

## FIVE EMBEDDINGS OF ONE SIMPLE GROUP

IVAN CHELTSOV AND CONSTANTIN SHRAMOV

ABSTRACT. We propose a new method to study birational maps between Fano varieties based on multiplier ideal sheaves. Using this method, we prove equivariant birational rigidity of four Fano threefolds acted on by the group  $A_6$ . As an application, we obtain that  $\text{Bir}(\mathbb{P}^3)$  has at least five non-conjugate subgroups isomorphic to  $A_6$ .

### CONTENTS

1. Introduction	1289
2. Preliminaries	1298
3. Alternating group	1303
4. Projective space	1306
5. Segre cubic	1315
6. Quadric threefold	1320
Appendix A. Klein cubic threefold	1323
Appendix B. Del Pezzo fibrations	1326
References	1329

We assume that all varieties are projective, normal, and defined over  $\mathbb{C}$ .

### 1. INTRODUCTION

The group of birational automorphisms of the projective plane  $\mathbb{P}^2$ , also known as the Cremona group  $\text{Cr}_2(\mathbb{C})$ , is a classical object in algebraic geometry. Classifying finite subgroups in  $\text{Cr}_2(\mathbb{C})$  up to conjugacy is an important research direction originating in the works of Kantor, Bertini, Wiman and others. This classification has been almost completed in [16]. In particular, we know how to decide whether or not two finite isomorphic subgroups in  $\text{Cr}_2(\mathbb{C})$  are conjugate.

The group of birational automorphisms of the projective space  $\mathbb{P}^3$ , the Cremona group  $\text{Cr}_3(\mathbb{C})$ , is poorly understood. We know so little about finite subgroups in  $\text{Cr}_3(\mathbb{C})$  that Serre asked

**Question 1.1** ([40, Question 6.0]). Does there exist a finite group which is not embeddable in  $\text{Cr}_3(\mathbb{C})$ ?

---

Received by the editors February 1, 2011 and, in revised form, November 27, 2011.

2010 *Mathematics Subject Classification*. Primary 14J30, 14J70, 13F15, 14B05.

The first author was supported by the grants NSF DMS-0701465 and EPSRC EP/E048412/1.

The second author was supported by the grants RFFI 08-01-00395-a, NSh-1987.2008.1 and EPSRC EP/E048412/1.

It is well known that most finite groups are not embeddable in  $\mathrm{Cr}_2(\mathbb{C})$ . For example, the only finite simple non-abelian subgroups in  $\mathrm{Cr}_2(\mathbb{C})$  are  $A_5, A_6, \mathrm{PSL}_2(\mathbb{F}_7)$  (see [16]). Prokhorov explicitly answered Question 1.1 by proving the following.

**Theorem 1.2** ([38, Theorem 1.3]). *Let  $\bar{G}$  be a finite simple non-abelian subgroup in  $\mathrm{Cr}_3(\mathbb{C})$ . Then  $\bar{G}$  is one of the following groups:  $A_5, A_6, \mathrm{PSL}_2(\mathbb{F}_7), A_7, \mathrm{SL}_2(\mathbb{F}_8), \mathrm{PSp}_4(\mathbb{F}_3)$ . All the possibilities occur.*

During a workshop “Subgroups of Cremona groups” at Edinburgh in Spring 2010, Serre posed

**Question 1.3.** What are normalizers in  $\mathrm{Cr}_3(\mathbb{C})$  of finite simple non-abelian subgroups in  $\mathrm{Cr}_3(\mathbb{C})$ ?

An answer to Question 1.3 depends on the classification of finite non-abelian simple subgroups in  $\mathrm{Cr}_3(\mathbb{C})$  up to conjugation. But Theorem 1.2 gives such a classification only up to isomorphism. So before trying to answer Question 1.3, we must first consider

**Question 1.4.** How do we decide whether or not two finite isomorphic subgroups in  $\mathrm{Cr}_3(\mathbb{C})$  are conjugate?

With very few exceptions (see [31], [8], [38]), we do not know how to answer Question 1.4 in full generality. In this paper, we show a new technique that can be used to answer both Questions 1.3 and 1.4 in some cases. In particular, we use this technique to prove

**Theorem 1.5.** *Up to conjugation, there are at least 5 subgroups in  $\mathrm{Cr}_3(\mathbb{C})$  that are isomorphic to  $A_6$ . For three of these non-conjugated subgroups, the normalizer in  $\mathrm{Cr}_3(\mathbb{C})$  is  $S_6$ , and for one it is the free product of  $S_6$  and  $S_6$  with an amalgamated subgroup  $A_6$ .*

Let us show how to translate Questions 1.4 and 1.3 into a geometric language. Let  $\bar{G}$  be a finite subgroup, and let  $\tau: \bar{G} \rightarrow \mathrm{Cr}_3(\mathbb{C})$  be a monomorphism. It is well known (see [38, Section 4.2]) that there is a birational map  $\xi: V \dashrightarrow \mathbb{P}^3$  such that

- the threefold  $V$  is normal and has terminal singularities,
- there exists a monomorphism  $v: \bar{G} \rightarrow \mathrm{Aut}(V)$ ,
- for every element  $g \in \bar{G}$ , we have  $\tau(g) = \xi \circ v(g) \circ \xi^{-1}$ ,
- there exists a  $v(\bar{G})$ -Mori fibration  $\pi: V \rightarrow S$ , i.e. a non-birational  $v(\bar{G})$ -equivariant surjective morphism with connected fibers such that the divisor  $-K_V$  is  $\pi$ -ample, and for every  $v(\bar{G})$ -invariant Weil divisor  $D$  on  $V$ , there is  $\delta \in \mathbb{Q}$  such that

$$\delta K_V + D \sim_{\mathbb{Q}} \pi^*(H)$$

for some  $\mathbb{Q}$ -Cartier divisor  $H$  on the variety  $S$ .

**Definition 1.6.** The quadruple  $(V, \xi, v, \pi)$  is a Mori regularization of the pair  $(\bar{G}, \tau)$ .

**Theorem 1.7.** *The normalizer of the group  $\tau(\bar{G})$  in  $\mathrm{Cr}_3(\mathbb{C})$  is isomorphic to the group of all  $v(\bar{G})$ -equivariant birational automorphisms of the variety  $V$ .*

*Proof.* The proof is obvious and left to the reader. □

Let  $\tau': \bar{G} \rightarrow \mathrm{Cr}_3(\mathbb{C})$  be another monomorphism, and let  $(V', \xi', v', \pi')$  be a Mori regularization of the pair  $(\bar{G}, \tau')$ .

**Theorem 1.8** ([16, Lemma 3.4]). *The following assertions are equivalent:*

- the subgroups  $\tau(\bar{G})$  and  $\tau'(\bar{G})$  are conjugate in  $\text{Cr}_3(\mathbb{C})$ ,
- there is a birational map  $\rho: V \dashrightarrow V'$  such that for every  $g \in \bar{G}$  there is  $g' \in \bar{G}$  such that

$$v'(g') = \rho \circ v(g) \circ \rho^{-1} \in \text{Aut}(V').$$

Thus, the subgroups  $\tau(\bar{G})$  and  $\tau'(\bar{G})$  are conjugate in  $\text{Cr}_3(\mathbb{C})$  if and only if there exists a  $\bar{G}$ -equivariant birational map  $V \dashrightarrow V'$  with respect to the actions of the group  $\bar{G}$  on  $V$  and  $V'$  induced by the monomorphisms  $v$  and  $v'$ . However, to prove or disprove the existence of such a birational map is not easy in general. The following two definitions help to deal with this in the case when  $\pi(V) = S$  is a point, i.e. in the case when  $V$  is a normal Fano variety with terminal singularities such that the  $v(\bar{G})$ -invariant subgroup in  $\text{Cl}(V) \otimes \mathbb{Q}$  is generated by  $-K_V$ .

**Definition 1.9** (cf. [7, Definition 0.3.3]). If  $S$  is a point, then  $V$  is called  $v(\bar{G})$ -birationally rigid if for every Mori regularization  $(V', \xi', v', \pi')$  of the pair  $(\bar{G}, \tau)$ , we have  $V' \cong V$ ,  $\pi'(V')$  is a point, and the subgroups  $v(\bar{G})$  and  $v'(\bar{G})$  are conjugate in  $\text{Aut}(V) \cong \text{Aut}(V')$ .

**Definition 1.10** (cf. [7, Definition 0.3.4]). If  $S$  is a point, then  $V$  is called  $v(\bar{G})$ -birationally superrigid if for every Mori regularization  $(V', \xi', v', \pi')$  of the pair  $(\bar{G}, \tau)$ , the map  $\xi^{-1} \circ \xi'$  is biregular.

From now on we will identify the group  $\bar{G}$  with its image  $v(\bar{G})$  to simplify the notation. Note that if  $S$  is a point and  $V$  is  $\bar{G}$ -birationally superrigid, then it follows from Definitions 1.9 and 1.10 that  $V$  is  $\bar{G}$ -birationally rigid. Denote by  $\text{Aut}^{\bar{G}}(V)$  (by  $\text{Bir}^{\bar{G}}(V)$ , respectively) the groups of  $\bar{G}$ -equivariant biregular (birational, respectively) self-maps of  $V$ . Note also that the group  $\text{Aut}^{\bar{G}}(V)$  (the group  $\text{Bir}^{\bar{G}}(V)$ , respectively) coincides with the normalizer of  $\bar{G}$  in  $\text{Aut}(V)$  (in  $\text{Bir}(V)$ , respectively).

**Corollary 1.11.** *Suppose that  $S$  is a point, the threefold  $V$  is  $\bar{G}$ -birationally rigid, and  $V \not\cong V'$ . Then  $\tau(\bar{G})$  and  $\tau'(\bar{G})$  are not conjugate in  $\text{Cr}_3(\mathbb{C})$ .*

**Corollary 1.12.** *Suppose that  $S$  is a point and  $V$  is  $\bar{G}$ -birationally superrigid. Then the normalizer of the group  $\tau(\bar{G})$  in  $\text{Cr}_3(\mathbb{C})$  is isomorphic to  $\text{Aut}^{\bar{G}}(V)$ .*

Let us consider examples of Mori regularizations in the case when  $\bar{G}$  is one of the following groups:  $A_7$ ,  $\text{SL}_2(\mathbb{F}_8)$ , and  $\text{PSP}_4(\mathbb{F}_3)$ . Note that in the latter case, the variety  $S$  must be a point, since  $A_7$ ,  $\text{SL}_2(\mathbb{F}_8)$ , and  $\text{PSP}_4(\mathbb{F}_3)$  are not embeddable in  $\text{Cr}_2(\mathbb{C})$ .

**Example 1.13.** If  $\bar{G} = A_7$  or  $\bar{G} \cong \text{PSP}_4(\mathbb{F}_3)$ , then there is a monomorphism  $v: \bar{G} \rightarrow \text{Aut}(\mathbb{P}^3)$  (see [15]).

**Example 1.14** ([38, Example 2.11]). Put  $\bar{G} = \text{SL}_2(\mathbb{F}_8)$ . One can show that there is a monomorphism  $\alpha: \bar{G} \rightarrow \text{Aut}(\text{LGr}(4, 9))$  and that  $\text{Pic}(\text{LGr}(4, 9)) = \mathbb{Z}[H]$ , where  $H$  is an ample divisor such that  $|H|$  gives an embedding  $\zeta: \text{LGr}(4, 9) \rightarrow \mathbb{P}^{15}$ , which implies that the monomorphism  $\alpha$  induces a monomorphism  $\beta: \bar{G} \rightarrow \text{Aut}(\mathbb{P}^{15})$ . Put  $V = \zeta(\text{LGr}(4, 9)) \cap \Pi$ , where  $\Pi$  is the unique  $\beta(\bar{G})$ -invariant linear subspace  $\Pi \subset \mathbb{P}^{15}$  such that  $\dim(\Pi) = 8$ . Then  $V$  is a smooth Fano threefold such that  $\text{Pic}(V) \cong \mathbb{Z}$  and  $-K_V^3 = 12$  (see [35]), and  $V$  is rational (see [25, Corollary 4.4.12]). The monomorphism  $\beta$  induces an isomorphism  $v: \bar{G} \rightarrow \text{Aut}(V)$ .

**Example 1.15.** Suppose that  $V$  is a complete intersection in  $\mathbb{P}^5$  that is given by

$$\sum_{i=0}^5 x_i = \sigma_4(x_0, x_1, x_2, x_3, x_4, x_5) = 0 \subset \mathbb{P}^5 \cong \text{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]),$$

where  $\sigma_4$  is the elementary symmetric form of degree 4. Put  $\bar{G} = \text{PSP}_4(\mathbb{F}_3)$ . Then there is a monomorphism  $v: \bar{G} \rightarrow \text{Aut}(V)$  (see [23, Chapter 5]). It is known that the threefold  $V$  is rational (see [23, Section 5.2.7]) and the  $\bar{G}$ -invariant subgroup in the group  $\text{Cl}(V)$  is  $\mathbb{Z}$  (cf. Theorem 1.20).

The threefold constructed in Example 1.15 is known as the Burkhardt quartic (see [23]).

**Theorem 1.16** ([38, Theorem 1.5], [4]). *Suppose that  $\bar{G}$  is one of the following subgroups:  $A_7$ ,  $\text{SL}_2(\mathbb{F}_8)$ , or  $\text{PSP}_4(\mathbb{F}_3)$ . Let  $V$  be a normal rational threefold with at most terminal singularities that admits a faithful action of the group  $\bar{G}$  such that there exists a  $\bar{G}$ -Mori fibration  $\pi: V \rightarrow S$ . Then  $S$  is a point and*

- if  $\bar{G} \cong A_7$ , then  $V \cong \mathbb{P}^3$  (see Example 1.13),
- if  $\bar{G} \cong \text{SL}_2(\mathbb{F}_8)$ , then  $V$  is the threefold constructed in Example 1.14,
- if  $\bar{G} \cong \text{PSP}_4(\mathbb{F}_3)$ , then either  $V \cong \mathbb{P}^3$  or  $V$  is the threefold constructed in Example 1.15.

Now let us consider examples of Mori regularizations in the case when  $\bar{G} \cong A_6$ .

**Example 1.17.** Put  $V = \mathbb{P}^3$  and  $\bar{G} = A_6$ . Then there exists a monomorphism  $2.A_6 \rightarrow \text{SL}_4(\mathbb{C})$  (see [15]), which induces a monomorphism  $v: \bar{G} \rightarrow \text{Aut}(V) \cong \text{PGL}_4(\mathbb{C})$ . Furthermore, up to conjugation, this is the unique subgroup isomorphic to  $A_6$  in  $\text{PGL}_4(\mathbb{C})$ .

**Example 1.18.** Put  $\bar{G} = A_6$ . Then there exists a monomorphism  $\bar{G} \rightarrow \text{SO}_5(\mathbb{R})$ , which implies the existence of a monomorphism  $v: \bar{G} \rightarrow \text{Aut}(V)$ , where  $V$  is a smooth quadric threefold.

**Example 1.19.** Put  $\bar{G} = A_6$ . Suppose that  $V$  is the complete intersection

$$\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^3 = 0 \subset \mathbb{P}^5 \cong \text{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]).$$

Then  $V$  has exactly 10 isolated ordinary double points (and hence  $V$  is rational) and there exists a natural monomorphism  $v: \bar{G} \rightarrow \text{Aut}(V)$ . Furthermore, one has  $\text{Cl}(V) \cong \mathbb{Z}^6$  (see [18], [19]), which implies that the  $\bar{G}$ -invariant subgroup of the group  $\text{Cl}(V)$  is  $\mathbb{Z}$ .

The threefold constructed in Example 1.19 is known as the Segre cubic (see [23, Section 3.2]).

Let us show how to prove that  $V$  is  $\bar{G}$ -birational superrigid in the case when  $S$  is a point.

**Theorem 1.20.** *Put  $\bar{G} = A_6$ . Suppose that  $V$  is the threefold constructed in Example 1.15. Then  $\text{rkCl}^{\bar{G}}(V) = 1$ , the variety  $V$  is  $\bar{G}$ -birationally superrigid, and  $\text{Bir}^{\bar{G}}(V) \cong S_6$ .*

*Proof.* The required assertions are probably well known to experts, but we failed to find any relevant reference. The fact that the  $\bar{G}$ -invariant subgroup of the group

$\text{Cl}(V)$  is  $\mathbb{Z}$  follows from the isomorphism  $V/\bar{F} \cong \mathbb{P}(1, 2, 2, 3)$ , where  $\bar{F}$  is a subgroup in  $\bar{G}$  such that  $\bar{F} \cong S_4$  and  $\bar{F}$  fixes a point in  $\text{Sing}(V)$ . The isomorphism  $\text{Bir}^{\bar{G}}(V) \cong S_6$  follows from the  $\bar{G}$ -birational superrigidity of the threefold  $V$ , Corollary 1.12, and the classification of primitive subgroups in  $\text{SL}_5(\mathbb{C})$  (see [17]). The fact that  $V$  is  $\bar{G}$ -rationally superrigid easily follows from the proof of [33, Theorem 5] (cf. [42, Section 9]), but we decided to prove this here to illustrate how to prove  $\bar{G}$ -birational superrigidity using only a classical technique that goes back to [24].

Suppose that  $V$  is not  $\bar{G}$ -rationally superrigid. Then [12, Theorem 4.2] implies the existence of a  $\bar{G}$ -invariant linear system  $\mathcal{M}$  on the threefold  $V$  such that  $\mathcal{M}$  does not have fixed components, and the log pair  $(V, \lambda\mathcal{M})$  is not canonical, where  $\lambda \in \mathbb{Q}$  is such that  $K_V + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$ .

Let  $M_1$  and  $M_2$  be sufficiently general surfaces in the linear system  $\mathcal{M}$ , and let  $H$  be a general surface in  $|-K_V|$ . Consider  $V$  as a quartic threefold in  $\mathbb{P}^4$ .

Suppose that there is an irreducible curve  $C \subset V$  such that the log pair  $(V, \lambda\mathcal{M})$  is not canonical along the curve  $C$ . Then the multiplicity of the linear system  $|\mathcal{M}|$  along the curve  $C$  is greater than  $1/\lambda$  (see e.g. [14, Exercise 6.18]). Let  $Z$  be the  $\bar{G}$ -orbit of the curve  $C$ . Put  $d = -K_V \cdot Z$ . Then

$$4/\lambda^2 = M_1 \cdot M_2 \cdot H \geq d \text{mult}_C(M_1)\text{mult}_C(M_2) > d/\lambda^2,$$

which implies that  $d \leq 3$ . So, the curve  $Z$  is contained in a hyperplane in  $\mathbb{P}^4$ , which is impossible, because the corresponding five-dimensional representation of the group  $\bar{G}$  is irreducible.

Thus, the centers of canonical singularities (see [7, Definition 1.3.8]) of the log pair  $(V, \lambda\mathcal{M})$  are points. Take any point  $P \in V$  such that the singularities of the log pair  $(V, \lambda\mathcal{M})$  are not canonical at the point  $P$ . Suppose that  $P \in V$  is a smooth point, and let  $H_P$  be a general hyperplane section of  $V \subset \mathbb{P}^4$  passing through  $P$ . Then

$$4/\lambda^2 = M_1 \cdot M_2 \cdot H_P \geq \text{mult}_P(M_1 \cdot M_2) > 4/\lambda^2$$

by [13, Corollary 3.4]. The obtained contradiction shows that the quartic  $V$  is singular at  $P$ .

Let  $\Sigma$  be the  $\bar{G}$ -orbit of the point  $P \in V$ . Then there is a subset  $\Gamma \subset \Sigma$  such that  $|\Gamma| = 4$ , and the set  $\Gamma$  is not contained in any two-dimensional linear subspace of  $\mathbb{P}^4$ .

Let  $\mathcal{Q} \subset |-2K_V|$  be a linear subsystem that consists of all surfaces in  $|-2K_V|$  that pass through every point of the set  $\Gamma$ . Then the base locus of the linear system  $\mathcal{Q}$  is the set  $\Gamma$ .

