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Gluon condensates from the Hamiltonian formalism

VLADIMIR PROCHAZKA ^{a1} & ROMAN ZWICKY ^{b2}

^a *Higgs centre for theoretical physics
School of Physics and Astronomy,
University of Edinburgh, Edinburgh EH9 3JZ, Scotland*

Abstract

We derive recently obtained relations, relating the logarithmic gauge coupling derivative of the hadron mass and the cosmological constant to the matter and vacuum gluon condensates, within a Hamiltonian framework. The key idea is a canonical transformation which brings the relevant part of the Hamiltonian into a suitable form. Furthermore we illustrate the relations within the Schwinger model and $\mathcal{N} = 2$ super Yang Mills theory (Seiberg-Witten theory).

¹v.prochazka@sms.ed.ac.uk

²roman.zwicky@ed.ac.uk

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1 Introduction

The Feynman-Hellmann theorem [1], originally derived in quantum mechanics, applies straightforwardly to quantum field theory in the case where the relevant part of the Hamiltonian is known. One such example is the fermion mass term of a gauge theory $\mathcal{H}_m = m\bar{q}q$ e.g. [2]. The Hamiltonian formalism of gauge theories is not straightforward because of the elimination of two degrees of freedom from the vector potential one of which is associated with the gauge freedom.

In [3] a Feynman Hellman relation for the gauge coupling constant was obtained by combining the trace anomaly, renormalization group equation (RGE) and the Feynman Hellmann theorem for the fermion mass. The relations read [3]:

$$\bar{g} \frac{\partial}{\partial \bar{g}} E_\varphi^2 = -\frac{1}{2} \langle \varphi | \frac{1}{\bar{g}^2} \bar{G}^2 | \varphi \rangle_c, \quad (1)$$

$$\bar{g} \frac{\partial}{\partial \bar{g}} \Lambda_{\text{GT}} = -\frac{1}{2} \langle \frac{1}{\bar{g}^2} \bar{G}^2 \rangle_0 \quad (2)$$

where $G^2 = G_{\mu\nu} G^{\mu\nu}$ is the field strength tensor squared, the subscript c stands for the connected part, φ denotes a physical state (normalisation to be specified be-

low) and $\langle X \rangle_0 \equiv \langle 0|X|0 \rangle$ corresponds to the vacuum expectation value throughout. The scheme dependence of the matrix elements on the right hand side is determined by the scheme dependence of the couplings on the left hand side. The barred symbols denote *renormalized* quantities to distinguish them from unrenormalized quantities. The partial derivatives are understood in the sense of the RGE. That is to say implicit dependencies of other parameters on the coupling are not considered by definition. In Eq. (1) the momentum is taken to be independent of M_φ as in [3].¹ Relation (1) is valid for the following normalisation of states,

$$\langle \varphi(E', \vec{p}') | \varphi(E, \vec{p}) \rangle = 2E_\varphi (2\pi)^{D-1} \delta^{(D-1)}(\vec{p} - \vec{p}') , \quad (3)$$

where D stands for the space-time dimension. The cosmological constant Λ_{GT} contribution in (2) was defined as $\langle T^\mu_\mu \rangle_0 = D\Lambda_{\text{GT}}$. The goal of this paper is to derive these relations, after all, using a Hamiltonian formalism. The key observation is that by a canonical transformation (rescalings in the gauge coupling constant), one can obtain a suitable form of the Hamiltonian.

The paper is organised as follows. In section 2 we pursue the derivation of relations (1,2) within the Hamiltonian formalism. In sections 3.1, 3.2 and 3.3 we illustrate the formula within the Schwinger Model and the $\mathcal{N} = 2$ super Yang Mills theory (Seiberg-Witten theory). We end the paper with summary and conclusions in section 4. Relevant comments on the transformation of the measure under the canonical transformation can be found in appendix A.

