar{c})^{d_t(n, \alpha)})$ acts naturally on $\text{Hilb}_{\mathcal{E}_n^{\alpha}, \alpha}$ and Q_n^{α} .

The quotient stacks $\underline{\text{Coh}}_{\mathbb{X}}^{\alpha, \geq n} := [Q_n^{\alpha}/G_n^{\alpha}]$ for $n \in \mathbb{Z}$ are open substacks of $\underline{\text{Coh}}_{\mathbb{X}}^{\alpha}$ which form an atlas, i.e.:

$$\underline{\text{Coh}}_{\mathbb{X}}^{\alpha} = \bigcup_{n \in \mathbb{Z}} \underline{\text{Coh}}_{\mathbb{X}}^{\alpha, \geq n} = \bigcup_{n \in \mathbb{Z}} [Q_n^{\alpha}/G_n^{\alpha}]$$

We also introduce the stack $\underline{\text{Bun}}_{\mathbb{X}} = \bigcup_{\alpha} \underline{\text{Bun}}_{\mathbb{X}}^{\alpha}$ of vector bundles on \mathbb{X} , which is an open substack of $\underline{\text{Coh}}_{\mathbb{X}}$. We have an atlas given by the open substacks

$$\underline{\text{Bun}}_{\mathbb{X}}^{\alpha, \geq n} = [U_n^{\alpha}/G_n^{\alpha}], \text{ where } U_n^{\alpha} = \{(\phi : \mathcal{E}_n^{\alpha} \twoheadrightarrow \mathcal{F}) \in Q_n^{\alpha} \mid \mathcal{F} \text{ is locally free}\} / \sim.$$

3.2. The stack of Higgs bundles. We want to describe the cotangent stack $T^*(\underline{\text{Coh}}_{\mathbb{P}^1}^{\alpha})$ by giving it an atlas obtained by symplectic reduction of the varieties $T^*(Q_n^{\alpha})$.

First recall that the tangent space at a point $\phi : \mathcal{E} \twoheadrightarrow \mathcal{F}$ of Q_n^{α} is canonically isomorphic to $\text{Hom}(\text{Ker } \phi, \mathcal{F})$ (see [Le]).

The group G_n^{α} acts on $T^*Q_n^{\alpha}$ in a Hamiltonian fashion. Let \mathfrak{g}_n^{α} be the Lie algebra of G_n^{α} . The corresponding moment map $\mu_n : T^*Q_n^{\alpha} \rightarrow (\mathfrak{g}_n^{\alpha})^*$ is described as follows: over a point $z = (\phi, f) \in T^*Q_n^{\alpha}$ with $\phi : \mathcal{E}_n^{\alpha} \twoheadrightarrow \mathcal{F}$ and $f \in \text{Hom}(\text{Ker } \phi, \mathcal{F})^*$, we have

$$\begin{aligned} \mu_n^{\alpha}(z) : \mathfrak{g}_n^{\alpha} &\rightarrow \mathbb{C} \\ g &\mapsto \langle f, g|_{\text{Ker } \phi} \rangle, \end{aligned}$$

where \mathfrak{g}_n^{α} is identified with $\bigoplus_{t \in \Pi} \text{Hom}(\mathcal{L}_t(n\bar{c})^{d_t(n, \alpha)}, \mathcal{F})$ by means of the isomorphisms $\phi_{t*} : \mathbb{C}^{d_t(n, \alpha)} \simeq \text{Hom}(\mathcal{L}_t(n\bar{c}), \mathcal{F})$.

We want to describe the subvariety $(\mu_n^{\alpha})^{-1}(0) \subseteq T^*Q_n^{\alpha}$. To do this, fix some point $\phi : \mathcal{E}_n^{\alpha} \twoheadrightarrow \mathcal{F}$ in Q_n^{α} and write the short exact sequence:

$$0 \rightarrow \text{Ker}(\phi) \rightarrow \mathcal{E}_n^{\alpha} \rightarrow \mathcal{F} \rightarrow 0.$$

Applying the functor $\text{Hom}(_, \mathcal{F})$ yields

$$0 \rightarrow \text{Hom}(\mathcal{F}, \mathcal{F}) \rightarrow \text{Hom}(\mathcal{E}_n^{\alpha}, \mathcal{F}) \rightarrow \text{Hom}(\text{Ker } \phi, \mathcal{F}) \rightarrow \text{Ext}^1(\mathcal{F}, \mathcal{F}) \rightarrow \text{Ext}^1(\mathcal{E}_n^{\alpha}, \mathcal{F}) \rightarrow \dots$$

Since ϕ belongs to Q_n^{α} , we have $\langle \mathcal{L}_t(n\bar{c}), \mathcal{F} \rangle = d_t(n, \alpha) = \dim \text{Hom}(\mathcal{L}_t(n\bar{c}), \mathcal{F})$, and so $\text{Ext}^1(\mathcal{E}_n^{\alpha}, \mathcal{F}) = 0$. By dualizing, we get :

$$0 \rightarrow \text{Ext}^1(\mathcal{F}, \mathcal{F})^* \rightarrow \text{Hom}(\text{Ker } \phi, \mathcal{F})^* \xrightarrow{a} \text{Hom}(\mathcal{E}_n^{\alpha}, \mathcal{F})^* \rightarrow \dots$$

One can easily check that the map a is precisely the moment map μ_n . So if (ϕ, f) is in $(\mu_n^{\alpha})^{-1}(0)$ then f defines a unique element in $\text{Ext}^1(\mathcal{F}, \mathcal{F})^*$, which, abusing notation, we still denote by f .

Serre duality gives a canonical isomorphism: $\text{Ext}^1(\mathcal{F}, \mathcal{F})^* \simeq \text{Hom}(\mathcal{F}, \mathcal{F}(\bar{\omega}))$, where we write $\mathcal{F}(\bar{\omega})$ for $\mathcal{F} \otimes$

$\mathcal{O}(\bar{\omega})$.

We finally have :

$$(\mu_n^\alpha)^{-1}(0) = \{(\phi : \mathcal{E}_n^\alpha \rightarrow \mathcal{F}, f) \in T^*Q_n^\alpha \mid f \in \text{Hom}(\mathcal{F}, \mathcal{F}(\bar{\omega}))\} / \sim$$

By symplectic reduction, the cotangent bundle stack of the quotient stack Q_n^α/G_n^α is the quotient $[(\mu_n^\alpha)^{-1}(0)/G_n^\alpha]$. This gives us an atlas of $T^*(\underline{\text{Coh}}_{\mathbb{X}}^\alpha)$:

$$T^*(\underline{\text{Coh}}_{\mathbb{X}}^\alpha) = \bigcup_{n \in \mathbb{Z}} [(\mu_n^\alpha)^{-1}(0)/G_n^\alpha].$$

We then can write elements of $T^*\underline{\text{Coh}}_{\mathbb{X}}^\alpha$ as follows:

$$T^*\underline{\text{Coh}}_{\mathbb{X}}^\alpha = \{(\mathcal{F}, f) \mid \mathcal{F} \in \underline{\text{Coh}}_{\mathbb{X}}^\alpha, f \in \text{Hom}(\mathcal{F}, \mathcal{F}(\bar{\omega}))\}.$$

3.3. The global nilpotent cone. Let us now introduce the nilpotent part of the cotangent bundle:

$$S_n^\alpha := (\mu_n^\alpha)^{-1}(0)^{\text{nilp}} = \{(\phi : \mathcal{E}_n^\alpha \rightarrow \mathcal{F}, f) \in \mu_n^{-1}(0) \mid f \text{ nilpotent}\}$$

where we say that f is *nilpotent* if there exists m such that

$$f((m-1)\bar{\omega}) \circ \dots \circ f(\bar{\omega}) \circ f = 0$$

as an element of $\text{Hom}(\mathcal{F}, \mathcal{F}(m\bar{\omega}))$.

The quotient stacks $\underline{\Delta}_{\mathbb{X}}^{\alpha, \geq n} = [S_n^\alpha/G_n^\alpha]$ are closed substacks of $T^*\underline{\text{Coh}}_{\mathbb{X}}^{\alpha, \geq n}$, and form a compatible family with respect to the inductive system $T^*\underline{\text{Coh}}_{\mathbb{X}}^{\alpha, \geq n}$. They give rise in the limit to a closed substack

$$\underline{\Delta}_{\mathbb{X}}^\alpha = \varinjlim [S_n^\alpha/G_n^\alpha] = \bigcup_{n \in \mathbb{Z}} [S_n^\alpha/G_n^\alpha] \subseteq T^*\underline{\text{Coh}}_{\mathbb{X}}^\alpha.$$

We will then write elements of the global nilpotent cone as follows:

$$\underline{\Delta}_{\mathbb{X}}^\alpha = \{(\mathcal{F}, f) \mid \mathcal{F} \in \underline{\text{Coh}}_{\mathbb{X}}^\alpha, f \in \text{Hom}(\mathcal{F}, \mathcal{F}(\bar{\omega})), f \text{ is nilpotent}\}.$$

4. FIRST PROPERTIES OF IRREDUCIBLE COMPONENTS OF $\underline{\Delta}_{\mathbb{X}}$

In this section we describe how the classification of irreducible of the global nilpotent cone splits into its locally free part and its torsion part, and then we describe the irreducible component of the torsion part.

4.1. Vector bundles and torsion sheaves. We first separate the study of irreducible components $\underline{\Delta}_{\mathbb{X}}$ into their torsion and locally free parts. For an element $\mathcal{F} \in \underline{\text{Coh}}_{\mathbb{X}}$ we denote by \mathcal{F}^{tor} its torsion part. Let $\alpha, \beta \in K^+(\underline{\text{Coh}}_{\mathbb{X}})$, with $\text{rk}(\beta) = 0$, and define

$$\underline{\Delta}_{\mathbb{X}}^{\alpha, \beta} = \{(\mathcal{F}, f) \in \underline{\Delta}_{\mathbb{X}}^\alpha \mid [\mathcal{F}^{\text{tor}}] = \beta\}.$$

We also define the stack parametrizing isomorphism classes of objects:

$$\underline{L}^{\alpha, \beta} = \left\{ (\mathcal{V}, \tau, f_1, f_2, f_3) \mid \begin{array}{l} \mathcal{V} \in \underline{\text{Coh}}_{\mathbb{X}}^{\alpha-\beta, 0}, \tau \in \underline{\text{Coh}}_{\mathbb{X}}^\beta, f_1 \in \text{Hom}(\mathcal{V}, \mathcal{V}(\bar{\omega}))^{\text{nilpotent}} \\ f_2 \in \text{Hom}(\tau, \tau(\bar{\omega}))^{\text{nilpotent}}, f_3 \in \text{Hom}(\mathcal{V}, \tau(\bar{\omega})) \end{array} \right\},$$

an isomorphism between two objects $(\mathcal{V}, \tau, f_1, f_2, f_3)$ and $(\mathcal{V}', \tau', f'_1, f'_2, f'_3)$ being a couple of isomorphisms (ψ_1, ψ_2) where $\psi_1 : \mathcal{V} \simeq \mathcal{V}'$ and $\psi_2 : \tau \simeq \tau'$ such that the following diagrams commute:

$$\begin{array}{ccccc} \mathcal{V} & \xrightarrow{\psi_1} & \mathcal{V}' & & \tau & \xrightarrow{\psi_2} & \tau' & & \mathcal{V} & \xrightarrow{\psi_1} & \mathcal{V}' \\ \downarrow f_1 & & \downarrow f'_1 & & \downarrow f_2 & & \downarrow f'_2 & & \downarrow f_3 & & \downarrow f'_3 \\ \mathcal{V}(\bar{\omega}) & \xrightarrow{\psi_1(\bar{\omega})} & \mathcal{V}'(\bar{\omega}) & & \tau(\bar{\omega}) & \xrightarrow{\psi_2(\bar{\omega})} & \tau'(\bar{\omega}) & & \tau & \xrightarrow{\psi_2} & \tau' \end{array}$$

We have a natural diagram:

$$(2) \quad \begin{array}{ccc} & \underline{L}^{\alpha, \beta} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \underline{\Delta}_{\mathbb{X}}^{\alpha, \beta} & & \underline{\Delta}_{\mathbb{X}}^{\alpha-\beta, 0} \times \underline{\Delta}_{\mathbb{X}}^\beta \end{array}$$

where π_1 is defined from the functor of groupoids:

$$\begin{aligned} \pi_1 : (\mathcal{V}, \tau, f_1, f_2, f_3) &\mapsto \left(\mathcal{V} \oplus \tau, \begin{pmatrix} f_1 & 0 \\ f_3 & f_2 \end{pmatrix} \right) \\ (\psi_1, \psi_2) &\mapsto \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix} \end{aligned}$$

and π_2 is defined from the functor:

$$\begin{aligned} \pi_2 : (\mathcal{V}, \tau, f_1, f_2, f_3) &\mapsto ((\mathcal{V}, f_1), (\tau, f_2)) \\ (\psi_1, \psi_2) &\mapsto (\psi_1, \psi_2) \end{aligned}$$

Lemma 4.1. *The map π_1 is an affine fibration and π_2 is a vector bundle. Both π_1 and π_2 are of relative dimension $\langle \alpha - \beta, \beta \rangle$ and with connected fibers. This induces a bijection between irreducible components:*

$$\text{Irr}(\underline{\Delta}_{\mathbb{X}}^{\alpha, \beta}) \leftrightarrow \text{Irr}(\underline{\Delta}_{\mathbb{X}}^{\alpha - \beta, 0}) \times \text{Irr}(\underline{\Delta}_{\mathbb{X}}^{\beta}).$$

Moreover this correspondence preserves dimensions, i.e. if we have $Z \leftrightarrow Z_1 \times Z_2$ under this correspondence, then $\dim Z = \dim Z_1 + \dim Z_2$.