Let  $\pi: U \rightarrow V$  be a blow up of the set  $\Gamma$ , let  $\bar{\mathcal{M}}$  be the proper transforms of the linear system  $\mathcal{M}$  on the variety  $U$ , and let  $E_1, \dots, E_4$  be exceptional divisors of  $\pi$ . Then there is  $m \in \mathbb{Z}$  such that  $\bar{\mathcal{M}} \sim \pi^*(\mathcal{M}) - m \sum_{i=1}^4 E_i$ . Moreover, it follows from [7, Theorem 1.7.20] that  $m > 1/\lambda$  (cf. [13, Theorem 3.10]). Let  $\Upsilon$  be the intersection of two sufficiently general surfaces in  $\mathcal{Q}$ , and let  $\bar{\Upsilon}$  be the proper transform of the curve  $\Upsilon$  on the threefold  $U$ . Then

$$0 > 8/\lambda - 8m > 8/\lambda - m \sum_{O \in \Gamma} \text{mult}_O(\Upsilon) = \bar{M} \cdot \bar{\Upsilon} \geq 0,$$

where  $\bar{M}$  is a general surface in  $\bar{\mathcal{M}}$ . The obtained contradiction completes the proof. □

**Corollary 1.21.** *Up to conjugation, there are at least 2 subgroups in  $\text{Cr}_3(\mathbb{C})$  isomorphic to  $A_6$ .*

**Corollary 1.22.** *Suppose that  $\bar{G} = \text{PSP}_4(\mathbb{F}_3)$  and  $V$  is the threefold constructed in Example 1.15. Then the  $\bar{G}$ -invariant subgroup of the group  $\text{Cl}(V)$  is  $\mathbb{Z}$ , and  $V$  is  $\bar{G}$ -birationally superrigid.*

Now applying Theorem 1.8, Corollary 1.11, and Theorem 1.16, we obtain

**Corollary 1.23.** *Up to conjugation, the group  $\text{Cr}_3(\mathbb{C})$  contains exactly 1 subgroup that is isomorphic to  $\text{SL}_2(\mathbb{F}_8)$ , exactly 2 subgroups that are isomorphic to  $\text{PSP}_4(\mathbb{F}_3)$ , and exactly 1 subgroup that is isomorphic to  $A_7$ .*

The proof of Theorem 1.20 goes back to the classical result of Iskovskikh and Manin about the non-rationality of every smooth quartic threefold (see [24]). Of course, the quartic threefold constructed in Example 1.15 is not smooth, but this problem is dealt with by using technique introduced in [13] and [33]. So the proof of Theorem 1.20 is a nice illustration of how to answer Question 1.4 in one particular case. Unfortunately, this approach is hard to apply to Fano varieties of large anti-canonical degree. The main purpose of this paper is to introduce a new technique to prove  $\bar{G}$ -birational rigidity or  $\bar{G}$ -birational superrigidity of the threefold  $V$  in the case when  $S$  is a point. This technique does not always work, but sometimes it does. We will use this technique to prove

**Theorem 1.24.** *Suppose that  $\bar{G} = A_6$ . If  $V \cong \mathbb{P}^3$  (cf. Example 1.17), then the threefold  $V$  is  $\bar{G}$ -birationally rigid (but not  $\bar{G}$ -birationally superrigid) and  $\text{Bir}^{\bar{G}}(V)$  is a free product of two copies of  $S_6$  with amalgamated subgroup  $A_6$ . If  $V$  is either the Segre cubic or a smooth quadric threefold, then  $V$  is  $\bar{G}$ -birationally superrigid and  $\text{Bir}^{\bar{G}}(V) \cong S_6$ .*

**Corollary 1.25.** *Up to conjugation, there are at least 5 subgroups in  $\text{Cr}_3(\mathbb{C})$  isomorphic to  $A_6$ .*

Unfortunately, we are unable to classify all subgroups in  $\text{Cr}_3(\mathbb{C})$  that are isomorphic to  $A_6$  up to conjugation, similar to what was done in Corollary 1.23 for  $A_7, \text{SL}_2(\mathbb{F}_8), \text{PSP}_4(\mathbb{F}_3)$ . The main problem is that  $S$  is not necessarily a point if  $\bar{G} = A_6$ , since  $A_6$  is embeddable in  $\text{Cr}_2(\mathbb{C})$ . If  $\bar{G} = A_6$  and  $\dim(S) = 1$ , i.e.  $\pi: V \rightarrow S$  is a del Pezzo fibration, then  $V \cong \mathbb{P}^2 \times \mathbb{P}^1$  by Theorem B.1. However, if  $\bar{G} = A_6$  and  $\dim(S) = 2$ , i.e.  $\pi: V \rightarrow S$  is a conic bundle, then we do not have any decent description of the threefold  $V$ , and this is the main reason why we are unable to classify all subgroups in  $\text{Cr}_3(\mathbb{C})$  that are isomorphic to  $A_6$  up to conjugation.

*Remark 1.26.* Suppose that  $\bar{G} = A_6$  and  $\dim(S) = 2$ . Then the monomorphism  $v: \bar{G} \rightarrow \text{Aut}(V)$  induces a monomorphism  $\iota: \bar{G} \rightarrow \text{Aut}(S)$ , because  $\pi$  is  $\bar{G}$ -equivariant and  $\bar{G}$  is simple. On the other hand, the group  $\bar{G}$  faithfully acts on  $\mathbb{P}^2 \times \mathbb{P}^1$  so that its action of the first factor is induced by a three-dimensional representation of the group  $3.A_6$ , and its action on the second factor is trivial. Keeping in mind that  $V$  is rational, it is very tempting to expect that there always exists a commutative diagram

$$\begin{array}{ccc}
 V & \xrightarrow{\alpha} & \mathbb{P}^2 \times \mathbb{P}^1 \\
 \downarrow \pi & & \downarrow pr_1 \\
 S & \xrightarrow{\beta} & \mathbb{P}^2,
 \end{array}$$

where  $\alpha$  is some  $\bar{G}$ -equivariant birational map,  $\beta$  is some  $\iota(\bar{G})$ -equivariant birational map, and  $pr_1: \mathbb{P}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$  is the projection to the first factor. We do not have any strong evidence for such expectation, but we failed to construct a counterexample either.

Let us describe the strategy for proving Theorem 1.24. Let  $X$  be a Fano threefold with Gorenstein terminal singularities, and let  $\bar{G}$  be a subgroup in  $\text{Aut}(X)$  such that  $\text{rkCl}^{\bar{G}}(X) = 1$ . We suppose that  $X$  is not  $\bar{G}$ -birationally superrigid and seek a contradiction. It is well known that in this case there are a movable  $\bar{G}$ -invariant linear system  $\mathcal{M}$  without fixed components and  $\lambda \in \mathbb{Q}$  such that  $K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$  and the log pair  $(X, \lambda\mathcal{M})$  has non-canonical singularities (see [12, Theorem 4.2], [7, Theorem 1.4.1]). Since  $X$  is Gorenstein, the singularities of the log pair  $(X, 2\lambda\mathcal{M})$  must be worse than log canonical by Corollary 2.3. Now, although the last condition is much weaker than non-canonicity of the log pair  $(X, \lambda\mathcal{M})$ , we are in a position to use the machinery of multiplier ideal sheaves to obtain a contradiction. Namely, we choose  $\mu < 2\lambda$  such that the log pair  $(X, \mu\mathcal{M})$  is strictly log canonical and pick up a minimal center  $S$  of log canonical singularities of  $(X, \mu\mathcal{M})$  (see Section 2 for definitions). The minimality of the center  $S$  implies that the  $\bar{G}$ -orbit of  $S$  is either a finite set or a disjoint union of irreducible curves. We use Lemma 2.13 to observe that one may assume that every center of log canonical singularities of the log pair  $(X, \mu\mathcal{M})$  is  $g(S)$  for some  $g \in \bar{G}$ . Then applying the Nadel–Shokurov vanishing theorem (see Theorem 2.4, [30, Theorem 9.4.8]) we obtain an upper bound on the number of irreducible components of the  $\bar{G}$ -orbit of  $S$ . If  $S$  is a point, the latter appears to be mostly enough to obtain a contradiction with the structure of  $\bar{G}$ -orbits on  $X$ . If  $S$  is a curve, the Kawamata subadjunction theorem (see Theorem 2.8, [28, Theorem 1]) implies that  $S$  is smooth, and we proceed with applying the Nadel–Shokurov vanishing theorem, the Riemann–Roch theorem, the Clifford theorem (see Theorem 2.19, [22, Theorem 5.4], [2, Section III.1]), and the Castelnuovo bound (see Theorem 2.20, [22, Theorem 6.4], [2, Section III.2]).

Using these, we obtain various restrictions on the number of irreducible components of  $S$ , the degree of the curve  $S$ , and the genus of the curve  $S$ . In many cases these restrictions appear to be incompatible with a faithful action of  $\bar{G}$  on  $X$ . Namely, suppose that  $X$  is one of the three threefolds from Theorem 1.24 and  $\bar{G} \cong A_6$ . Then all possibilities for  $S$  are ruled out by the above arguments, except for the cases when either  $X$  is the Segre cubic and  $S$  is its singular point, or when  $X \cong \mathbb{P}^3$  and  $S$  is a line whose  $\bar{G}$ -orbit consists of 6 lines. In the former case the contradiction is obtained by brute force (see Lemma 5.7). So we have to deal with the case when  $X \cong \mathbb{P}^3$  and  $S$  is a line whose  $\bar{G}$ -orbit consists of 6 lines. If  $S$  is not a center of canonical singularities of the log pair  $(\mathbb{P}^3, \lambda\mathcal{M})$ , then we use Lemma 2.17 to obtain a contradiction. However, we cannot obtain a contradiction here in general, because  $\mathbb{P}^3$  is not  $\bar{G}$ -birationally superrigid! Indeed, for every line  $L \subset \mathbb{P}^3$  whose  $\bar{G}$ -orbit consists of 6 lines, there exists a non-biregular  $\bar{G}$ -equivariant birational involution  $\mathbb{P}^3 \rightarrow \mathbb{P}^3$  discovered by Todd in [45] that is undefined along every line in the  $\bar{G}$ -orbit of  $L$  (see Lemma 4.5 for a description). There are 12 such lines in  $\mathbb{P}^3$  in total. These 12 lines split in two  $\bar{G}$ -orbits, and each  $\bar{G}$ -orbit gives us a non-biregular  $\bar{G}$ -equivariant birational involution  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ , which we denote by  $\iota$  and  $\iota'$ , respectively. Applying  $\iota$  and  $\iota'$  to  $\mathcal{M}$  if necessary, we may assume that lines whose  $\bar{G}$ -orbit consist of 6 lines are not centers of canonical singularities of the log pair  $(X, \lambda\mathcal{M})$  (see Lemma 4.7). Combining this with what we have

already proved, we see that the singularities of the log pair  $(X, \lambda\mathcal{M})$  are canonical up to the action of  $\langle \bar{G}, \iota, \iota' \rangle$ , which implies that  $\mathbb{P}^3$  is  $\bar{G}$ -birationally rigid and  $\text{Bir}^{\bar{G}}(\mathbb{P}^3) = \langle \text{Aut}^{\bar{G}}(\mathbb{P}^3), \iota, \iota' \rangle$ . Finally, we observe that

$$\text{Aut}^{\bar{G}}(\mathbb{P}^3) \cong \langle \bar{G}, \iota \rangle \cong \langle \bar{G}, \iota' \rangle \cong S_6$$

and use this to prove that  $\langle \text{Aut}^{\bar{G}}(\mathbb{P}^3), \iota, \iota' \rangle$  is the free product of two copies of  $S_6$  with amalgamated subgroup  $A_6$ . Let us illustrate the described strategy by proving

**Theorem 1.27.** *Suppose that  $\bar{G}$  is one of the following subgroups:  $A_7, \text{SL}_2(\mathbb{F}_8), \text{PSP}_4(\mathbb{F}_3)$ . Then  $V$  is  $\bar{G}$ -birationally superrigid.*

*Proof.* If  $V \cong \mathbb{P}^3$ , then either  $\bar{G} \cong A_7$  or  $\bar{G} \cong \text{PSP}_4(\mathbb{F}_3)$ , and the proof of Theorem 1.24 implies that  $V$  is  $\bar{G}$ -birationally superrigid, since  $A_7$  and  $\text{PSP}_4(\mathbb{F}_3)$  contain subgroups isomorphic to  $A_6$ . In fact, the  $\bar{G}$ -birational superrigidity of the threefold  $V$  in the latter case is much easier to prove than Theorem 1.24, but we leave the details to the reader.

It follows from Theorem 1.16 and Corollary 1.22 that we may assume that  $\bar{G} \cong \text{SL}_2(\mathbb{F}_8)$  and  $V$  is the threefold constructed in Example 1.14. Suppose that  $V$  is not  $\bar{G}$ -birationally superrigid. Then it follows from [12, Theorem 4.2] or [7, Theorem 1.4.1] that there is a (non-empty)  $\bar{G}$ -invariant linear system  $\mathcal{M}$  on  $V$  such that  $\mathcal{M}$  does not have fixed components, and  $(V, \lambda\mathcal{M})$  is not canonical, where  $\lambda$  is a positive rational number such that  $\lambda\mathcal{M} \sim_{\mathbb{Q}} -K_V$ . By Corollary 2.3 there is  $\mu \in \mathbb{Q}$  such that  $\mu < 2\lambda$  and  $(V, \mu\mathcal{M})$  is log canonical and not Kawamata log terminal (see [29, Definition 3.5]).

Let  $S$  be a subvariety in  $V$  that is a minimal center of log canonical singularities of the log pair  $(V, \mu\mathcal{M})$  (see [27], [28], [10, Definition 2.8]), and let  $Z$  be the  $\bar{G}$ -orbit of the subvariety  $S$ . Then  $S$  is either a point or a curve, since  $\mathcal{M}$  does not have fixed components. Moreover, the subvariety  $Z \subset V$  is a disjoint union of its components (see Lemma 2.7, [27, Proposition 1.5]).

Arguing as in the proof of [27, Theorem 1.10] or [28, Theorem 1], we can construct a  $\bar{G}$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on  $V$  such that  $D \sim_{\mathbb{Q}} -\delta K_V$  for some positive rational number  $\delta < 2$ , the log pair  $(V, D)$  is log canonical, the subvariety  $S$  is a minimal center of log canonical singularities of the log pair  $(V, D)$ , and the only centers of log canonical singularities (see [27, Definition 1.3]) of the log pair  $(V, D)$  are irreducible components of  $Z$  (see Corollary 2.3 and Lemma 2.13).

Let  $\mathcal{I}_Z$  be its ideal sheaf and put  $q = h^0(\mathcal{O}_V(-K_V) \otimes \mathcal{I}_Z) = 0$ . Then

$$(1.28) \quad h^0(\mathcal{O}_Z \otimes \mathcal{O}_V(-K_V)) = h^0(\mathcal{O}_V(-K_V)) - q = 9 - q \leq 9$$

by the Nadel–Shokurov vanishing theorem (see Theorem 2.4, [30, Theorem 9.4.8]) and the Riemann–Roch theorem. Let  $r$  be the number of irreducible components of the curve  $Z$ . Then  $r \notin \{2, 3, \dots, 8\}$ , since  $\bar{G}$  cannot act transitively on a finite set consisting of at least 2 and at most 8 elements (see [15]). Note that  $Z = S$  if  $r = 1$ .

Suppose that  $S$  is a point. Then either  $Z = S$  or  $|Z| = r = 9$  by (1.28). If  $Z = S$ , then  $\bar{G}$  must act faithfully on the tangent space to  $V$  at the point  $S$ , which is impossible, since  $\text{SL}_2(\mathbb{F}_8)$  has no non-trivial three-dimensional representations (see [15]). Thus, we see that  $|Z| = 9$ . Let  $F_{56}$  be the stabilizer subgroup in  $\bar{G}$  of the point  $S$ . Then  $|F_{56}| = |\bar{G}|/9 = 56$ , and hence one has  $F_{56} \cong \mathbb{Z}_2^3 : \mathbb{Z}_7$  (see [15]). On the other hand, the group  $F_{56}$  must faithfully act on the tangent space to  $V$



at the point  $S$ , which is impossible, since  $\mathbb{Z}_2^3 : \mathbb{Z}_7$  has no faithful three-dimensional representations.

Thus, we see that  $S$  is a curve. Put  $d = -K_V \cdot C$ . By the Kawamata subadjunction theorem (see Theorem 2.8, [28, Theorem 1]), the curve  $S$  is a smooth curve of genus  $g$  such that  $d > 2g - 2$ . In particular, the divisor  $-K_V|_S$  is not special. Thus, it follows from (1.28) and the Riemann–Roch theorem that

$$(1.29) \quad 9 - q = r(d - g + 1),$$

where either  $r = 1$  or  $r = 9$ . If  $r = 9$ , then it follows from (1.29) that  $d = g$  and  $q = 0$ , which is impossible, since  $d > 2g - 2$ . Thus, we see that  $r = 1$ , which simply means that  $Z = S$ . Then  $\bar{G}$  acts faithfully on  $S$ , since  $\bar{G}$  is simple and there are no points in  $V$  that are  $\bar{G}$ -invariant. Moreover, it follows from (1.29) that  $d = 8 + g - q$ . Keeping in mind that  $d > 2g - 2$ , we see that  $g < 10$ . Then  $g = 7$  by the refined Hurwitz bound (see [6, Theorem 3.17], Theorem 2.18). In particular, we see that  $d = 15 - q$ , since  $d = 8 + g - q$ .

Let  $M_1$  and  $M_2$  be general surfaces in  $\mathcal{M}$ , and let  $H$  be a general surface in  $|-K_V|$ . Then it follows from [13, Theorem 3.1] that  $\text{mult}_S(M_1 \cdot M_2) \geq 4/\mu^2$ , where  $\mu < 2\lambda$ . Thus, one has

$$\frac{12}{\lambda^2} = M_1 \cdot M_2 \cdot H \geq \sum_{O \in H \cap S} \text{mult}_S(M_1 \cdot M_2) \geq \frac{4|H \cap S|}{\mu^2} > \frac{|H \cap S|}{\lambda^2},$$

which implies that  $|H \cap S| < 12$ . But  $|H \cap S| = d = 15 - q$ , which implies that  $q \geq 4$ . The latter is impossible, since  $\text{SL}_2(\mathbb{F}_8)$  has no non-trivial representations of dimension at most 6. □

The reader may find it useful to compare the proof of Theorem 1.20 (that uses a classical approach to birational rigidity) with the proof of Theorem 1.27 (that uses our new approach). Note that the proof of Theorem 1.27 does not use any information about the geometry of the variety  $V$  acted on by the group  $\text{SL}_2(\mathbb{F}_8)$ . Applying Theorem 1.7 and Corollary 1.12 and using the classification of primitive subgroups in  $\text{SL}_4(\mathbb{C})$  and  $\text{SL}_5(\mathbb{C})$  (see [17]), we obtain

**Corollary 1.30.** *Let  $\bar{G}$  be a subgroup in  $\text{Cr}_3(\mathbb{C})$  that is isomorphic to either  $A_7$  or  $\text{PSp}_4(\mathbb{F}_3)$ . If  $\bar{G} \cong A_7$ , then its normalizer in  $\text{Cr}_3(\mathbb{C})$  is isomorphic to  $S_7$ . If  $\bar{G} \cong \text{PSp}_4(\mathbb{F}_3)$ , then its normalizer in  $\text{Cr}_3(\mathbb{C})$  is  $\bar{G}$  itself.*

The plan of the paper is as follows. In Section 2 we collect the well-known preliminary results on the log canonical singularities, make some remarks on varieties with group action and recall some general classical theorems. In Section 3 we collect the necessary facts about the structure of the groups  $A_6$  and  $2.A_6$  and the relevant representation theory. In Sections 4, 5 and 6 we study the projective space  $\mathbb{P}^3$ , the Segre cubic and the smooth quadric threefold according to the plan explained above, proving Theorems 4.8, 4.9, 5.5 and 6.2, respectively. These four theorems imply Theorem 1.24. In Appendix A, we answer (negatively) a question on the existence of equivariant birational maps between two certain remarkable threefolds acted on by the group  $\bar{\Gamma} \cong \text{PSL}_2(\mathbb{F}_{11})$  posed in [38, Remark 2.10]. Appendix B, written by Yu. Prokhorov, classifies terminal threefolds  $X$  acted on by the group  $\bar{G} \cong A_6$  such that  $X$  admits a  $\bar{G}$ -Mori fibration  $\pi : X \rightarrow \mathbb{P}^1$ .