2 (Re)derivation in the Hamiltonian formalism

2.1 The suitable canonical transformation of the Hamiltonian

In the Hamiltonian formalism of a (non-abelian) gauge theory $\vec{\pi} = \vec{E}$ and \vec{A} are the independent canonically conjugate variables. (e.g. [5]).² The Hamiltonian reads,

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_g + \mathcal{H}_C + \mathcal{H}_G , \\ \mathcal{H}_g &= \frac{1}{2}(\vec{E}^2 + \vec{B}^2) - \bar{q}(i\vec{\gamma} \cdot \vec{D} - m)q , \end{aligned} \quad (4)$$

¹The latter is of significance (appendix B) for the derivation of the trace anomaly matrix element from an RGE for the Energy.

²The variable A_0 is degraded to be a Lagrangian multiplier imposing Gauss' law in (c.f. \mathcal{H}_G below) and $\pi_0 = 0$ is at the heart of all the difficulties with the Hamiltonian formalism of gauge theories (parameterised by \mathcal{H}_C below).

where $\vec{D} = \vec{\partial} + ig\vec{A}$ is the gauge covariant derivative and q stands for fermions (quarks) in some representation of the gauge group. The magnetic field is defined as $2B_k = \epsilon_{kij}G_{ij} = \epsilon_{kij}(\partial_i A_j - \partial_j A_i + ig[A_i, A_j])$. The term $\mathcal{H}_G = A_0^a \mathcal{G}^a$ with $\mathcal{G}^a = ((\vec{D} \cdot \vec{E})^a + \bar{q} t^a \gamma_0 q)$ corresponds to Gauss' law (i.e. one of Maxwell's equations). The expression \mathcal{H}_C is associated with primary and secondary constraints (resulting in gauge transformation). Both \mathcal{H}_G and \mathcal{H}_C vanish on matrix elements of physical states and shall therefore be omitted hereafter.

Our strategy is to make the dependence on the coupling g as simple as possible through the canonical transformation,

$$\begin{aligned} \vec{A} &\rightarrow \frac{1}{g} \vec{A} \\ \vec{E} &\rightarrow g \vec{E} . \end{aligned} \quad (5)$$

This leads to a Hamiltonian of the form,

$$\mathcal{H}_g = \frac{1}{2}(g^2 \vec{E}^2 + \frac{1}{g^2} \vec{B}^2) - \bar{q}(i\vec{\gamma} \cdot \vec{D} + m)q , \quad (6)$$

where, crucially, the only g -dependence is in front of the electric and magnetic field terms. It is important to note that the transformation in Eq. (5) leaves the measure of the path integral $\mathcal{D}\vec{E}\mathcal{D}\vec{A}$ invariant. First the transformation (5) does not affect the equal time canonical commutation relation, $[A^k(x_0, \vec{x}), E_l(x_0, \vec{y})] = i\delta_l^k \delta^{(D-1)}(\vec{x} - \vec{y})$; the (simple) Jacobian is therefore trivial. Second the measure is not affected by a rescaling anomaly of the type [7] since the two transformations in (5) exactly cancel each other (as outlined in appendix A).

2.2 Gluon condensates from Hamiltonian

The Feynman-Hellmann theorem [1] in quantum mechanics (here $\langle \varphi | \varphi \rangle = 1$) states that

$$\frac{\partial}{\partial \lambda} E_\varphi(\lambda) = \langle \varphi | \frac{\partial}{\partial \lambda} H(\lambda) | \varphi \rangle , \quad (7)$$

where λ is a parameter. It is crucial that $|\varphi\rangle$ is an eigenstate of the Hamiltonian H . The rest follows from the normalisation being independent on the parameter λ . The adaption to quantum field theory solely involves the incorporation of the specific normalisation convention (3). The right hand side of (7), in our case, is obtained by differentiating (6)

$$g \frac{\partial}{\partial g} \mathcal{H}_g = g^2 \vec{E}^2 - \frac{1}{g^2} \vec{B}^2 = -\frac{1}{2} \frac{1}{g^2} G_{\mu\nu} G^{\mu\nu} . \quad (8)$$