Proof. The result is obvious for π_2 since $\dim \text{Hom}(\mathcal{V}, \tau) = \langle \alpha - \beta, \beta \rangle$.

We introduce the following natural stack parametrizing objects:

$$\underline{\mathcal{S}}^{\alpha, \beta} = \{(\mathcal{F}, f) \mid \mathcal{F} \in \underline{\text{Coh}}_{\mathbb{X}}^{\alpha, \beta}, f \in \text{Hom}(\mathcal{F}, \mathcal{F}(\tilde{\omega})), f|_{\mathcal{F}^{\text{tor}}} = 0, f(\mathcal{F}) \subseteq \mathcal{F}^{\text{tor}}(\tilde{\omega})\}$$

and a morphism ψ between objects (\mathcal{F}, f) and (\mathcal{F}', f') is an isomorphism $\psi : \mathcal{F} \simeq \mathcal{F}'$ such that the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\psi} & \mathcal{F}' \\ \downarrow f & & \downarrow f' \\ \mathcal{F}(\tilde{\omega}) & \xrightarrow{\psi(\tilde{\omega})} & \mathcal{F}'(\tilde{\omega}) \end{array}$$

is commutative.

We have a natural map $\pi : \underline{\mathcal{S}}^{\alpha, \beta} \rightarrow \underline{\text{Coh}}_{\mathbb{X}}^{\alpha, \beta}$, which makes $\underline{\mathcal{S}}^{\alpha, \beta}$ a vector bundle over $\underline{\text{Coh}}_{\mathbb{X}}^{\alpha, \beta}$.

Define its pullback over $\underline{\Delta}_{\mathbb{X}}^{\alpha, \beta}$:

$$\underline{\tilde{\mathcal{S}}}^{\alpha, \beta} = \underline{\mathcal{S}}^{\alpha, \beta} \times_{\underline{\text{Coh}}_{\mathbb{X}}^{\alpha, \beta}} \underline{\Delta}_{\mathbb{X}}^{\alpha, \beta}$$

This is a vector bundle over $\underline{\Delta}_{\mathbb{X}}^{\alpha, \beta}$ of rank $\langle \alpha - \beta, \beta \rangle$.

We have a natural action of $\underline{\tilde{\mathcal{S}}}^{\alpha, \beta}$ on $\underline{L}^{\alpha, \beta}$ over $\underline{\Delta}_{\mathbb{X}}^{\alpha, \beta}$ defined as follows. Take a point $P = (\mathcal{V}, \tau, f_1, f_2, f_3) \in \underline{\Delta}_{\mathbb{X}}^{\alpha, \beta}$.

The fiber of $\underline{\tilde{\mathcal{S}}}^{\alpha, \beta}$ over $\pi_1(P)$ is by construction canonically identified with

$$\{g \in \text{End}(\mathcal{V} \oplus \tau) \mid g(\tau) = 0, g(\mathcal{V} \oplus \tau) \subseteq \tau\}.$$

We define the action of such elements g as follows:

$$g.P = (\mathcal{V}, \tau, f_1, f_2, f_3 - gf_1 + f_2g);$$

this corresponds to the action of $(Id + g)$ by conjugation on $\begin{pmatrix} f_1 & 0 \\ f_3 & f_2 \end{pmatrix}$.

As we have $\text{Aut}(\mathcal{V} \oplus \tau) = (\text{Aut}(\mathcal{V}) \times \text{Aut}(\tau)) \rtimes \text{Hom}(\mathcal{V}, \tau)$, we can identify $\underline{\Delta}_{\mathbb{X}}^{\alpha, \beta}$ as the quotient of $\underline{L}^{\alpha, \beta}$ by the action of $\underline{\tilde{\mathcal{S}}}^{\alpha, \beta}$.

◇

4.2. The cyclic quiver \mathcal{C}_p . In order to study the irreducible components of the torsion part, we need to introduce the nilpotent variety for the space of representations of the cyclic quiver \mathcal{C}_p , where p is a positive integer with $p \geq 2$. The cyclic quiver \mathcal{C}_p is the quiver with vertices indexed by $i \in \mathbb{Z}/p\mathbb{Z}$, and arrows $\phi_i : i \rightarrow i+1$, for $i \in \mathbb{Z}/p\mathbb{Z}$. A nilpotent representation of \mathcal{C}_p is the following data:

- a set of finite dimensional \mathbb{C} -vector spaces $(V_i)_{i \in \mathbb{Z}/p\mathbb{Z}}$ indexed by the vertices,
- for any $i \in \mathbb{Z}/p\mathbb{Z}$, a \mathbb{C} -linear map $\phi_i : V_i \rightarrow V_{i+1}$,

- there exists a positive integer k such that for any $i \in \mathbb{Z}/p\mathbb{Z}$, we have

$$\phi_{i+k} \circ \phi_{i+k-1} \circ \cdots \circ \phi_{i+1} \circ \phi_i = 0,$$

as an element of $\text{Hom}(V_i, V_{i+k+1})$.

Denote by $\text{RepNilp}(\mathcal{C}_p)$ the stack of isomorphism classes of nilpotent representations of \mathcal{C}_p , and by (V, ϕ) a representation of \mathcal{C}_p . The space of nilpotent representations of the cyclic quiver forms an abelian category and its Grothendieck group is isomorphic to \mathbb{Z}^p . The class of a representation in the Grothendieck group is given by its vector dimension $(\dim V_i)_{i \in \mathbb{Z}/p\mathbb{Z}}$, so the semi-Grothendieck group $K^+(\text{Rep}(\mathcal{C}_p))$ is isomorphic to \mathbb{N}^p . The Grothendieck group is equipped with the Euler form.

We have the decomposition:

$$\text{RepNilp}(\mathcal{C}_p) = \bigsqcup_{\alpha \in \mathbb{N}^p} \text{RepNilp}^\alpha(\mathcal{C}_p),$$

where $\text{RepNilp}^\alpha(\mathcal{C}_p)$ is the space of nilpotent representations of dimension vector α .

For any $i \in \mathbb{Z}/p\mathbb{Z}$ and $l \in \mathbb{N}$, define the *cyclic multisegment* $[i; l]$ to be the image of the projection to $\mathbb{Z}/p\mathbb{Z}$ of the segment $[i' - (l-1), i']$ for any $i' \in \mathbb{Z}$, $i' \equiv i \pmod{p}$. A cyclic multisegment is a finite linear combination $\mathbf{m} = \sum_{i,l} a_{i,l} [i; l]$ with $a_{i,l} \in \mathbb{N}$. The objects of the category $\text{RepNilp}(\mathcal{C}_p)$ are in natural correspondence with cyclic multisegments. Under this correspondence, the simple representation with dimension 1 at the vertex i and 0 elsewhere is mapped to $[i; 1]$, and the indecomposable representation of length l and head S_i is mapped to the cyclic segment $[i; l]$. We will also say that a cyclic multisegment $\mathbf{m} = \sum_{i,l} a_{i,l} [i; l]$ is *aperiodic* if for any $l \in \mathbb{N}$ there exists at least one i such that $a_{i,l}$ is zero.

Introduce the doubled quiver $\bar{\mathcal{C}}_p$: it is obtained from \mathcal{C}_p by adding an arrow from i to $i-1$ for any $i \in \mathbb{Z}/p\mathbb{Z}$. A representation of the doubled quiver $\bar{\mathcal{C}}_p$ will be denoted by $(V, \phi, \bar{\phi})$, where (V, ϕ) is a representation of \mathcal{C}_p and $(\bar{\phi}_i)_{i \in \mathbb{Z}/p\mathbb{Z}}$ are maps

$$\bar{\phi}_i : V_i \rightarrow V_{i-1}.$$

Fix a dimension vector α , and introduce the cotangent stack, which is obtained by symplectic reduction:

$$T^* \text{RepNilp}(\mathcal{C}_p)^\alpha := \{(V, \phi, \bar{\phi}) \in \text{Rep}(\bar{\mathcal{C}}_p)^\alpha \mid [\phi, \bar{\phi}] = 0\},$$

where $[\phi, \phi'] = ([\phi, \phi']_i)_{i \in \mathbb{Z}/p\mathbb{Z}}$, and $[\phi, \phi']_i = \phi'_{i+1} \phi_i - \phi_{i-1} \phi'_i \in \text{End}(V_i)$.

The *nilpotent variety*, introduced by Lusztig ([L1],[L5]), is defined as:

$$\underline{\Delta}_{\mathcal{C}_p}^\alpha := \{(V, \phi, \bar{\phi}) \in T^* \text{Rep}(\mathcal{C}_p)^\alpha, (V, \phi, \bar{\phi}) \text{ is nilpotent}\}.$$

Theorem 4.1. (Lusztig, [L2],[L3],[L5])

The stack $\underline{\Delta}_{\mathcal{C}_p}^\alpha$ is pure of dimension $-\langle \alpha, \alpha \rangle$.

Remark also that the space $\underline{\Delta}_{\mathcal{C}_p}$ has a rotation automorphism \mathbf{r} : $\mathbf{r}((V_i, \phi_i, \bar{\phi}_i)_i) = (V_{i+1}, \phi_{i+1}, \bar{\phi}_{i+1})$.

Theorem 4.2. ([LTV],[KS])

The irreducible components of the space $\underline{\Delta}_{\mathcal{C}_p} = \bigsqcup_{\alpha \in \mathbb{N}^p} \underline{\Delta}_{\mathcal{C}_p}^\alpha$ are in natural bijection with the aperiodic cyclic multisegments.

4.3. Ordinary torsion sheaves. Fix $\alpha \in K^+(\mathbb{X})$ such that $\text{rank}(\alpha) = 0$.

We first study the case of the projective line without any weight. Define, for a positive integer d , $\mathcal{P}(d)$ to be the set of *partitions* of d , i.e. $\nu \in \mathcal{P}(d)$ is a set of non-negative integers (ν_1, ν_2, \dots) such that $\sum_i \nu_i = d$ and $\nu_{i+1} \leq \nu_i$. The *length* $l(\nu)$ of a partition ν is the biggest integer l such that $\nu_l \neq 0$. For a partition ν of d define

$$\mathcal{O}_{\mathbb{P}^1}^\nu := \{\mathcal{O}_{x_1}^{(\nu_1)} \oplus \cdots \oplus \mathcal{O}_{x_d}^{(\nu_d)} \mid x_i \text{ distinct points}\},$$

where $\mathcal{O}_x^{(d)}$ is the indecomposable torsion sheaf supported on x of degree d . This is a smooth strata of $\underline{\text{Coh}}_{\mathbb{P}^1}^{d\delta}$. Let $T_{\mathcal{O}_{\mathbb{P}^1}, \nu}^* \underline{\text{Coh}}_{\mathbb{P}^1}^{d\delta}$ be the conormal bundle to this strata.

Lemma 4.2 ([La1], theorem 3.3.13). We have the following decomposition into irreducible components:

$$\underline{\Delta}_{\mathbb{P}^1}^{d\delta} = \bigcup_{\nu \in \mathcal{P}(d)} T_{\mathcal{O}_{\mathbb{P}^1}, \nu}^* \overline{\underline{\text{Coh}}_{\mathbb{P}^1}^{d\delta}}$$

Each irreducible component has dimension 0.

Now we go back to the case of an arbitrary weighted projective line \mathbb{X} and we define for any positive integer k the stack of torsion Higgs bundles of ordinary support:

$$\underline{\Delta}_{\mathbb{X},\text{ord}}^{k\delta} := \{(\mathcal{F}, f) \in \underline{\Delta}_{\mathbb{X}}^{d\delta} \mid \forall i = 1, \dots, n, x_i \notin \text{Supp}(\mathcal{F})\}$$

Now remark that by definition the sheaf $\mathcal{O}_{\mathbb{X}}$ restricted to the open subset $\mathbb{X} - \{\lambda_i\}_i$ is obviously the same as the sheaf $\mathcal{O}_{\mathbb{P}^1}$ restricted to the open subset $\mathbb{P}^1 - \{\lambda_i\}_i$. Then the stacks of torsion sheaves on $\mathbb{X} - \{\lambda_i\}_i$ and $\mathbb{P}^1 - \{\lambda_i\}_i$ are isomorphic. The stack of torsion sheaves on $\mathbb{P}^1 - \{\lambda_i\}_i$ is easily seen to be a dense open substack of the stack of torsion sheaves on \mathbb{P}^1 . We then deduce the following:

Lemma 4.3. *The stack $\underline{\Delta}_{\mathbb{X},\text{ord}}^{k\delta}$ is pure of dimension 0. Its irreducible components are indexed by partitions of k .*

4.4. Exceptional torsion sheaves. For an element $\alpha^{(i)}$ such that $\alpha^{(i)} = \sum_{j=0}^{p_i-1} l_{(i,j)} \alpha_{i,j}$, define the stack of Higgs bundles supported on λ_i of class $\alpha^{(i)}$:

$$\underline{\Delta}_{\mathbb{X},\lambda_i}^{\alpha^{(i)}} = \{(\mathcal{F}, f) \in \underline{\Delta}_{\mathbb{X}}^{\alpha^{(i)}} \mid \text{supp}(\mathcal{F}) = \lambda_i\}$$

The category of coherent sheaves supported at x_i is equivalent to the category of nilpotent representations of the cyclic quiver \mathcal{C}_{p_i} (see [Sc2]). This correspondence maps indecomposable sheaves $S_j^{(i)}(l)$ to the indecomposable element $S_j(l)$. This equivalence induces an isomorphism of stacks, and so we also deduce an isomorphism between the cotangent stacks:

$$G: \begin{array}{ccc} \underline{\Delta}_{\mathbb{X},\lambda_i}^{\alpha_i^i} & \rightarrow & \Lambda_{\mathcal{C}_{p_i}}^{(l_{i,j})} \\ (\mathcal{F}, f) & \mapsto & (V, \phi, \phi'). \end{array}$$