We would like to thank T. Dokchitser, I. Dolgachev, Yu. Prokhorov, L. Rybnikov and E. Smirnov for very useful discussions and comments. We would like to thank

the referee for many valuable and useful comments that helped us to improve the paper significantly.

2. PRELIMINARIES

Throughout the paper we use the standard language of the singularities of pairs (see [29] and [14, Section 6]). By *strictly log canonical* singularities we mean log canonical singularities that are not Kawamata log terminal (see [29, Definition 3.5]).

Let  $X$  be a variety with at most Kawamata log terminal singularities (see [29, Definition 3.5]), and let  $B_X$  be a formal linear combination with non-negative rational coefficients of non-empty (but not necessarily complete) linear systems on  $X$ . Then we may write  $B_X = \sum_{i=1}^r a_i B_i$ , where  $a_i \in \mathbb{Q}$ , and  $B_i$  is either a prime Weil divisor on the variety  $X$  or a linear system without fixed components. We assume for simplicity that  $B_i \neq B_j$  if  $i \neq j$ .

*Remark 2.1.* Usually  $B_X$  is assumed to be an effective  $\mathbb{Q}$ -divisor. But sometimes it is necessary to assume that some or all components of  $B_X$  are linear systems without fixed components (see [14, Section 6]). We will need such a log pair throughout the paper, but do not want to stress on using them, since all properties of standard log pairs we plan to use hold for these generalized log pairs (see [29, Definition 4.6] and [14, Definition 6.16]). Moreover, we can always treat  $B_X$  as a  $\mathbb{Q}$ -divisor by replacing each its mobile components by a weighted sum of its sufficiently general members (see [29, Theorem 4.8]).

Suppose that  $K_X + B_X$  is  $\mathbb{Q}$ -Cartier. Let  $\pi: \bar{X} \rightarrow X$  be a birational morphism such that  $\bar{X}$  is smooth. Then

$$K_{\bar{X}} + \sum_{i=1}^r a_i \bar{B}_i \sim_{\mathbb{Q}} \pi^* (K_X + B_X) + \sum_{i=1}^m d_i E_i,$$

where  $\bar{B}_i$  is the proper transforms of  $B_i$  on the variety  $\bar{X}$ , and  $E_i$  is an exceptional divisor of the morphism  $\pi$ , and  $d_i$  is a rational number. We may assume that  $\sum_{i=1}^r \bar{B}_i + \sum_{i=1}^m E_i$  is a divisor with simple normal crossing and that every mobile linear system among  $\bar{B}_1, \dots, \bar{B}_r$  (if any) is free from base points.

**Lemma 2.2.** *Suppose that  $X$  has terminal singularities. Let  $n$  be a positive integer such that  $nK_X$  is a Cartier divisor. Then  $(X, (1 + n)\lambda B_X)$  is not log canonical if  $(X, \lambda B_X)$  is not canonical.*

*Proof.* There are positive rational numbers  $c_1, \dots, c_r$  such that  $K_{\bar{X}} \sim_{\mathbb{Q}} \pi^*(K_X) + \sum_{i=1}^m c_i E_i$ , because  $X$  has terminal singularities and, in particular,  $K_X$  is a  $\mathbb{Q}$ -Cartier divisor. Moreover, the numbers  $nc_1, \dots, nc_r$  are positive integers, since  $nK_X$  is a Cartier divisor. On the other hand, we have

$$\sum_{i=1}^r a_i \bar{B}_i \sim_{\mathbb{Q}} \pi^*(B_X) - \sum_{i=1}^m m_i E_i$$

for some non-negative rational numbers  $m_1, \dots, m_r$ . Then  $d_i = c_i - m_i$  for every  $i \in \{1, \dots, r\}$ .

Suppose that  $(X, \lambda B_X)$  is not canonical. Then there exists  $s \in \{1, \dots, m\}$  such that  $c_s - m_s = d_s < 0$  (see [14, Definition 6.16]). Thus, we have

$$c_s - (1 + n)m_s = c_s - m_s - nm_s < -nm_s < -1,$$

because  $m_s > c_s \geq 1/n$ , since  $nc_s$  is a positive integer. On the other hand, we have

$$K_{\bar{X}} + (1+n) \sum_{i=1}^r a_i \bar{B}_i \sim_{\mathbb{Q}} \pi^* \left( K_X + (1+n)B_X \right) + \sum_{i=1}^m (c_i - (1+n)m_i) E_i,$$

where  $c_s - (1+n)m_s < -1$ . Therefore, the log pair  $(X, (1+n)\lambda B_X)$  is not log canonical. □

**Corollary 2.3.** *Suppose that  $X$  has terminal Gorenstein singularities and  $(X, \lambda B_X)$  is not canonical. Then  $(X, 2\lambda B_X)$  is not log canonical.*

Suppose that  $(X, B_X)$  is log canonical. Put  $\hat{a}_i = a_i$  if  $B_i$  is a prime Weil divisor and  $\hat{a}_i = 0$  if  $B_i$  is a linear system without fixed components. Put  $\mathcal{I}(X, B_X) = \pi_* (\sum_{i=1}^m [d_i] E_i - \sum_{i=1}^r [\hat{a}_i] \bar{B}_i)$ .

**Theorem 2.4** ([30, Theorem 9.4.8]). *Let  $H$  be a nef and big  $\mathbb{Q}$ -divisor on  $X$ , and let  $D$  be a Cartier divisor on  $X$  such that  $D \equiv K_X + B_X + H$ . Then  $h^i(\mathcal{I}(X, B_X) \otimes D) = 0$  for every  $i \geq 1$ .*

Let  $\mathcal{L}(X, B_X)$  be a subscheme of  $X$  that corresponds to the ideal sheaf  $\mathcal{I}(X, B_X)$ . Put  $\text{LCS}(X, B_X) = \text{Supp}(\mathcal{L}(X, B_X))$ . Note that  $\mathcal{L}(X, B_X)$  is reduced, because  $(X, B_X)$  is log canonical.

**Theorem 2.5** ([41, Lemma 5.7], [27, Theorem 1.4]). *Let  $\zeta: X \rightarrow Z$  be a surjective morphism with connected fibers, and let  $F$  be a fiber of  $\zeta$ . Then the locus  $\text{LCS}(X, B_X) \cap F$  is connected if  $-(K_X + B_X)$  is  $\eta$ -nef and  $\eta$ -big.*

Recall that there are standard names for  $\mathcal{I}(X, B_X)$ ,  $\mathcal{L}(X, B_X)$  and  $\text{LCS}(X, B_X)$ . Namely, the ideal  $\mathcal{I}(X, B_X)$  is known as the multiplier ideal sheaf (see [30, Section 9.2]), the subscheme  $\mathcal{L}(X, B_X)$  is known as the log canonical singularities subscheme (see [7, Definition 1.7.5]), and the locus  $\text{LCS}(X, B_X)$  is known as the locus of log canonical singularities (see [41, Definition 3.14]).

**Theorem 2.6** ([14, Theorem 6.29]). *Suppose that  $a_1 = 1$  and  $B_1$  is a prime Weil divisor. Suppose that  $B_1$  is a Cartier divisor on  $X$  and  $B_1$  is a normal variety that has at most Kawamata log terminal singularities. Then  $(X, B_X)$  is log canonical along  $B_1$  if and only if the log pair  $(B_1, \sum_{i=2}^r a_i B_i|_{B_1})$  is log canonical.*

Let  $Z$  be a center of log canonical singularities of the log pair  $(X, B_X)$  (see [27, Definition 1.3]), and let  $\text{LCS}(X, B_X)$  be the set of all centers of log canonical singularities of the log pair  $(X, B_X)$ .

**Lemma 2.7** ([27, Proposition 1.5]). *Let  $Z$  and  $Z'$  be elements of the set  $\text{LCS}(X, B_X)$ , and let  $Y$  be any irreducible reduced component of  $Z \cdot Z'$ . Then  $Y \in \text{LCS}(X, B_X)$ .*

Suppose that  $Z$  is a minimal center in  $\text{LCS}(X, B_X)$  (see [27], [28], [10, Definition 2.8]).

**Theorem 2.8** ([28, Theorem 1]). *Let  $\Delta$  be an ample  $\mathbb{Q}$ -Cartier  $\mathbb{Q}$ -divisor on  $X$ . Then*

- *the variety  $Z$  is normal and has at most rational singularities,*
- *there exists an effective  $\mathbb{Q}$ -divisor  $B_Z$  on  $Z$  such that  $(K_X + B_X + \Delta)|_Z \sim_{\mathbb{Q}} K_Z + B_Z$ , and  $(Z, B_Z)$  has Kawamata log terminal singularities.*

Let  $\bar{G} \subseteq \text{Aut}(X)$  be a finite subgroup.

**Lemma 2.9.** *Suppose that  $X$  is a curve. Let  $\Sigma \subset X$  be a  $\bar{G}$ -orbit. If  $|\Sigma| = 1$ , then  $\bar{G}$  is cyclic. If  $\bar{G} \cong A_5$ , then  $|\Sigma| \in \{12, 20, 30, 60\}$ . If  $\bar{G} \cong A_6$ , then  $|\Sigma| \in \{60, 72, 90, 120, 180, 360\}$ .*

*Proof.* If  $\Sigma$  is a point, then  $\bar{G}$  acts faithfully on the tangent space to  $X$  at the point  $\Sigma$ . □

**Lemma 2.10.** *Suppose that  $X$  is a curve of genus  $g$  and  $\bar{G} \cong A_6$ . If  $g \leq 34$ , then  $g \in \{10, 16, 19, 25, 31\}$ . If  $g = 10$ , then  $X$  does not contain  $\bar{G}$ -orbits of length 120.*

*Proof.* Let  $\bar{F} \subset \bar{G}$  be a stabilizer of a point in  $X$ . Then  $\bar{F} \cong \mathbb{Z}_k$  for some  $k \in \{1, 2, 3, 4, 5, 6\}$  by Lemma 2.9. Put  $\bar{X} = X/\bar{G}$ . Then  $\bar{X}$  is a smooth curve of genus  $\bar{g}$ . The Riemann–Hurwitz formula gives

$$2g - 2 = 360(2\bar{g} - 2) + 180a_2 + 240a_3 + 270a_4 + 288a_5 + 300a_6,$$

where  $a_k$  is the number of  $\bar{G}$ -orbits in  $X$  with a stabilizer of a point isomorphic to  $\mathbb{Z}_k$ .

Suppose that  $g \leq 34$ . Note that  $g \neq 0$  by the classification of finite subgroups of  $\text{PGL}_2(\mathbb{C})$  and  $g \neq 1$  since  $\bar{G}$  is non-solvable. Since  $a_k \geq 0$ , one has  $\bar{g} = 0$ , and

$$(2.11) \quad 2g - 2 = -720 + 180a_2 + 240a_3 + 270a_4 + 288a_5 + 300a_6,$$

which implies that  $g \in \{10, 16, 19, 25, 31\}$ . The only solution to (2.11) for  $g = 10$  is  $(a_2, a_3, a_4, a_5, a_6) = (1, 0, 1, 1, 0)$ , which completes the proof. □

*Remark 2.12.* We do not claim that every case listed in Lemma 2.10 is realized. The obtained restrictions on the genus are enough for our purposes.

Suppose, in addition, that  $B_X$  is  $\bar{G}$ -invariant. Then  $g(Z) \in \text{LCS}(X, B_X)$  for every  $g \in \bar{G}$ , and the locus  $\text{LCS}(X, B_X)$  is  $\bar{G}$ -invariant. It follows from Lemma 2.7 that  $Z \cap g(Z) \neq \emptyset$  if and only if  $Z = g(Z)$  for every  $g \in \bar{G}$ , because  $Z$  is a minimal center in  $\text{LCS}(X, B_X)$ .

**Lemma 2.13.** *Suppose that the divisor  $B_X$  is ample. Let  $\epsilon$  be an arbitrary rational number such that  $\epsilon > 0$ . Then there exists an effective  $\bar{G}$ -invariant  $\mathbb{Q}$ -divisor  $D$  on the variety  $X$  such that the set  $\text{LCS}(X, D)$  consists of all components of the  $\bar{G}$ -orbit of  $Z$ , the log pair  $(X, D)$  is log canonical, and the equivalence  $D \sim_{\mathbb{Q}} (1 + \epsilon)B_X$  holds.*

*Proof.* See the proofs of [27, Theorem 1.10], [28, Theorem 1], [10, Lemma 2.11]. □

Suppose, in addition, that  $X$  is a Fano variety. Put

$$\text{lct}(X, \bar{G}) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ has log canonical singularities} \\ \text{for every } \bar{G}\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X \end{array} \right. \right\} \in \mathbb{R}.$$

*Remark 2.14.* If  $X$  is smooth, then it follows from [9, Theorem A.3] that  $\text{lct}(X, \bar{G}) = \alpha_{\bar{G}}(X)$ , where  $\alpha_{\bar{G}}(X)$  is the  $\bar{G}$ -invariant  $\alpha$ -invariant of the variety  $X$  introduced in [43] and [44].

Suppose that  $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ .

**Lemma 2.15.** *Suppose that  $\bar{G} \cong A_5$ . Let  $\Sigma \subset X$  be a  $\bar{G}$ -invariant subset. Then  $|\Sigma| \geq 12$ .*

*Proof.* The required assertion follows from Lemma 2.9. □

Let  $L_1$  and  $L_2$  be fibers of the projection to the first and the second factor, respectively.

**Lemma 2.16.** *The inequality  $\text{lct}(X, \bar{G}) \geq 1$  holds if and only if the linear systems  $|L_1|$ ,  $|L_2|$ ,  $|L_1 + L_2|$  contain no  $\bar{G}$ -invariant curves.*

*Proof.* If there is a  $\bar{G}$ -invariant curve in  $|L_1|$  or  $|L_2|$  or  $|L_1 + L_2|$ , then  $\text{lct}(X, \bar{G}) = 1/2$  (see [8, Theorem 1.7]).

Suppose that  $|L_1|$ ,  $|L_2|$ ,  $|L_1 + L_2|$  contain no  $\bar{G}$ -invariant curves, but  $\text{lct}(X, \bar{G}) < 1$ . There are  $\lambda \in \mathbb{Q}$  and an effective  $\bar{G}$ -invariant  $\mathbb{Q}$ -divisor  $D$  on  $X$  such that  $\lambda < 1$  and  $D \sim_{\mathbb{Q}} 2(L_1 + L_2)$ , but  $(X, \lambda D)$  is not Kawamata log terminal. We may assume that  $(X, \lambda D)$  is log canonical.

Suppose that there is a  $\bar{G}$ -invariant curve  $C \subset X$  such that  $\lambda D = \mu C + \Omega$ , where  $\mu \geq 1$ , and  $\Omega$  is an effective  $\bar{G}$ -invariant  $\mathbb{Q}$ -divisor, whose support does not contain any component of the curve  $C$ . Then  $C \sim aL_1 + bL_2$  for some non-negative integers  $a$  and  $b$ . Then either  $a \geq 2$  or  $b \geq 2$ . But

$$2 > 2\lambda = \lambda(D \cdot L_1) = (\mu C + \Omega) \cdot L_1 \geq C \cdot L_1 = b,$$

which implies that  $b \leq 1$ . Similarly, we see that  $a = C \cdot L_2 \leq 1$ , which is a contradiction.

We see that the locus  $\text{LCS}(X, \lambda D)$  is a finite  $\bar{G}$ -invariant set. Thus, the locus  $\text{LCS}(X, \lambda D)$  consists of a single point  $O \in X$  by Theorem 2.5. Let  $H$  be the unique curve in  $|L_1 + L_2|$  that is singular at  $O$ . Then  $H$  must be  $\bar{G}$ -invariant, which is a contradiction, because the linear system  $|L_1 + L_2|$  contains no  $\bar{G}$ -invariant divisors. □

Let us identify  $X \cong \mathbb{P}^1 \times \mathbb{P}^1$  with a smooth quadric surface in  $\mathbb{P}^3$ , and let us identify  $\bar{G}$  with a subgroup in  $\text{Aut}(\mathbb{P}^3)$  such that  $X$  is  $\bar{G}$ -invariant. Let  $\phi: \text{SL}_4(\mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}^3)$  be the natural projection. Then there is a finite subgroup  $G \subset \text{SL}_4(\mathbb{C})$  such that  $\bar{G} = \phi(G)$ . Put  $W = \mathbb{C}^4$ .

**Lemma 2.17.** *Suppose that  $\bar{G} \cong G \cong A_5$  and that  $W \cong \mathbb{C}^4$  is an irreducible representation of the group  $G$ . Assume that  $B_X \sim_{\mathbb{Q}} mL_1 + (8 - m)L_2$ , where  $m$  is a rational number such that  $1 < m < 2$ . Then the locus  $\text{LCS}(X, B_X)$  consists of finitely many points.*

*Proof.* Let  $U$  be a two-dimensional representation of the group  $2.A_5$ . Then the center of the group  $2.A_5$  acts trivially on  $U \otimes U^\vee$ , and  $U \otimes U^\vee \cong W$  as representations of  $A_5$ . Therefore, one has

$$H^0(X, \mathcal{O}_X(L_1 + nL_2)) \cong U \otimes \text{Sym}^n(U^\vee)$$

as representations of  $A_5$ .

Suppose that the locus  $\text{LCS}(X, B_X)$  does not consist of finitely many points. Then there is a  $\bar{G}$ -invariant curve  $C \subset X$  such that  $B_X = \mu C + \Omega$ , where  $\mu \geq 1$  and  $\Omega$  is an effective  $\bar{G}$ -invariant  $\mathbb{Q}$ -divisor, whose support does not contain any component of the curve  $C$ . Then  $C \sim aL_1 + bL_2$  for some non-negative integers  $a$  and  $b$ . Then either  $a \geq 2$  or  $b \geq 2$ , since the linear systems  $|L_1|$ ,  $|L_2|$  and  $|L_1 + L_2|$  do not contain  $\bar{G}$  invariant divisors (indeed,  $W$  is an irreducible  $\bar{G}$  representation, so that  $\mathbb{P}(W)$  does not contain  $\bar{G}$ -invariant hyperplanes, and both  $U$  and  $U^\vee$  are

irreducible  $G$ -representations, so that  $\mathbb{P}(U) \cong \mathbb{P}^1$  and  $\mathbb{P}(U^\vee) \cong \mathbb{P}^1$  do not contain  $\bar{G}$ -invariant points). On the other hand,

$$2 > m = B_X \cdot L_2 = (\mu C + \Omega) \cdot L_2 \geq C \cdot L_2 = a,$$

which implies that either  $a = 0$  or  $a = 1$ . Similarly, we see that

$$7 > 8 - m = B_X \cdot L_1 = (\mu C + \Omega) \cdot L_1 \geq C \cdot L_1 = b,$$

which implies that  $b \leq 6$ . If  $a = 0$ , then  $C \in |bL_2|$ , and thus  $C$  is  $\bar{G}$ -invariant. The latter is implied by Lemma 2.9 (actually, Lemma 2.9 shows that the linear system  $|bL_2|$  does not contain  $\bar{G}$ -invariant curves for  $b \leq 11$ ). If  $a = 1$ , then  $C \in |L_1 + bL_2|$ . Now a direct computation\* shows that  $U \otimes \text{Sym}^n(U^\vee)$  does not contain one-dimensional  $G$ -invariant subspaces for  $b \leq 6$ . The obtained contradiction completes the proof. □

Let  $C$  be a smooth irreducible curve of genus  $g$ .

