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NONCOMMUTATIVE LOCALIZATION IN ALGEBRAIC L -THEORY

ANDREW RANICKI

ABSTRACT. Given a noncommutative (Cohn) localization $A \rightarrow \sigma^{-1}A$ which is injective and stably flat we obtain a lifting theorem for induced f.g. projective $\sigma^{-1}A$ -module chain complexes and localization exact sequences in algebraic L -theory, matching the algebraic K -theory localization exact sequence of Neeman-Ranicki [3] and Neeman [2].

CONTENTS

Introduction	1
1. Lifting chain complexes	3
2. Algebraic L -theory	9
References	18

INTRODUCTION

The series of papers [3], [2], studied the algebraic K -theory of the noncommutative (Cohn) localization $\sigma^{-1}A$ of a ring A inverting a collection σ of morphisms of f.g. projective left A -modules. By definition, $\sigma^{-1}A$ is stably flat if

$$\mathrm{Tor}_i^A(\sigma^{-1}A, \sigma^{-1}A) = 0 \quad (i \geq 1) .$$

An (A, σ) -module is an A -module T which admits a f.g. projective A -module resolution

$$0 \longrightarrow P \xrightarrow{s} Q \longrightarrow T \longrightarrow 0$$

with $s : \sigma^{-1}P \rightarrow \sigma^{-1}Q$ an isomorphism of the induced $\sigma^{-1}A$ -modules. For $A \rightarrow \sigma^{-1}A$ which is injective and stably flat we obtained an algebraic K -theory localization exact sequence

$$\dots \rightarrow K_n(A) \rightarrow K_n(\sigma^{-1}A) \rightarrow K_{n-1}(H(A, \sigma)) \rightarrow K_{n-1}(A) \rightarrow \dots$$

with $H(A, \sigma)$ the exact category of (A, σ) -modules.

Let C be a bounded $\sigma^{-1}A$ -module chain complex such that each $C_i = \sigma^{-1}P_i$ is induced from a f.g. projective A -module P_i . The *chain complex lifting problem* is to decide if C is chain equivalent to $\sigma^{-1}D$ for a bounded chain complex D of f.g. projective A -modules. The problem has a trivial affirmative solution for a commutative or Ore localization, by

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the clearing of denominators, when C is actually isomorphic to $\sigma^{-1}D$. In general, it is not possible to lift chain complexes: the injective noncommutative localizations $A \rightarrow \sigma^{-1}A$ which are not stably flat constructed in Neeman, Ranicki and Schofield [4, Remark 2.13] provide examples of induced f.g. projective $\sigma^{-1}A$ -module chain complexes of dimensions ≥ 3 which cannot be lifted.

In §1 we solve the chain complex lifting problem in the injective stably flat case, obtaining the following results (Theorems 1.4,1.5) :

Theorem 0.1. *For a stably flat injective noncommutative localization $A \rightarrow \sigma^{-1}A$ every bounded chain complex C of induced f.g. projective $\sigma^{-1}A$ -modules is chain equivalent to $\sigma^{-1}D$ for a bounded chain complex D of f.g. projective A -modules. Moreover, if C is n -dimensional*

$$C : \cdots \rightarrow 0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0 \rightarrow \cdots$$

then D can be chosen to be n -dimensional. □

In §2 we consider the algebraic L -theory of a noncommutative localization, obtaining the following results (Theorems 2.4, 2.5, 2.9) :

Theorem 0.2. *Let $A \rightarrow \sigma^{-1}A$ be a noncommutative localization of a ring with involution A , such that σ is invariant under the involution.*

(i) *There is a localization exact sequence of quadratic L -groups*

$$\cdots \longrightarrow L_n(A) \longrightarrow L_n^I(\sigma^{-1}A) \xrightarrow{\partial} L_n(A, \sigma) \longrightarrow L_{n-1}(A) \longrightarrow \cdots$$

with $I = \text{im}(K_0(A) \rightarrow K_0(\sigma^{-1}A))$, and $L_n(A, \sigma)$ the cobordism group of $\sigma^{-1}A$ -contractible $(n-1)$ -dimensional quadratic Poincaré complexes over A .

(ii) *If $\sigma^{-1}A$ is stably flat over A there is a localization exact sequence of symmetric L -groups*

$$\cdots \longrightarrow L^n(A) \longrightarrow L_I^n(\sigma^{-1}A) \xrightarrow{\partial} L^n(A, \sigma) \longrightarrow L^{n-1}(A) \longrightarrow \cdots$$

with $L^n(A, \sigma)$ the cobordism group of $\sigma^{-1}A$ -contractible $(n-1)$ -dimensional symmetric Poincaré complexes over A .

(iii) *If $A \rightarrow \sigma^{-1}A$ is injective then $L^n(A, \sigma)$ (resp. $L_n(A, \sigma)$) is the cobordism group of n -dimensional symmetric (resp. quadratic) Poincaré complexes of (A, σ) -modules. □*

The L -theory exact sequences of Theorem 0.2 for an injective Ore localization $A \rightarrow \sigma^{-1}A$ (which is flat and hence stably flat) were obtained in Ranicki [5]. The quadratic L -theory exact sequence of 0.2 (i) for arbitrary injective $A \rightarrow \sigma^{-1}A$ was obtained by Vogel [8], [9]. The symmetric L -theory exact sequence of 0.2 (ii) is new.

We refer to [6, 7] for some of the applications of the algebraic L -theory of noncommutative localizations to topology.

Amnon Neeman used to be a coauthor of the paper, but decided to withdraw in May 2007.

1. LIFTING CHAIN COMPLEXES

If $A \rightarrow \sigma^{-1}A$ is a stably flat localization, we know from [3, Theorem 0.4, Proposition 4.5 and Theorem 3.7] that the functor $Ti : \frac{D^{\text{perf}}(A)}{\mathfrak{R}^c} \rightarrow D^{\text{perf}}(\sigma^{-1}A)$ is just an idempotent completion; it is fully faithful and all objects in $D^{\text{perf}}(\sigma^{-1}A)$ are, up to isomorphisms, direct summands of objects in the image of Ti . A fairly easy consequence of this is the following. Let $C \in D^{\text{perf}}(\sigma^{-1}A)$ be the complex

$$0 \rightarrow \sigma^{-1}C^m \rightarrow \sigma^{-1}C^{m+1} \rightarrow \dots \rightarrow \sigma^{-1}C^{n-1} \rightarrow \sigma^{-1}C^n \rightarrow 0,$$

with C^i all finitely generated, projective A -modules. Then there is complex $X \in D^{\text{perf}}(A)$ with $C \simeq \{\sigma^{-1}A\}^L \otimes_A X$. That is, C is homotopy equivalent to the tensor product with $\sigma^{-1}A$ of a perfect complex over the ring A . In Section 1 we prove this (Theorem 1.4), and then refine the result to show that X may be chosen to be a complex of the form

$$0 \rightarrow X^m \rightarrow X^{m+1} \rightarrow \dots \rightarrow X^{n-1} \rightarrow X^n \rightarrow 0.$$

(Proof in Theorem 1.5).