It may be explicitly described as follows:

- the map $\mathcal{F} \rightarrow (V, \phi)$ is deduced from the equivalence $\underline{\text{Coh}}_{\mathbb{X},x_i}^{\alpha_i^i} \simeq \text{Reprnil}_{\mathcal{C}_{p_i}}^{\alpha_i^i}$.
- tensoring by $\bar{\omega}$ on $\underline{\text{Coh}}_{\mathbb{X},x_i}^{\alpha_i^i}$ corresponds via Φ to the diagram automorphism \mathbf{r} acting on $\text{Reprnil}_{\mathcal{C}_{p_i}}^{\alpha_i^i}$, i.e. $\mathbf{r}((V_i, \phi_k)_{k \in \mathbb{Z}/p_i\mathbb{Z}}) = (V_{k+1}, \phi_{k+1})_{k \in \mathbb{Z}/p_i\mathbb{Z}}$.
- the element $f \in \text{Hom}(\mathcal{F}, \mathcal{F}(\bar{\omega}))$ is mapped via Φ to an element $\phi' \in \text{Hom}(\phi, \mathbf{r}_*^{-1}\phi)$, so that if $\phi = (V_k, \phi_k : V_k \rightarrow V_{k+1})_{k \in \mathbb{Z}/p_i\mathbb{Z}}$, then the element ϕ' is some element:

$$\phi' = (\phi'_k : V_k \rightarrow V_{k-1})$$

which is nilpotent and commutes with ϕ , i.e. $\phi_{k-1}\phi'_k = \phi'_{k+1}\phi_k$ for $k \in \mathbb{Z}/p_i\mathbb{Z}$. So (ϕ, ϕ') is actually an element of:

$$\Lambda_{\mathcal{C}_{p_i}}^{(l_{i,j})} = \{(\phi, \phi') \in \text{Reprnil}(\bar{\mathcal{C}}_{p_i}^{\alpha_i^i}) \mid (\phi, \phi') \text{ nilpotent}, [\phi, \phi'] = 0\}.$$

Reciprocally, a nilpotent element $\phi' \in \text{Hom}(\phi, \mathbf{r}_*\phi)$ is easily seen to correspond to an element $f \in \text{Hom}_{\mathbb{C}}(\mathcal{F}, \mathcal{F}(\bar{\omega}))$, and the commutativity implies that it is in fact a morphism of $\mathcal{O}_{\mathbb{X}}$ -modules, which is nilpotent.

As a consequence, we have:

Lemma 4.4. *The stack $\underline{\Delta}_{\mathbb{X},\lambda_i}^{\alpha_i^i}$ is isomorphic to $\Lambda_{\mathcal{C}_{p_i}}^{(l_{i,j})}$. Each irreducible component is of dimension $-\langle \alpha_i^i, \alpha_i^i \rangle$, and they are indexed by aperiodic cyclic p_i -multisegments of dimension α_i^i .*

4.5. Irreducible components of the torsion part. In this subsection we prove that the space $\underline{\Delta}_{\mathbb{X}}^{\alpha}$, with $\text{rk}(\alpha) = 0$, is pure of dimension $-\langle \alpha, \alpha \rangle$, and give a description of the irreducible components. Define for an element $\alpha \in K^+(\underline{\text{Coh}}_{\mathbb{X}})$ such that $\text{rk}(\alpha) = 0$ the set

$$W(\alpha) = \{(l, (l_{i,j})_{i=1 \dots n, j \in \mathbb{Z}/p_i\mathbb{Z}}) \mid l\delta + \sum_{i,j} l_{i,j} \alpha_{i,j} = \alpha\}.$$

Any $\mathcal{F} \in \underline{\text{Coh}}_{\mathbb{X}}^{\alpha}$ is canonically decomposed as

$$\mathcal{F} = \mathcal{F}_{\delta} \oplus \bigoplus_{i=1}^n \mathcal{F}_i,$$

where $\text{supp}(\mathcal{F}_{\delta}) \subseteq \mathbb{X} - \{\lambda_i\}_{i=1 \dots n}$ and for any $i = 1, \dots, n, \text{supp}(\mathcal{F}_i) = \lambda_i$.

We introduce the following stratification of $\underline{\Lambda}_{\mathbb{X}}^{\alpha}$: for $w = (l, l_{(i,j)}) \in W(\alpha)$, define $\alpha_w^{(i)} := \sum_j l_{(i,j)} \alpha_{i,j}$ and

$$\underline{\Lambda}_{\mathbb{X}}^{\alpha,w} = \{(\mathcal{F}, f) \in \underline{\Lambda}_{\mathbb{X}}^{\alpha} \mid [\mathcal{F}_{\delta}] = l\delta, \forall i [\mathcal{F}_i] = \alpha_w^{(i)}\}$$

For such sheaves \mathcal{F} , the elements f split as a sum $f = f_{\delta} \oplus \bigoplus_i f_i$, where $f_{\delta} \in \text{Hom}(\mathcal{F}_{\delta}, \mathcal{F}_{\delta}(\bar{\omega}))$ and $f_i \in \text{Hom}(\mathcal{F}_i, \mathcal{F}_i(\bar{\omega}))$, all of which are nilpotent.

By definition we have

$$\underline{\Lambda}_{\mathbb{X}}^{\alpha,w} \simeq \underline{\Lambda}_{\mathbb{X},\text{ord}}^{l\delta} \times \prod_{i=1}^n \underline{\Lambda}_{\mathbb{X},\lambda_i}^{\alpha_w^{(i)}}, \quad \text{and} \quad \underline{\Lambda}_{\mathbb{X}}^{\alpha} = \bigsqcup_{w \in W(\alpha)} \underline{\Lambda}_{\mathbb{X}}^{\alpha,w}.$$

Combining the preceding lemmas, we obtain the following description of the irreducible component of the torsion part.

Theorem 4.3. *Let $\alpha \in K^+(\text{Coh}_{\mathbb{X}})$ such that $\text{rk}(\alpha) = 0$. We have a natural bijection:*

$$\text{Irr}(\underline{\Lambda}_{\mathbb{X}}^{\alpha}) \leftrightarrow \bigsqcup_{(l, l_{(i,j)}) \in W(\alpha)} \mathcal{P}(l) \times \text{Irr}(\Lambda_{\mathcal{C}_{p_i}}^{\alpha_i})$$

Each irreducible component is of dimension $-\langle \alpha, \alpha \rangle$.

5. THE LOOP CRYSTAL

In this section we define natural correspondences between irreducible components, indexed by indecomposable rigid coherent sheaves.

5.1. Geometric correspondences. Fix an indecomposable rigid element \mathcal{I} in the category $\text{Coh}_{\mathbb{X}}$. For a coherent sheaf \mathcal{F} on \mathbb{X} , define

$$\text{rk}_{\mathcal{I}}(\mathcal{F}) = \max\{i \in \mathbb{N} \mid \text{Inj}(\mathcal{I}^{\oplus i}, \mathcal{F}) \neq \emptyset\}$$

where $\text{Inj}(\mathcal{I}^{\oplus i}, \mathcal{F})$ is the set of injections in $\text{Hom}(\mathcal{I}^{\oplus i}, \mathcal{F})$.

Fix \mathcal{I} an indecomposable rigid element of $\text{Coh}_{\mathbb{X}}$, two non-negative integers s and n , an element $\alpha \in K^+(\text{Coh}_{\mathbb{X}})$ and define the following locally closed substacks:

$$\begin{aligned} \underline{\Lambda}_{\mathcal{I},s}^{\alpha} &:= \{(\mathcal{F}, f) \in \underline{\Lambda}_{\mathbb{X}}^{\alpha} \mid \text{rk}_{\mathcal{I}}(\text{Ker}(f)) = s\}, \\ \underline{\Lambda}_{\mathcal{I},n,s}^{\alpha} &= \{(\mathcal{F}, f) \in \underline{\Lambda}_{\mathbb{X}}^{\alpha} \mid \text{rk}_{\mathcal{I}}(\text{Ker}(f)) = s, \dim(\text{Hom}(\mathcal{I}, \text{Ker}(f))) = n\}. \end{aligned}$$

We have that:

$$\underline{\Lambda}_{\mathbb{X}}^{\alpha} = \bigcup_{s \geq 0} \underline{\Lambda}_{\mathcal{I},s}^{\alpha},$$

a finite union of constructible substacks, and

$$\underline{\Lambda}_{\mathbb{X}}^{\alpha} = \bigcup_{s,n \geq 0} \underline{\Lambda}_{\mathcal{I},n,s}^{\alpha},$$

a locally finite union of constructible substacks (i.e. the number of non-empty intersections of these strata with a substack of $\Lambda_{\mathbb{X}}$ of finite type is finite).

In the following write $\gamma = s[\mathcal{I}]$ and $\beta = \alpha - \gamma$, three elements of $K(\text{Coh}_{\mathbb{X}})$.

Define the following stack:

$$\underline{\mathcal{E}}_{\mathcal{I},n,s}^{\alpha} = \{(\mathcal{F}, f, i) \mid (\mathcal{F}, f) \in \underline{\Lambda}_{\mathcal{I},n,s}^{\alpha}, i \in \text{Inj}(\mathcal{I}^{\oplus s}, \text{Ker}(f))\}$$

representing the functor from the category of affine schemes to the category of groupoids:

$$\Sigma \mapsto \left\{ (\mathcal{F}, f, i) \mid \begin{array}{l} \mathcal{F} \text{ is a coherent } \Sigma\text{-flat sheaf on } \mathbb{X} \times \Sigma, f \in \text{Hom}(\mathcal{F}, \mathcal{F} \otimes \mathcal{O}_{\Sigma \times \mathbb{X}}(\bar{\omega})), i : \mathcal{O}_{\Sigma \times \mathbb{X}}^s \rightarrow \text{Ker } f, \\ \mathcal{F}_{\sigma} \text{ is of class } \alpha \text{ and } \text{rk}_{\mathcal{I}}(\text{Ker } f_{\sigma}) = s, i_{\sigma} : \mathcal{I}^s \hookrightarrow \text{Ker } f_{\sigma} \text{ for all closed point } \sigma \in \Sigma \end{array} \right\}$$

where a morphism ψ between objects (\mathcal{F}, f, i) and (\mathcal{F}', f', i') is an isomorphism $\psi : \mathcal{F} \simeq \mathcal{F}'$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\psi} & \mathcal{F}' \\ \downarrow f & & \downarrow f' \\ \mathcal{F}(\bar{\omega}) & \xrightarrow{\psi(\bar{\omega})} & \mathcal{F}'(\bar{\omega}) \end{array} \quad \begin{array}{ccc} \mathcal{I}_{\Sigma}^s & \xrightarrow{i} & \text{Ker } f \\ & \searrow i' & \downarrow \psi \\ & & \text{Ker } f' \end{array}$$

Now consider the following diagram:

(3)

$$\begin{array}{ccc} & \underline{\mathcal{E}}_{\mathcal{I},n,s}^\alpha & \\ p_1 \swarrow & & \searrow p_2 \\ \underline{\mathcal{A}}_{\mathcal{I},n,s}^\alpha & & \underline{\mathcal{A}}^\beta \end{array}$$

where the maps p_1 and p_2 are defined on objects as follows:

$$\begin{aligned} p_1 : (\mathcal{F}, f, i) &\mapsto (\mathcal{F}, f) \\ p_2 : (\mathcal{F}, f, i) &\mapsto (\mathcal{F}/i(\mathcal{I}^s), f|_{\mathcal{F}/i(\mathcal{I}^s)}), \end{aligned}$$

We begin with the following lemma:

Lemma 5.1. $p_2(\underline{\mathcal{E}}_{\mathcal{I},n,s}^\alpha) \subseteq \underline{\mathcal{A}}_{\mathcal{I},n-s,0}^\beta$.

Proof. Take an element (\mathcal{F}, i, f) in $\underline{\mathcal{E}}_{\mathcal{I},n,s}^\alpha$ and define $(\mathcal{G}, g) = p_2(\mathcal{F}, i, f)$. We have a short exact sequence:

$$(4) \quad 0 \longrightarrow \mathcal{I}^s \xrightarrow{i} \mathcal{F} \xrightarrow{j} \mathcal{G} \longrightarrow 0$$

As $\text{Ext}^1(\mathcal{I}, \mathcal{I}) = 0$, we also have

$$(5) \quad 0 \longrightarrow \text{Hom}(\mathcal{I}, \mathcal{I}^s) \xrightarrow{i'} \text{Hom}(\mathcal{I}, \mathcal{F}) \xrightarrow{j'} \text{Hom}(\mathcal{I}, \mathcal{G}) \longrightarrow 0$$

Now the following diagram commutes for any $a \in \text{Hom}(\mathcal{I}, \text{Ker } f)$

(6)

$$\begin{array}{ccccc} \mathcal{I} & \longrightarrow & \mathcal{G} & \xrightarrow{g} & \mathcal{G}(\bar{\omega}) \\ & \searrow a & \uparrow j & & \uparrow j \\ & & \mathcal{F} & \xrightarrow{f} & \mathcal{F}(\bar{\omega}), \end{array}$$

so we see that $f \circ a = 0$ implies $g \circ j'(a) = 0$, whence $j'(\text{Hom}(\mathcal{I}, \text{Ker } f)) \subseteq \text{Hom}(\mathcal{I}, \text{Ker } g)$. In the other way, if $g \circ j'(a) = 0$, then as the kernel of the morphism $\mathcal{F}(\bar{\omega}) \rightarrow \mathcal{G}(\bar{\omega})$ is $\mathcal{I}(\bar{\omega})^s$, we have that $\text{Im}(f \circ a) \subseteq \mathcal{I}(\bar{\omega})^s$. The morphism $f \circ a$ lies inside $\text{Hom}(\mathcal{I}, \mathcal{I}(\bar{\omega})^s) = (\text{Ext}^1(\mathcal{I}, \mathcal{I})^s)^* = 0$, so that $a \in \text{Hom}(\mathcal{I}, \text{Ker } f)$. We just proved that we have a short exact sequence:

$$0 \rightarrow \text{Hom}(\mathcal{I}, \mathcal{I}^s) \rightarrow \text{Hom}(\mathcal{I}, \text{Ker } f) \rightarrow \text{Hom}(\mathcal{I}, \text{Ker } g) \rightarrow 0$$

It follows that $\dim \text{Hom}(\mathcal{I}, \text{Ker } g) = n - s$.