**Theorem 2.18.** *Suppose that  $g \geq 2$ . Then*

$$|\text{Aut}(C)| \leq \begin{cases} 320 & \text{if } g = 9, \\ 432 & \text{if } g = 10, \\ 240 & \text{if } g = 11, \\ 120 & \text{if } g = 12, \\ 360 & \text{if } g = 13, \\ 504 & \text{if } g = 15, \end{cases}$$

and the inequality  $|\text{Aut}(C)| \leq 84(g - 1)$  holds for any  $g \geq 2$ .

*Proof.* The inequality  $|\text{Aut}(C)| \leq 84(g - 1)$  is the famous Hurwitz bound (see [6, Theorem 3.17]). The exact bounds for particular genera may be found in [6, Table 13]. □

Let  $D$  be an effective divisor on the curve  $C$ .

**Theorem 2.19** ([22, Theorem 5.4]). *If  $h^1(\mathcal{O}_C(D)) \neq 0$ , then*

$$h^0(\mathcal{O}_C(D)) \leq \frac{\text{deg}(D)}{2} + 1.$$

Suppose that there is an embedding  $\zeta: C \rightarrow \mathbb{P}^n$  such that  $\zeta(C)$  is a curve of degree  $d$ .

**Theorem 2.20** ([2, Section III.2]). *If  $\zeta(C)$  is not contained in any hyperplane in  $\mathbb{P}^n$ , then*

$$g \leq \frac{m(m - 1)(n - 1)}{2} + m\varepsilon,$$

where  $m = \lfloor (d - 1)/(n - 1) \rfloor$  and  $\varepsilon = d - 1 - m(n - 1)$ .

---

\*We used Magma software [5] to carry it out.

3. ALTERNATING GROUP

Let  $S_6$  be the group of permutations of the set  $\{1, 2, 3, 4, 5, 6\}$ .

**Definition 3.1.** Let  $H \subset A_6$  be a subgroup that is isomorphic to  $A_5$ ,  $S_4$  or  $A_4$ . Then we say that the embedding  $H \subset A_6$  is standard if one of the following conditions is satisfied:

- if  $H \cong A_5$  and  $H$  is conjugate to the subgroup of even permutations of  $\{1, 2, 3, 4, 5\}$ ,
- if  $H \cong S_4$  and  $H$  is conjugate to the subgroup of permutations of the set  $\{1, 2, 3, 4\}$  such that the odd permutations of  $\{1, 2, 3, 4\}$  are twisted by the transposition  $(5, 6)$ ,
- if  $H \cong A_4$  and  $H$  is conjugate to the subgroup of even permutations of  $\{1, 2, 3, 4\}$ .

We say that the embedding  $H \subset A_6$  is non-standard if it is not standard.<sup>†</sup>

Let  $2.A_6$  be the group such that there exists a non-splitting exact sequence of groups

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\alpha} 2.A_6 \xrightarrow{\beta} A_6 \longrightarrow 1.$$

Note that  $\alpha(\mathbb{Z}_2)$  is the center of the group  $2.A_6$ .

**Definition 3.2.** Let  $H \subset 2.A_6$  such that  $\beta(H)$  is isomorphic to  $A_5$ ,  $S_4$  or  $A_4$ . Then we say that the embedding  $H \subset 2.A_6$  is standard if  $\beta(H) \subset A_6$  is standard, and we say that the embedding  $H \subset 2.A_6$  is non-standard if  $\beta(H) \subset A_6$  is non-standard.

Let  $\bar{G}$  be a finite subgroup in  $\text{Aut}(\mathbb{P}^3)$  such that  $\bar{G} \cong A_6$ .

**Lemma 3.3** ([15]). *Every maximal proper subgroup of the group  $\bar{G} \cong A_6$  is isomorphic to either  $A_5$ ,  $S_4$ , or  $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ . Moreover, up to conjugation, the group  $\bar{G}$  contains one subgroup isomorphic to  $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ , one subgroup isomorphic to  $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ , and two subgroups isomorphic to  $S_4$  (respectively,  $A_5$ ,  $A_4$ ).*

Put  $W = \mathbb{C}^4$ . There exists a finite subgroup  $G \subset \text{SL}_4(\mathbb{C})$  such that  $G \cong 2.A_6$  and  $\bar{G} = \phi(G) \subset \text{Aut}(\mathbb{P}^3) \cong \text{PGL}_4(\mathbb{C})$ , where  $\phi: \text{SL}_4(\mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}^3)$  is the natural projection.

*Remark 3.4.* The group  $G$  has two irreducible four-dimensional representations (see [15]), which implies that we may identify  $W$  with one of them, because another one differs from  $W$  by an outer automorphism of the group  $G$ . Thus, we may assume that the natural action of the group  $G$  on  $\Lambda^2(W) \cong \mathbb{C}^6$  arises from the permutation representation of the group  $\bar{G} \cong A_6$ .

Let  $\bar{F} \subset \bar{G}$  be a subgroup, and let  $F \subset G$  be a subgroup such that  $\phi(F) = \bar{F}$ .

*Remark 3.5.* Suppose that  $F \cong 2.A_5 \subset 2.A_6$ . Then

$$\mathbb{C}^4 \cong W \cong \begin{cases} U \oplus U' & \text{if } F \subset G \text{ is standard,} \\ \text{Sym}^3(U) \cong \text{Sym}^3(U') & \text{if } F \subset G \text{ is non-standard,} \end{cases}$$

where  $U$  and  $U'$  are different two-dimensional representations of the group  $F \cong 2.A_5$ .

---

<sup>†</sup>Note that the non-standard and standard embeddings of the group  $H$  into  $A_6$  are interchanged by  $\text{Aut}(A_6)$ .

**Lemma 3.6.** *Suppose that  $F \cong 2.A_5$  and  $F \subset G$  is a non-standard embedding. Then there exists an irreducible  $\bar{F}$ -invariant smooth rational cubic curve  $Z \subset \mathbb{P}^3$ .*

*Proof.* The required assertion follows from Remark 3.5. □

Let  $z$  and  $e$  be the non-trivial element in the center of  $G$  and the identity element, respectively.

**Lemma 3.7** (cf. [10, Lemma 4.7]). *Any semi-invariant of the group  $G$  is its invariant. The group  $G$  does not have invariants of odd degree, as well as invariants of degree at most 7. On the other hand, the group  $G$  has two linearly independent invariants of degree 8.*

*Proof.* Semi-invariants of the group  $G$  are its invariants because the center of the group  $G$  is contained in its commutator, and the group  $\bar{G}$  is a simple non-abelian group.

The group  $G$  does not have invariants of odd degree because  $G$  contains a scalar matrix whose non-zero entries are  $-1$ . Therefore, to prove that  $G$  has no invariants of degree at most 7, it is enough to show that  $G$  does not have invariants of degree 4 and 6.

Let  $\chi_m$  be the character of the representation  $\text{Sym}^m(W)$  (cf. Remark 3.4). Put  $\chi = \chi_1$ .

The values of the characters  $\chi, \chi_4, \chi_6$  and  $\chi_8$  are listed in the following table:

	$[5, 1]_{10}$	$[5, 1]_5$	$[4, 2]_8$	$[3, 3]_6$	$[3, 3]_3$	$[3, 1, 1, 1]_6$	$[3, 1, 1, 1]_3$	$[2, 2, 1, 1]_4$	$z$	$e$
#	144	144	180	40	40	40	40	90	1	1
$\chi$	1	-1	0	-1	1	2	-2	0	-4	4
$\chi_4$	0	0	-1	2	2	-4	-4	3	35	35
$\chi_6$	-1	-1	0	3	3	3	3	-4	84	84
$\chi_8$	0	0	1	3	3	3	3	5	165	165

where the first row lists the types of the elements in  $G$  (for example, the symbol  $[5, 1]_{10}$  denotes the set<sup>‡</sup> of elements of order 10 whose image in  $A_6$  is a product of disjoint cycles of length 5 and 1).

Recall that there is a natural inner product  $\langle \cdot, \cdot \rangle$  defined for the characters  $\theta$  and  $\theta'$  by

$$\langle \theta, \theta' \rangle = \frac{1}{|G|} \sum_{g \in G} \theta(g) \overline{\theta'(g)}.$$

Let  $\chi_0$  be the trivial character of  $G$ . Then  $\langle \chi_4, \chi_0 \rangle = \langle \chi_6, \chi_0 \rangle = 0$  so that  $G$  has no invariants of degree 4 and 6. On the other hand,  $\langle \chi_8, \chi_0 \rangle = 2$ , which means that the group  $G$  has exactly two linearly independent invariants of degree 8. □

Suppose that  $\bar{F}$  is a stabilizer of a point  $P \in \mathbb{P}^3$ .

**Lemma 3.8.** *Let  $\Sigma \subset \mathbb{P}^3$  be the  $\bar{G}$ -orbit of the point  $P \in \mathbb{P}^3$ . Then  $|\Sigma| \geq 36$ .*

---

<sup>‡</sup>Note that these sets do not coincide with conjugacy classes. For example, the image of the set of the elements of type  $[5, 1]_{10}$  under the natural projection  $2.A_6 \rightarrow A_6$  is a union of two different conjugacy classes in  $A_6$ .



*Proof.* It follows from Lemma 3.7 that  $|\Sigma| \geq 8$  and  $|\Sigma|$  is even. Suppose that  $|\Sigma| \leq 35$ . Then

$$45 = \frac{|\bar{G}|}{8} \geq |\bar{F}| \geq \frac{|\bar{G}|}{35} > 10,$$

which implies that  $|\bar{F}| \in \{12, 18, 36\}$  by Lemma 3.3.

Let us consider the vector space  $W$  as a representation of the group  $F$ , and let  $\chi$  be its character. There is a homomorphism  $\theta: F \rightarrow \mathbb{C}^*$  such that the inner product  $\langle \theta, \chi \rangle \neq 0$ .

Suppose that  $|\bar{F}| = 36$ . Then  $\bar{F} \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$  by Lemma 3.3.

The structure of the group  $F$  and the values of  $\chi$  are given in the following table:

	$[4, 2]_8$	$[3, 3]_6$	$[3, 3]_3$	$[3, 1, 1, 1]_6$	$[3, 1, 1, 1]_3$	$[2, 2, 1, 1]_4$	$z$	$e$
#	36	4	4	4	4	18	1	1
$\chi$	0	-1	1	2	-2	0	-4	4

where we use the notation that are used in the proof of Lemma 3.7.

We have  $[F, F] \cong 2 \cdot (\mathbb{Z}_3 \times \mathbb{Z}_3)$ , which gives  $\theta(g) = 1$  for any  $g \in F$  that is not of type  $[4, 2]_8$  and  $[2, 2]_4$ . Hence  $\langle \theta, \chi \rangle = 0$ , which is a contradiction.

Suppose that  $|\bar{F}| = 18$ . Then  $\bar{F} \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$  by Lemma 3.3. Arguing as above, we get  $\langle \theta, \chi \rangle = 0$ .

Suppose that  $|\bar{F}| = 12$ . Then  $\bar{F} \cong A_4$  by Lemma 3.3.

Up to conjugation, the group  $\bar{G}$  contains two subgroups isomorphic to  $A_4$ .

If  $\bar{F} \subset \bar{G}$  is a standard embedding, then the values of  $\chi$  are given in the following table:

	$[3, 1, 1, 1]_6$	$[3, 1, 1, 1]_3$	$[2, 2, 1, 1]_4$	$z$	$e$
#	8	8	6	1	1
$\chi$	2	-2	0	-4	4

If  $\bar{F} \subset \bar{G}$  is a non-standard embedding, then the values of  $\chi$  are given in the following table:

	$[3, 3]_6$	$[3, 3]_3$	$[2, 2, 1, 1]_4$	$z$	$e$
#	8	8	6	1	1
$\chi$	-1	1	0	-4	4

We have  $[F, F] \cong 2 \cdot (\mathbb{Z}_2 \times \mathbb{Z}_2)$ . So  $\theta(g) = 1$  for any  $g \in F$  of order different from 3 and 6, and  $\theta(g) = \theta(g^2)$  for all  $g \in F$  of order 6. Now we can check that  $\langle \theta, \chi \rangle = 0$ , which is a contradiction. □

**Lemma 3.9.** *Let  $C$  be a smooth irreducible  $\bar{G}$ -invariant curve in  $\mathbb{P}^3$  of genus  $g \geq 13$ , and let  $\mathcal{I}_C$  be its ideal sheaf. Then  $h^0(\mathcal{O}_{\mathbb{P}^3}(i) \otimes \mathcal{I}_C) = 0$  for any  $i \in \{1, 2, 3\}$ .*

*Proof.* It follows from Theorem 2.20 and Lemma 2.10 that  $d > 4$ . Hence  $h^0(\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{I}_C) = 0$ , since  $G$  does not have semi-invariants of degree 2 by Lemma 3.7.

Suppose that there is a cubic surface  $X \subset \mathbb{P}^3$  such that  $C \subset X$ . Then  $h^0(\mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{I}_C) \geq 2$ , since  $G$  does not have semi-invariants of degree 3 by Lemma 3.7. Thus, there is a cubic surface  $X' \subset \mathbb{P}^3$  such that  $C \subset X' \neq X$ . Thus  $C \subset X \cap X'$ ,

and  $X$  and  $X'$  are irreducible, because  $C$  is contained neither in a quadric nor in a plane. We see that  $d \leq 9$ . Hence, we have  $g \leq 12$  by Theorem 2.20, which is a contradiction.  $\square$

4. PROJECTIVE SPACE

Let  $\bar{G}$  be a subgroup in  $\text{Aut}(\mathbb{P}^3)$  such that  $\bar{G} \cong A_6$ . Then there is a subgroup  $\hat{G} \subset \text{Aut}(\mathbb{P}^3)$  such that  $\bar{G} \subset \hat{G} \cong S_6$ , which implies that  $\hat{G} \subseteq \text{Aut}^{\bar{G}}(\mathbb{P}^3)$ , because  $\bar{G}$  is a normal subgroup of the group  $\hat{G}$  (recall that  $\text{Aut}^{\bar{G}}(\mathbb{P}^3)$  is the normalizer of the group  $\bar{G}$  in the group  $\text{Aut}(\mathbb{P}^3)$ ).

The main purpose of this section is to prove that  $\mathbb{P}^3$  is  $\bar{G}$ -birationally rigid and to describe the group  $\text{Bir}^{\bar{G}}(\mathbb{P}^3)$  (see Theorem 1.24). We will do this in several steps. But first of all, we must study lines in  $\mathbb{P}^3$  whose  $\bar{G}$ -orbits consists of 6 lines (there are 12 such lines in total, the groups  $\hat{G}$  acts transitively on them, and they form two  $\bar{G}$ -orbits consisting of 6 lines each).

Let  $\bar{G}_1 \subset \bar{G}$  be a subgroup such that  $\bar{G}_1 \cong A_5$  and the embedding  $A_5 \cong \bar{G}_1 \subset \bar{G} \cong A_6$  is standard (see Definition 3.1). Then there are two disjoint  $\bar{G}_1$ -invariant lines  $L_1$  and  $L'_1$  in  $\mathbb{P}^3$  by Remark 3.5. Denote by  $L_1, \dots, L_6$  ( $L'_1, \dots, L'_6$ , respectively) the lines in  $\mathbb{P}^3$  that are the images of  $L_1$  ( $L'_1$ , respectively) under the action of  $\bar{G}$ . Put  $L = \{L_1, \dots, L_6\}$  and  $L' = \{L'_1, \dots, L'_6\}$ .

**Lemma 4.1.** *The curve  $\sum_{i=1}^6 L_i + \sum_{i=1}^6 L'_i$  is a  $\hat{G}$ -orbit of the line  $L_1$ . The 12 lines of  $L \cup L'$  are pairwise disjoint. For any 4 lines among the lines  $L_1, \dots, L_6$  ( $L'_1, \dots, L'_6$ , respectively), there are 2 lines in  $\mathbb{P}^3$  that intersect them. There are no lines in  $\mathbb{P}^3$  that intersect 5 lines among  $L_1, \dots, L_6$  ( $L'_1, \dots, L'_6$ , respectively).*

*Proof.* The first assertion of the lemma follows from the construction of the lines  $L_i$  and  $L'_i$ . To prove the second assertion note that the stabilizers of any two of the 12 lines of  $L \cup L'$  except for the case of the lines  $L_i$  and  $L'_i$  corresponding to one and the same stabilizer  $A_5 \subset \bar{G}$  (i.e. two standard subgroups isomorphic to  $A_5$ ) generate together the whole group  $\bar{G} \cong A_6$ . In particular, all 12 lines of  $L \cup L'$  are pairwise distinct, since otherwise there would exist a  $\bar{G}$ -invariant line in  $\mathbb{P}^3$ . Similarly, if some two of the 12 lines of  $L \cup L'$  intersected at a point  $P$ , then there would exist a  $\bar{G}$ -invariant point in  $\mathbb{P}^3$ .

Suppose that there exist more than 2 lines in  $\mathbb{P}^3$  that intersect some 4 of the lines of  $L$  (say,  $L_1, L_2, L_3$  and  $L_4$ ). Then the lines  $L_1, \dots, L_4$  are contained in a (unique) smooth quadric  $X \subset \mathbb{P}^3$ . Note that there is an element  $g \in \bar{G} \cong A_6$  that preserves the lines  $L_1$  and  $L_6$ , interchanges the lines  $L_2$  and  $L_3$  and interchanges the lines  $L_4$  and  $L_5$ . The quadric  $X$  is invariant under  $g$  (indeed, an intersection of two distinct smooth quadrics in  $\mathbb{P}^3$  cannot contain three skew lines). Hence  $X$  contains the line  $L_5$ . Similarly,  $X$  contains the line  $L_6$ , and thus  $X$  is  $\bar{G}$ -invariant. The latter contradicts Lemma 3.7. Therefore, to prove the third assestion of the lemma we may suppose that there exists exactly 1 line  $L_{1234}$  in  $\mathbb{P}^3$  that intersects the lines  $L_1, \dots, L_4$ . Note that the group  $\bar{G}$  acts transitively on the 4-tuples of the lines of  $L$ . Thus for any 4 lines  $L_{i_1}, \dots, L_{i_4}$  there exists a unique line  $L_{i_1 \dots i_4}$  that intersects  $L_{i_1}, \dots, L_{i_4}$ . Put  $P_{i_2 i_3 i_4} = L_1 \cap L_{i_2 i_3 i_4}$ . Then the set  $\{P_{i_2 i_3 i_4}\}$  is invariant under the group  $\bar{G}_1 \cong A_5$  and consists of at most  $\binom{5}{3} = 10$  points. This is impossible by Lemma 2.9.

Finally, suppose that for some 5 of the lines of  $L$  (say, for  $L_2, \dots, L_6$ ) there exists a line  $L$  that intersects all 5 of them. The above argument implies that there are at

most 2 lines intersecting  $L_2, \dots, L_6$  so that  $L$  is invariant under the group  $\bar{G}_1 \cong A_5$ . Since the lines of  $L \cup L'$  are pairwise disjoint, the line  $L$  coincides with neither  $L_1$  nor  $L'_1$ . Hence there are at least 3 lines in  $\mathbb{P}^3$  that are invariant under  $\bar{G}_1$ , which contradicts Remark 3.5.

The same arguments apply if one replaces the lines of  $L$  by the lines of  $L'$ .  $\square$

Recall that several lines in  $\mathbb{P}^3$  are said to lie in a linear complex if the corresponding points of the Grassmannian  $\text{Gr}(2, 4) \subset \mathbb{P}^5$  lie in a hyperplane section.