This form is very close to Eqs. (1,2). In particular a Lorentz invariant result has emerged from the non-covariant Hamilton formalism as is usually the case. Note, the Hamiltonian is a physical quantity and is therefore not renormalized. Below we shall write the Hamiltonian in terms of renormalized quantities (denoted by bars) which is natural since the physical quantities are matrix elements thereof. Identifying $\langle \mathcal{H} \rangle_0 = \Lambda_{\text{GT}}$ one gets (2) from (8):

$$\bar{g} \frac{\partial}{\partial \bar{g}} \Lambda_{\text{GT}} = \langle \bar{g} \frac{\partial}{\partial \bar{g}} \mathcal{H} \rangle_0 + \Lambda_{\text{GT}} \bar{g} \underbrace{\frac{\partial}{\partial \bar{g}} \langle 0|0 \rangle}_{=0} \stackrel{(8)}{=} -\frac{1}{2} \langle \frac{1}{\bar{g}^2} \bar{G}^2 \rangle_0 . \quad (9)$$

For the derivation of (1) the factor E_φ in the normalisation (3) complicates the algebra and we shall use $\sqrt{2E_\varphi} |\tilde{\varphi}\rangle = |\varphi\rangle$ below restoring the factor in the end.

$$\bar{g} \frac{\partial}{\partial \bar{g}} E_\varphi = \bar{g} \frac{\partial}{\partial \bar{g}} \langle \tilde{\varphi} | \mathcal{H} | \tilde{\varphi} \rangle_c = \langle \tilde{\varphi} | \bar{g} \frac{\partial}{\partial \bar{g}} \mathcal{H} | \tilde{\varphi} \rangle_c + \frac{E_\varphi}{V} \bar{g} \underbrace{\frac{\partial}{\partial \bar{g}} \langle \tilde{\varphi} | \tilde{\varphi} \rangle_c}_{=0} = \langle \tilde{\varphi} | \frac{1}{\bar{g}^2} \bar{G}^2 | \tilde{\varphi} \rangle_c$$

where V is the volume. Above we have identified $(2\pi)^{D-1} \delta^{(D-1)}(\vec{p} - \vec{p}') = \int d^{D-1}x$ (in the sense of distributions) since the Hamiltonian is given by $H = \int d^{D-1}x \mathcal{H}$. Restoring the normalisation (3) we get an expression,

$$2E_\varphi \bar{g} \frac{\partial}{\partial \bar{g}} E_\varphi = \langle \varphi | \frac{1}{\bar{g}^2} \bar{G}^2 | \varphi \rangle_c , \quad (10)$$

which is equivalent to (1). We have therefore rederived Eqs. (1,2) in a Hamiltonian framework which was the main goal of our work. We proceed to illustrate the formula in three models where exact results are known.

3 Examples

The relation (1) was used [4] to derive the scaling corrections to the hadron masses in two alternative ways. It therefore constitutes one independent check. Below we provide three further examples.

3.1 Photon mass in the Schwinger Model

Two dimensional quantum electrodynamics, known as the Schwinger model [8, 9] (for a review c.f. [10]), has served as a test ground for many formal approaches

and lattice simulations. A curious feature of the Schwinger model is that the photon acquires a mass through the chiral anomaly as the η' in quantum chromodynamics. This is sometimes referred to as a dynamical Higgs mechanism. The photon mass is:

$$M_\gamma^2 = \frac{e^2}{\pi} . \quad (11)$$

The relation (1) adapted to the Schwinger model, for a massive photon state at rest, reads:

$$e \frac{\partial}{\partial e} M_\gamma^2 = -\frac{1}{2} \langle \gamma | F^2 | \gamma \rangle_c . \quad (12)$$

Above $F^2 = F_{\mu\nu} F^{\mu\nu}$ is the electromagnetic field strength tensor squared and e is the charge of mass dimension one. The latter does not receive any renormalization (vanishing beta function).