Now we prove that there are no injections from \mathcal{I} into $\text{Ker } g$. Assuming that such an injection h exists, consider an element $h' \in \text{Hom}(\mathcal{I}, \text{Ker } f)$ such that $j'(h') = h$. Define $h'' = h' \oplus i \in \text{Hom}(\mathcal{I} \oplus \mathcal{I}^s, \text{Ker } f)$. From the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}^s & \longrightarrow & \mathcal{I}^{s+1} & \xrightarrow{pr} & \mathcal{I} \longrightarrow 0 \\ & & & & \downarrow h'' & & \downarrow h \\ & & & & \mathcal{F} & \xrightarrow{j} & \mathcal{G} \end{array},$$

we deduce that $pr(\text{Ker } h'') = 0$, i.e. $\text{Ker } h'' \subseteq \mathcal{I}^s$. But the restriction of h'' to \mathcal{I}^s is i , which is injective. We have thus proved that h'' is an injection from \mathcal{I}^{s+1} into $\text{Ker } f$, which is impossible since $\text{rk}_{\mathcal{I}}(\text{Ker } f) = s$. \diamond

The diagram (3) is then refined to the diagram:

(7)

$$\begin{array}{ccc} & \underline{\mathcal{E}}_{\mathcal{I},n,s}^\alpha & \\ p_1 \swarrow & & \searrow p_2 \\ \underline{\mathcal{A}}_{\mathcal{I},n,s}^\alpha & & \underline{\mathcal{A}}_{\mathcal{I},n-s,0}^\beta \end{array}$$

The main theorem of this section is the following:

Theorem 5.1. *We have a natural bijection between irreducible components of $\underline{\Lambda}_{\mathcal{I},n,s}^\alpha$ and irreducible components of $\underline{\Lambda}_{\mathcal{I},n-s,0}^\beta$. If $Z_1 \leftrightarrow Z_2$ under this correspondence then we have $\dim Z_1 = \dim Z_2 - \langle \beta, \gamma \rangle - \langle \gamma, \beta \rangle - \langle \gamma, \gamma \rangle$. We denote the two applications in the following way:*

$$\text{Irr}(\underline{\Lambda}_{\mathcal{I},n,s}^\alpha) \begin{array}{c} \xrightarrow{f_{\mathcal{I}}^{\max}} \\ \xleftarrow{e_{\mathcal{I}}^s} \end{array} \text{Irr}(\underline{\Lambda}_{\mathcal{I},n-s,0}^\beta)$$

We have to study the maps p_1 and p_2 . We start with p_2 .

Proposition 5.1. *The map p_2 is smooth with connected fibers of dimension $-\langle \beta, \gamma \rangle - \langle \gamma, \beta \rangle - \langle \gamma, \gamma \rangle$.*

Corollary 5.1. *The map p_2 induces a natural bijection between irreducible components of $\underline{\Lambda}_{\mathcal{I},n-s,0}^\beta$ and irreducible components of $\underline{\mathcal{X}}_{\mathcal{I},n,s}^\alpha$. Moreover, if $Z_2 \leftrightarrow Z_3$ under this correspondence we have $\dim Z_3 = \dim Z_2 - \langle \beta, \gamma \rangle - \langle \gamma, \beta \rangle - \langle \gamma, \gamma \rangle$.*

As usual, to prove proposition (5.1) we need to study locally the diagram (3). To do so, define the locally closed subvariety $S_{m,(\mathcal{I},n,s)}^\alpha$ of S_m^α by

$$S_{m,(\mathcal{I},n,s)}^\alpha = \{(\phi, \mathcal{F}, f) \in S_m^\alpha \mid (\mathcal{F}, f) \in \underline{\Lambda}_{\mathcal{I},n,s}^\alpha\}.$$

For an integer $m \ll 0$ such that \mathcal{I} and \mathcal{F} are generated in degree m set $d_1^{(t)} = \langle \mathcal{L}_t(m\bar{c}), \mathcal{I}^s \rangle (= \dim \text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{I}^s))$ for $t \in \Pi$, $d'^{(t)} = \langle \mathcal{L}_t(m\bar{c}), \mathcal{F} \rangle (= \dim \text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{F}))$ and $d_2^{(t)} = d'^{(t)} - d_1^{(t)}$. Define

$$E_{\mathcal{I},s,n}^{\alpha, \geq m} = \{(\phi, \mathcal{F}, f, i, h_1, h_2) \mid (\phi, \mathcal{F}, f) \in S_{m,(\mathcal{I},n,s)}^\alpha, i : \mathcal{I}^s \hookrightarrow \text{Ker } f, h_1 : \mathbb{C}^{d_1^{(t)}} \simeq \text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{I}^s), \\ h_2 : \mathbb{C}^{d_2^{(t)}} \simeq \text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{F}/i(\mathcal{I}^s))\}$$

The group $G = \prod_{t \in \Pi} GL_{d^{(t)}} \times GL_{d_1^{(t)}} \times GL_{d_2^{(t)}}$ acts naturally on $E_{\mathcal{I},s,n}^{\alpha, \geq m}$ and the quotient stack is $\underline{\mathcal{X}}_{\mathcal{I},n,s}^{\alpha, \geq m}$. Introduce

$$C = \{(V^{(t)}, a^{(t)}, b^{(t)})_{t \in \Pi} \mid V^{(t)} \subseteq \mathbb{C}^{d'^{(t)}}, a^{(t)} : V^{(t)} \simeq \mathbb{C}^{d_1^{(t)}}, b^{(t)} : \mathbb{C}^{d'^{(t)}}/V^{(t)} \simeq \mathbb{C}^{d_2^{(t)}}\}.$$

We define q_2 as follows:

$$q_2 : \begin{array}{ccc} E_{\mathcal{I},n,s}^{\alpha, \geq m} & \rightarrow & S_{m,(\mathcal{I},n-s,0)}^\beta \times S_{m,(\mathcal{I},s,s)}^\gamma \times C \\ (\phi, \mathcal{F}, f, i, h_1, h_2) & \mapsto & ((\psi_1, \mathcal{G}, g), (\psi_2, \mathcal{I}^s, 0), (V^{(t)}, a^{(t)}, b^{(t)})) \end{array}$$

where:

- (1) $\mathcal{G} := \mathcal{F}/i(\mathcal{I}^s)$,
- (2) $\psi_1 : \bigoplus_{t \in \Pi} \mathcal{L}(m\bar{c})^{d_2^{(t)}} \twoheadrightarrow \mathcal{G}$ is deduced from ϕ and $h_2 = (h_2^{(t)})_{t \in \Pi}$,
- (3) $\psi_2 : \bigoplus_{t \in \Pi} \mathcal{L}_t(m\bar{c})^{d_1^{(t)}} \twoheadrightarrow \mathcal{I}^s$ is deduced from ϕ and $h_1 = (h_1^{(t)})_{t \in \Pi}$,
- (4) $(V^{(t)}, a^{(t)}, b^{(t)}) \in Z$ is defined by $V^{(t)} = \phi_{t*}(\text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{I}^s)) \subseteq \mathbb{C}^{d_1^{(t)} + d_2^{(t)}} = \phi_{t*}(\text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{F}))$ (via i) and $a^{(t)}$ and $b^{(t)}$ are deduced from the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{I}^s) & \xrightarrow{i} & \text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{F}) & \longrightarrow & \text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{G}) \longrightarrow 0 \\ & & h_1^{(t)} \downarrow & & \phi_{t*} \downarrow & & h_2^{(t)} \downarrow \\ 0 & \longrightarrow & \mathbb{C}^{d_1^{(t)}} & \longrightarrow & \mathbb{C}^{d'^{(t)}} & \longrightarrow & \mathbb{C}^{d_2^{(t)}} \longrightarrow 0 \end{array}$$

Lemma 5.2. *The map q_2 is an affine fibration, with fibers of dimension $-\langle \gamma, \beta \rangle - \langle \beta, \gamma \rangle + d_1 d_2 + s(n-s)$.*

Proof. Let us fix some notation. We will write $\mathcal{O}_{\mathcal{I}} = \bigoplus_{t \in \Pi} \mathcal{L}_t(m\bar{c})^{d_1^{(t)}}$, $\mathcal{O}_{\mathcal{G}} = \bigoplus_{t \in \Pi} \mathcal{L}_t(m\bar{c})^{d_2^{(t)}}$, $\mathcal{O}_{\mathcal{F}} = \bigoplus_{t \in \Pi} \mathcal{L}_t(m\bar{c})^{d'^{(t)}}$.

For maps

$$\begin{cases} \psi_1 : \mathcal{O}_{\mathcal{G}} \twoheadrightarrow \mathcal{G} \\ \psi_2 : \mathcal{O}_{\mathcal{I}} \twoheadrightarrow \mathcal{I}^s \\ \phi : \mathcal{O}_{\mathcal{F}} \twoheadrightarrow \mathcal{F} \end{cases}$$

write $\mathcal{K}_{\mathcal{I}}$, $\mathcal{K}_{\mathcal{G}}$ and $\mathcal{K}_{\mathcal{F}}$ for the corresponding kernels. We denote $i_{\mathcal{I}} : \mathcal{K}_{\mathcal{I}} \hookrightarrow \mathcal{O}_{\mathcal{I}}$ and $i_{\mathcal{G}} : \mathcal{K}_{\mathcal{G}} \hookrightarrow \mathcal{O}_{\mathcal{G}}$ the corresponding injections.

Let us describe the morphism q_2 in terms of some diagrams. Elements in the space $E_{\mathcal{I},n,s}^{\alpha,\geq n}$ are in canonical bijection with commutative diagrams

$$(8) \quad \begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ \mathcal{O}_{\mathcal{I}} & \xrightarrow{\psi_2} & \mathcal{I}^s & \longrightarrow & 0 \\ & \downarrow a' & & \downarrow i & \\ \mathcal{O}_{\mathcal{F}} & \xrightarrow{\phi} & \mathcal{F} & \longrightarrow & 0 \\ & \downarrow b' & & \downarrow & \\ \mathcal{O}_{\mathcal{G}} & \xrightarrow{\psi_1} & \mathcal{G} & \longrightarrow & 0 \\ & \downarrow & & \downarrow & \\ & 0 & & 0 & \end{array}$$

together with a map $f \in \text{Hom}(\mathcal{F}, \mathcal{F}(\bar{\omega}))$ such that $i : \mathcal{I}^s \hookrightarrow \text{Ker } f$. Indeed, the maps ψ_2, ψ_1 are deduced from h_1, h_2 by the formulas:

$$\begin{aligned} \psi_2 &= \text{can} \circ h_1, \\ \psi_1 &= \text{can} \circ h_2 \end{aligned}$$

where can is the evaluation map and a', b' are defined uniquely in order to make (8) commute. Recall that in the construction of $\text{Hilb}_{\mathcal{E}_n^\alpha}$, two maps $\phi : \oplus_{t \in \Pi} \mathcal{L}_t(m)^{d^{(t)}} \rightarrow \mathcal{F}$, $\phi' : \oplus_{t \in \Pi} \mathcal{L}_t(m)^{d^{(t)}} \rightarrow \mathcal{F}'$ are equivalent if $\text{Ker } \phi = \text{Ker } \phi'$. We use the same equivalence relation for diagrams.

Similarly, points in $S_{m,(\mathcal{I},n-s,0)}^\beta \times S_{m,(\mathcal{I},s,s)}^\gamma \times C$ correspond bijectively to diagrams

$$(9) \quad \begin{array}{ccccc} & 0 & & & \\ & \downarrow & & & \\ \mathcal{O}_{\mathcal{I}} & \xrightarrow{\psi_2} & \mathcal{I}^s & \longrightarrow & 0 \\ & \downarrow a' & & & \\ \mathcal{O}_{\mathcal{F}} & & & & \\ & \downarrow b' & & & \\ \mathcal{O}_{\mathcal{G}} & \xrightarrow{\psi_1} & \mathcal{G} & \longrightarrow & 0 \\ & \downarrow & & & \\ & 0 & & & \end{array}$$

together with an element $g \in \text{Hom}(\mathcal{G}, \mathcal{G}(\bar{\omega}))$.

The horizontal sequences are the elements of $S_{m,(\mathcal{I},n-s,0)}^\beta$ and $S_{m,(\mathcal{I},s,s)}^\gamma$, and the vertical sequence is deduced from $(V^{(t)}, a^{(t)}, b^{(t)})$ via

$$\begin{array}{ccccccc} 0 & \longrightarrow & V^{(t)} & \longrightarrow & \mathbb{C}^{d_1^{(t)}+d_2^{(t)}} & \longrightarrow & \mathbb{C}^{d_1^{(t)}+d_2^{(t)}}/V^{(t)} \longrightarrow 0 \\ & & \downarrow a^{(t)} & & \parallel & & \downarrow b^{(t)} \\ 0 & \longrightarrow & \mathbb{C}^{d_1^{(t)}} & \longrightarrow & \mathbb{C}^{d_1^{(t)}+d_2^{(t)}} & \longrightarrow & \mathbb{C}^{d_2^{(t)}} \longrightarrow 0 \end{array}$$

and then tensoring by $\mathcal{L}_t(m\bar{c})$.