**Lemma 4.2.** *Neither  $L_1, L_2, L_3, L_4, L_5, L_6$  nor  $L'_1, L'_2, L'_3, L'_4, L'_5, L'_6$  lie in a linear complex.*

*Proof.* It follows from Remark 3.4 that the space  $\Lambda^2(V) \cong \mathbb{C}^6$  is the permutation representation of the group  $\bar{G} \cong A_6$ . The natural action of the group  $\bar{G}$  on  $\text{Gr}(2, 4) \subset \mathbb{P}^5 \cong \mathbb{P}(\Lambda^2(V))$  arises from this representation. Let us identify  $\bar{G}$  with a subgroup in  $\text{Aut}(\mathbb{P}^5)$ . Then there is a unique  $\bar{G}$ -invariant hyperplane  $H \subset \mathbb{P}^5$ . We may assume that  $\text{Gr}(2, 4) \cap H$  is given by

$$\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^2 = 0 \subset \mathbb{P}^5 \cong \text{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]),$$

and  $H$  is given by  $\sum_{i=0}^5 x_i = 0$ . Let  $P \in \text{Gr}(2, 4)$  be a point that corresponds to the line  $L_1 \subset \mathbb{P}^3$ , and let  $\bar{G}_1$  be the stabilizer subgroup in  $\bar{G}$  of the point  $P$ . Then  $\bar{G}_1 \cong A_5$ , and the embedding  $\bar{G}_1 \subset \bar{G}$  is standard. If the  $\bar{G}$ -orbit of the point  $P$  is contained in a hyperplane, then this hyperplane must be  $H$ , which implies that the  $\bar{G}$ -orbit of the point  $P$  must contain a point in  $\mathbb{P}^5$  that is given by  $x_0 = \dots = x_4 = -x_5/5$ , which is impossible, because this point does not belong to the intersection  $\text{Gr}(2, 4) \cap H$ .  $\square$

Now we are ready to formulate the main technical result of this section.

**Theorem 4.3.** *Let  $\mathcal{M}$  be a (non-empty)  $\bar{G}$ -invariant linear system on  $\mathbb{P}^3$  that does not have fixed components, and let  $\lambda$  be a positive rational number such that  $\lambda\mathcal{M} \sim_{\mathbb{Q}} -K_{\mathbb{P}^3}$ . Suppose that  $(\mathbb{P}^3, \lambda\mathcal{M})$  is canonical at a general point of every line  $L_1, \dots, L_6, L'_1, \dots, L'_6$ . Then  $(\mathbb{P}^3, \lambda\mathcal{M})$  is canonical.*

Before proving Theorem 4.3, let us show that it implies that  $\mathbb{P}^3$  is  $\bar{G}$ -birationally rigid and  $\text{Bir}^{\bar{G}}(\mathbb{P}^3)$  is isomorphic to a free product of  $S_6$  and  $S_6$  with an amalgamated subgroup  $A_6$ . To do this, will use the lines  $L_1, \dots, L_6, L'_1, \dots, L'_6$  to prove that  $\text{Aut}^{\bar{G}}(\mathbb{P}^3) = \hat{G}$  and to construct two  $\bar{G}$ -equivariant birational non-biregular involutions of  $\mathbb{P}^3$  that were described in [45]. The proof of  $\bar{G}$ -birational rigidity of  $\mathbb{P}^3$  and the description of the group  $\text{Bir}^{\bar{G}}(\mathbb{P}^3)$  crucially depend on these birational involutions.

**Lemma 4.4.** *The equality  $\text{Aut}^{\bar{G}}(\mathbb{P}^3) = \hat{G}$  holds.*

*Proof.* Put  $\tilde{G} = \text{Aut}^{\bar{G}}(\mathbb{P}^3)$ . Then the curve  $\sum_{i=1}^6 L_i + \sum_{i=1}^6 L'_i$  is  $\tilde{G}$ -invariant, and the group  $\tilde{G}$  is a normal subgroup of the group  $\bar{G}$ . There is a subgroup  $\check{G} \subset \tilde{G}$  of index 2 such that  $\sum_{i=1}^6 L_i$  and  $\sum_{i=1}^6 L'_i$  are  $\check{G}$ -invariant.

Let  $Y$  be the Hierholzer surface of the lines  $L_1, L_2, L_3, L_4, L_5, L_6$  (see [46, Section 2.1]). Then the surface  $Y$  is  $\check{G}$ -invariant and there exists a monomorphism  $\xi: \check{G} \rightarrow \text{Aut}(Y)$ . Moreover, the surface  $Y$  is birational to a surface of general type

(see [46, Theorem 2.1]), which implies that  $\check{G}$  is a finite group (see e. g. [21, Theorem 1.1]). Thus  $\tilde{G}$  is a finite group. Hence, we have  $\text{lct}(\mathbb{P}^3, \tilde{G}) \geq \text{lct}(\mathbb{P}^3, \bar{G}) \geq 5/4$  by [10, Theorem 4.13].

Let  $\phi: \text{SL}_4(\mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}^3)$  be the natural projection. Then there is a finite subgroup  $G \subset \text{SL}_4(\mathbb{C})$  such that  $\tilde{G} = \phi(G)$ . Note that  $G$  may not be uniquely defined. However, by [10, Theorem 4.13], we may assume that  $G$  is one of the eight groups described in [10, Lemma 4.12]. Suppose that  $\hat{G} \subsetneq \tilde{G}$ . Then  $G \not\cong 2.S_6$ .

Recall that  $\bar{G}$  is a (non-trivial) normal subgroup in  $\tilde{G}$ . Hence  $G$  cannot be isomorphic to any of the groups  $2.A_7$  and  $\text{PSP}_4(\mathbb{F}_3)$  because the images of these groups in  $\text{PGL}_4(\mathbb{C})$  are simple groups. Moreover,  $G \not\cong 2.A_6$ , since there is a subgroup  $\hat{G} \subset \bar{G}$  isomorphic to  $S_6$ .

For each of the four remaining groups from [10, Lemma 4.12] one has (see [36]) an exact sequence of groups

$$1 \longrightarrow \mathbb{Z}_2^4 \xrightarrow{\alpha} \tilde{G} \xrightarrow{\beta} \Gamma \longrightarrow 1,$$

where  $\Gamma$  is a subgroup of the group  $S_6$ . Thus  $\Gamma \cong S_6$  because  $\tilde{G} \supset \hat{G} \cong S_6$ . Let  $e$  be the identity element in  $\tilde{G}$ . Then  $\bar{G} \cap \text{im}(\alpha) = e$  because  $\bar{G}$  is a simple group, and  $\bar{G}$  is a normal subgroup of the group  $\tilde{G}$ . Hence the subgroup of  $\tilde{G}$  generated by  $\bar{G} \cong A_6$  and  $\text{im}(\alpha) \cong \mathbb{Z}_2^4$  is a direct product of the latter groups. This leads to a contradiction because the action of  $\Gamma$  on  $\mathbb{Z}_2^4$  is faithful (see [36]).  $\square$

Let  $H$  be a general hyperplane in  $\mathbb{P}^3$ .

**Lemma 4.5.** *There is a  $\bar{G}$ -equivariant birational non-biregular involution  $\iota \in \text{Bir}(\mathbb{P}^3)$  such that  $\iota(H)$  is a surface of degree 19 that has singularity of multiplicity 5 in a general point of every line  $L_1, \dots, L_6$ , and the group generated by  $\bar{G}$  and  $\iota$  is isomorphic to  $S_6$ .*

*Proof.* Let  $\mathcal{H}$  be a linear subsystem of the linear system  $|\mathcal{O}_{\mathbb{P}^3}(4)|$  consisting of surfaces that pass through the lines  $L_1, L_2, L_3, L_4, L_5, L_6$ . Then it follows from [45] and [46, Theorem 2.4] that  $\mathcal{H}$  does not have fixed components, and  $\mathcal{H}$  gives a rational map  $\psi: \mathbb{P}^3 \dashrightarrow V$ , where  $V$  is a quartic threefold in  $\mathbb{P}^4$ . Let  $\alpha: U \rightarrow \mathbb{P}^3$  be a blow up along  $L_1, L_2, L_3, L_4, L_5, L_6$ . Then there is a commutative diagram

$$\begin{array}{ccc} & U & \\ \alpha \swarrow & & \searrow \beta \\ \mathbb{P}^3 & \dashrightarrow \phi \dashrightarrow & V, \end{array}$$

where  $\beta$  is a birational morphism that contracts finitely many curves. It follows from [46, Theorem 2.4] that  $V \subset \mathbb{P}^4$  can be given by the equation<sup>§</sup>

$$\begin{aligned} & 5(yw - zt + xy + xw - xz - xt - x^2)^2 - 20xyw(w - x + y - z - t) \\ & \parallel \\ & (3x^2 + 2y^2 + 2z^2 + 2t^2 + 2w^2 - 3xy + 3xz + 3xt - 3xw - 2yz - 2yt + yw + zt - 2zw - 2tw)^2 \end{aligned}$$

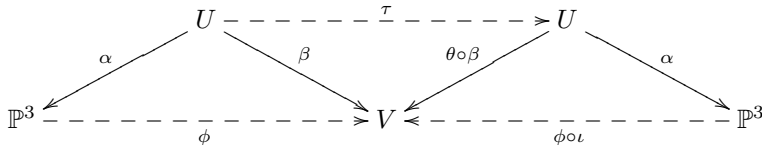
<sup>§</sup>Note that the quartic threefold  $V \subset \mathbb{P}^4$  is determinantal (see [45], [37, Example 6.4.2]).

in appropriate homogeneous coordinates  $[x : y : z : t : w]$  in  $\mathbb{P}^4$ . By [45] we know that the singular locus of the threefold  $V$  consists of 36 nodes, and the morphism  $\beta$  contracts the proper transforms of 30 lines in  $\mathbb{P}^3$ , each of whom intersects 4 lines among  $L_1, L_2, L_3, L_4, L_5, L_6$ , and proper transforms of 6 twisted cubics in  $\mathbb{P}^3$ , each of whom has the lines  $L_1, L_2, L_3, L_4, L_5, L_6$  as chords.

The map  $\phi$  is  $\bar{G}$ -equivariant. We can identify  $\bar{G}$  with a subgroup in  $\text{Aut}(V)$  and with a subgroup in  $\text{Aut}(\mathbb{P}^4)$ . By Lemma 3.7, there are no  $\bar{G}$ -fixed points in  $\mathbb{P}^4$ . So, the action of the group  $\bar{G}$  on  $\mathbb{P}^4$  arises from its irreducible five-dimensional representation (cf. Remark 3.4). Hence, there is a subgroup  $\bar{\Gamma} \subset \text{Aut}(V)$  such that  $\bar{G} \subset \bar{\Gamma} \cong S_6$ . Let  $\theta$  be an involution in  $\bar{\Gamma}$  such that  $\theta \notin \bar{G}$ . Put  $\iota = \phi^{-1} \circ \theta \circ \phi \in \text{Bir}^{\bar{G}}(\mathbb{P}^3)$ . Then it follows from the construction of  $\iota$  that  $\langle \iota, \bar{G} \rangle \cong S_6$ .

Let us show that  $\iota$  is not biregular. Suppose that this is not true. Let us identify the subgroup  $\langle \iota, \bar{G} \rangle$  with  $\bar{\Gamma} \cong S_6$ . Then  $\sum_{i=1}^6 L_i$  must be  $\bar{\Gamma}$ -invariant. Let  $\bar{G}_1 \subset \bar{G}$  and  $\bar{\Gamma}_1 \subset \bar{\Gamma}$  be stabilizers of the line  $L_1$ . Then  $A_5 \cong \bar{G}_1 \subset \bar{\Gamma}_1 \cong S_5$  and there are natural homomorphisms  $\zeta: \bar{G}_1 \rightarrow \text{Aut}(L_1)$  and  $\eta: \bar{\Gamma}_1 \rightarrow \text{Aut}(L_1)$ . Thus, we have  $A_5 \cong \text{im}(\zeta) \subseteq \text{im}(\eta)$ , which gives  $\text{im}(\eta) \cong S_5$ . But  $\text{Aut}(L_1) \cong \text{PGL}_2(\mathbb{C})$  contains no subgroups isomorphic to  $S_5$ . The obtained contradiction shows that  $\iota$  is not biregular.

Put  $\tau = \beta^{-1} \circ \theta \circ \beta \in \text{Bir}(U)$ . Then  $\tau$  is a composition of flops, and the commutative diagram



is a  $\bar{G}$ -equivariant Sarkisov link of type II (see [13, Definition 3.4]). The involution  $\tau$  is biregular outside of the curves contracted by  $\beta$ . Then  $\tau$  naturally acts on the group  $\text{Pic}(U)$ . Moreover, this action is non-trivial because  $\iota$  is not biregular.

Put  $\hat{H} = \alpha^*(H)$ . Let  $E_k$  be the  $\alpha$ -exceptional divisor such that  $\alpha(E_k) = L_k$ . Then

$$\tau^*(\hat{H}) \cdot \tau^*(\hat{H}) \cdot \left( 4\hat{H} - \sum_{i=1}^6 E_i \right) = 4$$

because  $\tau^*(K_U) \sim K_U$  and  $\beta$  contracts finitely many curves. Similarly, we see that

$$\tau^*(\hat{H}) \cdot \tau^*(E_k) \cdot \left( 4\hat{H} - \sum_{i=1}^6 E_i \right) = 1, \quad \tau^*(E_i) \cdot \tau^*(E_k) \cdot \left( 4\hat{H} - \sum_{i=1}^6 E_i \right) = \begin{cases} -2 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}$$

which immediately implies that either  $\tau^*(\hat{H}) \sim \hat{H}$  and  $\tau^*(E_k) = E_k$  or

$$(4.6) \quad \begin{cases} \tau^*(\hat{H}) \sim 19\hat{H} - 5 \sum_{i=1}^6 E_i, \\ \tau^*(E_k) \sim 12\hat{H} - 4E_k - \sum_{i \neq k} 3E_i. \end{cases}$$

Since  $\iota$  is not biregular, the involution  $\tau$  must act non-trivially on  $\text{Pic}(U)$ . Thus, we see that (4.6) holds, which implies, in particular, that  $\iota(H)$  is a surface of degree 19 that has singularity of multiplicity 5 in a general point of every line  $L_1, \dots, L_6$  because  $\iota(H) = \alpha \circ \tau(\hat{H})$ . □

Put  $\iota' = \nu \circ \iota \circ \nu^{-1}$  for any  $\nu \in \hat{G} \cong S_6$  such that  $\nu \notin \bar{G}$ . Then  $\iota'$  is a  $\bar{G}$ -equivariant birational non-biregular involution of  $\mathbb{P}^3$  such that  $\iota'(T)$  is a surface of degree 19 that has singularity of multiplicity 5 in a general point of every line  $L'_1, \dots, L'_6$ , and the group generated by  $\bar{G}$  and  $\iota'$  is also isomorphic to  $S_6$ . Note that the choice of  $\iota$  and  $\iota'$  is not unique. Put  $\Gamma = \langle \iota, \iota', \hat{G} \rangle$ . Let us use birational involutions  $\iota$  and  $\iota'$  together with Theorem 4.3 to prove

**Lemma 4.7.** *Let  $\mathcal{M}$  be a (non-empty)  $\bar{G}$ -invariant linear system on  $\mathbb{P}^3$  that does not have fixed components. Then there are  $\rho \in \Gamma$  and  $\mu \in \mathbb{Q}_{>0}$  such that  $\mu\rho(\mathcal{M}) \sim_{\mathbb{Q}} -K_{\mathbb{P}^3}$ , and  $(\mathbb{P}^3, \mu\rho(\mathcal{M}))$  is canonical.*

*Proof.* Take  $\lambda \in \mathbb{Q}_{>0}$  such that  $\lambda\mathcal{M} \sim_{\mathbb{Q}} -K_{\mathbb{P}^3}$ . If  $(\mathbb{P}^3, \lambda\mathcal{M})$  is canonical, then we are done. Thus, we may assume that  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not canonical. By Theorem 4.3, the log pair  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not canonical at a general point of one line among  $L_1, \dots, L_6, L'_1, \dots, L'_6$ .

Without loss of generality, we may assume that  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not canonical at a general point of the line  $L_1$ . Then  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not canonical at a general point of the lines  $L_2, \dots, L_6$ , since  $\mathcal{M}$  is  $\bar{G}$ -invariant. Let  $M$  be a general surface in  $\mathcal{M}$ . Then  $\text{mult}_{L_i}(M) > 1/\lambda$  for every  $i \in \{1, \dots, 6\}$  (see [14, Exercise 6.18]). Note that  $M$  is a surface of degree  $d = 4/\lambda$ . Let  $\tilde{d}$  be the degree of the surface  $\iota(M)$ . Put  $\tilde{\lambda} = 4/\tilde{d}$ ; then  $\tilde{\lambda}\iota(\mathcal{M}) \sim_{\mathbb{Q}} -K_{\mathbb{P}^3}$ . Moreover, it follows from Lemma 4.5 that

$$\tilde{d} = 19d - 12 \sum_{i=1}^6 \text{mult}_{L_i}(M) < 19d - 18d = d.$$

If  $(\mathbb{P}^3, \tilde{\lambda}\iota(\mathcal{M}))$  is canonical, then we are done. If it is not canonical, then it follows from Theorem 4.3 that  $(\mathbb{P}^3, \tilde{\lambda}\iota(\mathcal{M}))$  is not canonical at a general point of one line among  $L_1, \dots, L_6, L'_1, \dots, L'_6$ . Thus, we can iterate the above process. Since  $\tilde{d} < d$ , our iterations must terminate in at most  $d-1$  steps, which completes the proof.  $\square$

Now we ready to prove

**Theorem 4.8.** *The variety  $\mathbb{P}^3$  is  $\bar{G}$ -birationally rigid and  $\text{Bir}^{\bar{G}}(\mathbb{P}^3) = \Gamma$ .*

*Proof.* Suppose that there exists a rational normal threefold  $V$  with at most terminal singularities that admits a faithful action of the group  $\bar{G}$  such that there are a  $\bar{G}$ -equivariant Mori fibration  $\pi: V \rightarrow S$  and a  $\bar{G}$ -equivariant birational map  $\xi: V \dashrightarrow \mathbb{P}^3$ . Let  $D$  be a sufficiently general very ample divisor on  $V$ . Put  $\mathcal{M} = \xi(|D|)$ . Then  $\mathcal{M}$  does not have fixed components and is  $\bar{G}$ -invariant. Thus, it follows from Lemma 4.7 that there are  $\rho \in \Gamma$  and  $\mu \in \mathbb{Q}_{>0}$  such that  $\mu\rho(\mathcal{M}) \sim_{\mathbb{Q}} -K_{\mathbb{P}^3}$ , and  $(\mathbb{P}^3, \mu\rho(\mathcal{M}))$  is canonical. Then  $\rho \circ \xi$  is biregular by [12, Theorem 4.2]. In particular, we see that  $V \cong \mathbb{P}^3$  and  $S$  is a point. Since all subgroups in  $\text{Aut}(\mathbb{P}^3)$  that are isomorphic to  $\bar{G}$  are conjugate to each other, we see that  $\mathbb{P}^3$  is  $\bar{G}$ -birationally rigid and  $\text{Bir}^{\bar{G}}(\mathbb{P}^3) = \Gamma$ .  $\square$

Before proving Theorem 4.3, let us use it one more time to prove

**Theorem 4.9.** *The group  $\Gamma$  is the free product of  $\hat{G}$  and  $\langle \iota, \bar{G} \rangle$  with amalgamated subgroup  $\bar{G}$ .*

*Proof.* Let  $\xi$  be a birational automorphism in  $\text{Bir}^{\bar{G}}(\mathbb{P}^3)$  such that  $\xi \notin \hat{G}$ . Put  $\mathcal{M} = \xi(|T|)$  and take  $\lambda \in \mathbb{Q}$  such that  $\lambda\mathcal{M} \sim_{\mathbb{Q}} -K_{\mathbb{P}^3}$ . Then  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not canonical by [12, Theorem 4.2]. By Theorem 4.3, the log pair  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not

canonical at a general point of one line among  $L_1, \dots, L_6, L'_1, \dots, L'_6$ . Arguing as in the proof of Lemma 4.7, we see that there exists a combination

$$\zeta = \underbrace{\dots \circ \iota \circ \iota' \circ \iota \circ \iota' \circ \iota \circ \dots}_{m \text{ times}}$$

such that  $(\mathbb{P}^3, \mu\zeta(\mathcal{M}))$  is canonical, where  $\mu$  is a positive rational number such that one has  $\mu\zeta(\mathcal{M}) \sim_{\mathbb{Q}} -K_{\mathbb{P}^3}$ . Then  $\zeta \circ \xi$  is biregular by [12, Theorem 4.2], which implies that  $\zeta \circ \xi \in \hat{G}$  by Lemma 4.4.