In order to obtain (11) from (12) we have to evaluate the matrix element $\langle \gamma | F^2 | \gamma \rangle_c$ for which we resort to the operator solution of the Schwinger model [11] (e.g. chapter 10 [10]). The Field strength tensor is given by

$$F_{\mu\nu} = \frac{\sqrt{\pi}}{e} \epsilon_{\mu\nu} \square \Sigma , \quad (13)$$

where $\square = \partial_\mu \partial^\mu$ is the Laplacian and Σ is a canonically normalised free field of mass e^2/π . Choosing the connected part automatically fixes the scheme of the matrix element, which incidentally corresponds to normal ordering as used in ordinary perturbation theory: $\langle F^2 \rangle_0 = 0$. This is not surprising since there is no scheme ambiguity on the left hand side as the coupling does not run. Through an explicit computation in terms of creation and annihilation operators one gets,

$$\langle \gamma | F^2 | \gamma \rangle_c = \frac{\pi^2}{e^2} \epsilon_{\mu\nu} \epsilon^{\mu\nu} 2(-M_\gamma^2)^2 = -4 \frac{e^2}{\pi} , \quad (14)$$

where the factor of 2 is of combinatorial nature and we have replaced $\square \rightarrow -q^2 = -M_\gamma^2$. Inserting (14) into (12) we get:

$$e \frac{\partial}{\partial e} M_\gamma^2 = 2 \frac{e^2}{\pi} \Rightarrow M_\gamma^2 = \frac{e^2}{\pi} + C , \quad (15)$$

where C is a constant. From the limit $e \rightarrow 0$, where we expect $M_\gamma \rightarrow 0$, we infer $C = 0$ and therefore (15) corresponds to the exact result (11) known in the literature. In essence we have shown that (13) and (12) implies the Photon mass (11).

As an additional, but not necessary test, we can verify whether (12) is compatible with an RGE. The trace of the energy momentum tensor in massless QED, in terms of bare quantities, reads $T^\mu_\mu = -(D - 4)\mathcal{L} + \text{EOM}$, where EOM stands for terms which vanish by the equation of motions. The latter are not of interest for us as we shall evaluate the trace on physical states. Using $D = 2$ we get

$$\langle \gamma | T^\mu_\mu | \gamma \rangle_c = -\frac{1}{2} \langle \gamma | F^2 | \gamma \rangle_c, \quad (16)$$

and since $2M_\gamma^2 = \langle \gamma | T^\mu_\mu | \gamma \rangle_c$ it can be combined with (12) into

$$\left(e \frac{\partial}{\partial e} - 2 \right) M_\gamma^2 = 0 \quad \Rightarrow \quad M_\gamma^2 = C' e^2 \quad (17)$$

where C' is a constant ($C' = 1/\pi$ according to (11)) and the equation on the right hand side corresponds to an RGE. In fact the latter is equivalent to an equation based on dimensional analysis on grounds of the fact that there are no running quantities in the Schwinger model.

3.2 Vacuum energy in massive mutliflavour Schwinger model

The Schwinger Model with N_f massive fermions has aspects which are known exactly (c.f. [12] and references therein). The model has got a global $SU_L(N_f) \times SU_R(N_f)$ flavour symmetry which is explicitly broken down to $SU_V(N_f)$ by the fermion mass term. The spectrum consists of one massive boson (the massive photon of the proceeding section) and $N_f^2 - 1$ quasi Goldstone boson, similar to the η' and the octet π, K, η in quantum chromodynamics. The situation is though distinct in that the quark condensate does not form in the massless case and the quasi Goldstone bosons show scaling behaviour of a critical theory. The vacuum energy is proportional to the mass gap squared (for $m \ll e$ c.f. [12] and references therein):

$$\Lambda_{\text{GT}} \propto M_{\text{gap}}^2 \propto m^{\eta_m} e^{\eta_e}, \quad \eta_m = \frac{2N_f}{N_f + 1}, \quad \eta_e = \frac{2}{N_f + 1}. \quad (18)$$

From the trace anomaly equation one gets:

$$2\Lambda_{\text{GT}} = -\frac{1}{2} \langle F^2 \rangle_0 + N_f m \langle \bar{q}q \rangle_0. \quad (19)$$