The map q_2 assigns to a diagram as in (8) its subdiagram (9) and the element g deduced from f . We may

complete the diagrams (8), (9) by adding kernels of ψ_2, ψ_1, ϕ :

$$(10) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{K}_{\mathcal{I}} & \xrightarrow{i_{\mathcal{I}}} & \mathcal{O}_{\mathcal{I}} & \xrightarrow{\psi_2} & \mathcal{I}^s \longrightarrow 0 \\ & & \downarrow & & \downarrow a' & & \downarrow i \\ 0 & \longrightarrow & \mathcal{K}_{\mathcal{F}} & \xrightarrow{i_{\mathcal{F}}} & \mathcal{O}_{\mathcal{F}} & \xrightarrow{\phi} & \mathcal{F} \longrightarrow 0 \\ & & \downarrow & & \downarrow b' & & \downarrow \\ 0 & \longrightarrow & \mathcal{K}_{\mathcal{G}} & \xrightarrow{i_{\mathcal{G}}} & \mathcal{O}_{\mathcal{G}} & \xrightarrow{\psi_1} & \mathcal{G} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

$$(11) \quad \begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{K}_{\mathcal{I}} & \xrightarrow{i_{\mathcal{I}}} & \mathcal{O}_{\mathcal{I}} & \xrightarrow{\psi_2} & \mathcal{I}^s \longrightarrow 0 \\ & & & & \downarrow a' & & \\ & & & & \mathcal{O}_{\mathcal{F}} & & \\ & & & & \downarrow b' & & \\ 0 & \longrightarrow & \mathcal{K}_{\mathcal{G}} & \xrightarrow{i_{\mathcal{G}}} & \mathcal{O}_{\mathcal{G}} & \xrightarrow{\psi_1} & \mathcal{G} \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Let us fix a point $x = (\psi_1, \mathcal{G}, g, \psi_2, \mathcal{I}^s, V, a, b) \in S_{m, (\mathcal{I}, n-s, 0)}^\beta \times S_{m, (\mathcal{I}, s, s)}^\gamma \times C$, and denote the fiber $F = q_2^{-1}(x)$. We will use the following lemma:

Lemma 5.3. *We have a (non-canonical) isomorphism:*

$$\mathcal{K}_{\mathcal{F}} \simeq \mathcal{K}_{\mathcal{G}} \oplus \mathcal{K}_{\mathcal{I}}$$

Moreover, any injective morphism $\mathcal{K}_{\mathcal{G}} \oplus \mathcal{K}_{\mathcal{I}} \hookrightarrow \mathcal{O}_{\mathcal{F}}$ such that:

- the restriction to $\mathcal{K}_{\mathcal{I}}$ is $i_{\mathcal{I}}$.
 - the induced morphism $\mathcal{K}_{\mathcal{G}} \rightarrow \mathcal{O}_{\mathcal{G}}$ is $i_{\mathcal{G}}$
- induces an element $\phi: \mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{O}_{\mathcal{F}}/\mathcal{K}_{\mathcal{G}} \oplus \mathcal{K}_{\mathcal{I}}$ in Q_n^α .

Proof. We first apply the functor $\text{Hom}(\mathcal{L}_t(n\bar{c}), _)$, where $t \in \Pi$, to the short exact sequence:

$$0 \rightarrow \mathcal{K}_{\mathcal{I}} \rightarrow \mathcal{O}_{\mathcal{I}} \rightarrow \mathcal{I}^s \rightarrow 0$$

in order to have the long exact sequence:

$$(12) \quad 0 \rightarrow \text{Hom}(\mathcal{L}_t(n\bar{c}), \mathcal{K}_{\mathcal{I}}) \rightarrow \text{Hom}(\mathcal{L}_t(n\bar{c}), \mathcal{O}_{\mathcal{I}}) \xrightarrow{\kappa} \text{Hom}(\mathcal{L}_t(n\bar{c}), \mathcal{I}^s) \rightarrow \text{Ext}^1(\mathcal{L}_t(n\bar{c}), \mathcal{K}_{\mathcal{I}}) \rightarrow 0$$

But the map κ restricted to $\text{Hom}(\mathcal{L}_t(n\bar{c}), \mathcal{L}_t(n\bar{c}))^{d_t(n, \alpha)} \simeq \mathbb{C}^{d_t(n, \alpha)}$ is exactly the map ξ_{t^*} , so the condition for ξ in Q_n^γ implies that κ is surjective, hence $\text{Ext}^1(\mathcal{L}_t(n\bar{c}), \mathcal{K}_{\mathcal{I}}) = 0$ for any t , which gives that $\text{Ext}^1(\mathcal{O}_{\mathcal{G}}, \mathcal{K}_{\mathcal{I}}) = 0$. Now we apply the functor $\text{Hom}(_, \mathcal{K}_{\mathcal{I}})$ to the short exact sequence:

$$0 \rightarrow \mathcal{K}_{\mathcal{G}} \rightarrow \mathcal{O}_{\mathcal{G}} \rightarrow \mathcal{G} \rightarrow 0$$

which gives a long exact sequence whose end is:

$$\text{Ext}^1(\mathcal{O}_{\mathcal{G}}, \mathcal{K}_{\mathcal{I}}) \rightarrow \text{Ext}^1(\mathcal{K}_{\mathcal{G}}, \mathcal{K}_{\mathcal{I}}) \rightarrow 0$$

which then gives $\text{Ext}^1(\mathcal{K}_{\mathcal{G}}, \mathcal{K}_{\mathcal{I}}) = 0$. We deduce that the short exact sequence

$$0 \rightarrow \mathcal{K}_{\mathcal{I}} \rightarrow \mathcal{K}_{\mathcal{F}} \rightarrow \mathcal{K}_{\mathcal{G}} \rightarrow 0$$

splits, as required.

For the second part of the lemma, take a morphism a as in the lemma. Its cokernel \mathcal{F} has obviously the right class in $K(\text{Coh}_{\mathbb{X}})$. It remains to prove that for any $t \in \Pi$, we have:

$$\phi_{t*} : \text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{F}) \simeq \mathbb{C}^{d^{(t)}}.$$

But as $\text{Ext}^1(\mathcal{L}_t(m\bar{c}), \mathcal{I}) = 0$, we have the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{I}) & \longrightarrow & \text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{F}) & \longrightarrow & \text{Hom}(\mathcal{L}_t(m\bar{c}), \mathcal{G}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow b^{(t)} \\ 0 & \longrightarrow & \mathbb{C}^{d_1^{(t)}} & \longrightarrow & \mathbb{C}^{d^{(t)}} & \longrightarrow & \mathbb{C}^{d_2^{(t)}} \longrightarrow 0 \end{array}$$

where the left and right vertical arrows are isomorphisms by hypothesis. The result follows. \diamond

By the preceding lemma, the set of classes of maps $\phi : \mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{F}$ making 8 commutative is in bijection with the set of subsheaves $\mathcal{K}_{\mathcal{F}} \subseteq \mathcal{O}_{\mathcal{F}}$ satisfying:

$$(13) \quad \begin{cases} \mathcal{K}_{\mathcal{F}} \cap a'(\mathcal{O}_{\mathcal{I}}) = a'(\mathcal{K}_{\mathcal{I}}) \\ b'(\mathcal{K}_{\mathcal{F}}) = \mathcal{K}_{\mathcal{G}} \end{cases}$$

Subsheaves $\mathcal{K}_{\mathcal{F}} \subseteq \mathcal{O}_{\mathcal{F}}$ satisfying 13 form a principal $\text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s)$ -space. Indeed

$$\{\mathcal{K}_{\mathcal{F}} \subseteq \mathcal{O}_{\mathcal{F}} \mid 13 \text{ satisfied}\} = \{\mathcal{K}'_{\mathcal{F}} \subseteq \mathcal{O}_{\mathcal{F}}/\mathcal{K}_{\mathcal{I}} \mid \mathcal{K}'_{\mathcal{F}} \cap \mathcal{I}^s = 0, b'(\mathcal{K}'_{\mathcal{F}}) = \mathcal{K}_{\mathcal{G}}\} = \{s : \mathcal{K}_{\mathcal{G}} \rightarrow \mathcal{O}_{\mathcal{F}}/\mathcal{K}_{\mathcal{I}} \mid b' \circ s = \text{Id}_{\mathcal{K}_{\mathcal{G}}}\}$$

and if s, s' are two sections $\mathcal{K}_{\mathcal{G}} \rightarrow \mathcal{O}_{\mathcal{F}}/\mathcal{K}_{\mathcal{I}}$ as above then the difference $s - s'$ is in $\text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{O}_{\mathcal{I}}/\mathcal{K}_{\mathcal{I}}) = \text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s)$.

For convenience, let us choose a section s_0 as above. This corresponds to an identification $\mathcal{O}_{\mathcal{F}}/\mathcal{K}_{\mathcal{I}} \simeq \mathcal{I}^s \oplus \mathcal{O}_{\mathcal{G}}$. Then to $u \in \text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s)$ we associate the diagram

$$(14) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{K}_{\mathcal{I}} & \xrightarrow{i_{\mathcal{I}}} & \mathcal{O}_{\mathcal{I}} & \xrightarrow{\psi_2} & \mathcal{I}^s \longrightarrow 0 \\ & & \downarrow & & \downarrow a' & & \downarrow i \\ 0 & \longrightarrow & \mathcal{K}_{\mathcal{F}} & \xrightarrow{i_{\mathcal{F}}} & \mathcal{O}_{\mathcal{F}} & \xrightarrow{\phi_u} & \mathcal{F} \longrightarrow 0 \\ & & \downarrow & & \downarrow b' & & \downarrow \\ 0 & \longrightarrow & \mathcal{K}_{\mathcal{G}} & \xrightarrow{i_{\mathcal{G}}} & \mathcal{O}_{\mathcal{G}} & \xrightarrow{\psi_1} & \mathcal{G} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Note that $\mathcal{F} \simeq \text{Coim}(\mathcal{K}_{\mathcal{G}} \xrightarrow{(i_{\mathcal{G}}, u)} \mathcal{O}_{\mathcal{G}} \oplus \mathcal{I}^s)$.

It remains to describe the possible choices for the map f in the fiber. Such an element verifies two conditions:

$$(*) \quad f|_{\mathcal{I}^s} = 0$$

$$(**) \quad f'|_{\mathcal{G}} = g, \text{ where } f' \in \text{Hom}(\mathcal{G}, \mathcal{G}(\bar{\omega})) \text{ is deduced from } f.$$

We have the short exact sequence derived from u :

$$0 \rightarrow \mathcal{I}^s \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$$

as the composition of the two dotted arrows and the map $\text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{G}) \rightarrow \text{Ext}^1(\mathcal{G}, \mathcal{G})$, which is exactly the dual of our claim.

So for $h \in \text{Hom}(\mathcal{I}^s, \mathcal{G})$, we have $\theta'_u(a_g)(h) = a_g(h \circ (i_{\mathcal{G}}, u))$. Then θ_u is obtained by evaluating a_g on the projection of $h \circ (i_{\mathcal{G}}, u)$ into $\text{Hom}(\mathcal{I}^s, \mathcal{G})$, i.e. $\theta_u(g)(h) = a_g(h \circ u)$.

◇

We can now consider a new linear map θ defined from θ_u , this time considering the dependence on u :

$$\begin{array}{ccc} \theta : \text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s) & \rightarrow & \text{Hom}(\mathcal{I}^s, \mathcal{G})^* \\ u & \mapsto & \theta_u(g) \end{array}$$

We have proved the following statement:

Lemma 5.5. *The fiber F is isomorphic to $\text{Ext}^1(\mathcal{I}^s, \mathcal{G})^* \oplus \text{Ker } \theta$.*

It remains to give the dimension of the fiber F . We need the following lemma:

Lemma 5.6. *We have $(\text{Im } \theta)^\perp = \text{Hom}(\mathcal{I}^s, \text{Ker } g)$.*

Proof. Take $h \in \text{Hom}(\mathcal{I}^s, \mathcal{G})$, and define $\mathcal{H} := \text{Im } h$. Now $h \in (\text{Im } \theta)^\perp$ is equivalent to:

$$\forall u \in \text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s), \quad a_g(h \circ u) = 0.$$

We have a natural map $\text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s) \xrightarrow{p} \text{Ext}^1(\mathcal{G}, \mathcal{I}^s)$, and by definition of a_g , we have $a_g(v) = g(p(v))$ for any $v \in \text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s)$.

$$\begin{array}{ccc} \text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s) & \xrightarrow{p} & \text{Ext}^1(\mathcal{G}, \mathcal{I}^s) \\ \downarrow a_g & & \downarrow g \\ \mathbb{C} & & \mathbb{C} \end{array}$$

From the surjection $h : \mathcal{I}^s \twoheadrightarrow \mathcal{H}$, we have a surjection $\text{Ext}^1(\mathcal{G}, \mathcal{I}^s) \twoheadrightarrow \text{Ext}^1(\mathcal{G}, \mathcal{H})$, and we can make use of the following commutative diagram:

$$\begin{array}{ccccc} \text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s) & \xrightarrow{h'_*} & \text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{H}) & & \\ \downarrow p' & & \downarrow p & & \\ \text{Ext}^1(\mathcal{G}, \mathcal{I}^s) & \xrightarrow{h_*} & \text{Ext}^1(\mathcal{G}, \mathcal{H}) & \longrightarrow & 0 \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

So that we have the following chain of equivalence:

$$\begin{aligned} h \in (\text{Im } \theta)^\perp &\Leftrightarrow \forall u \in \text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s), \quad a_g(h \circ u) = 0 = g(p(h \circ u)) \\ &\Leftrightarrow g(p(h'_*(\text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s)))) = 0 \\ &\Leftrightarrow g(h_*(p'(\text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s)))) = 0 \\ &\Leftrightarrow g(h_*(\text{Ext}^1(\mathcal{G}, \mathcal{I}^s))) = 0 \text{ (by surjectivity of } p') \\ &\Leftrightarrow g|_{\text{Ext}^1(\mathcal{G}, \mathcal{H})} = 0 \text{ (by surjectivity of } h_*). \end{aligned}$$

Now the restriction $g|_{\text{Ext}^1(\mathcal{G}, \mathcal{H})}$ is equal to the image of g by the morphism $\text{Ext}^1(\mathcal{G}, \mathcal{G})^* \rightarrow \text{Ext}^1(\mathcal{G}, \mathcal{H})^*$ which is just the restriction morphism, as we see from Serre duality:

$$\begin{array}{ccc} \text{Ext}^1(\mathcal{G}, \mathcal{G})^* & \longrightarrow & \text{Ext}^1(\mathcal{G}, \mathcal{H})^* \\ \parallel & & \parallel \\ \text{Hom}(\mathcal{G}, \mathcal{G}(\bar{\omega})) & \xrightarrow{|\mathcal{H}|} & \text{Hom}(\mathcal{H}, \mathcal{G}(\bar{\omega})) \end{array}$$

so that $g|_{\text{Ext}^1(\mathcal{G}, \mathcal{H})} = g|_{\mathcal{H}}$, this time considered as an element of $\text{Hom}(\mathcal{G}, \mathcal{G}(\bar{\omega}))$.