Let us show that  $\zeta$  is uniquely determined by  $\xi$  and the algorithm hidden in the proof of Lemma 4.7.

Let  $M$  be a general surface in  $\mathcal{M}$ , and let  $d$  be its degree. Then  $d = 4/\lambda$ . Arguing as in the proof of Lemma 4.7, we see that if  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not canonical at a general point of one line among  $L_1, \dots, L_6$ , then the degree of  $\iota(M)$  is smaller than  $d$ . Similarly, we see that if  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not canonical at a general point of one line among  $L'_1, \dots, L'_6$ , then the degree of  $\iota'(M)$  is smaller than  $d$ . We can use this to construct  $\xi$  step by step. Thus, to show that  $\zeta$  is uniquely determined by  $\xi$ , we must show that  $(\mathbb{P}^3, \lambda\mathcal{M})$  cannot be non-canonical at a general point of one line among  $L_1, \dots, L_6$  and at the same time at a general point of one line among  $L'_1, \dots, L'_6$ .

Suppose that  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not canonical at a general point of one line among  $L_1, \dots, L_6$  and at the same time at a general point of one line among  $L'_1, \dots, L'_6$ . Then  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not canonical at a general point of every line among  $L_1, \dots, L_6, L'_1, \dots, L'_6$ , since  $\mathcal{M}$  is  $\bar{G}$ -invariant. Then  $\text{mult}_{L_i}(M) > 1/\lambda$  and  $\text{mult}_{L'_i}(M) > 1/\lambda$  for every  $i \in \{1, 2, \dots, 6\}$ . Let  $\Pi$  be a general plane in  $\mathbb{P}^3$  that contains the line  $L_1$ . Put  $\mathcal{M}|_{\Pi} = \text{mult}_{L_1}(M)L_1 + \mathcal{B}$ , where  $\mathcal{B}$  is a linear system on  $\Pi$  that does not have fixed components. Let  $B_1$  and  $B_2$  be general curves in  $\mathcal{B}$ , let  $O_i$  be the point  $\Pi \cap L_i$  for every  $i \in \{2, \dots, 6\}$ , and let  $O'_i$  be the point  $\Pi \cap L'_i$  for every  $i \in \{1, \dots, 6\}$ . Then  $\text{mult}_{O_i}(B_1) = \text{mult}_{O_i}(B_2) = \text{mult}_{L_1}(M)$  for every  $i \in \{2, \dots, 6\}$  and  $\text{mult}_{O'_i}(B_1) = \text{mult}_{O'_i}(B_2) = \text{mult}_{L'_1}(M)$  for every  $i \in \{1, \dots, 6\}$ . Then

$$B_1 \cdot B_2 \geq \sum_{i=2}^6 \text{mult}_{O_i}(B_1)\text{mult}_{O_i}(B_2) + \sum_{i=1}^6 \text{mult}_{O'_i}(B_1)\text{mult}_{O'_i}(B_2) > \frac{11}{\lambda^2},$$

because  $\text{mult}_{L_1}(M) > 1/\lambda$  and  $\text{mult}_{L'_1}(M) > 1/\lambda$ . On the other hand, we have

$$B_1 \cdot B_2 = \left(\frac{4}{\lambda} \Pi|_{\Pi} - \text{mult}_{L_1}(M)L_1\right)^2 = \left(\frac{4}{\lambda} - \text{mult}_{L_1}(M)\right)^2 < \frac{9}{\lambda^2},$$

which is a contradiction. Thus we see that  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not canonical either at a general point of every line among  $L_1, \dots, L_6$  or at a general point of every line among  $L'_1, \dots, L'_6$ . But it cannot be non-canonical at a general point of all these 12 lines.

Recall that we defined  $\iota'$  ad  $\nu \circ \iota \circ \nu^{-1}$  for some  $\nu \in \hat{G}$  such that  $\nu \notin \bar{G}$ . Without loss of generality we may assume that  $\nu$  is an involution. Thus, we see that every  $\alpha \in \text{Bir}^{\bar{G}}(\mathbb{P}^3)$  can be uniquely written as

$$\alpha = \underbrace{\dots \circ \iota \circ \nu \circ \iota \circ \nu \circ \iota \circ \nu \circ \iota \circ \nu \circ \iota \circ \nu \circ \iota \circ \nu \circ \iota \circ \nu \circ \dots}_{r \text{ times}} \beta$$

for some  $\beta \in \hat{G}$  such that  $r = 0$  if and only if  $\alpha \in \hat{G}$ . The latter implies that  $\Gamma$  is the free product of the groups  $\hat{G} = \langle \nu, \bar{G} \rangle \cong S_6$  and  $\langle \iota, \bar{G} \rangle \cong S_6$  with amalgamated subgroup  $\bar{G} \cong A_6$ .  $\square$

*Remark 4.10.* Note that  $\hat{G} \not\subset \langle \iota, \iota', \bar{G} \rangle$  (see the proof of Theorem 4.9).

In the rest of this section, we prove Theorem 4.3. Let  $\mathcal{M}$  be a (non-empty)  $\bar{G}$ -invariant linear system on  $\mathbb{P}^3$  that does not have fixed components, and let  $\lambda$  be a positive rational number such that  $\lambda\mathcal{M} \sim_{\mathbb{Q}} -K_{\mathbb{P}^3}$ . Suppose that  $(\mathbb{P}^3, \lambda\mathcal{M})$  is canonical at a general point of every line  $L_1, \dots, L_6, L'_1, \dots, L'_6$ . We must prove that  $(\mathbb{P}^3, \lambda\mathcal{M})$  is canonical.

Suppose that the log pair  $(\mathbb{P}^3, \lambda\mathcal{M})$  is not canonical. Let us seek a contradiction. By Corollary 2.3 there is  $\mu \in \mathbb{Q}$  such that  $\mu < 2\lambda$  and  $(\mathbb{P}^3, \mu\mathcal{M})$  is strictly log canonical. Let  $S \subset X$  be a minimal center in  $\text{LCS}(\mathbb{P}^3, \mu\mathcal{M})$ . Then  $S$  is not a surface, since  $\mathcal{M}$  has no fixed components.

Recall that we denote by  $H$  a general hyperplane in  $\mathbb{P}^3$ . By Lemma 2.13, there is  $\delta \in \mathbb{Q}$  and there is a  $\bar{G}$ -invariant effective  $\mathbb{Q}$ -divisor  $D \sim_{\mathbb{Q}} \delta H$  on the threefold  $\mathbb{P}^3$  such that  $0 < \delta < 8$ , the log pair  $(\mathbb{P}^3, D)$  is log canonical and the set  $\text{LCS}(\mathbb{P}^3, D)$  consists of all irreducible components of the  $\bar{G}$ -orbit of  $S$ .

*Remark 4.11.* It follows from the proof of Lemma 2.13 that we may assume that  $\text{mult}_{L_i}(D) < 2$  and  $\text{mult}_{L'_i}(D) < 2$  for every  $i \in \{1, 2, \dots, 6\}$ , since  $(\mathbb{P}^3, \lambda\mathcal{M})$  is canonical at a general point of every line  $L_1, \dots, L_6, L'_1, \dots, L'_6$ .

Let  $\mathcal{I}$  be the multiplier ideal sheaf of the log pair  $(\mathbb{P}^3, D)$ , and let  $\mathcal{L}$  be the log canonical singularities subscheme of the log pair  $(\mathbb{P}^3, D)$ . Then it follows from Theorem 2.4 that

$$(4.12) \quad h^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_{\mathbb{P}^3}(4H)) = h^0(\mathcal{O}_{\mathbb{P}^3}(4H)) - h^0(\mathcal{O}_{\mathbb{P}^3}(4H) \otimes \mathcal{I}) = 35 - h^0(\mathcal{O}_{\mathbb{P}^3}(4H) \otimes \mathcal{I}).$$

**Lemma 4.13.** *The center  $S$  is a curve.*

*Proof.* If  $S$  is a point, then  $|\text{LCS}(\mathbb{P}^3, D)| \leq 35$  by (4.12), which contradicts Lemma 3.8.  $\square$

By Theorem 2.8, the curve  $S$  is a smooth curve in  $\mathbb{P}^3$  of degree  $d$  and genus  $g$  such that

$$(4.14) \quad g \leq 2d.$$

Put  $q = h^0(\mathcal{O}_{\mathbb{P}^3}(4H) \otimes \mathcal{I})$ . Let  $Z$  be the  $\bar{G}$ -orbit of the curve  $S$ , and let  $r$  be the number of irreducible components of  $Z$ . Then  $Z = \mathcal{L}$ , since  $(\mathbb{P}^3, D)$  is log canonical. Moreover, the curve  $Z$  is smooth by Lemma 2.7, and  $Z$  is not contained in a hyperplane in  $\mathbb{P}^3$  by Remark 3.4.

**Lemma 4.15.** *The equality  $r(4d - g + 1) = 35 - q$  holds.*

*Proof.* The equality follows from (4.12) and the Riemann–Roch theorem, because  $2d \geq g$ .  $\square$

**Corollary 4.16.** *The inequality  $g \leq 34$  holds.*

**Lemma 4.17.** *Suppose that  $1 \leq q \leq 7$ . Then  $q \in \{5, 6\}$ .*



*Proof.* Put  $W_4 = H^0(\mathcal{O}_{\mathbb{P}^3}(4H) \otimes \mathcal{I})$ . Since the center of  $G$  acts trivially on polynomials of even degree, the space  $W_4$  has a natural structure of a  $\tilde{G}$ -representation. Suppose that  $q \notin \{5, 6\}$ . Then  $W_4$  has a trivial subrepresentation of the group  $\tilde{G}$  by dimension reasons (see [15]), which is impossible by Lemma 3.7.  $\square$

**Lemma 4.18.** *Suppose that  $r \neq 1$ . Then  $r = 6$  and  $d = 1$ .*

*Proof.* Since  $r \leq 35$  by Lemma 4.15, one has  $r \in \{6, 10, 15, 20, 30\}$  by Lemma 3.3. In particular, the inequality  $q \geq 1$  holds by Lemma 4.15.

Suppose that  $q \geq 6$ . Then  $r(4d - g + 1) < 30$  by Lemma 4.15. We see that  $g + 1 \leq 4d - g + 1 < 5$ , which implies that  $g \leq 3$ . Then  $d = 1$  and  $g = 0$  so that  $4d - g + 1 = 5$ , which is a contradiction.

By Lemma 4.17 we may assume that  $q = 5$ . Then  $r \in \{6, 10, 15, 30\}$  by Lemma 4.15.

If  $r = 30$ , then  $g + 1 \leq 4d - g + 1 = 1$  by Lemma 4.15, which is a contradiction.

If  $r = 15$ , then  $g + 1 \leq 4d - g + 1 = 2$  by Lemma 4.15, which leads to a contradiction.

If  $r = 10$ , then  $g + 1 \leq 4d - g + 1 = 3$  by Lemma 4.15, which leads to a contradiction.

If  $r = 6$ , then  $g + 1 \leq 4d - g + 1 = 5$  by Lemma 4.15, which gives  $g = 0$  and  $d = 1$ .  $\square$

**Lemma 4.19.** *The equality  $r = 1$  holds.*

*Proof.* Suppose that  $r \neq 1$ . Then  $Z$  is a disjoint union of 6 lines by Lemma 4.18, which implies that either  $Z = L_1 + L_2 + L_3 + L_4 + L_5 + L_6$  or  $Z = L'_1 + L'_2 + L'_3 + L'_4 + L'_5 + L'_6$ .

Without loss of generality, we may assume that  $S = L_1$  and  $Z = L_1 + L_2 + L_3 + L_4 + L_5 + L_6$ .

Let  $\bar{F} \subset \tilde{G}$  be the stabilizer of  $L_1$ . Then  $\bar{F} \cong A_5$ . Let  $\pi: U \rightarrow \mathbb{P}^3$  be the blow up of  $L_1$ . Then

$$K_U + \bar{D} + (\text{mult}_{L_1}(D) - 1)E \sim_{\mathbb{Q}} \pi^*(K_{\mathbb{P}^3} + D),$$

where  $E$  is the  $\pi$ -exceptional divisor and  $\bar{D}$  is the proper transform of  $D$  on  $U$ . One has  $2 > \text{mult}_{L_1}(D) > 1$  by Remark 4.11. The group  $\bar{F}$  naturally acts on  $E$  so that the divisor  $\bar{D}|_E$  is  $\bar{F}$ -invariant.

We can identify the surface  $E$  with a smooth quadric in  $\mathbb{P}^3$ . The action of the group  $\bar{F}$  extends to the ambient space  $\mathbb{P}^3$ . Note that this action arises from the standard four-dimensional representation of the group  $\bar{F} \cong A_5$ .

It follows from the inequality  $\text{mult}_{L_1}(D) < 2$  that the set  $\mathbb{LCS}(U, \bar{D} + (\text{mult}_{L_1}(D) - 1)E)$  contains an irreducible reduced curve  $C \subset E$  such that  $\pi(C) = L_1$ . One has  $C \in \mathbb{LCS}(E, \bar{D}|_E)$  by Theorem 2.6, which is impossible by Lemma 2.17.  $\square$

We see that  $r = 1$  so that  $Z = S$ . By Lemma 2.10 one has  $g \in \{31, 25, 19, 16, 10\}$ .

**Lemma 4.20.** *The inequality  $g \neq 31$  holds.*

*Proof.* Suppose that  $g = 31$ . Then it follows from Lemma 4.15 that

$$(4.21) \quad d = \frac{65 - q}{4},$$

which gives  $q > 0$ . Hence  $q \geq 5$  by Lemma 4.17, and thus  $d \leq 15$  by (4.21). By (4.14) we have  $g/2 \leq d \leq 15$ , which is impossible.  $\square$

**Lemma 4.22.** *The inequality  $g \neq 25$  holds.*

*Proof.* Suppose that  $g = 25$ . Then it follows from Lemma 4.15 that

$$(4.23) \quad d = \frac{59 - q}{4},$$

which gives  $q > 0$ . Hence  $q \geq 11$  by Lemma 4.17, and thus  $d \leq 15$  by (4.23). By (4.14) we have  $g/2 \leq d \leq 12$ , which is impossible.  $\square$

**Lemma 4.24.** *The inequality  $g \neq 19$  holds.*

*Proof.* Suppose that  $g = 19$ . Then  $S$  is not contained in a cubic surface by Lemma 3.9. We have  $h^0(\mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{O}_S) \geq h^0(\mathcal{O}_{\mathbb{P}^3}(3)) = 20$  because there is an exact sequence of the cohomology groups,

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{I}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{O}_S).$$

By the Riemann–Roch theorem, we have

$$20 \leq h^0(\mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{O}_S) = 3d - g + 1 + h^1(\mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{O}_S) = 3d - 18 + h^1(\mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{O}_S),$$

which implies that  $h^1(\mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{O}_S) \neq 0$  because  $d \leq 12$  by Lemma 4.15. Then it follows from Theorem 2.19 that  $19 \geq 3d/2 + 1 \geq h^0(\mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{O}_S) \geq 20$ , which is a contradiction.  $\square$

**Lemma 4.25.** *The inequality  $g \neq 16$  holds.*

*Proof.* Suppose that  $g = 16$ . Arguing as in the proof of Lemma 4.24 and keeping in mind that  $d \leq 11$  by Lemma 4.15, we see that  $18 \geq 3d/2 + 1 \geq h^0(\mathcal{O}_{\mathbb{P}^3}(3) \otimes \mathcal{O}_S) \geq 20$ , which is a contradiction.  $\square$

Therefore, we see that  $g = 10$ . Then  $d \geq 9$  by Theorem 2.20.

**Lemma 4.26.** *The inequality  $d \neq 9$  holds.*

*Proof.* Suppose that  $d = 9$ . It follows from Lemma 3.7 that  $h^0(\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{I}) \neq 1$ , which implies that  $S = F_1 \cap F_2$ , where  $F_1$  and  $F_2$  are cubic surfaces in  $\mathbb{P}^3$  (see [22, Example 6.4.3]).

The group  $\bar{G}$  cannot act non-trivially on the pencil generated by  $F_1$  and  $F_2$ , which implies that the surfaces  $F_1$  and  $F_2$  must be  $\bar{G}$ -invariant. The latter is impossible by Lemma 3.7.  $\square$

**Lemma 4.27.** *The inequality  $d \neq 10$  holds.*

*Proof.* Suppose that  $d = 10$ . Then  $q = 4$  by Lemma 4.15, which is impossible by Lemma 4.17.  $\square$

Thus we see that  $d \geq 11$ . Then  $q = 0$  and  $d = 11$  by Lemma 4.15.

Take a subgroup  $\bar{F} \subset \bar{G}$  such that  $\bar{F} \cong A_5$  and the embedding  $\bar{F} \subset \bar{G}$  is non-standard (see Definition 3.1). Then there is an  $\bar{F}$ -invariant twisted cubic curve  $C \subset \mathbb{P}^3$  by Lemma 3.6.

Let  $R$  be the quartic surface in  $\mathbb{P}^3$  that is swept out by the lines that are tangent to  $C$ . Then the surface  $R$  is  $\bar{F}$ -invariant, and the curve  $S$  is not contained in

the surface  $R$  because  $q = 0$ . Put  $\Sigma = R \cap S$ . We have  $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_r$ , where  $\Sigma_i$  is a  $\bar{F}$ -orbit. Hence, we have

$$44 = R \cdot S = \sum_{i=1}^r a_i |\Sigma_i|$$

for some positive integers  $a_1, \dots, a_r$ . We may assume that  $|\Sigma_1| \geq \dots \geq |\Sigma_r|$ , but  $|\Sigma_i| \in \{12, 20, 30, 60\}$  for every  $i \in \{1, \dots, r\}$  by Lemma 2.9. Thus, we see that  $|\Sigma_1| = 20$ .

Let  $O \in \Sigma_1$  be a point, and let  $\bar{F}_O \subset \bar{F}$  be the stabilizer of the point  $O$ . Then  $\bar{F}_O \cong \mathbb{Z}_3$ .

Let  $\Gamma$  be the  $\bar{G}$ -orbit of the point  $O$ , and let  $\bar{G}_O \subset \bar{G}$  be the stabilizer of the point  $O$ . Then the order of the group  $\bar{G}_O$  must divide 18. Moreover, the group  $\bar{G}_O$  is cyclic by Lemma 2.9, which implies that  $\bar{G}_O \cong \mathbb{Z}_3$ . Hence, we see that  $|\Gamma| = 120$ , which contradicts Lemma 2.10.

The obtained contradiction completes the proof of Theorem 4.3.

### 5. SEGRE CUBIC

Let  $G \subset \text{SL}_5(\mathbb{C})$  be a subgroup such that  $G \cong A_6$ . Let  $\phi: \text{SL}_5(\mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}^4)$  be the natural projection. Put  $W = \mathbb{C}^5$  and  $\bar{G} = \phi(G) \subset \text{Aut}(\mathbb{P}^4)$ . Then the space  $W$  is an irreducible representation of the group  $G \cong \bar{G} \cong A_6$ , and there is a unique cubic hypersurface  $X \subset \mathbb{P}^4$  that is  $\bar{G}$ -invariant. Let us identify  $X$  with a complete intersection in  $\mathbb{P}^5$  that is given by the equation

$$\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^3 = 0 \subset \mathbb{P}^5 \cong \text{Proj}\left(\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]\right),$$

and let us identify  $\bar{G}$  with a subgroup of the group  $\text{Aut}(X)$  (cf. Example 1.19).