Remark 1.1. The proof of Theorem 1.4 relies on the following fact about triangulated categories. Suppose \mathcal{A} is a full, triangulated subcategory of a triangulated category \mathcal{B} , and suppose all objects in \mathcal{B} are direct summands of objects of \mathcal{A} . An object $X \in \mathcal{B}$ belongs to $\mathcal{A} \subset \mathcal{B}$ if and only if $[X] \in K_0(\mathcal{B})$ lies in the image of $K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B})$. This fact may be found, for example, in [1, Proposition 4.5.11], but for the reader's convenience its proof is included here in Lemma 1.2 and Proposition 1.3. □

We begin by reminding the reader of some basic facts about Grothendieck groups. For any additive category \mathcal{A} we define $K_0^{\text{add}}(\mathcal{A})$ to be the Grothendieck group of the split exact category \mathcal{A} . This means that the short exact sequences in \mathcal{A} are precisely the split sequences. It is well known that every element of $K_0^{\text{add}}(\mathcal{A})$ can be expressed as

$$[X] - [Y]$$

for X and Y objects of \mathcal{A} . The expressions $[X] - [Y]$ and $[X'] - [Y']$ are equal in $K_0^{\text{add}}(\mathcal{A})$ if and only if there exists an object $P \in \mathcal{A}$ and an isomorphism

$$X \oplus Y' \oplus P = X' \oplus Y \oplus P.$$

If \mathcal{A} happens to be a triangulated category, then $K_0(\mathcal{A})$ means the quotient of $K_0^{\text{add}}(\mathcal{A})$ by a subgroup we will denote $T(\mathcal{A})$. The subgroup $T(\mathcal{A})$ is defined as the group generated by all

$$[X] - [Y] + [Z],$$

where there exists a distinguished triangle in \mathcal{A}

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X.$$

We prove:

Lemma 1.2. *Suppose \mathcal{B} is a triangulated category. Let \mathcal{A} be a full, triangulated subcategory of \mathcal{B} . Assume further that every object of \mathcal{B} is a direct summand of an object in $\mathcal{A} \subset \mathcal{B}$.*

Then the map $f : K_0^{\text{add}}(\mathcal{A}) \longrightarrow K_0^{\text{add}}(\mathcal{B})$ induces a surjection $T(\mathcal{A}) \longrightarrow T(\mathcal{B})$. In symbols: $f(T(\mathcal{A})) = T(\mathcal{B})$.

Proof. Let $[X] - [Y] + [Z]$ be a generator of $T(\mathcal{B}) \subset K_0^{\text{add}}(\mathcal{B})$. We need to show it lies in the image of $T(\mathcal{A}) \subset K_0^{\text{add}}(\mathcal{A})$. Suppose therefore that

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

is a distinguished triangle in \mathcal{B} . Because every object of \mathcal{B} is a direct summand of an object in \mathcal{A} , we can choose objects C and D with

$$X \oplus C, \quad Z \oplus D$$

both lying in \mathcal{A} . But then we have a two distinguished triangles in \mathcal{B}

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ C & \longrightarrow & C \oplus D & \longrightarrow & D & \xrightarrow{0} & \Sigma C \end{array}$$

and their direct sum is a distinguished triangle

$$X \oplus C \longrightarrow Y \oplus C \oplus D \longrightarrow Z \oplus D \longrightarrow \Sigma(X \oplus C).$$

Two of the objects lie in \mathcal{A} . Since the subcategory $\mathcal{A} \subset \mathcal{B}$ is full and triangulated, the entire distinguished triangle lies in \mathcal{A} . Thus

$$[X \oplus C] - [Y \oplus C \oplus D] + [Z \oplus D] = [X] - [Y] + [Z]$$

lies in the image of $T(\mathcal{A})$. □

The next proposition is well-known; again, the proof is included for the convenience of the reader.

Proposition 1.3. *Let the hypotheses be as in Lemma 1.2. That is, suppose \mathcal{B} is a triangulated category. Let \mathcal{A} be a full, triangulated subcategory of \mathcal{B} . Assume further that every object of \mathcal{B} is a direct summand of an object in $\mathcal{A} \subset \mathcal{B}$.*

If X is an object of \mathcal{B} and $[X]$ lies in the image of the natural map $f : K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B})$, then $X \in \mathcal{A}$.

Proof. If we consider $[X]$ as an element of $K_0^{\text{add}}(\mathcal{B})$, then saying that its image in $K_0(\mathcal{B})$ lies in the image of $K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B})$ is equivalent to saying that, modulo $T(\mathcal{B})$, $[X]$ lies in the image of $K_0^{\text{add}}(\mathcal{A})$. That is,

$$[X] \in T(\mathcal{B}) + f(K_0^{\text{add}}(\mathcal{A})) \subset K_0^{\text{add}}(\mathcal{B}).$$

By Lemma 1.2 we have that $f(T(\mathcal{A})) = T(\mathcal{B})$. Thus

$$\begin{aligned} T(\mathcal{B}) + f(K_0^{\text{add}}(\mathcal{A})) &= f(T(\mathcal{A})) + f(K_0^{\text{add}}(\mathcal{A})) \\ &= f(K_0^{\text{add}}(\mathcal{A})). \end{aligned}$$

That means there exist objects C and D in $\mathcal{A} \subset \mathcal{B}$ and an identity in $K_0^{\text{add}}(\mathcal{B})$

$$[X] = [C] - [D].$$

There must therefore be an object $P \in \mathcal{B}$ and an isomorphism

$$X \oplus D \oplus P \simeq C \oplus P.$$

But P is an object of \mathcal{B} , hence a direct summand of an object of \mathcal{A} . There is an object $P' \in \mathcal{B}$ with $P \oplus P' \in \mathcal{A}$. We have an isomorphism

$$X \oplus D \oplus P \oplus P' \simeq C \oplus P \oplus P'.$$

Putting $D' = D \oplus P \oplus P'$ and $C' = C \oplus P \oplus P'$ we have objects C', D' in \mathcal{A} , and a (split) distinguished triangle

$$D' \longrightarrow C' \longrightarrow X \longrightarrow \Sigma D'.$$

Since $\mathcal{A} \subset \mathcal{B}$ is triangulated we conclude that $X \in \mathcal{A}$. □

The relevance of these results to our work here is

Theorem 1.4. *Let $A \longrightarrow \sigma^{-1}A$ be a stably flat localization of rings. Suppose we are given a perfect complex C over $\sigma^{-1}A$. Suppose further that $C \in D^{\text{perf}}(\sigma^{-1}A)$ is of the form*

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \dots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0$$

where each C^i is a finitely generated, projective A -module. Then C is homotopy equivalent to $\{\sigma^{-1}A\}^L \otimes_A X$, for some $X \in D^{\text{perf}}(A)$.