The analogous equation for four dimension is given in [3]. The adaption of the F^2 -term to two dimensions has been discussed in the previous section and the

anomalous dimension of the mass is zero. Using (2) and $N_f m_f \langle \bar{q}q \rangle_0 = m \frac{\partial}{\partial m} \Lambda_{\text{GT}}$ one gets

$$2\Lambda_{\text{GT}} = e \frac{\partial}{\partial e} \Lambda_{\text{GT}} + m \frac{\partial}{\partial m} \Lambda_{\text{GT}} = \underbrace{(\eta_e + \eta_m)}_{=2} \Lambda_{\text{GT}} , \quad (20)$$

a consistent result. Summarising we obtain $\langle F^2 \rangle_0 = -2\eta_e \Lambda_{\text{GT}}$ and $N_f m \langle \bar{q}q \rangle_0 = \eta_m \Lambda_{\text{GT}}$. Again (20) reveals itself directly equivalent to an RGE for $\Lambda_{\text{GT}} = \Lambda_{\text{GT}}(m, e)$

$$\left(e \frac{\partial}{\partial e} + m \frac{\partial}{\partial m} - \Delta_{\Lambda_{\text{GT}}} \right) \Lambda_{\text{GT}}(m, e) = 0 . \quad (21)$$

Above $\Delta_{\Lambda_{\text{GT}}} = 2$ is the scaling dimension of the Λ_{GT} which is free from anomalous scaling as it is an observable. As (17) Eq. (21) is merely an equation that follows from dimensional analysis since all the scale breaking is explicit and not anomalous.

3.3 Magnetic monopole in Seiberg-Witten theory

The $\mathcal{N} = 2$ pure super Yang-Mills theory (with gauge group $SU(2)$), known as Seiberg-Witten theory [13], has features which are known exactly. In particular it is known that BPS states obey [13],

$$M_{(n_e, n_m)} = 2|Z|^2 \quad \text{with} \quad Z = n_e a + n_m a_D , \quad (22)$$

where n_e and n_m count the units of electric and magnetic charges. Exact solutions for a and a_D along with the effective coupling constant $\tau(a)$ constitute part of the work of Seiberg and Witten [13]. First we are going to derive Eq. (8) for the BPS sector. In the magnetic BPS sector the relevant part of the Hamiltonian reads [13]

$$\mathcal{H}_{\text{BPS}} = \frac{1}{g^2} \vec{D}\phi \cdot \vec{D}\phi + \frac{1}{2} \frac{1}{g^2} \vec{B}^2 , \quad (23)$$

where we shall comment on the (non-)significance of the extra $1/g^2$ -factor in front of the scalar kinetic term shortly below. Note, Maxwell's equations imply $\vec{E} = 0$ for static solution with $\vec{B} \neq 0$ (magnetic monopole). The fermionic terms are absent by construction of what is known as a BPS state in supersymmetry. Using the BPS equation,

$$\vec{D}\phi|_{\text{BPS}} = \frac{1}{\sqrt{2}} \vec{B}|_{\text{BPS}} , \quad (24)$$

the total Hamiltonian becomes,

$$\mathcal{H}_{\text{BPS}} = \frac{1}{g^2} \vec{B}^2, \quad (25)$$

and the $\mathcal{N} = 2$ supersymmetry, which is responsible for the $1/g^2$ -factor in front of the kinetic term in (23), effectively introduces a factor of 2 in the relation (1). This can be seen explicitly by differentiating, with respect to the coupling constant (8),

$$g \frac{\partial}{\partial g} \mathcal{H}_{\text{BPS}} = -2 \frac{1}{g^2} \vec{B}^2 \stackrel{\vec{E}=0}{=} -\frac{1}{g^2} G^2 \quad (26)$$

and comparing with Eq. (26). In summary we have shown that in Seiberg-Witten theory (1) holds on the BPS subspace. Conversely assuming that the formula (1) is true we know that (24) has to hold for \mathcal{H}_{BPS} in (23).