We have proved that $h \in (\text{Im } \theta)^\perp \Leftrightarrow g|_{\mathcal{H}} = 0$, which by definition is equivalent to $\mathcal{H} \subseteq \text{Ker } g$.

◇

Lemma 5.5 gives that q_2 is an affine fibration with connected fibers. Lemma 5.6 allows us to compute the dimension of the fiber. As $\dim \text{Hom}(\mathcal{I}^s, \text{Ker } g) = s \dim \text{Hom}(\mathcal{I}, \text{Ker } g) = s(n-s)$, we have:

$$\begin{aligned}
\dim q_2^{-1}(x) &= \dim \text{Ext}^1(\mathcal{I}^s, \mathcal{G}) + \dim \text{Ker } \theta \\
&= \dim \text{Ext}^1(\mathcal{I}^s, \mathcal{G}) + \dim \text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s) - (\dim \text{Hom}(\mathcal{I}^s, \mathcal{G}) - s(n-s)) \\
&= \dim \text{Ext}^1(\mathcal{I}^s, \mathcal{G}) - \dim \text{Hom}(\mathcal{I}^s, \mathcal{G}) + \dim \text{Hom}(\mathcal{K}_{\mathcal{G}}, \mathcal{I}^s) + s(n-s) \\
&= -\langle \mathcal{I}^s, \mathcal{G} \rangle + \langle \mathcal{K}_{\mathcal{G}}, \mathcal{I}^s \rangle + s(n-s) \\
&= -\langle \mathcal{I}^s, \mathcal{G} \rangle + \langle \mathcal{O}_{\mathcal{G}}, \mathcal{I}^s \rangle - \langle \mathcal{G}, \mathcal{I}^s \rangle + s(n-s) \\
&= -\langle \gamma, \beta \rangle - \langle \beta, \gamma \rangle + \sum_t d_1^{(t)} d_2^{(t)} + s(n-s).
\end{aligned}$$

◇

As the map q_2 is G -equivariant, we can pass to the quotient to obtain a map q'_2 , which is also an affine fibration with connected fibers:

$$q'_2 : \underline{\mathcal{E}}_{\mathcal{I}, n, s}^{\alpha, \geq m} \rightarrow \underline{\Lambda}_{\mathcal{I}, n-s, 0}^{\beta, \geq m} \times \underline{\Lambda}_{\mathcal{I}, s, s}^{\gamma, \geq m} \times [C/GL_{d'}]$$

The variety C is an homogenous $GL_{d'}$ -variety, where $GL_{d'} = \prod_t GL_{d'(t)}$, (hence smooth) of dimension $\sum_t (d'(t))^2 - d_1^{(t)} d_2^{(t)}$, so the quotient is a smooth (connected) stack of dimension $-\sum_t d_1^{(t)} d_2^{(t)}$.

We have the following diagram:

$$\begin{array}{ccc}
\underline{\mathcal{E}}_{\mathcal{I}, n, s}^{\alpha, \geq m} & \xrightarrow{q'_2} & \underline{\Lambda}_{\mathcal{I}, n-s, 0}^{\beta, \geq m} \times \underline{\Lambda}_{\mathcal{I}, s, s}^{\gamma, \geq m} \times [C/GL_{d'}] \\
& \searrow p_2 & \downarrow \text{proj} \\
& & \underline{\Lambda}_{\mathcal{I}, n-s, 0}^{\beta, \geq m}
\end{array}$$

As the stack $\underline{\Lambda}_{\mathcal{I}, s, s}^{\gamma}$ is smooth connected of dimension $-\langle \gamma, \gamma \rangle = -\langle s[\mathcal{I}], s[\mathcal{I}] \rangle = -s^2$, the morphism p_2 is smooth with connected fibers of dimension $-\langle \beta, \gamma \rangle - \langle \gamma, \beta \rangle - \langle \gamma, \gamma \rangle + s(n-s)$. The proposition (5.1) follows.

Now we study the map p_1 .

Proposition 5.2. *There is a natural bijection between irreducible components Z_1 of $\underline{\Lambda}_{\mathcal{I}, s, n}^{\alpha}$ and irreducible components Z_3 of $\underline{\mathcal{E}}_{\mathcal{I}, n, s}^{\alpha}$. Under this correspondence we have $\dim Z_3 = \dim Z_1 + s(n-s)$.*

Proof. We enlarge our stack $\underline{\mathcal{E}}_{\mathcal{I}, n, s}^{\alpha}$. Define a stack classifying isomorphism classes of objects:

$$\underline{F}_{\mathcal{I}, n, s}^{\alpha} = \{(\mathcal{F}, f, i) \mid (\mathcal{F}, f) \in \underline{\Lambda}_{\mathcal{I}, n, s}^{\alpha}, i \in Gr_s^{\text{Hom}(\mathcal{I}, \text{Ker } f)}\}$$

where $Gr_s^{\text{Hom}(\mathcal{I}, \text{Ker } f)}$ is the Grassmannian of s -dimensional subspaces of $\text{Hom}(\mathcal{I}, \text{Ker } f)$, and where as usual morphisms between objects (\mathcal{F}, f, i) and (\mathcal{F}', f', i') are isomorphisms $\psi : \mathcal{F} \simeq \mathcal{F}'$ such that the following diagrams commute:

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\psi} & \mathcal{F}' \\
\downarrow f & & \downarrow f' \\
\mathcal{F}(\bar{\omega}) & \xrightarrow{\psi} & \mathcal{F}'(\bar{\omega})
\end{array}
\quad
\begin{array}{ccc}
\mathcal{I}^s & \xrightarrow{i} & \text{Ker } f \\
\searrow i' & & \downarrow \psi \\
& & \text{Ker } f'
\end{array}$$

The substack $\underline{\mathcal{E}}_{\mathcal{I}, n, s}^{\alpha}$ is easily seen to be an open dense substack of $\underline{F}_{\mathcal{I}, n, s}^{\alpha}$, as the condition i injective is open in the irreducible variety $Gr_s^{\text{Hom}(\mathcal{I}, \text{Ker } f)}$, and the map p_1 naturally extends to $\underline{F}_{\mathcal{I}, n, s}^{\alpha, \beta}$.

Define the stacks

$$\underline{G}_{\mathcal{I}, n, s}^{\alpha} = \{(\mathcal{F}, f, i, h) \mid (\mathcal{F}, f, i) \in \underline{F}_{\mathcal{I}, n, s}^{\alpha}, h : \mathbb{C}^n \simeq \text{Hom}(\mathcal{I}, \text{Ker } f)\}$$

and

$$\underline{G}'_{\mathcal{I}, n, s}^{\alpha} = \{(\mathcal{F}, f, i, h) \mid i \in Gr_s^n\}$$

with the rest of the data as in $\underline{G}_{\mathcal{I},n,s}^\alpha$. We have natural maps which lead to the following commutative diagram:

$$\begin{array}{ccccc} \underline{G}_{\mathcal{I},n,s}^\alpha & \xrightarrow{\nu_1} & \underline{F}_{\mathcal{I},n,s}^\alpha & \xrightarrow{p_1} & \underline{\Lambda}_{\mathcal{I},n,s}^\alpha \\ \downarrow u & & & \nearrow \text{proj} & \\ \underline{G}'_{\mathcal{I},n,s}^\alpha & \xrightarrow{\nu_2} & \underline{\Lambda}_{\mathcal{I},n,s}^\alpha \times Gr_n^s & & \end{array}$$

where we have:

- (1) The map u is defined by $u(\mathcal{F}, f, i, h) = (\mathcal{F}, f, i', h)$ where $i' \in Gr_n^s$ is deduced from i via h . It is clearly an isomorphism.
- (2) The maps ν_1 and ν_2 , defined by $\nu(\mathcal{F}, f, i, h) = (\mathcal{F}, f, i)$, are GL_n principal bundles.

Consequently, this diagram induces a bijection between $\text{Irr}(\underline{F}_{\mathcal{I},n,s}^\alpha)$ and $\text{Irr}(\underline{\Lambda}_{\mathcal{I},n,s}^\alpha)$. But as $\underline{\mathcal{E}}_{\mathcal{I},s,n}^\alpha$ is an open dense substack of $\underline{F}_{\mathcal{I},s,n}^\alpha$, it also gives a bijection between $\text{Irr}(\underline{\mathcal{E}}_{\mathcal{I},s,n}^\alpha)$ and $\text{Irr}(\underline{\Lambda}_{\mathcal{I},s,n}^\alpha)$. Moreover, under this correspondence $Z_1 \leftrightarrow Z_2$ we have $\dim Z_1 = \dim Z_2 + s(n-s)$. ◇

Now Theorem 5.1 is a consequence of propositions (5.1) and (5.2)

5.2. Definition, consequences and first properties. The following result is a consequence of Theorem 5.1:

Theorem 5.2. *For any $\alpha \in K^+(\text{Coh}_{\mathbb{X}})$ and any line bundle \mathcal{L} , we have the following:*

- the space $\underline{\Lambda}_{\mathcal{L},0}^\alpha$ is open in $\underline{\Lambda}_{\mathbb{X}}^\alpha$,
- the stack $\underline{\Lambda}_{\mathbb{X}}^\alpha$ is pure of dimension $-\langle \alpha, \alpha \rangle$.

Proof. We first prove the first point. Let us fix a point $(\mathcal{F}, f) \in \underline{\Lambda}_{\mathbb{X}}^\alpha$. Define a map $\psi_{\mathcal{F},f}$ as follows:

$$\psi_{\mathcal{F},f}: \begin{array}{ccc} \text{Hom}(\mathcal{L}, \mathcal{F}^{\text{vec}}) & \rightarrow & \text{Hom}(\mathcal{L}, \mathcal{F}(\tilde{\omega})) \\ h & \mapsto & f \circ h. \end{array}$$

Now remark that if $s \in \text{Hom}(\mathcal{L}, \mathcal{F}^{\text{vec}})$ is non-zero, it is an injection, and then $\psi_{\mathcal{F},f}(s) = s|_{\mathcal{L}}$. So the condition that there is no injection $\mathcal{L} \hookrightarrow \text{Ker } f$ is equivalent to the fact that for any injection $s: \mathcal{L} \hookrightarrow \mathcal{F}$, the restriction $s|_{\mathcal{L}} = \psi_{\mathcal{F},f}(s)$ is non-zero. We then have the equivalence:

$$\psi_{\mathcal{F},f} \text{ is injective} \iff (\mathcal{F}, f) \in \underline{\Lambda}_{\mathcal{L},0}^\alpha.$$

The condition on the left is open, so we have that $\underline{\Lambda}_{\mathcal{L},0}^\alpha$ is open in $\underline{\Lambda}_{\mathbb{X}}^\alpha$.

Now we prove the second point. We already know the result for any α such that $\text{rk}(\alpha) = 0$ (see Section 4.5). Now we proceed by induction on the rank; take a positive integer r and let us suppose that the theorem is true for any β with $\text{rk}(\beta) < r$. Take α with $\text{rk}(\alpha) = r$. We have the following:

$$(15) \quad \underline{\Lambda}_{\mathbb{X}}^\alpha = \bigcup_{\mathcal{L} \text{ linebundle}, s>0} \underline{\Lambda}_{\mathcal{L},s}^\alpha.$$

Now consider an irreducible component Z of $\underline{\Lambda}_{\mathbb{X}}^\alpha$ of dimension d . Then there exists a line bundle \mathcal{L} and two positive integers s and n such that $Z \cap \underline{\Lambda}_{\mathcal{L},n,s}^\alpha$ is dense in Z . From the diagram (7), it corresponds to an irreducible component of $\underline{\Lambda}_{\mathcal{L},n,s}^{\alpha-s[\mathcal{L}]}$ of dimension $d' = d + \langle \alpha, s[\mathcal{L}] \rangle + \langle s[\mathcal{L}], \alpha \rangle - \langle s[\mathcal{L}], s[\mathcal{L}] \rangle$. Its closure Z' in $\underline{\Lambda}_{\mathbb{X}}^{\alpha-s[\mathcal{L}]}$ is irreducible, so by induction hypothesis we have $d' = \dim Z' \leq \langle \alpha - s[\mathcal{L}], \alpha - s[\mathcal{L}] \rangle$.