Proof. The localization is stably flat. By [3, Theorem 0.4] the functor $T : \mathcal{T}^c \longrightarrow D^{\text{perf}}(\sigma^{-1}A)$ is an equivalence of categories. By [3, Proposition 4.5 and Theorem 3.7] we also know that the functor $i : \frac{D^{\text{perf}}(A)}{\mathcal{T}^c} \longrightarrow \mathcal{T}^c$ is fully faithful, and that every object in \mathcal{T}^c is isomorphic to a direct summand of an object in the image of i . Next we apply Proposition 1.3, with $\mathcal{B} = D^{\text{perf}}(\sigma^{-1}A)$ and \mathcal{A} the full subcategory containing all objects isomorphic to $Ti(x)$, for any $x \in \frac{D^{\text{perf}}(A)}{\mathcal{T}^c}$.

Now C is an object of $D^{\text{perf}}(\sigma^{-1}A)$, and in $K_0(D^{\text{perf}}(\sigma^{-1}A))$ we have an identity

$$[C] = \sum_{\ell=-\infty}^{\infty} (-1)^\ell [\sigma^{-1}C^\ell]$$

with

$$[\sigma^{-1}C^\ell] = [\{\sigma^{-1}A\} \otimes_A C^\ell] = [TiC^\ell]$$

certainly lying in the image of the map

$$K_0(Ti) : K_0\left(\frac{D^{\text{perf}}(A)}{\mathcal{R}^c}\right) \longrightarrow K_0(D^{\text{perf}}(\sigma^{-1}A)).$$

Proposition 1.3 therefore tells us that C is isomorphic to an object in the image of the functor Ti . There exists a perfect complex $X \in D^{\text{perf}}(A)$ and a homotopy equivalence $C \simeq \{\sigma^{-1}A\}^L \otimes_A X$. \square

The problem with Theorem 1.4 is that it gives us no bound on the length of the complex X with $\{\sigma^{-1}A\}^L \otimes_A X \simeq C$. We really want to know

Theorem 1.5. *Let $A \longrightarrow \sigma^{-1}A$ be a stably flat localization of rings. Suppose $C \in D^{\text{perf}}(\sigma^{-1}A)$ is the complex*

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \dots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0.$$

Then the complex $X \in D^{\text{perf}}(A)$ with $C \simeq \{\sigma^{-1}A\}^L \otimes_A X$, whose existence is guaranteed by Theorem 1.4, may be chosen to be a complex

$$0 \longrightarrow X^m \longrightarrow X^{m+1} \longrightarrow \dots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0.$$

If $m = n$ this is easy. For $m < n$ we need to prove something. Our proof will appeal to the results of [3, Section 4]. We remind the reader that this was the section which dealt with the subcategories $\mathcal{K}[m, n]$ of complexes in \mathcal{R}^c vanishing outside the range $[m, n]$. First we need a lemma.

Lemma 1.6. *Let M and N be any finitely generated projective A -modules. We may view M and N as objects in the derived category $D^{\text{perf}}(A)$, concentrated in degree 0. Then any map in $\mathcal{T}^c(\pi M, \pi N)$ can be represented as $\pi(\alpha)^{-1}\pi(\beta)$, for some α, β morphisms in $D^{\text{perf}}(A)$ as below*

$$M \xrightarrow{\beta} Y \xleftarrow{\alpha} N.$$

The map $\alpha : N \longrightarrow Y$ fits in a triangle

$$X \longrightarrow N \xrightarrow{\alpha} Y \longrightarrow \Sigma X$$

and X may be chosen to lie in $\mathcal{K}[0, 1]$.

Proof. By [3, Proposition 4.5 and Theorem 3.7] we know that the map

$$i : \frac{D^{\text{perf}}(A)}{\mathcal{R}^c} \longrightarrow \mathcal{T}^c$$

is fully faithful. Therefore

$$\mathcal{T}^c(\pi M, \pi N) = \frac{D^{\text{perf}}(A)}{\mathcal{R}^c}(M, N).$$

That is, any map $\pi M \longrightarrow \pi N$ can be written as $\pi(\alpha)^{-1}\pi(\beta)$, for some α, β morphisms in $D^{\text{perf}}(A)$ as below

$$M \xrightarrow{\beta} Y \xleftarrow{\alpha} N.$$

The map $\alpha : N \longrightarrow Y$ fits in a triangle

$$X \longrightarrow N \xrightarrow{\alpha} Y \xrightarrow{\beta} \Sigma X$$

and X may be chosen to lie in \mathcal{R}^c . What is not clear is that we may choose X in $\mathcal{K}[0, 1] \subset \mathcal{R}^c$.

The easy observation is that we may certainly modify our choice of X to lie in $\mathcal{K} \subset \mathcal{R}^c$. This follows from [2, Lemma 4.5], which tells us that for any choice of X as above there exists an X' with $X \oplus X'$ isomorphic to an object in \mathcal{K} . We have a distinguished triangle

$$X \oplus X' \longrightarrow N \xrightarrow{\begin{pmatrix} \alpha \\ 0 \end{pmatrix}} Y \oplus \Sigma X' \xrightarrow{\beta \oplus 1} \Sigma(X \oplus X')$$

and a diagram

$$M \xrightarrow{\begin{pmatrix} \beta \\ 0 \end{pmatrix}} Y \oplus \Sigma X' \xleftarrow{\begin{pmatrix} \alpha \\ 0 \end{pmatrix}} N,$$

and replacing our original choices by these we may assume $X \in \mathcal{K}$. Now we have to shorten X .

By [2, Lemma 4.7], there exists a triangle in \mathcal{R}^c

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X'$$

with $X' \in \mathcal{K}[1, \infty)$ and $X'' \in \mathcal{K}(-\infty, 1]$. The composite $X' \longrightarrow X \longrightarrow N$ is a map from $X' \in \mathcal{K}[1, \infty)$ to $N \in \mathcal{S}^{\leq 0}$, which must vanish. Hence we have that $X \longrightarrow N$ factors as $X \longrightarrow X'' \longrightarrow N$. We complete to a morphism of triangles

$$\begin{array}{ccccccc} X & \longrightarrow & N & \xrightarrow{\alpha} & Y & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'' & \longrightarrow & N & \xrightarrow{\gamma\alpha} & Y'' & \longrightarrow & \Sigma X'' \end{array}$$

and another representative of our morphism is the diagram

$$M \xrightarrow{\gamma\beta} Y'' \xleftarrow{\gamma\alpha} N$$

We may, on replacing Y by Y'' , assume $X \in \mathcal{K}(-\infty, 1]$.

Applying [2, Lemma 4.7] again, we have that any $X \in \mathcal{K}(-\infty, 1]$ admits a triangle

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X'$$

with $X' \in \mathcal{K}[0, 1]$ and $X'' \in \mathcal{K}(-\infty, 0]$. Form the octahedron

$$\begin{array}{ccccccc}
X' & \longrightarrow & N & \xrightarrow{\alpha'} & Y' & \longrightarrow & \Sigma X' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & N & \xrightarrow{\alpha} & Y & \longrightarrow & \Sigma X \\
& & & & \downarrow & & \downarrow \\
& & & & \Sigma X'' & \xrightarrow{1} & \Sigma X''
\end{array}$$

The composite $M \longrightarrow Y \longrightarrow \Sigma X''$ is a map from the projective module M , viewed as a complex concentrated in degree 0, to $\Sigma X'' \in \mathcal{K}(\infty, -1]$. This composite must vanish.