Unlike in the Schwinger model we cannot compute the matrix elements in (26) on the BPS states directly. We may turn things around and use the formula to express the matrix elements for the magnetic monopole in terms of a_D which is known explicitly in terms of the coupling constant. Formula (1) adapted for $\mathcal{N} = 2$ supersymmetry (with factor of two difference as explained above) reads:

$$\langle (0, n_m) | \frac{1}{g^2} G^2 | (0, n_m) \rangle_c = -g \frac{\partial M_{(0, n_m)}^2}{\partial g}, \quad (27)$$

In order to evaluate the right hand side we use $M_{(0, n_m)}^2 = 2n_m^2 |a_D|^2$ (22) and $g \frac{\partial}{\partial g} = -\frac{1}{2} \omega \frac{\partial}{\partial \omega}$ where $\omega \equiv \frac{1}{g^2}$,³

$$\begin{aligned} \frac{1}{n_m^2} \frac{\partial M_{(0, n_m)}^2}{\partial \omega} &= 2 \left[a_D^* \frac{\partial a_D}{\partial \omega} + a_D \frac{\partial a_D^*}{\partial \omega} \right] = 8\pi i \left[a_D^* \frac{\partial a_D}{\partial \tau} - a_D \frac{\partial a_D^*}{\partial \tau^*} \right] \\ &= -16\pi \text{Im} \left[a_D^* \frac{\partial a_D}{\partial \tau} \right]. \end{aligned} \quad (28)$$

This leads to

$$\langle (0, n_m) | \frac{1}{g^2} G^2 | (0, n_m) \rangle_c = 8\pi \frac{n_m^2}{g^2} \text{Im} \left[a_D^* \frac{\partial a_D}{\partial \tau} \right]. \quad (29)$$

The function a_D is known [13]

$$a_D(\tau) = \frac{\sqrt{2}\Lambda}{\pi} \int_1^{v(\tau)} \frac{dx \sqrt{x - v(\tau)}}{\sqrt{x^2 - 1}}, \quad v(\tau) = -1 + \frac{2}{\lambda(\tau)}, \quad (30)$$

³In doing so use the fact that a_D is a holomorphic function of holomorphicity in $\tau = 4\pi i/g^2 + \frac{1}{2\pi}\theta$.

with $v = u/\Lambda^2$ where $u = \langle \phi^2 \rangle_0$ is a modulus and Λ is a dynamical scale and constitute important parameters of the theory. The function $\lambda(\tau)$ is given in [14]. We have checked numerically that the condensate is zero for $g_D \propto 1/g \rightarrow 0$ and increases monotonically as a function of g_D . The coupling g_D corresponds to the magnetic coupling and is dual to the electric coupling g . Loosely speaking the magnetic monopole condensate is governed by the magnetic coupling g_D .

4 Summary and conclusions

We have derived the relations in Eqs. (1,2), previously obtained in [3] through the trace anomaly, the Feynman-Hellmann theorem and an RGE, in a Hamiltonian formulation of gauge theories. The derivation contains two ingredients. Eliminate the terms which vanish as matrix elements from the Hamiltonian. In this way we bypass the notoriously difficult problem of gauge fixing. The second step is a canonical transformation which arranges the Hamiltonian in such a way that only the \vec{E}^2 and \vec{B}^2 -terms depend on the gauge coupling. The derivative with respect to the gauge coupling then gives rise to the explicitly Lorentz invariant result. A subtle point, which we have verified in appendix A, is that the canonical transformation is free from rescaling anomalies of the Konishi type. One possible advantage of the Hamiltonian derivation is that it makes it clear that the relations holds for gauge theories with more than one gauge coupling. Furthermore we have tested the relation within the Schwinger Model and the $\mathcal{N} = 2$ super Yang Mills theory (Seiberg-Witten theory).

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A The rescaling anomaly in Hamiltonian language

In section 2 we have used a particular canonical transformation (5) and one might wonder whether the measure is anomalous under this transformation. Generally any rescaling of a field which is gauged, produces anomalous term proportional to the kinetic term of the corresponding gauge field [15]. We shall see that for the transformation (5) the effect cancels.