Let  $O \in X$  be a point, and let  $\bar{F} \subset \bar{G}$  be its stabilizer.

*Remark 5.1.* Let  $\tilde{T}$  be the affine tangent space to  $\mathbb{P}^4$  at the point  $O \in \mathbb{P}^4$ . Then  $\bar{F}$  naturally acts on the space  $\tilde{T}$ . Let us consider  $W$  as a representation of the group  $\bar{F}$ . One has

$$\tilde{T} \cong W/W_O \otimes W_O^*,$$

where  $W_O$  is the one-dimensional subrepresentation of  $\bar{F}$  that corresponds to the point  $O \in \mathbb{P}^4$ .

Let  $\Sigma$  be the  $\bar{G}$ -orbit of the point  $O \in X$ .

**Lemma 5.2.** *Suppose that  $|\Sigma| \leq 15$ . Then  $|\Sigma| \in \{10, 15\}$ .*

*Proof.* One has  $|\Sigma| \neq 1$ , because  $W$  is an irreducible representation of the group  $\bar{G}$ . Hence  $|\Sigma| \in \{6, 10, 15\}$  by Lemma 3.3. Suppose that  $|\Sigma| = 6$ . Then  $\bar{F} \cong A_5$  by Lemma 3.3.

Let us consider  $W$  as a representation of the group  $\bar{F}$ . Then  $W$  is reducible and  $W \cong W_t \oplus W_4$ , where  $W_t$  and  $W_4$  are the trivial and four-dimensional representations of the group  $\bar{F}$ , respectively. The embedding  $\bar{F} \subset \bar{G}$  is standard (see Definition 3.1) because  $W$  is reducible.

Note that  $W_t$  is the only one-dimensional subrepresentation of the representation  $W$  because the representation  $W_4$  is irreducible. Hence, the set  $\Sigma$  must contain a point in  $\mathbb{P}^5$  that is given by  $x_0 = \dots = x_4 = -x_5/5$ , which is impossible because this point does not belong to the hypersurface  $X \subset \mathbb{P}^5$ . □

Let  $T$  be the affine tangent space to  $X$  at the point  $O$ . Then  $\bar{F}$  naturally acts on  $T$ .

**Lemma 5.3.** *Suppose that  $|\Sigma| = 10$ . Then  $\Sigma = \text{Sing}(X)$  and  $\bar{F} \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ . Moreover,  $T$  is an irreducible four-dimensional representation of the group  $\bar{F}$ .*

*Proof.* One has  $\bar{F} \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$  by Lemma 3.3.

Up to conjugation, the group  $\bar{G}$  has a unique subgroup that is isomorphic to  $\bar{F}$ , which implies that we may assume that  $\bar{F}$  fixes the point  $[1 : -1 : 1 : -1 : 1 : -1] \in \text{Sing}(X)$ .

Let us consider  $W$  as a representation of the group  $\bar{F}$ , and let  $W_1 \subset W$  be the one-dimensional subrepresentation of the group  $\bar{G}$  that corresponds to the point  $[1 : -1 : 1 : -1 : 1 : -1]$ . Then  $W \cong W_1 \oplus W_4$ , where  $W_4$  is some four-dimensional representation of the group  $\bar{F}$ .

Let  $\chi_1$  and  $\chi$  be the characters of the representations  $W_1$  and  $W$ , respectively. The values of the characters  $\chi_1$  and  $\chi$  and the structure of the subgroup  $\bar{F}$  are given in the following table:

	[4, 2]	[3, 3]	[3, 1, 1, 1]	[2, 2, 1, 1]	$e$
#	18	4	4	9	1
$\chi$	-1	-1	2	1	5
$\chi_1$	-1	1	1	1	1

where we use notation similar to the ones used in the proof of Lemma 3.7.

We see that  $W_4$  is an irreducible representation of the group  $\bar{F}$ . Thus, without loss of generality, we may assume that  $O = [1 : -1 : 1 : -1 : 1 : -1]$ . Then  $T \cong W_4 \otimes W_1^*$  by Remark 5.1, which implies that  $T$  is an irreducible representation of the group  $\bar{F}$ . □

Let  $\Gamma$  be the  $\bar{G}$ -orbit of the point  $[0 : 0 : 0 : 0 : 1 : -1] \in X \setminus \text{Sing}(X)$ . Then  $|\Gamma| = 15$ .

**Lemma 5.4.** *Suppose that  $|\Sigma| = 15$ . Then  $\Sigma = \Gamma$  and  $\bar{F} \cong S_4$ . Moreover,  $T$  is an irreducible three-dimensional representation of the group  $\bar{F}$ .*

*Proof.* One has  $\bar{F} \cong S_4$  by Lemma 3.3. Up to conjugation, the group  $\bar{G}$  contains exactly two non-conjugate subgroups that are isomorphic to the group  $\bar{F} \cong S_4$  (see Lemma 3.3).

Let us consider  $W$  as a representation of the group  $\bar{F}$ . Then  $W$  contains some one-dimensional subrepresentation  $U$  of the group  $\bar{F}$  that corresponds to the point  $O \in X$ .

Let  $W_t$  be the trivial one-dimensional representations of the group  $\bar{F}$ , and let  $W_1$  be the non-trivial one-dimensional representations of the group  $\bar{F}$ . Then either  $U \cong W_t$  or  $U \cong W_1$ .

Let  $\chi$  and  $\chi_1$  be the characters of  $W$  and  $W_1$ , respectively.

Suppose that  $S_4 \cong \bar{F} \subset \bar{G} \cong A_6$  is a non-standard embedding. The values of  $\chi_1$  and  $\chi$  and the structure of the subgroup  $\bar{F}$  are given in the following table:

	[4, 2]	[3, 3]	[2, 2, 1, 1]	[2, 2, 1, 1]	$e$
#	6	8	3	6	1
$\chi$	-1	-1	1	1	5
$\chi_1$	-1	1	1	-1	1

where we use notation similar to the ones used in the proof of Lemma 3.7, and we divide the elements of type [2, 2, 1, 1] that are contained in the subgroup  $\bar{F}$  into two classes with respect to the values of the character  $\chi_1$ . We see that  $W$  contains no one-dimensional subrepresentations.

Thus  $S_4 \cong \bar{F} \subset \bar{G} \cong A_6$  is a standard embedding. The values of  $\chi_1$  and  $\chi$  and the structure of the subgroup  $\bar{F}$  are given in the following table:

	[4, 2]	[3, 1, 1, 1]	[2, 2, 1, 1]	[2, 2, 1, 1]	$e$
#	6	8	3	6	1
$\chi$	-1	2	1	1	5
$\chi_1$	-1	1	1	-1	1

which implies that  $W \cong W_t \oplus W_1 \oplus W_3$ , where  $W_3$  is an irreducible three-dimensional representation of the group  $\bar{F}$ . Hence  $T$  is an irreducible representation of the group  $\bar{F}$  by Remark 5.1.

Suppose that  $\Sigma \neq \Gamma$ . Then  $[1 : 1 : 1 : 1 : -2 : -2] \in \Sigma$ . But  $[1 : 1 : 1 : 1 : -2 : -2] \notin X$ , which gives a contradiction. □

Let  $H$  be a general hyperplane section of the cubic  $X \subset \mathbb{P}^4$ . Then the  $\bar{G}$ -invariant subgroup of the group  $\text{Cl}(X)$  is generated by  $H$  (see Example 1.19). The main purpose of this section is to prove

**Theorem 5.5.** *The threefold  $X$  is  $\bar{G}$ -birationally superrigid and  $\text{Bir}^{\bar{G}}(X) \cong S_6$ .*

By Corollary 1.12, the isomorphism  $\text{Bir}^{\bar{G}}(X) \cong S_6$  follows from  $\bar{G}$ -birational superrigidity of  $X$  and the classification of primitive subgroups in  $\text{SL}_5(\mathbb{C})$  (see [17]). In the remaining part of this section, we will prove that  $X$  is  $\bar{G}$ -birationally superrigid.

Suppose that  $X$  is not  $\bar{G}$ -birationally superrigid. Let us derive a contradiction. It follows from [12, Theorem 4.2] or [7, Theorem 1.4.1] that there is a (non-empty)  $\bar{G}$ -invariant linear system  $\mathcal{M}$  on  $X$  such that  $\mathcal{M}$  does not have fixed components, and  $(X, \lambda\mathcal{M})$  is not canonical, where  $\lambda \in \mathbb{Q}$  such that  $\lambda\mathcal{M} \sim_{\mathbb{Q}} -K_X$ . By Corollary 2.3 there is  $\mu \in \mathbb{Q}$  such that  $\mu < 2\lambda$  and  $(X, \mu\mathcal{M})$  is strictly log canonical. Let  $S \subset X$  be a minimal center in  $\text{LCS}(X, \mu\mathcal{M})$ . Then it follows from Lemma 2.7 that  $S \cap g(S) \neq \emptyset$  if and only if  $S = g(S)$  for every  $g \in \bar{G}$ . Moreover, the center  $S$  is not a surface, since  $\mathcal{M}$  has no fixed components.

By Lemma 2.13, there is a  $\bar{G}$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on  $X$  such that the set  $\text{LCS}(X, D)$  consists of irreducible components of the  $\bar{G}$ -orbit of  $S$ , the log pair  $(X, D)$  is log canonical and  $D \sim_{\mathbb{Q}} -\delta K_X$  for some  $\delta \in \mathbb{Q}$  such that  $0 < \delta < 2$ . Arguing as in the proof of [10, Lemma 2.11], we see that we can replace  $D$  by

$(\mu - \epsilon)\mathcal{M} + \epsilon'\mathcal{H}$ , where  $\epsilon \in \mathbb{Q}_{>0} \ni \epsilon'$  such that  $\epsilon \ll 1$  and  $\epsilon' \ll 1$ ,  $\mathcal{H}$  is a  $\bar{G}$ -invariant linear system such that  $\mathcal{H} \sim -nK_X$  holds for some  $n \gg 0$ , the base locus of the linear system  $\mathcal{H}$  coincides with  $\text{LCS}(X, D)$ , and  $\delta = (\mu - \epsilon)/\lambda + \epsilon'n$ . Thus, without loss of generality, we can replace  $D$  by  $\mu\mathcal{M}$  (cf. Lemma 2.13). Therefore, without loss of generality, we may assume that the set  $\text{LCS}(X, \mu\mathcal{M})$  also consists of irreducible components of the  $\bar{G}$ -orbit of  $S$ .

Let  $\mathcal{I}$  be the multiplier ideal sheaf of  $(X, \mu\mathcal{M})$ , and let  $\mathcal{L}$  be the subscheme given by  $\mathcal{I}$ . Then

$$(5.6) \quad h^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_X(2H)) = h^0(\mathcal{O}_X(2H)) - h^0(\mathcal{O}_X(2H) \otimes \mathcal{I}) = 15 - h^0(\mathcal{O}_X(2H) \otimes \mathcal{I})$$

by Theorem 2.4.

**Lemma 5.7.** *The center  $S$  is not a singular point of the threefold  $X$ .*

*Proof.* Suppose that  $S$  is a singular point of the threefold  $X$ . Then  $\text{LCS}(X, \mu\mathcal{M}) = \text{Sing}(X)$ , because  $\bar{G}$  acts transitively on the set  $\text{Sing}(X)$ . Recall that  $|\text{Sing}(X)| = 10$ .

Let  $\bar{F} \subset \bar{G}$  be a stabilizer of the point  $S$ . Then  $\bar{F} \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$  by Lemma 5.3.

Let  $\pi: U \rightarrow X$  be a blow up of the points  $\text{Sing}(X)$ , and let  $E_1, \dots, E_{10}$  be the  $\pi$ -exceptional divisors. Then there is a positive rational number  $m$  such that

$$K_U + \mu\bar{\mathcal{M}} + (m - 1) \sum_{i=1}^{10} E_i \sim_{\mathbb{Q}} \pi^*(K_X + \mu\mathcal{M}),$$

where  $\bar{\mathcal{M}}$  is the proper transform of the linear system  $\mathcal{M}$  on the variety  $U$ .

Note that it follows from [7, Theorem 1.7.20] that  $m \geq 1$  (cf. [13, Theorem 3.10]).

We may assume that  $\pi(E_1) = S$ . There is a natural homomorphism  $v: \bar{F} \rightarrow \text{Aut}(E_1)$ . Then  $\text{lct}(E_1, v(\bar{F})) \geq 1$  by Lemmas 2.16 and 5.3, because  $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ .

Let us show that  $\text{lct}(E_1, v(\bar{F})) < 1$  to derive a contradiction.

One can easily check that there exists a two-dimensional linear subspace  $\Pi \subset \mathbb{P}^4$  such that  $|\Pi \cap \text{Sing}(X)| = 4$  and  $\Pi \subset X$  (see [23, Section 3.2]). Let  $C$  be a general conic in  $\Pi$  that contains  $\Pi \cap \text{Sing}(X)$ . Then  $C$  is not contained in the base locus of the linear system  $\mathcal{M}$ , and  $C$  is irreducible. Let  $\bar{C}$  be a proper transform of the curve  $C$  on the threefold  $U$ . Then

$$\frac{8}{\mu} - 4\frac{m}{\mu} > \frac{4}{\lambda} - 4\frac{m}{\mu} = \bar{\mathcal{M}} \cdot \bar{C} \geq 0,$$

which implies that  $m < 2$ . In particular, the log pair  $(U, \mu\bar{\mathcal{M}} + (m - 1) \sum_{i=1}^{10} E_i)$  is not Kawamata log terminal along  $E_1$ . Hence  $\text{lct}(E_1, v(\bar{F})) < 1$  by Theorem 2.6.  $\square$

**Lemma 5.8.** *The center  $S$  is a curve.*

*Proof.* Suppose that  $S$  is a point. Then  $S \notin \text{Sing}(X)$  by Lemma 5.8. It follows from (5.6) that  $|\text{LCS}(X, \mu\mathcal{M})| \leq 15$  so that  $\text{LCS}(X, \mu\mathcal{M})$  is the  $\bar{G}$ -orbit of the point  $[0 : 0 : 0 : 0 : 1 : -1]$  by Lemma 5.2. There are 15 two-dimensional linear subspaces  $\Pi_1, \dots, \Pi_{15}$  in  $\mathbb{P}^4$  such that  $|\Pi_i \cap \text{LCS}(X, \mu\mathcal{M})| = 6$  and  $X \cap \Pi_i = L_i^1 + L_i^2 + L_i^3$  for any  $i \in \{1, \dots, 15\}$ , where  $L_i^j$  is a line such that

$$\left( (L_i^1 \cap L_i^2) \cup (L_i^1 \cap L_i^3) \cup (L_i^2 \cap L_i^3) \right) \cap \text{LCS}(X, \mu\mathcal{M}) = \emptyset$$

and  $L_i^j = L_r^s$  if and only if  $(i, j) = (r, s)$ . Note that the curve  $\sum_{i=1}^{15} (L_i^1 + L_i^2 + L_i^3)$  is a  $\bar{G}$ -orbit of the line  $L_1^1$ . Without loss of generality, we may assume that  $S \in L_1^1$ .

Let  $M_1$  and  $M_2$  be general surfaces in the linear system  $\mathcal{M}$ . Put

$$M_1 \cdot M_2 = \gamma \sum_{i=1}^{15} (L_i^1 + L_i^2 + L_i^3) + \Omega,$$

where  $\Omega$  is an effective cycle such that  $L_i^j \not\subseteq \text{Supp}(\Omega)$  for every  $i \in \{1, \dots, 15\}$  and  $j \in \{1, 2, 3\}$ , and  $\gamma$  is a non-negative rational number. Put  $m = \text{mult}_S(M_1 \cdot M_2)$ , and let  $D$  be a general surface in  $|H|$  that contains the lines  $L_1^1, L_1^2$  and  $L_1^3$ . Then

$$\frac{12}{\lambda^2} - 3\gamma = D \cdot \left( \Omega + \gamma \sum_{i=2}^{15} (L_i^1 + L_i^2 + L_i^3) \right) \geq 6(m - \gamma),$$

which implies that  $\gamma \geq 2m - 4/\lambda^2$ . Therefore, we see that

$$\frac{12}{\lambda^2} = H \cdot M_1 \cdot M_2 = 45\gamma + H \cdot \Omega \geq 45\gamma \geq 45\left(2m - \frac{4}{\lambda^2}\right),$$

which implies that  $m \leq 32/(15\lambda^2)$ . In particular, we see that

$$(5.9) \quad \text{mult}_S(\mathcal{M}) \leq \sqrt{m} \leq \frac{2\sqrt{2}}{\sqrt{15}\lambda} < \frac{3}{2\lambda} < \frac{3}{\mu}.$$

Let  $\pi: U \rightarrow X$  be a blow up of the point  $S$ , and let  $E$  be the  $\pi$ -exceptional divisor. Then

$$K_U + \mu\bar{\mathcal{M}} + \left(\mu \text{mult}_S(\mathcal{M}) - 2\right)E \sim_{\mathbb{Q}} \pi^*(K_X + \mu\mathcal{M}),$$

where  $\bar{\mathcal{M}}$  is the proper transform of  $\mathcal{M}$  on the variety  $U$ .

Let  $\bar{F} \subset \bar{G}$  be a stabilizer of the point  $S$ . Then  $\bar{F} \cong S_4$  by Lemma 5.4, and there is a natural homomorphism  $v: \bar{F} \rightarrow \text{Aut}(E)$ . Note that  $v$  is a monomorphism by Lemma 5.4.

There is an irreducible proper subvariety  $C \subsetneq E \cong \mathbb{P}^2$  such that

$$g(C) \in \mathbb{LCS}(U, \mu\bar{\mathcal{M}} + (\mu \text{mult}_S(\mathcal{M}) - 2)E)$$

for every  $g \in v(\bar{F})$ . Then  $C$  is a curve by Theorem 2.5 and Lemma 5.4.

Let  $\bar{M}_1$  and  $\bar{M}_2$  be general surfaces in  $\bar{\mathcal{M}}$ . Then it follows from [13, Theorem 3.1] that

$$\text{mult}_{g(C)}(\bar{M}_1 \cdot \bar{M}_2) \geq \frac{4}{\mu^2} \left(3 - \mu \text{mult}_S(\mathcal{M})\right)$$

for every  $g \in v(\bar{F})$ . Let  $\delta$  be the degree in  $E \cong \mathbb{P}^2$  of the  $v(\bar{F})$ -orbit of the curve  $C$ . Then

$$\begin{aligned} \frac{128}{15\mu^2} &\geq \frac{32}{15\lambda^2} \geq m \geq \text{mult}_S^2(\mathcal{M}) + \delta \text{mult}_C(\bar{M}_1 \cdot \bar{M}_2) \\ &\geq \text{mult}_S^2(\mathcal{M}) + \frac{8}{\mu^2} \left(3 - \mu \text{mult}_S(\mathcal{M})\right), \end{aligned}$$

because  $\delta \geq 2$  by Lemma 5.4. The latter value is greater than  $9/\mu^2$ , which can be easily seen by using the elementary properties of quadratic forms and (5.9). The obtained contradiction completes the proof.  $\square$

By Theorem 2.8, the curve  $S$  is a smooth curve in  $\mathbb{P}^4$  of degree  $d$  and genus  $g \leq d$ . Put  $q = h^0(\mathcal{O}_X(2H) \otimes \mathcal{I})$ , let  $Z$  be the  $\bar{G}$ -orbit of the curve  $S$ , and let  $r$  be the number of irreducible components of the curve  $Z$ .

**Lemma 5.10.** *The equality  $r(2d - g + 1) = 15 - q$  holds.*

*Proof.* The equality follows from (5.6) and the Riemann–Roch theorem because  $d \geq g$ . □

**Corollary 5.11.** *The inequality  $g \leq 14$  holds.*

Note that  $Z$  is not contained in a hyperplane in  $\mathbb{P}^4$ , since  $W$  is an irreducible  $\bar{G}$ -representation.