The map $\beta : M \longrightarrow Y$ therefore factors as $M \xrightarrow{\beta'} Y' \xrightarrow{\gamma} Y$, and our morphism in \mathcal{T}^c has a representative

$$M \xrightarrow{\beta'} Y' \xleftarrow{\alpha'} N$$

so that in the triangle

$$X' \longrightarrow N \xrightarrow{\alpha'} Y' \longrightarrow \Sigma X'$$

X' may be chosen to lie in $\mathcal{K}[0, 1]$. □

Now we are ready for

Proof of Theorem 1.5. We are given a complex $C \in D^{\text{perf}}(\sigma^{-1}A)$ of the form

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \dots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0.$$

To eliminate the trivial case, assume $m \leq n + 1$. Shifting, we may assume $m = 0$ and $n \geq 1$. Theorem 1.4 guarantees that C is homotopy equivalent to $\{\sigma^{-1}A\}^L \otimes_A D$, with $D \in D^{\text{perf}}(A)$. But D need not be supported on the interval $[0, n]$. We need to show how to shorten D . Assume therefore that D is supported on $[-1, n]$. We will show how to replace D by a complex supported on $[0, n]$. Shortening a complex supported on $[0, n+1]$ is dual, and we leave it to the reader.

We may suppose therefore that $D \in D^{\text{perf}}(A)$ is the complex

$$\dots \longrightarrow 0 \longrightarrow D^{-1} \longrightarrow D^0 \longrightarrow \dots \longrightarrow D^n \longrightarrow 0 \longrightarrow \dots$$

and that there is a homotopy equivalence of $\sigma^{-1}D$ with a shorter complex, that is a commutative diagram

$$\begin{array}{ccccccccccc}
\longrightarrow & 0 & \longrightarrow & \sigma^{-1}D^{-1} & \xrightarrow{\partial} & \sigma^{-1}D^0 & \longrightarrow & \dots & \longrightarrow & \sigma^{-1}D^n & \longrightarrow & 0 & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \sigma^{-1}C^0 & \longrightarrow & \dots & \longrightarrow & \sigma^{-1}C^n & \longrightarrow & 0 & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & \sigma^{-1}D^{-1} & \xrightarrow{\partial} & \sigma^{-1}D^0 & \longrightarrow & \dots & \longrightarrow & \sigma^{-1}D^n & \longrightarrow & 0 & \longrightarrow
\end{array}$$

so that the composite is homotopic to the identity. In particular, there is a map $d : \sigma^{-1}D^0 \rightarrow \sigma^{-1}D^{-1}$ so that $dd : \sigma^{-1}D^{-1} \rightarrow \sigma^{-1}D^{-1}$ is the identity.

By [2, Proposition 3.1] the map $d : \sigma^{-1}D^0 \rightarrow \sigma^{-1}D^{-1}$ lifts uniquely to a map $d' : \pi D^0 \rightarrow \pi D^{-1}$. By Lemma 1.6 the map d' can be represented as $\pi(\alpha)^{-1}\pi(\beta)$, where α and β are, respectively, the chain maps

$$\begin{array}{ccccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & D^{-1} & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{r} & Y & \longrightarrow & 0 & \longrightarrow \end{array}$$

and

$$\begin{array}{ccccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & D^0 & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & g \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{r} & Y & \longrightarrow & 0 & \longrightarrow \end{array}$$

The fact that $\sigma^{-1}\alpha$ is an equivalence tells us that the map $\sigma^{-1}r : \sigma^{-1}X \rightarrow \sigma^{-1}Y$ is injective, with cokernel $\sigma^{-1}D^{-1}$. The fact that $\alpha^{-1}\beta$ agrees with d' means that the composite

$$\sigma^{-1}D^0 \xrightarrow{\sigma^{-1}g} \sigma^{-1}Y \longrightarrow \text{Coker}(\sigma^{-1}r)$$

is just the map $d : \sigma^{-1}D^0 \rightarrow \sigma^{-1}D^{-1}$. Let X be the chain complex

$$\longrightarrow 0 \longrightarrow D^0 \oplus X \xrightarrow{\begin{pmatrix} \partial & 0 \\ g & r \end{pmatrix}} D^1 \oplus Y \longrightarrow \dots \longrightarrow D^n \longrightarrow 0 \longrightarrow$$

Let $f : X \rightarrow D$ be the natural map of chain complexes

$$\begin{array}{ccccccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & D^0 \oplus X & \xrightarrow{\begin{pmatrix} \partial & 0 \\ g & r \end{pmatrix}} & D^1 \oplus Y & \longrightarrow & \dots & \longrightarrow & D^n & \longrightarrow & 0 & \longrightarrow \\ & & & \downarrow & & \pi_1 \downarrow & & \downarrow \pi_1 & & & & & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & D^{-1} & \longrightarrow & D^0 & \xrightarrow{\partial} & D^1 & \longrightarrow & \dots & \longrightarrow & D^n & \longrightarrow & 0 & \longrightarrow \end{array}$$

where the vertical maps labelled π_1 are the projections to the first factor of the direct sum. The map $\sigma^{-1}f$ is easily seen to be homotopy equivalence. Thus $\sigma^{-1}X$ is homotopy equivalent to $\sigma^{-1}D \cong C$. \square

2. ALGEBRAIC L -THEORY

An *involution* on a ring A is an anti-automorphism

$$A \longrightarrow A ; r \mapsto \bar{r} .$$

The involution is used to regard a left A -module M as a right A -module by

$$M \times A \longrightarrow M ; (x, r) \mapsto \bar{r}x .$$

The *dual* of a (left) A -module M is the A -module

$$M^* = \text{Hom}_A(M, A), \quad A \times M^* \longrightarrow M^*; \quad (r, f) \mapsto (x \mapsto f(x)\bar{r}).$$

The *dual* of an A -module morphism $s : P \longrightarrow Q$ is the A -module morphism

$$s^* : Q^* \longrightarrow P^*; \quad f \mapsto (x \mapsto f(s(x))).$$

If M is f.g. projective then so is M^* , and

$$M \longrightarrow M^{**}; \quad x \mapsto (f \mapsto \overline{f(x)})$$

is an isomorphism which is used to identify $M^{**} = M$.

Hypothesis 2.1. *In this section, we assume that*

- (i) A is a ring with involution,
- (ii) the duals of morphisms $s : P \longrightarrow Q$ in σ are morphisms $s^* : Q^* \longrightarrow P^*$ in σ ,
- (iii) $\epsilon \in A$ is a central unit such that $\bar{\epsilon} = \epsilon^{-1}$ (e.g. $\epsilon = \pm 1$).