We want to prove that $d' = \langle \alpha - s[\mathcal{L}], \alpha - s[\mathcal{L}] \rangle$. For this, consider an irreducible component \tilde{Z}' of $\underline{\Lambda}_{\mathcal{L},0}^{\alpha-s[\mathcal{L}]}$ containing Z' . By induction hypothesis and the first part of the theorem, \tilde{Z}' is of dimension $\langle \alpha - s[\mathcal{L}], \alpha - s[\mathcal{L}] \rangle$. Choose the integer n' such that $\tilde{Z}' \cap \underline{\Lambda}_{\mathcal{L},n',0}^{\alpha-s[\mathcal{L}]}$ is dense in \tilde{Z}' , and use diagram (7) to obtain a corresponding irreducible component of $\underline{\Lambda}_{\mathcal{L},n'+s,s}^\alpha$. Let us denote by \tilde{Z} the closure of this irreducible component in $\underline{\Lambda}_{\mathcal{L},s}^\alpha$. We have that \tilde{Z} is irreducible of dimension $-\langle \alpha, \alpha \rangle$. We obviously have that $Z \subseteq \tilde{Z}$, and as Z is supposed to be an irreducible component, we have the quality $Z = \tilde{Z}$. Hence we have $d = -\langle \alpha, \alpha \rangle$. ◇

We now consider the set $\mathbf{B} := \text{Irr}(\underline{\Lambda}_{\mathbb{X}})$ and endow it with the following structure. First, the decomposition as connected components $\underline{\Lambda}_{\mathbb{X}} = \bigsqcup_{\alpha \in K^+(\underline{\text{Coh}}_{\mathbb{X}})} \underline{\Lambda}_{\mathbb{X}}^{\alpha}$ gives rise to a weight map:

$$wt : \mathbf{B} \rightarrow K^+(\underline{\text{Coh}}_{\mathbb{X}}).$$

Given an element $Z \in \mathbf{B}$ and an indecomposable rigid element \mathcal{I} , there is only one stratum $\underline{\Lambda}_{\mathcal{I},n,s}^{\alpha}$ such that $Z \cap \underline{\Lambda}_{\mathcal{I},n,s}^{\alpha}$ is dense in Z . We define two operators $e_{\mathcal{I}}$ and $f_{\mathcal{I}}$ on the set \mathbf{B} the following way:

$$\begin{aligned} e_{\mathcal{I}}(Z) &= e_{\mathcal{I}}^{s+1}(f_{\mathcal{I}}^{\max}(Z)) \\ f_{\mathcal{I}}(Z) &= \begin{cases} e_{\mathcal{I}}^{s-1}(f_{\mathcal{I}}^{\max}(Z)) & \text{if } s > 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We have already defined the function $\epsilon_{\mathcal{I}} : \underline{\Lambda}_{\mathbb{X}} \rightarrow \mathbb{Z}^+$, and we define a function, with the same notation, on the set \mathbf{B} by taking the generic value on an irreducible component. Finally define $\phi_{\mathcal{I}} : \mathbf{B} \rightarrow \mathbb{Z}$ by the formula:

$$\phi_{\mathcal{I}}(Z) = \epsilon_{\mathcal{I}}(Z) + \langle [\mathcal{I}], wt(Z) \rangle$$

Definition 5.1. The collection $(\mathbf{B}, wt, e_{\mathcal{I}}, f_{\mathcal{I}}, \epsilon_{\mathcal{I}}, \phi_{\mathcal{I}})$, where \mathcal{I} describes the set of indecomposable rigid elements in $\text{Coh}_{\mathbb{X}}$ is called the *loop crystal* associated to the weighted projective line \mathbb{X} .

As in the case of usual crystals, we can see this data on a colored graph: the vertices correspond to irreducible components, and we draw an arrow from Z to Z' with color \mathcal{I} if $f_{\mathcal{I}}(Z) = Z'$. The following result is immediate from the construction of the loop crystal.

Proposition 5.3. *We have the following properties, for any indecomposable rigid object \mathcal{I} and any element Z, Z' in \mathbf{B} :*

- (1) $wt(e_{\mathcal{I}}(Z)) = wt(Z) + \alpha_{\mathcal{I}}$ and $wt(f_{\mathcal{I}}(Z)) = wt(Z) - \alpha_{\mathcal{I}}$ if $f_{\mathcal{I}}(Z) \neq 0$,
- (2) $\epsilon_{\mathcal{I}}(e_{\mathcal{I}}(Z)) = \epsilon_{\mathcal{I}}(Z) + 1$ and $\phi_{\mathcal{I}}(e_{\mathcal{I}}(Z)) = \phi_{\mathcal{I}}(Z) - 1$,
- (3) $\epsilon_{\mathcal{I}}(f_{\mathcal{I}}(Z)) = \epsilon_{\mathcal{I}}(Z) - 1$ (resp. $\phi_{\mathcal{I}}(f_{\mathcal{I}}(Z)) = \phi_{\mathcal{I}}(Z) + 1$) if $\epsilon_{\mathcal{I}}(Z) \neq 0$, otherwise $f_{\mathcal{I}}(Z) = 0$,
- (4) $Z' = f_{\mathcal{I}}(Z)$ if and only if $e_{\mathcal{I}}(Z') = Z$.

We can also prove the following:

Proposition 5.4. *The loop crystal graph is connected.*

Proof. First remark that if Z is an irreducible component of class α with α of rank strictly positive, then there exist a line bundle \mathcal{L} such that $f_{\mathcal{L}}(Z)$ is non zero by (15). By an easy induction on the rank, any irreducible component Z is connected to an irreducible component Z' of rank 0.

Now from the description of the irreducible components of rank 0, we can reduce ourselves to irreducible components of type Z_{λ} , for a partition λ . Introduce the conjugate partition μ such that $\mu_i = |\{j \mid \lambda_j \geq i\}|$. Let $k = \text{length}(\mu)$. Introduce the following sequence of partitions:

$$\mu^{(i)} = (\mu_1, \mu_2, \dots, \mu_i), \quad 1 \leq i \leq k,$$

and $\lambda^{(i)}$ the conjugate of $\mu^{(i)}$. For $l \in \mathbb{N}$, define the following vector bundle on \mathbb{X} :

$$\mathcal{V}_l := \bigoplus_{k=1}^l \mathcal{O}(2k\bar{c}).$$

Consider the space:

$$Z_l = \{(\mathcal{V}_l, f) \mid f \in \text{Hom}(\mathcal{V}, \mathcal{V}(\bar{\omega})) \text{ such that } \mathcal{O}(2k\bar{c}) \rightarrow \mathcal{O}(2(k+1)\bar{c} + \bar{\omega}) \text{ is non-zero for } 1 \leq k \leq l-1\}.$$

Then the closure \overline{Z}_l is an irreducible component of $\underline{\Lambda}_{\mathbb{X}}$ as it is the closure of the conormal to the point \mathcal{V}_l . Indeed the nilpotency condition is automatically checked for any element $f \in \text{Hom}(\mathcal{V}_l, \mathcal{V}_l(\bar{\omega}))$: as there is no non-zero map from $\mathcal{O}(2(k+1)\bar{c})$ to any $\mathcal{O}(2(j+1)\bar{c} + \bar{\omega})$ for $j < k$, the kernel should contain $\mathcal{O}(2(l+1)\bar{\omega})$, and by an easy induction $f^l = 0$. Now we introduce for a partition ν and a positive integer l the following space:

$$Z_{l,\nu} := \{(\mathcal{V} \oplus \tau_{\nu}, f) \mid (\mathcal{V}, f|_{\mathcal{V}}) \in Z_l, \tau_{\nu} = \bigoplus_i \mathcal{O}_{y_i}^{(\nu_i)}, \text{Im } f \cap \tau(\bar{\omega}) = \tau(\bar{\omega})\},$$

where y_i are distinct ordinary points of \mathbb{X} .

Via the correspondence between vector bundles and torsion sheaves (in 4.1), the closure $\overline{Z}_{l,\nu}$ corresponds to an irreducible component associated with \mathcal{V}_l in the vector bundle part and the partition ν for the ordinary torsion sheaf part. It is thus an irreducible component of $\underline{\Lambda}_{\mathbb{X}}$.

Lemma 5.7. *For any $1 \leq j \leq k$ we have the following:*

$$f_{\mathcal{O}((2k-\sum_{m=1}^j \mu_m)\bar{c})}(\bar{Z}_{k-j+1, \mu^{(j-1)}}) = \bar{Z}_{k-j, \mu^{(j)}}$$

and

$$f_{\mathcal{O}((2k-j)\bar{c})}(\bar{Z}_{k-j+1}) = \bar{Z}_{k-j}$$

Proof. A generic element of the irreducible component $\bar{Z}_{k-j+1, \mu^{(j-1)}}$ is of the form $(\mathcal{V}_{k-j+1} \oplus \tau_\nu, f)$, where:

- the kernel of f is of rank 1, and the restriction of f to $\mathcal{O}(m)$, $m < k - j + 1$, is injective,
- $\text{Im } f \cap \tau_\nu(\bar{\omega}) = \tau_\nu(\bar{\omega})$.

It follows that the image of f is exactly $(\mathcal{V}_{k-j} \oplus \tau_\nu)(\bar{\omega})$. Generically, the torsion part of the kernel of f is the socle of τ_ν , i.e. $\bigoplus_i \mathcal{O}_{y_i}^{(1)}$. We then deduce that generically the kernel of f is:

$$\ker f = \mathcal{O}((2(k-j+1) - l(\nu))\bar{c}) \oplus \bigoplus_i \mathcal{O}_{y_i}.$$

Then in the case $l(\nu) = 0$ we have obviously $f_{\mathcal{O}((2k-j)\bar{c})}(\bar{Z}_{k-j+1}) = \bar{Z}_{k-j}$, as the only injection $\mathcal{O}((2k-j)\bar{c}) \rightarrow \text{Ker } f$ is an isomorphism into the locally free part of $\text{Ker } f$. Then $\mathcal{V}_{k-j+1}/\mathcal{O}(2(k-j)) = \mathcal{V}_{k-j}$.

In the case $l(\nu) > 0$, first compute the quotient $\mathcal{F}/\text{Ker } f^{\text{vec}} = \mathcal{F}/\mathcal{O}((2(k-j+1) - l(\nu))\bar{c})$. We have the exact sequence

$$0 \rightarrow \text{Ker } f \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\text{Ker } f \rightarrow 0.$$

By taking by $\text{Ker } f^{\text{vec}}$, we have:

$$0 \rightarrow \text{Ker } f/\text{Ker } f^{\text{vec}} \rightarrow \mathcal{F}/\text{Ker } f^{\text{vec}} \rightarrow \mathcal{F}/\text{Ker } f \rightarrow 0.$$

But $\text{Ker } f/\text{Ker } f^{\text{vec}}$ is exactly $\text{Ker } f^{\text{tor}} = \text{soc}(\tau_\nu)$. If we decompose $\mathcal{F}/\text{Ker } f^{\text{vec}}$ into its locally free part and its torsion part as $\mathcal{V}_{k-j} \oplus \tau$, we have by restricting to torsion parts:

$$0 \rightarrow \text{soc}(\tau_\nu) \rightarrow \tau \rightarrow \tau_\nu \rightarrow 0.$$

We deduce that τ has support $\{y_i\}$, with multiplicities $\nu_i + 1$. But as $Z_{l, \nu}$ is a dense subset of an irreducible component, the elements $(\mathcal{F}/\mathcal{O}((2(k-j+1) - l(\nu))\bar{c}), \tilde{f})$ (where \tilde{f} is deduced from f) describe, via the diagram (7), an open dense subset of $f_{\mathcal{O}((2(k-j+1) - l(\nu))\bar{c})}(\bar{Z}_{l, \nu})$. By the description of the torsion part of irreducible component given in Section (4), this implies that generically $\tau = \bigoplus_i \mathcal{O}_{y_i}^{(\nu_i+1)}$.

Now take an injection $I : \mathcal{O}(a\bar{c}) \rightarrow \text{Ker } f$. It factors through $\text{Ker } f^{\text{vec}}$, and generically the quotient $\text{Ker } f^{\text{vec}}/\mathcal{O}(a\bar{c})$ is of the form $\mathcal{O}_{z_i}^{(1)}$, where z_i are distinct ordinary points, different from the y_i 's. This implies that generically the quotient $\mathcal{F}/\mathcal{O}(a\bar{c})$ is of the form

$$\mathcal{V}_{k-j} \oplus \bigoplus_{i=1}^{|\mu^{(j-1)}|} \mathcal{O}_{y_i}^{(\nu_i+1)} \oplus \bigoplus_{i=1}^{|\mu^{(j)}| - |\mu^{(j-1)}|} \mathcal{O}_{z_i}^{(1)}.$$

The space of such elements form a dense subspace of the irreducible component $(\mathcal{V}_{k-j, \mu^{(j)}})$, which proves the second part of the lemma. ◇

In particular, as we have $\bar{Z}_{k, \nu^{(0)}} = \bar{Z}_k$, $\bar{Z}_{0, \nu^{(k)}} = Z_\lambda$ and $\bar{Z}_0 = \emptyset$, the irreducible components \emptyset and Z_λ are connected. ◇

5.3. Examples. In this subsection we give some exemples of computations and describe the irreducible components of the locally free part in the cases $g_{\mathbb{X}} < 1$ and $g_{\mathbb{X}} = 1$.

Case of the projective line $\mathbb{X} = \mathbb{P}^1$. The corresponding root system is the one of \widehat{sl}_2 . The irreducible components are indexed by pairs (\mathcal{V}, λ) , where \mathcal{V} is a vector bundle and λ a partition. The vector bundles split into sums of line bundles, and the only indecomposable rigid coherent sheaves are the line bundles $\mathcal{O}(k)$, $k \in \mathbb{Z}$.