Let us write (5) for a generic transformation

$$\begin{aligned}\vec{A} &\rightarrow \frac{1}{f(g)}\vec{A}, \\ \vec{E} &\rightarrow f(g)\vec{E}.\end{aligned}\tag{A.1}$$

The anomalous Jacobian of the $\mathcal{D}\vec{E}\mathcal{D}\vec{A}$ measure is

$$\begin{aligned}\ln \det \frac{\delta Q'(x)}{\delta Q(y)} &= \ln \det \begin{pmatrix} f(g)^{-1}\delta(x-y) & 0 \\ 0 & f(g)\delta(x-y) \end{pmatrix} = \\ \ln \det \begin{pmatrix} f(g)^{-1} & 0 \\ 0 & f(g) \end{pmatrix} \delta(x-y) &= \ln \det \delta(x-y),\end{aligned}\tag{A.2}$$

where we have used the compact notation $Q \equiv (\vec{A}, \vec{E})$. It is proportional to an expression independent of $f(g)$ and therefore justifies our manipulations in section 2. The second equality sign is the crucial step where we use the fact that the \vec{A} and \vec{E} can be expanded in the same set of eigenfunctions. For the chiral anomaly this is not the case since left and right handed fermions have different eigenfunction, or more precisely a different number of zero modes. For an arbitrary rescaling the two dimensional matrix on the second line is not of unit determinant and will therefore depend on the transformation [15].

B Trace anomaly and the Hamiltonian

In this appendix we show how the matrix element of the trace anomaly follows from an RGE of the Hamiltonian matrix elements. We consider

$$h(g, m, \mu, p) \equiv \langle \varphi(p) | \mathcal{H} | \varphi(p) \rangle_c \stackrel{(3)}{=} 2(E_\varphi(p))^2, \tag{A.3}$$

where $p = |\vec{p}|$ denotes the spatial angular momentum which is considered to be an external parameter. By the latter we mean that it is in particular *independent* on M_φ in accordance with the remark below Eq. (1). This type of matrix element satisfies an RGE of the form (e.g. [16])

$$\left(\bar{\beta} \frac{\partial}{\partial \bar{g}} - \bar{m}(1 + \bar{\gamma}) \frac{\partial}{\partial \bar{m}} + \Delta_h - p \frac{\partial}{\partial p}\right) h(\bar{g}, \bar{m}, \mu, p) = 0, \tag{A.4}$$

where $\Delta_h = 2$ is the scaling dimension of (A.3) which corresponds to the engineering dimension since E_φ is a physical observable. Using the fact that the

p -dependence is known exactly, $h = 2E_\varphi^2 = 2(M_\varphi^2 + \bar{p}^2)$, one can rewrite (A.4) as

$$\left(\bar{\beta} \frac{\partial}{\partial \bar{g}} - \bar{m}(1 + \bar{\gamma}) \frac{\partial}{\partial \bar{m}} + \Delta_{E^2}^{\text{eff}}\right) E_\varphi^2 = 0, \quad \Delta_{E^2}^{\text{eff}} \equiv 2 \frac{M_\varphi^2}{E_\varphi^2}. \quad (\text{A.5})$$

The two derivatives in (A.5) can be substituted by the relation (1) and $m \frac{\partial}{\partial m} E_\varphi^2 = \bar{m} \langle \varphi | \bar{q} \bar{q} | \varphi \rangle_c$ (e.g. eq (17) in [3]) with slight proliferation of notation in terms of barred quantities in the last expression. One obtains,

$$2M_\varphi^2 = \frac{\bar{\beta}}{2\bar{g}} \langle \varphi | \frac{1}{\bar{g}^2} \bar{G}^2 | \varphi \rangle_c + (1 + \bar{\gamma}) \bar{m} \langle \varphi | \bar{q} \bar{q} | \varphi \rangle_c, \quad (\text{A.6})$$

which corresponds to the well-known matrix element of the trace anomaly [17] between between a physical state (e.g. [3]).

We note that the derivation in this appendix corresponds to the, almost, backwards derivation of [3] where the Feynman-Hellmann relation (1) is derived from the trace anomaly. Furthermore it is also closely related to the heuristic derivation of the trace anomaly using $T_\alpha^\alpha \propto \frac{d}{d\mu} \mathcal{L}(\mu)$ where μ stands for some renormalization scale. The main reason for presenting the derivation is to clarify how matters work out for states with non-zero spatial momenta (i.e. $M_\varphi^2 \neq E_\varphi^2$). The latter necessitate an RGE (A.4) where the external momenta are taken into account.

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