The noncommutative localization $\sigma^{-1}A$ is then also a ring with involution, with $\epsilon \in \sigma^{-1}A$ a central unit such that $\bar{\epsilon} = \epsilon^{-1}$. \square

We review briefly the chain complex construction of the f.g. projective ϵ -quadratic L -groups $L_*(A, \epsilon)$ and the ϵ -symmetric L -groups $L^*(A, \epsilon)$. Given an A -module chain complex C let the generator $T \in \mathbb{Z}_2$ act on the \mathbb{Z} -module chain complex $C \otimes_A C$ by the ϵ -transposition duality

$$T_\epsilon : C_p \otimes_A C_q \longrightarrow C_q \otimes_A C_p : x \otimes y \mapsto (-1)^{pq} \epsilon y \otimes x.$$

Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$ -module resolution of \mathbb{Z}

$$W : \dots \longrightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2].$$

The ϵ -symmetric (resp. ϵ -quadratic) Q -groups of C are the \mathbb{Z}_2 -hypercohomology (resp. \mathbb{Z}_2 -hyperhomology) groups of $C \otimes_A C$

$$Q^n(C, \epsilon) = H^n(\mathbb{Z}_2; C \otimes_A C) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_A C)),$$

$$Q_n(C, \epsilon) = H_n(\mathbb{Z}_2; C \otimes_A C) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]}(C \otimes_A C)).$$

The Q -groups are chain homotopy invariants of C . There are defined forgetful maps

$$1 + T_\epsilon : Q_n(C, \epsilon) \longrightarrow Q^n(C, \epsilon); \quad \psi \mapsto (1 + T_\epsilon)\psi,$$

$$Q^n(C, \epsilon) \longrightarrow H_n(C \otimes_A C); \quad \phi \mapsto \phi_0.$$

For f.g. projective C the function

$$C \otimes_A C \longrightarrow \text{Hom}_A(C^*, C); \quad x \otimes y \mapsto (f \mapsto \overline{f(x)}y)$$

is an isomorphism of $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complexes, with $T \in \mathbb{Z}_2$ acting on $\text{Hom}_A(C^*, C)$ by $\theta \mapsto \epsilon\theta^*$. The element $\phi_0 \in H_n(C \otimes_A C) = H_n(\text{Hom}_A(C^*, C))$ is a chain homotopy class of A -module chain maps $\phi_0 : C^{n-*} \longrightarrow C$.

An n -dimensional ϵ -symmetric complex over A (C, ϕ) is a bounded f.g. projective A -module chain complex C together with an element $\phi \in Q^n(C, \epsilon)$. The complex (C, ϕ) is *Poincaré* if the A -module chain map $\phi_0 : C^{n-*} \rightarrow C$ is a chain equivalence.

Example 2.2. A 0-dimensional ϵ -symmetric Poincaré complex (C, ϕ) over A is essentially the same as a nonsingular ϵ -symmetric form (M, λ) over (A, σ) , with $M = (C_0)^*$ a f.g. projective A -module and

$$\lambda = \phi_0 : M \times M \rightarrow A$$

a sesquilinear pairing such that the adjoint

$$M \rightarrow M^* ; x \mapsto (y \mapsto \lambda(x, y))$$

is an A -module isomorphism. □

See pp. 210–211 of [6] for the notion of an ϵ -symmetric (*Poincaré*) pair. The *boundary* of an n -dimensional ϵ -symmetric complex (C, ϕ) is the $(n-1)$ -dimensional ϵ -symmetric Poincaré complex

$$\partial(C, \phi) = (\partial C, \partial \phi)$$

with $\partial C = C(\phi_0 : C^{n-*} \rightarrow C)_{*+1}$ and $\partial \phi$ as defined on p. 218 of [6]. The n -dimensional ϵ -symmetric L -group $L^n(A, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric Poincaré complexes (C, ϕ) over A with C n -dimensional. In particular, $L^0(A, \epsilon)$ is the Witt group of nonsingular ϵ -symmetric forms over A .

An n -dimensional ϵ -symmetric complex (C, ϕ) over A is $\sigma^{-1}A$ -Poincaré if the $\sigma^{-1}A$ -module chain map $\sigma^{-1}\phi_0 : \sigma^{-1}C^{n-*} \rightarrow \sigma^{-1}C$ is a chain equivalence, in which case $\sigma^{-1}(C, \phi)$ is an n -dimensional ϵ -symmetric Poincaré complex over $\sigma^{-1}A$.

The n -dimensional ϵ -symmetric Γ -group $\Gamma^n(A \rightarrow \sigma^{-1}A, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric $\sigma^{-1}A$ -Poincaré complexes (C, ϕ) over A such that $\sigma^{-1}C$ is chain equivalent to an n -dimensional induced f.g. projective $\sigma^{-1}A$ -module chain complex. The n -dimensional ϵ -symmetric L -group $L^n(A, \sigma, \epsilon)$ is the cobordism group of $(n-1)$ -dimensional ϵ -symmetric Poincaré complexes over A (C, ϕ) such that C is $\sigma^{-1}A$ -contractible, i.e. $\sigma^{-1}C \simeq 0$.

Similarly in the ϵ -quadratic case, with groups $L_n(A, \epsilon)$, $\Gamma_n(A \rightarrow \sigma^{-1}A, \epsilon)$, $L_n(A, \sigma, \epsilon)$. The ϵ -quadratic L - and Γ -groups are 4-periodic

$$L_n(A, \epsilon) = L_{n+2}(A, -\epsilon) = L_{n+4}(A, \epsilon) ,$$

$$\Gamma_n(A \rightarrow \sigma^{-1}A, \epsilon) = \Gamma_{n+2}(A \rightarrow \sigma^{-1}A, -\epsilon) = \Gamma_{n+4}(A \rightarrow \sigma^{-1}A, \epsilon) ,$$

$$L_n(A, \sigma, \epsilon) = L_{n+2}(A, \sigma, -\epsilon) = L_{n+4}(A, \sigma, \epsilon) .$$

Proposition 2.3. *For any ring with involution A and noncommutative localization $\sigma^{-1}A$ there is defined a localization exact sequence of ϵ -symmetric L -groups*

$$\cdots \longrightarrow L^n(A, \epsilon) \longrightarrow \Gamma^n(A \longrightarrow \sigma^{-1}A, \epsilon) \xrightarrow{\partial} L^n(A, \sigma, \epsilon) \longrightarrow L^{n-1}(A, \epsilon) \longrightarrow \cdots$$

Similarly in the ϵ -quadratic case, with an exact sequence

$$\cdots \longrightarrow L_n(A, \epsilon) \longrightarrow \Gamma_n(A \longrightarrow \sigma^{-1}A, \epsilon) \xrightarrow{\partial} L_n(A, \sigma, \epsilon) \longrightarrow L_{n-1}(A, \epsilon) \longrightarrow \cdots$$

Proof. The relative group of $L^n(A, \epsilon) \longrightarrow \Gamma^n(A \longrightarrow \sigma^{-1}A, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric $\sigma^{-1}A$ -Poincaré pairs over A ($f : C \longrightarrow D, (\delta\phi, \phi)$) with (C, ϕ) Poincaré. The effect of algebraic surgery on (C, ϕ) using this pair is a cobordant $(n-1)$ -dimensional ϵ -symmetric Poincaré complex (C', ϕ') with C' $\sigma^{-1}A$ -contractible. The function $(f : C \longrightarrow D, (\delta\phi, \phi)) \mapsto (C', \phi')$ defines an isomorphism between the relative group and $L^n(A, \sigma, \epsilon)$. \square