We have the obvious computation:

$$f_{\mathcal{O}}^{\max}((\mathcal{O}^n, 0)) = f_{\mathcal{O}}^n((\mathcal{O}^n, 0)) = \emptyset$$

from which we deduce that:

$$f_{\mathcal{O}}((\mathcal{O}^n, 0)) = (\mathcal{O}^{n-1}, 0) \text{ and } e_{\mathcal{O}}((\mathcal{O}^n, 0)) = (\mathcal{O}^{n+1}, 0).$$

Now a generic quotient $\mathcal{O}^n/\mathcal{O}(-1)^n$ is of the form $\bigoplus_{i=1}^n \mathcal{O}_{y_i}^{(1)}$, where y_i are distinct points of \mathbb{P}^1 , i.e. corresponding to the partition (1^n) . Then we have:

$$f_{\mathcal{O}(-1)}^{\max}((\mathcal{O}^n, 0)) = f_{\mathcal{O}(-1)}^n((\mathcal{O}^n, 0)) = (0, (1^n)),$$

and more generally for $n, l \geq 0$

$$f_{\mathcal{O}(-1)}^{\max}((\mathcal{O}(1)^l \oplus \mathcal{O}^n, 0)) = f_{\mathcal{O}(-1)}^{n+l}((\mathcal{O}(1)^l \oplus \mathcal{O}^n, 0)) = (0, (1^{n+2l}))$$

This gives us

$$f_{\mathcal{O}(-1)}((\mathcal{O}(1)^l \oplus \mathcal{O}^n, 0)) = (\mathcal{O}(1)^{l+1} \oplus \mathcal{O}^{n-2}, 0) \quad \text{for } n \geq 2,$$

$$f_{\mathcal{O}(-1)}((\mathcal{O}(1)^l \oplus \mathcal{O}, 0)) = (\mathcal{O}(2) \oplus \mathcal{O}(1)^{l-1}, 0) \quad \text{for } l \geq 1,$$

and

$$f_{\mathcal{O}(-1)}((\mathcal{O}, 0)) = (0, (1)).$$

We can sum up these in the following, where we only draw arrows corresponding to operators $f_{\mathcal{O}}$ and $e_{\mathcal{O}(1)}$, and some vertices corresponding to *stable* vector bundles, i.e. of the form $\mathcal{O}(k)^a \oplus \mathcal{O}(k-1)^b$, for some integer k and some non-negative integers a and b .

$$\begin{array}{ccccc} & \dots & & \dots & \\ & \swarrow & & \swarrow & \\ \dots & \xrightarrow{f_{\mathcal{O}}} & \mathcal{O}(1)^2 & \xrightarrow{f_{\mathcal{O}}} & \mathcal{O}(2) & \xrightarrow{f_{\mathcal{O}}} & (1^2) \\ & & \swarrow^{e_{\mathcal{O}(1)}} & & \swarrow^{e_{\mathcal{O}(1)}} & & \\ \dots & \xrightarrow{f_{\mathcal{O}}} & \mathcal{O}(1) \oplus \mathcal{O} & \xrightarrow{f_{\mathcal{O}}} & \mathcal{O}(1) & \xrightarrow{f_{\mathcal{O}}} & (1) \\ & & \swarrow^{e_{\mathcal{O}(1)}} & & \swarrow^{e_{\mathcal{O}(1)}} & & \\ \dots & \xrightarrow{f_{\mathcal{O}}} & \mathcal{O}^2 & \xrightarrow{f_{\mathcal{O}}} & \mathcal{O} & \xrightarrow{f_{\mathcal{O}}} & \emptyset \end{array}$$

Here for simplicity, for a locally free irreducible component $(\mathcal{V}, 0)$ we wrote just the vector bundle \mathcal{V} , and for a torsion component $(0, \tau)$ we wrote (τ) .

As for non-stable vector bundles, we also proved during lemma 5.7 that we have for instance:

$$\begin{array}{ccccccc} \mathcal{O}(4) \oplus \mathcal{O}(2) \oplus \mathcal{O} & \xrightarrow{f_{\mathcal{O}(4)}} & \mathcal{O}(2) \oplus \mathcal{O} & \xrightarrow{f_{\mathcal{O}(2)}} & \mathcal{O} & \xrightarrow{f_{\mathcal{O}}} & \emptyset \\ & \searrow^{f_{\mathcal{O}(3)}} & & & \searrow^{f_{\mathcal{O}(-1)}} & & \\ & & (\mathcal{O}(2) \oplus \mathcal{O}, (1)) & & & & (1) \\ & & \searrow^{f_{\mathcal{O}(1)}} & & \searrow^{f_{\mathcal{O}(-2)}} & & \\ & & & (\mathcal{O}, (2)) & & & (1^2) \\ & & \searrow^{f_{\mathcal{O}}} & & \searrow^{f_{\mathcal{O}(-1)}} & & \\ & & & (\mathcal{O}, (2, 1)) & & & (3) \\ & & & \searrow^{f_{\mathcal{O}(-2)}} & & & \\ & & & & & & (3, 2) \end{array}$$

Case $g_{\mathbb{X}} < 1$. In this case we have $p\tilde{\omega} < 0$, and it easily implies that any $f \in \text{Hom}(\mathcal{V}, \mathcal{V}(\tilde{\omega}))$, where \mathcal{V} is a vector bundle, is nilpotent. The closure of any conormal bundle to a vector bundle lies inside the global nilpotent cone, hence the irreducible components of $\underline{\Delta}_{\mathbb{X}}$ are indexed by pairs (\mathcal{V}, τ) , where \mathcal{V} is a vector bundle and τ is an irreducible of the torsion part.

The Kac-Moody algebra corresponding to the star-shaped diagram (1) is of finite Dynkin type. Its loop-Kac-Moody algebra is then an affine Lie algebra, corresponding to an affine Dynkin type.

The indecomposable rigid elements are of two kinds:

- (1) indecomposable vector bundles \mathcal{V} . Following Crawley-Boevey ([CB2]), there is a indecomposable vector bundle of class α if and only if α is a positive root, and it is unique up to isomorphism. For instance in type A (i.e. when $n \leq 2$), they are all line bundles, and as soon as $n > 2$ there are indecomposable vector bundles of rank > 1 .
- (2) indecomposable torsion sheaves. These are parametrized by positive roots of rank 0. The rigid ones are parametrized by *real* roots.

Case $g_{\mathbb{X}} = 1$. In this case the nilpotency condition is non-empty, as $p\tilde{\omega} = 0$. The corresponding Kac-Moody algebra is of affine Dynkin type, so the loop Kac-Moody algebra is a double-affine, or elliptic, Lie algebra. Unlike the previous case, some positive roots corresponding to (indecomposable) vector bundles are imaginary, meaning that these vector bundles are not rigid. Nevertheless, it is possible to describe explicitly the irreducible components. For this we need to introduce more notation. We will follow the one used in [Sc3].

First define the group homomorphism $\partial : L(\underline{\mathbb{P}}) \rightarrow \mathbb{Z}$ by $\partial(\tilde{x}_i) = \frac{p}{p_i}$. Then we define the *degree map* \mathbf{d} as the map of groups

$$\mathbf{d} : K(\text{Coh}_{\mathbb{X}}) \rightarrow \mathbb{Z}$$

defined on generators by $\mathbf{d}(\mathcal{O}_{\mathbb{X}}(\tilde{x})) = \partial(\tilde{x})$.

Define the *slope* $\nu(\mathcal{F})$ of a coherent sheaf \mathcal{F} as follows

$$\begin{aligned} \nu : \text{Coh}_{\mathbb{X}} &\rightarrow \mathbb{Z} \cup \{\infty\} \\ \mathcal{F} &\mapsto \mathbf{d}(\mathcal{F})/\text{rk}(\mathcal{F}). \end{aligned}$$

Here $\nu(\mathcal{F})$ is defined to be ∞ when \mathcal{F} is a torsion sheaf.

We need to introduce the *Harder-Narasimhan filtration* of a coherent sheaf, defined in [GL]. We say that a coherent sheaf \mathcal{F} is *semi-stable* if for any $\mathcal{G} \subseteq \mathcal{F}$ we have $\nu(\mathcal{G}) \leq \nu(\mathcal{F})$. We also define for $q \in \mathbb{Q}$ the category \mathcal{C}_q to be the full subcategory of $\text{Coh}_{\mathbb{X}}$ consisting of the zero sheaf together with sheaves of slope q . The following properties are proved in [GL] (for the first three ones) and [LM1] (for the last one):

- (1) for any $q \in \mathbb{Q}$, the category \mathcal{C}_q is abelian and closed under extensions,
- (2) if $\mathcal{F} \in \mathcal{C}_q$, $\mathcal{G} \in \mathcal{C}_{q'}$ and $q > q'$ then $\text{Hom}(\mathcal{F}, \mathcal{G}) = 0$,
- (3) (using $g_{\mathbb{X}} = 1$) if $\mathcal{F} \in \mathcal{C}_q$, $\mathcal{G} \in \mathcal{C}_{q'}$ and $q < q'$ then $\text{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$,
- (4) for any $q \in \mathbb{Q}$ the category \mathcal{C}_q is naturally isomorphic to \mathcal{C}_{∞} .

Any coherent sheaf \mathcal{F} has a unique filtration, the Harder-Narasimhan filtration,

$$0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_l = \mathcal{F},$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable of slope ν_i and $\nu_1 > \nu_2 > \dots > \nu_l$. The sequence of elements in $K^+(\mathbb{X})$:

$$\text{HN}(\mathcal{F}) = ([\mathcal{F}/\mathcal{F}_{l-1}], \dots, [\mathcal{F}_2/\mathcal{F}_1], [\mathcal{F}_1])$$

is called the *HN-type* of \mathcal{F} . For any sequence $\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of elements of $K^+(\mathbb{X})$, we can then define

$$\text{HN}^{-1}(\underline{\alpha}) = \{\mathcal{F} \in \underline{\text{Coh}}_{\mathbb{X}} \mid \text{HN}(\mathcal{F}) = \underline{\alpha}\},$$

a constructible substack of $\underline{\text{Coh}}_{\mathbb{X}}$. Set $|\underline{\alpha}| := \sum_i \alpha_i$. We define

$$T_{\text{HN}^{-1}(\underline{\alpha})}^* := \{(\mathcal{F}, f) \in T^* \underline{\text{Coh}}_{\mathbb{X}} \mid \text{HN}(\mathcal{F}) = \underline{\alpha}\}$$

a constructible substack of $T^* \underline{\text{Coh}}_{\mathbb{X}}$, and

$$\underline{\Delta}_{\text{HN}^{-1}(\underline{\alpha})} := \underline{\Delta}_{\mathbb{X}} \cap T_{\text{HN}^{-1}(\underline{\alpha})}^*$$

a constructible substack of $\underline{\Delta}_{\mathbb{X}}$. We also define, for any $\alpha \in K^+(\mathbb{X})$

$$\underline{\Delta}_{\mathbb{X}}^{(\alpha)} = \{(\mathcal{F}, f) \in \underline{\Delta}_{\mathbb{X}} \mid \mathcal{F} \text{ semistable of slope } \nu(\alpha)\}.$$

The proof of the following proposition is similar to the one of lemma 4.1.

Proposition 5.5. *The natural morphism*

$$P_{\underline{\alpha}} : \underline{\Delta}_{\text{HN}^{-1}(\underline{\alpha})} \rightarrow \prod_{i=1}^k \underline{\Delta}_{\mathbb{X}}^{(\alpha_i)}$$

given by $P_{\underline{\alpha}}(\mathcal{F}, f) = ((\mathcal{F}/\mathcal{F}_{l-1}, f|_{\mathcal{F}/\mathcal{F}_{l-1}}), \dots, (\mathcal{F}_2/\mathcal{F}_1, f|_{\mathcal{F}_2/\mathcal{F}_1}), (\mathcal{F}_1, f|_{\mathcal{F}_1}))$ is an affine fibration, hence it induces a correspondence between irreducible components.

We can now describe all irreducible components of $\underline{\Delta}_{\mathbb{X}}^{\alpha}$: from the previous proposition we have a bijection

$$\text{Irr}(\underline{\Delta}_{\text{HN}^{-1}(\underline{\alpha})}) \leftrightarrow \prod_{i=1}^k \text{Irr}(\underline{\Delta}_{\mathbb{X}}^{(\alpha_i)})$$

As the substacks $\underline{\Delta}_{\text{HN}^{-1}(\underline{\alpha})}$ stratify $\underline{\Delta}_{\mathbb{X}}^{\alpha}$, we have further bijections:

$$\text{Irr}(\underline{\Delta}_{\mathbb{X}}^{\alpha}) \leftrightarrow \bigcup_{|\underline{\alpha}|=\alpha} \text{Irr}(\underline{\Delta}_{\text{HN}^{-1}(\underline{\alpha})}) \leftrightarrow \bigcup_{\sum_{i=1}^l \alpha_i = \alpha} \prod_{i=1}^k \text{Irr}(\underline{\Delta}_{\mathbb{X}}^{(\alpha_i)})$$

We can use Property (4) in the previous discussion to determine the irreducible component of $\underline{\Delta}_{\mathbb{X}}^{(\alpha_i)}$. The group of automorphism $\text{Aut}(K^+(\mathbb{X}))$ has been determined in [LM2]: it is an extension of $PSL_2(\mathbb{C})$ with the product of the Picard group $\text{Pic}_0(\mathbb{X})$ and a finite group. For any $\alpha \in K^+(\mathbb{X})$ there is an element $\gamma \in \text{Aut}(K^+(\mathbb{X}))$ such that $\nu(\gamma(\alpha)) = \infty$; as proved in [LM2], this action lifts to a natural isomorphism from $\mathcal{C}_{\nu(\alpha)}^{\alpha}$, the stack classifying semi-stable sheaves of class α , to $\mathcal{C}_{\infty}^{\gamma(\alpha)}$. We thus have an isomorphism:

$$\underline{\Delta}_{\mathbb{X}}^{(\alpha)} \simeq \underline{\Delta}_{\mathbb{X}}^{\gamma(\alpha)}$$

and the irreducible components of $\underline{\Delta}_{\mathbb{X}}^{\gamma(\alpha)}$ are known from Section 4.

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