Define

$$I = \text{im}(K_0(A) \longrightarrow K_0(\sigma^{-1}A)) ,$$

the subgroup of $K_0(\sigma^{-1}A)$ consisting of the projective classes of the f.g. projective $\sigma^{-1}A$ -modules induced from f.g. projective A -modules. By definition, $L_I^n(\sigma^{-1}A, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric Poincaré complexes over $\sigma^{-1}A$ (B, θ) such that $[B] \in I$. There are evident morphisms of Γ - and L -groups

$$\begin{aligned} \sigma^{-1}\Gamma^* &: \Gamma^n(A \longrightarrow \sigma^{-1}A, \epsilon) \longrightarrow L_I^n(\sigma^{-1}A, \epsilon) ; (C, \phi) \mapsto \sigma^{-1}(C, \phi) , \\ \sigma^{-1}\Gamma_* &: \Gamma_n(A \longrightarrow \sigma^{-1}A, \epsilon) \longrightarrow L_n^I(\sigma^{-1}A, \epsilon) ; (C, \psi) \mapsto \sigma^{-1}(C, \psi) . \end{aligned}$$

In general, the morphisms $\sigma^{-1}\Gamma^*, \sigma^{-1}\Gamma_*$ need not be isomorphisms, since a bounded f.g. projective $\sigma^{-1}A$ -module chain complex D with $[D] \in I$ need not be chain equivalent to $\sigma^{-1}C$ for a bounded f.g. projective A -module chain complex C .

It was proved in Chapter 3 of Ranicki [5] that if $A \longrightarrow \sigma^{-1}A$ is an injective Ore localization then the morphisms $\sigma^{-1}Q^*, \sigma^{-1}Q_*, \sigma^{-1}\Gamma^*, \sigma^{-1}\Gamma_*$ are isomorphisms, so that there are defined localization exact sequences for both the ϵ -symmetric and the ϵ -quadratic L -groups

$$\begin{aligned} \cdots \longrightarrow L^n(A, \epsilon) \longrightarrow L_I^n(\sigma^{-1}A, \epsilon) \xrightarrow{\partial} L^n(A, \sigma, \epsilon) \longrightarrow L^{n-1}(A, \epsilon) \longrightarrow \cdots , \\ \cdots \longrightarrow L_n(A, \epsilon) \longrightarrow L_n^I(\sigma^{-1}A, \epsilon) \xrightarrow{\partial} L_n(A, \sigma, \epsilon) \longrightarrow L_{n-1}(A, \epsilon) \longrightarrow \cdots . \end{aligned}$$

Special cases of these sequences were obtained by Milnor-Husemoller, Karoubi, Pardon, Smith, Carlsson-Milgram.

Let $G\pi : D(A) \rightarrow D(A)$ be the functor of Proposition 6.1 of [3], with $D(A)$ the derived category of A . For any bounded f.g. projective A -module chain complex C the natural A -module chain map

$$\varinjlim_{(B, \beta)} B = G\pi(C) \longrightarrow \sigma^{-1}C$$

induces morphisms

$$\begin{aligned}\sigma^{-1}Q^* &: \varinjlim_{(B,\beta)} Q^n(B, \epsilon) = Q^n(G\pi(C), \epsilon) \longrightarrow Q^n(\sigma^{-1}C, \epsilon), \\ \sigma^{-1}Q_* &: \varinjlim_{(B,\beta)} Q_n(B, \epsilon) = Q_n(G\pi(C), \epsilon) \longrightarrow Q_n(\sigma^{-1}C, \epsilon)\end{aligned}$$

with the direct limits taken over all the bounded f.g. projective A -module chain complexes B with a chain map $\beta : C \longrightarrow B$ such that $\sigma^{-1}\beta : \sigma^{-1}C \longrightarrow \sigma^{-1}B$ is a $\sigma^{-1}A$ -module chain equivalence. The natural projection $D \otimes_A D \longrightarrow D \otimes_{\sigma^{-1}A} D$ is an isomorphism for any bounded f.g. projective $\sigma^{-1}A$ -module chain complex D (since this is already the case for $D = \sigma^{-1}A$), so the Q -groups of $\sigma^{-1}C$ are the same whether $\sigma^{-1}C$ is regarded as an A -module or $\sigma^{-1}A$ -module chain complex.

Theorem 2.4. (Vogel [9], Theorem 8.4) *For any ring with involution A and noncommutative localization $\sigma^{-1}A$ the morphisms*

$$\sigma^{-1}\Gamma_* : \Gamma_n(A \longrightarrow \sigma^{-1}A, \epsilon) \longrightarrow L_n^I(\sigma^{-1}A, \epsilon) ; (C, \psi) \mapsto \sigma^{-1}(C, \psi)$$

are isomorphisms, and there is a localization exact sequence of ϵ -quadratic L -groups

$$\cdots \longrightarrow L_n(A, \epsilon) \longrightarrow L_n^I(\sigma^{-1}A, \epsilon) \xrightarrow{\partial} L_n(A, \sigma, \epsilon) \longrightarrow L_{n-1}(A, \epsilon) \longrightarrow \cdots$$

Proof. By algebraic surgery below the middle dimension it suffices to consider only the special cases $n = 0, 1$. In effect, it was proved in [9] that $\sigma^{-1}Q_*$ is an isomorphism for 0- and 1-dimensional C . \square

It was claimed in Proposition 25.4 of Ranicki [6] that $\sigma^{-1}\Gamma^*$ is also an isomorphism, assuming (incorrectly) that the chain complex lifting problem can always be solved. However, we do have :

Theorem 2.5. *If $\sigma^{-1}A$ is a noncommutative localization of a ring with involution A which is stably flat over A , there is a localization exact sequence of ϵ -symmetric L -groups*

$$\cdots \longrightarrow L^n(A, \epsilon) \longrightarrow L_I^n(\sigma^{-1}A, \epsilon) \xrightarrow{\partial} L^n(A, \sigma, \epsilon) \longrightarrow L^{n-1}(A, \epsilon) \longrightarrow \cdots$$

Proof. For any bounded f.g. projective A -module chain complex C the natural A -module chain map $G\pi(C) \longrightarrow \sigma^{-1}C$ induces isomorphisms in homology

$$H_*(G\pi(C)) \cong H_*(\sigma^{-1}C).$$

Thus the natural $\mathbb{Z}[\mathbb{Z}_2]$ -module chain map

$$G\pi(C) \otimes_A G\pi(C) \longrightarrow \sigma^{-1}C \otimes_A \sigma^{-1}C = \sigma^{-1}C \otimes_{\sigma^{-1}A} \sigma^{-1}C$$

induces isomorphisms of ϵ -symmetric Q -groups

$$\sigma^{-1}Q^* : \varinjlim_{(B,\beta)} Q^n(B, \epsilon) \longrightarrow Q^n(\sigma^{-1}C, \epsilon)$$

(and also isomorphisms $\sigma^{-1}Q_*$ of ϵ -quadratic Q -groups). By Theorem 0.1 every n -dimensional induced f.g. projective $\sigma^{-1}A$ -module chain complex D is chain equivalent to $\sigma^{-1}C$ for an n -dimensional f.g. projective A -module chain complex C , with

$$Q^n(D, \epsilon) = Q^n(\sigma^{-1}C, \epsilon) = \varinjlim_{(B, \beta)} Q^n(B, \epsilon) .$$

It follows that the morphisms of ϵ -symmetric Γ - and L -groups

$$\sigma^{-1}\Gamma^* : \Gamma^n(A \longrightarrow \sigma^{-1}A, \epsilon) \longrightarrow L_I^n(\sigma^{-1}A, \epsilon) ; (C, \phi) \mapsto \sigma^{-1}(C, \phi)$$

are also isomorphisms, and the localization exact sequence is given by Proposition 2.3. \square

Hypothesis 2.6. *For the remainder of this section, we assume Hypothesis 2.1 and also that $A \longrightarrow \sigma^{-1}A$ is an injection.* \square

As in Proposition 2.2 of [2] it follows that all the morphisms in σ are injections.

We shall now generalize the results of Ranicki [5] and Vogel [8] to prove that under Hypotheses 2.1, 2.6 the relative L -groups $L^*(A, \sigma, \epsilon)$, $L_*(A, \sigma, \epsilon)$ in the L -theory localization exact sequences are the L -groups of $H(A, \sigma)$ with respect to the following duality involution.

Define the *torsion dual* of an (A, σ) -module M to be the (A, σ) -module

$$M^\wedge = \text{Ext}_A^1(M, A) ,$$

using the involution on A to define the left A -module structure. If M has f.g. projective A -module resolution

$$0 \longrightarrow P_1 \xrightarrow{s} P_0 \longrightarrow M \longrightarrow 0$$

with $s \in \sigma$ the torsion dual M^\wedge has the dual f.g. projective A -module resolution

$$0 \longrightarrow P_0^* \xrightarrow{s^*} P_1^* \longrightarrow M^\wedge \longrightarrow 0$$

with $s^* \in \sigma$.

Proposition 2.7. *Let $M = \text{coker}(s : P_1 \longrightarrow P_0)$, $N = \text{coker}(t : Q_1 \longrightarrow Q_0)$ be (A, σ) -modules.*

(i) *The adjoint of the pairing*

$$M \times M^\wedge \longrightarrow \sigma^{-1}A/A ; (g \in P_0, f \in P_1^*) \mapsto fs^{-1}g$$

defines a natural A -module isomorphism

$$M^\wedge \longrightarrow \text{Hom}_A(M, \sigma^{-1}A/A) ; f \mapsto (g \mapsto fs^{-1}g) .$$

(ii) *The natural A -module morphism*

$$M \longrightarrow M^{\wedge\wedge} ; x \mapsto (f \mapsto \overline{f(x)})$$

is an isomorphism.

(iii) There are natural identifications

$$\begin{aligned} M \otimes_A N &= \mathrm{Tor}_0^A(M, N) = \mathrm{Ext}_A^1(M^\wedge, N) = H_0(P \otimes_A Q), \\ \mathrm{Hom}_A(M^\wedge, N) &= \mathrm{Tor}_1^A(M, N) = \mathrm{Ext}_A^0(M^\wedge, N) = H_1(P \otimes_A Q). \end{aligned}$$

The functions

$$\begin{aligned} M \otimes_A N &\longrightarrow N \otimes_A M ; x \otimes y \mapsto y \otimes x, \\ \mathrm{Hom}_A(M^\wedge, N) &\longrightarrow \mathrm{Hom}_A(N^\wedge, M) ; f \mapsto f^\wedge \end{aligned}$$

determine transposition isomorphisms

$$T : \mathrm{Tor}_i^A(M, N) \longrightarrow \mathrm{Tor}_i^A(N, M) \quad (i = 0, 1).$$

(iv) For any finite subset $V = \{v_1, v_2, \dots, v_k\} \subset M \otimes_A N$ there exists an exact sequence of (A, σ) -modules

$$0 \longrightarrow N \longrightarrow L \longrightarrow \bigoplus_k M^\wedge \longrightarrow 0$$

such that $V \subset \ker(M \otimes_A N \longrightarrow M \otimes_A L)$.

Proof. (i) Apply the snake lemma to the morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(P_0, A) & \longrightarrow & \mathrm{Hom}_A(P_0, \sigma^{-1}A) & \longrightarrow & \mathrm{Hom}_A(P_0, \sigma^{-1}A/A) \longrightarrow 0 \\ & & \downarrow s^* & & \downarrow s_1^* & & \downarrow s_2^* \\ 0 & \longrightarrow & \mathrm{Hom}_A(P_1, A) & \longrightarrow & \mathrm{Hom}_A(P_1, \sigma^{-1}A) & \longrightarrow & \mathrm{Hom}_A(P_1, \sigma^{-1}A/A) \longrightarrow 0 \end{array}$$

with s^* injective, s_1^* an isomorphism and s_2^* surjective, to verify that the A -module morphism

$$M^\wedge = \mathrm{coker}(s^*) \longrightarrow \mathrm{Hom}_A(M, \sigma^{-1}A/A) = \ker(s_2^*)$$

is an isomorphism.

(ii) Immediate from the identification

$$s^{**} = s : (P_0)^{**} = P_0 \longrightarrow (P_1)^{**} = P_1.$$

(iii) Exercise for the reader.

(iv) Lift each $v_i \in M \otimes_A N$ to an element

$$v_i \in P_0 \otimes_A Q_0 = \mathrm{Hom}_A(P_0^*, Q_0) \quad (1 \leq i \leq k).$$

The A -module morphism defined by

$$u = \begin{pmatrix} s^* & 0 & 0 & \dots & 0 \\ 0 & s^* & 0 & \dots & 0 \\ 0 & 0 & s^* & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_1 & v_2 & v_3 & \dots & t \end{pmatrix} : U_1 = (\bigoplus_k P_0^*) \oplus Q_1 \longrightarrow U_0 = (\bigoplus_k P_1^*) \oplus Q_0$$

is in σ , so that $L = \text{coker}(u)$ is an (A, σ) -module with a f.g. projective A -module resolution

$$0 \longrightarrow U_1 \xrightarrow{u} U_0 \longrightarrow L \longrightarrow 0 .$$

The short exact sequence of 1-dimensional f.g. projective A -module chain complexes

$$0 \longrightarrow Q \longrightarrow U \longrightarrow \bigoplus_k P^{1-*} \longrightarrow 0$$

is a resolution of a short exact sequence of (A, σ) -modules

$$0 \longrightarrow N \longrightarrow L \longrightarrow \bigoplus_k M^\wedge \longrightarrow 0 .$$

The first morphism in the exact sequence

$$\text{Tor}_1^A(M, \bigoplus_k M^\wedge) \longrightarrow M \otimes_A N \longrightarrow M \otimes_A L \longrightarrow M \otimes_A (\bigoplus_k M^\wedge) \longrightarrow 0$$

sends $1_i \in \text{Tor}_1^A(M, \bigoplus_k M^\wedge) = \bigoplus_k \text{Hom}_A(M^\wedge, M^\wedge)$ to $v_i \in \ker(M \otimes_A N \longrightarrow M \otimes_A L)$. \square

Given an (A, σ) -module chain complex C define the ϵ -symmetric (resp. ϵ -quadratic) torsion Q -groups of C to be the \mathbb{Z}_2 -hypercohomology (resp. \mathbb{Z}_2 -hyperhomology) groups of the ϵ -transposition involution $T_\epsilon = \epsilon T$ on the \mathbb{Z} -module chain complex $\text{Tor}_1^A(C, C) = \text{Hom}_A(C^\wedge, C)$

$$Q_{\text{tor}}^n(C, \epsilon) = H^n(\mathbb{Z}_2; \text{Tor}_1^A(C, C)) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \text{Tor}_1^A(C, C))) ,$$

$$Q_n^{\text{tor}}(C, \epsilon) = H_n(\mathbb{Z}_2; \text{Tor}_1^A(C, C)) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (\text{Tor}_1^A(C, C))) .$$

There are defined forgetful maps

$$1 + T_\epsilon : Q_n^{\text{tor}}(C, \epsilon) \longrightarrow Q_{\text{tor}}^n(C, \epsilon) ; \psi \mapsto (1 + T_\epsilon)\psi ,$$

$$Q_{\text{tor}}^n(C, \epsilon) \longrightarrow H_n(\text{Tor}_1^A(C, C)) ; \phi \mapsto \phi_0 .$$

The element $\phi_0 \in H_n(\text{Tor}_1^A(C, C))$ is a chain homotopy class of A -module chain maps $\phi_0 : C^{n-\wedge} \longrightarrow C$.

An n -dimensional ϵ -symmetric complex over (A, σ) (C, ϕ) is a bounded (A, σ) -module chain complex C together with an element $\phi \in Q_{\text{tor}}^n(C, \epsilon)$. The complex (C, ϕ) is Poincaré if the A -module chain maps $\phi_0 : C^{n-\wedge} \longrightarrow C$ are chain equivalences.

Example 2.8. A 0-dimensional ϵ -symmetric Poincaré complex (C, ϕ) over (A, σ) is essentially the same as a nonsingular ϵ -symmetric linking form (M, λ) over (A, σ) , with $M = (C_0)^\wedge$ an (A, σ) -module and

$$\lambda = \phi_0 : M \times M \longrightarrow \sigma^{-1}A/A$$

a sesquilinear pairing such that the adjoint

$$M \longrightarrow M^\wedge ; x \mapsto (y \mapsto \lambda(x, y))$$

is an A -module isomorphism.

\square

The n -dimensional torsion ϵ -symmetric L -group $L_{\text{tor}}^n(A, \sigma, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric Poincaré complexes (C, ϕ) over (A, σ) , with C n -dimensional. In particular, $L_{\text{tor}}^0(A, \sigma, \epsilon)$ is the Witt group of nonsingular ϵ -symmetric linking forms over (A, σ) .

Similarly in the ϵ -quadratic case, with torsion L -groups $L_n^{\text{tor}}(A, \sigma, \epsilon)$. The ϵ -quadratic torsion L -groups are 4-periodic

$$L_n^{\text{tor}}(A, \sigma, \epsilon) = L_{n+2}^{\text{tor}}(A, \sigma, -\epsilon) = L_{n+4}^{\text{tor}}(A, \sigma, \epsilon) .$$

Theorem 2.9. *If $A \rightarrow \sigma^{-1}A$ is injective the relative L -groups in the localization exact sequences of Proposition 2.3*

$$\begin{aligned} \cdots &\longrightarrow L^n(A, \epsilon) \longrightarrow \Gamma^n(A \rightarrow \sigma^{-1}A, \epsilon) \xrightarrow{\partial} L^n(A, \sigma, \epsilon) \longrightarrow L^{n-1}(A, \epsilon) \longrightarrow \cdots \\ \cdots &\longrightarrow L_n(A, \epsilon) \longrightarrow \Gamma_n(A \rightarrow \sigma^{-1}A, \epsilon) \xrightarrow{\partial} L_n(A, \sigma, \epsilon) \longrightarrow L_{n-1}(A, \epsilon) \longrightarrow \cdots \end{aligned}$$

are the torsion L -groups

$$\begin{aligned} L^*(A, \sigma, \epsilon) &= L_{\text{tor}}^*(A, \sigma, \epsilon) , \\ L_*(A, \sigma, \epsilon) &= L_*^{\text{tor}}(A, \sigma, \epsilon) . \end{aligned}$$

Proof. For any bounded (A, σ) -module chain complex T there exists a bounded f.g. projective A -module chain complex C with a homology equivalence $C \rightarrow T$. Working as in [8] there is defined a distinguished triangle of $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complexes

$$\Sigma \text{Tor}_1^A(T, T) \longrightarrow C \otimes_A C \longrightarrow T \otimes_A T \longrightarrow \Sigma^2 \text{Tor}_1^A(T, T)$$

with \mathbb{Z}_2 acting by the ϵ -transposition T_ϵ on the \mathbb{Z} -module chain complex $\text{Tor}_1^A(T, T)$ and by the $(-\epsilon)$ -transpositions $T_{-\epsilon}$ on $C \otimes_A C$ and $T \otimes_A T$, inducing long exact sequences

$$\begin{aligned} \cdots &\longrightarrow Q_{\text{tor}}^n(T, \epsilon) \longrightarrow Q^{n+1}(C, -\epsilon) \longrightarrow Q^{n+1}(T, -\epsilon) \longrightarrow Q_{\text{tor}}^{n-1}(T, \epsilon) \longrightarrow \cdots \\ \cdots &\longrightarrow Q_n^{\text{tor}}(T, \epsilon) \longrightarrow Q_{n+1}(C, -\epsilon) \longrightarrow Q_{n+1}(T, -\epsilon) \longrightarrow Q_{n-1}^{\text{tor}}(T, \epsilon) \longrightarrow \cdots \end{aligned}$$

Passing to the direct limits over all the bounded (A, σ) -module chain complexes U with a homology equivalence $\beta : T \rightarrow U$ use Proposition 2.7 (iv) to obtain

$$\begin{aligned} \varinjlim_{(U, \beta)} Q^{n+1}(U, -\epsilon) &= 0 , \\ \varinjlim_{(U, \beta)} Q_{n+1}(U, -\epsilon) &= 0 \end{aligned}$$

and hence

$$\begin{aligned} \varinjlim_{(U, \beta)} Q_{\text{tor}}^n(U, \epsilon) &= Q^{n+1}(C, -\epsilon) , \\ \varinjlim_{(U, \beta)} Q_n^{\text{tor}}(U, \epsilon) &= Q_{n+1}(C, -\epsilon) . \end{aligned}$$

□

Remark 2.10. The identification $L_*(A, \sigma, \epsilon) = L_*^{\text{tor}}(A, \sigma, \epsilon)$ for noncommutative $\sigma^{-1}A$ was first obtained by Vogel [8].

□

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