**Lemma 5.12.** *The equality  $r = 1$  holds.*

*Proof.* Suppose that  $r \geq 2$ . Then  $r \leq 15$  by Lemma 5.10, which implies that  $r \in \{6, 10, 15\}$  by Lemma 3.3. If  $q = 0$ , then  $2d - g + 1 = 1$  by Lemma 5.10, which is impossible because  $g \leq d$ .

We have  $q \geq 1$ . Then  $r(g + 1) \leq 14$  and  $g \leq 1$ . If  $g = 0$ , then  $r(2d + 1) \leq 14$  by Lemma 5.10, which implies a contradictory inequality  $d \leq 0$ . We see that  $g = 1$ . Thus  $2rd \leq 14$  by Lemma 5.10, which implies that  $d = 1$  and  $g = 0$ , contradicting the equality  $g = 1$ . □

There is a natural monomorphism  $\theta: \bar{G} \rightarrow \text{Aut}(S)$  (see Lemma 5.2).

**Lemma 5.13.** *The equality  $g = 10$  holds.*

*Proof.* The required assertion follows from Lemmas 5.10 and 2.10. □

The equality  $g = 10$  and Lemma 5.10 imply that  $d \leq 12$ .

**Lemma 5.14.** *The equality  $q = 0$  holds.*

*Proof.* Let  $\bar{\Psi} \subset \bar{G}$  be a subgroup such that  $\bar{\Psi} \cong A_5$  and the embedding  $A_5 \cong \bar{\Psi} \subset \bar{G} \cong A_6$  is standard. There is a  $\bar{\Psi}$ -invariant hyperplane section  $H \subset X$ . Note that  $S \not\subset H$ . We have  $|H \cap S| \leq d \leq 12$ , which implies that  $|H \cap S| = 12$  by Lemma 2.9 because  $H \cap S$  is  $\bar{\Psi}$ -invariant. Then  $q = 0$  by Lemma 5.10. □

Let  $Q$  be the  $\bar{G}$ -invariant quadric in  $\mathbb{P}^4$  (cf. Example 1.18). Then  $S \not\subset Q$  because  $q = 0$ .

Put  $\Delta = Q \cap S$ . Then  $|\Delta| \leq 24$ . Let  $\bar{\Psi} \subset \bar{G}$  be a stabilizer of a point in  $\Delta$ . Then

$$|\bar{\Psi}| \geq \frac{|\bar{G}|}{|\Delta|} \geq \frac{360}{24} > 6,$$

which is impossible by Lemma 2.9.

The obtained contradiction completes the proof of Theorem 5.5.

### 6. QUADRIC THREEFOLD

Let  $G \subset \text{SL}_5(\mathbb{C})$  be a subgroup such that  $G \cong A_6$ , and let  $\phi: \text{SL}_5(\mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}^4)$  be the natural projection. Put  $W = \mathbb{C}^5$  and  $\bar{G} = \phi(G)$ . Then there is a smooth quadric hypersurface  $Q \subset \mathbb{P}^4$  that is  $\bar{G}$ -invariant. Let us identify  $Q$  with a smooth complete intersection in  $\mathbb{P}^5$  that is given by the equation

$$\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^2 = 0 \subset \mathbb{P}^5 \cong \text{Proj}\left(\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]\right),$$

and let us identify  $\bar{G}$  with a subgroup of the group  $\text{Aut}(Q)$  (cf. Example 1.18).

Let  $O \in Q$  be a point, let  $\bar{F} \subset \bar{G}$  be its stabilizer, and let  $\Sigma$  be its  $\bar{G}$ -orbit.

**Lemma 6.1.** *Suppose that  $|\Sigma| \leq 30$ . Then  $|\Sigma| = 30$ , and there exists a cubic hypersurface  $X \subset \mathbb{P}^4$  such that  $\Sigma \subset X$  and  $Q \not\subset X$ .*



*Proof.* One has  $|\Sigma| \neq 1$  because  $W$  is an irreducible representation of the group  $\bar{G}$ . Then  $|\Sigma| \in \{6, 10, 15, 20, 30\}$  by Lemma 3.3. Arguing as in the proof of Lemmas 4.2, 5.2, 5.3 and 5.4, we see that  $|\Sigma| \in \{20, 30\}$ .

Let us consider  $W$  as a representation of the group  $\bar{F}$ . Then  $W$  contains some one-dimensional subrepresentation  $U$  of the group  $\bar{F}$  corresponding to the point  $O \in Q$ .

Suppose that  $|\Sigma| = 20$ . Then  $\bar{F} \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$  by Lemma 3.3.

Let  $W_t$  and  $W_1$  be the trivial one-dimensional and the non-trivial one-dimensional representations of  $\bar{F}$  (see the proof of Lemma 5.3), respectively. Then either  $U \cong W_t$  or  $U \cong W_1$ .

Let  $\chi$  and  $\chi_1$  be the characters of the representations  $W$  and  $W_1$ , respectively. The values of the characters  $\chi_1$  and  $\chi$  and the structure of the subgroup  $\bar{F}$  are given in the following table:

	[3, 3]	[3, 1, 1, 1]	[2, 2, 1, 1]	$e$
#	4	4	9	1
$\chi$	-1	2	1	5
$\chi_1$	1	1	-1	1

where we use notation similar to the ones used in the proofs of Lemmas 3.7, 5.3 and 5.4.

We see that  $U \cong W_1$ . Thus  $[1 : -1 : 1 : -1 : 1 : -1] \in \Sigma$ . But  $[1 : -1 : 1 : -1 : 1 : -1] \notin Q$ .

Therefore, we see that  $|\Sigma| = 30$ . Then  $\bar{F} \cong A_4$  by Lemma 3.3. The embedding  $A_4 \cong \bar{F} \subset \bar{G} \cong A_6$  must be standard because otherwise the representation  $W$  would be an irreducible representation of the group  $\bar{F}$  (cf. the proof of Lemma 5.4).

There are exactly two  $\bar{F}$ -invariant points in  $Q$ . These points form a subset

$$\left\{ [1 : 1 : 1 : 1 : -2 + \sqrt{-2} : -2 - \sqrt{-2}], [1 : 1 : 1 : 1 : -2 - \sqrt{-2} : -2 + \sqrt{-2}] \right\} \subset \Sigma.$$

Let  $X$  be the cubic threefold in  $\mathbb{P}^5$  that is given by

$$\sum_{i=0}^5 x_i = (x_0 - x_1)(x_2 - x_3)(x_4 - x_5) = 0,$$

and let  $P$  be the point  $[1 : \omega : \omega^2 : 1 : \omega : \omega^2] \in \mathbb{P}^5$ , where  $\omega$  is a non-trivial cube root of unity. Then  $\Sigma \subset X \not\ni P \in Q$ , which completes the proof. □

The main purpose of this section is to prove the following result.

**Theorem 6.2.** *The quadric threefold  $Q$  is  $\bar{G}$ -birationally superrigid and  $\text{Bir}^{\bar{G}}(Q) \cong S_6$ .*

By Corollary 1.12, the isomorphism  $\text{Bir}^{\bar{G}}(Q) \cong S_6$  follows from  $\bar{G}$ -birational superrigidity of  $Q$  and the classification of primitive subgroups in  $\text{SL}_5(\mathbb{C})$  (see [17]). In the remaining part of this section, we will prove that  $Q$  is  $\bar{G}$ -birationally superrigid.

Suppose that  $Q$  is not  $\bar{G}$ -birationally superrigid. Arguing as in the proof of Theorem 5.5, we see that there is a  $\bar{G}$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on  $Q$  such that the set  $\text{LCS}(Q, D)$  consists of irreducible components of the  $\bar{G}$ -orbit of  $S$ , the log pair  $(Q, D)$  is log canonical, and  $D \sim_{\mathbb{Q}} -\delta K_Q$  for some positive rational

number  $\delta < 2$ , where  $S$  is a minimal center in  $\text{LCS}(Q, D)$  such that either  $S$  is a point or a smooth curve.

Let  $\mathcal{I}$  be the multiplier ideal sheaf of the log pair  $(Q, D)$ , let  $\mathcal{L}$  be subscheme that is given by the ideal sheaf  $\mathcal{I}$ , and let  $H$  be a general hyperplane section of the threefold  $Q \subset \mathbb{P}^3$ . Then

$$(6.3) \quad h^0(\mathcal{O}_{\mathcal{L}} \otimes \mathcal{O}_Q(3H)) = h^0(\mathcal{O}_Q(3H)) - h^0(\mathcal{O}_Q(3H) \otimes \mathcal{I}) = 30 - h^0(\mathcal{O}_Q(3H) \otimes \mathcal{I})$$

by Theorem 2.4.

**Lemma 6.4.** *The center  $S$  is a curve.*

*Proof.* The required assertion follows from (6.3) and Lemma 6.1. □

By Theorem 2.8, the curve  $S$  is a smooth curve of degree  $d$  and genus  $g$  such that

$$(6.5) \quad g \leq \frac{3d + 1}{2}.$$

Put  $q = h^0(\mathcal{O}_Q(3H) \otimes \mathcal{I})$ , let  $Z$  be the  $\bar{G}$ -orbit of the curve  $S$ , and let  $r$  be the number of irreducible components of the curve  $Z$ .

**Lemma 6.6.** *The equality  $r(3d - g + 1) = 30 - q$  holds.*

*Proof.* The equality follows from (6.3) and the Riemann–Roch theorem because  $3d \geq 2g - 1$ . □

Let  $X$  be the cubic threefold in  $\mathbb{P}^5$  that is given by  $\sum_{i=0}^5 x_i = \sum_{i=0}^5 x_i^3 = 0$  (cf. Example 1.19).

**Lemma 6.7.** *Suppose that  $q \geq 2$ . Then  $q \geq 5$ .*

*Proof.* Suppose that  $2 \leq q \leq 4$ . It follows from [15] that  $\bar{G}$  acts trivially on the  $q$ -dimensional vector space  $H^0(\mathcal{O}_Q(3H) \otimes \mathcal{I})$ . But  $Q \cap X$  is the only  $\bar{G}$ -invariant surface in  $|\mathcal{O}_Q(3H)|$ . □

Note that  $Z$  is not contained in a hyperplane in  $\mathbb{P}^4$  since  $W$  is an irreducible  $\bar{G}$ -representation.

**Lemma 6.8.** *The curve  $S$  is irreducible.*

*Proof.* Suppose that  $r \geq 2$ . By Lemma 3.3, either  $r = 6$  or  $r \geq 10$ .

Suppose that  $q = 0$ . If  $r = 6$ , then it follows from Lemma 6.6 that

$$(6.9) \quad 3d - g + 1 = 5.$$

From (6.5) and (6.9) we obtain that  $g \leq 5$ . Applying (6.9) we see that  $d \leq 3$  and thus  $g \leq 1$ . Applying (6.9) once again, we obtain a contradiction. Therefore, we see that  $r \geq 10$  and

$$(6.10) \quad 3d - g + 1 \leq 3$$

by Lemma 6.6. By (6.5) and (6.10) we have  $g \leq 3$ . Applying (6.10) again, we obtain  $d = 1$  and thus  $g = 0$ , which is incompatible with (6.10).

Suppose that  $q = 1$ . Then  $r(3d - g + 1) = 29$  by Lemma 6.6, which is impossible by Lemma 3.3.

Therefore, we see that  $q \geq 2$ . Hence  $q \geq 5$  by Lemma 6.7. It follows from Lemma 6.6 that  $r(3d - g + 1) \leq 25$ , which implies that  $3d - g \leq 3$ . We have  $g \leq 4$  because  $2g - 1 \leq 3d$ . Thus  $d \leq 2$  and  $g = 0$ .

Applying Lemma 6.6, we get  $d = 1$  and  $r = 6$ . Then  $S$  is a line and  $Z$  is a union of six lines.

Let  $\bar{\Psi} \subset \bar{G}$  be a stabilizer of the line  $S$ . Then  $\bar{\Psi} \cong A_5$  by Lemma 3.3.

Let us consider  $W$  as a representation of the group  $\bar{\Psi} \cong A_5$ . Then either  $W$  is irreducible or  $W \cong W_t \oplus W_4$ , where  $W_t$  and  $W_4$  are the trivial one-dimensional and the standard four-dimensional representations of the group  $\bar{\Psi} \cong A_5$ , respectively. In both cases, the line  $S$  cannot be  $\bar{\Psi}$ -invariant.  $\square$

We see that  $r = 1$ . Hence  $g \leq 30$  and  $d \leq 19$  by (6.3) because  $3d \geq 2g - 1$ .

**Lemma 6.11.** *The equalities  $d = 12$  and  $g = 10$  hold.*

*Proof.* Let  $\Pi_i \subset Q$  be a hyperplane section that is cut out by  $x_i = 0$ , where  $i \in \{0, \dots, 5\}$ . Then  $\Pi_0 \cap \dots \cap \Pi_5 = \emptyset$ , which implies that, without loss of generality, we may assume that  $S \not\subset \Pi_0$ .

Let  $\bar{\Psi} \subset \bar{G}$  be a stabilizer of the surface  $\Pi_0$ . Then  $\bar{\Psi} \cong A_5$  and the embedding  $\bar{\Psi} \subset \bar{G}$  must be standard. Hence we have

$$19 \geq d = \Pi_0 \cdot S \geq |\Pi_0 \cap S|,$$

which implies that  $d = \Pi_0 \cdot S = |\Pi_0 \cap S| = 12$  by Lemma 2.15, because  $\Pi_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ .

It follows from Theorem 2.20 that  $g \leq 15$ . Thus  $g = 10$  by Lemma 2.10.  $\square$

Thus, it follows from Lemma 6.6 that  $q = 3$ , which is impossible by Lemma 6.7. The obtained contradiction completes the proof of Theorem 6.2.

### APPENDIX A. KLEIN CUBIC THREEFOLD

Put  $\bar{\Gamma} = \text{PSL}_2(\mathbb{F}_{11})$ . Let  $V$  be a Fano threefold with terminal singularities such that  $V$  admits a non-trivial action of the group  $\bar{\Gamma}$  and that the  $\bar{\Gamma}$ -invariant subgroup of the group  $\text{Cl}(V)$  is  $\mathbb{Z}$ .

**Example A.1.** If  $V$  is a smooth hypersurface in  $\mathbb{P}^4$  that is given by the equation

$$x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0 = 0 \subset \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]),$$

then  $V$  is non-rational by [11, Theorem 0.12], and  $\text{Aut}(V_3) \cong \bar{\Gamma}$  by [1].

**Example A.2** ([38, Example 2.9]). There is a non-trivial action of the group  $\bar{\Gamma}$  on  $\text{Gr}(2, 6)$ , and  $\text{Pic}(\text{Gr}(2, 6)) = \mathbb{Z}[H]$  for some very ample divisor  $H$ . The linear system  $|H|$  gives an embedding  $\zeta: \text{Gr}(2, 6) \rightarrow \mathbb{P}^{14}$ , which induces a non-trivial action of the group  $\bar{\Gamma}$  on  $\mathbb{P}^{14}$ . Put  $V = \zeta(\text{Gr}(2, 6)) \cap \Pi$ , where  $\Pi$  is the unique  $\bar{\Gamma}$ -invariant linear subspace  $\Pi \subset \mathbb{P}^{14}$  such that  $\dim(\Pi) = 9$ . Then the variety  $V$  is a smooth Fano threefold such that  $\text{Pic}(V) \cong \mathbb{Z}$  and  $-K_V^3 = 14$  (see [25]). Furthermore, the variety  $V$  admits a non-trivial action of the group  $\bar{\Gamma}$ .

Let  $V_3$  and  $V_{14}$  be the threefolds that are constructed in Examples A.1 and A.2, respectively.

**Theorem A.3** ([38, Theorem 1.5]). *There is a  $\bar{\Gamma}$ -equivariant birational map  $\chi: V \dashrightarrow U$  such that either  $U \cong V_3$  or  $U \cong V_{14}$ .*

*Remark A.4* (see [38, Remark 2.10]). The varieties  $V_3$  and  $V_{14}$  are birationally equivalent.

The main purpose of this section is to prove the following result (cf. [38, Remark 2.10]).

**Theorem A.5.** *The varieties  $V_3$  and  $V_{14}$  are  $\bar{\Gamma}$ -birationally superrigid.*

**Corollary A.6.** *The birational map  $\chi: V \dashrightarrow U$  in Theorem A.3 is biregular.*

**Corollary A.7.** *There exists no  $\bar{\Gamma}$ -equivariant birational map  $V_{14} \dashrightarrow V_3$ .*

**Corollary A.8.** *Up to conjugation, the group  $\text{Bir}(V_3) \cong \text{Bir}(V_{14})$  contains exactly 2 subgroups that are isomorphic to the simple group  $\text{PSL}_2(\mathbb{F}_{11})$ .*

Let us prove that  $V_{14}$  is  $\bar{\Gamma}$ -birationally superrigid. Suppose that  $V_{14}$  is not  $\bar{\Gamma}$ -birationally superrigid. There is a (non-empty)  $\bar{\Gamma}$ -invariant linear system  $\mathcal{M}$  without fixed components such that  $\lambda\mathcal{M} \sim_{\mathbb{Q}} -K_{V_{14}}$  for some  $\lambda \in \mathbb{Q}_{>0}$ . Then  $(V_{14}, \lambda\mathcal{M})$  is not canonical (see [7, Theorem 1.4.1]).

There is  $\mu \in \mathbb{Q}$  such that  $\mu < 2\lambda$  and  $(V_{14}, \mu\mathcal{M})$  is strictly log canonical.

Let  $S \subset V_{14}$  be a minimal center in  $\text{LCS}(V_{14}, \mu\mathcal{M})$ . Then  $\dim(S) \in \{0, 1\}$ .

By Lemma 2.13, there is a  $\bar{\Gamma}$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on the threefold  $V_{14}$  such that the set  $\text{LCS}(V_{14}, D)$  consists of irreducible components of the  $\Gamma$ -orbit of  $S$  and that  $D \sim_{\mathbb{Q}} -\epsilon K_{V_{14}}$ , where  $\epsilon$  is a positive rational number such that  $\epsilon < 2$ .

Let  $\mathcal{I}$  be the multiplier ideal sheaf of the log pair  $(V_{14}, D)$ , and let  $\mathcal{L}(V_{14}, D)$  be the log canonical singularities subscheme of the log pair  $(V_{14}, D)$ . Then it follows from Theorem 2.4 that

$$\begin{aligned} \text{(A.9)} \quad h^0\left(\mathcal{O}_{\mathcal{L}(V_{14}, D)} \otimes \mathcal{O}_{V_{14}}(H)\right) &= h^0\left(\mathcal{O}_{V_{14}}(H)\right) - h^0\left(\mathcal{O}_{V_{14}}(H) \otimes \mathcal{I}\right) \\ &= 10 - h^0\left(\mathcal{O}_{V_{14}}(H) \otimes \mathcal{I}\right), \end{aligned}$$

where  $H \in |-K_{V_{14}}|$ .

**Lemma A.10.** *The equality  $\dim(S) = 0$  is impossible.*

*Proof.* Suppose that  $\dim(S) = 0$ . Let  $\bar{F} \subset \bar{\Gamma}$  be a stabilizer of a point in  $\text{LCS}(V_{14}, D)$ . Then  $|\text{LCS}(V_{14}, D)| \leq 10$  by (A.9). Thus, we see that  $|\bar{F}| \geq |\bar{\Gamma}|/10 = 66$ . Hence, we must have  $\text{LCS}(V_{14}, D) = S$  and  $\bar{F} = \bar{\Gamma}$  because there are no proper subgroups of  $\bar{\Gamma}$  of order greater than 60 (see [15]).

The action of  $\bar{\Gamma}$  on the tangent space to  $V_{14}$  at the point  $S$  gives a faithful three-dimensional representation of the group  $\bar{\Gamma}$ , which does not exist (see [15]).  $\square$

It follows from Theorem 2.8 that  $S$  is a smooth curve of genus  $g$  such that  $2g - 2 < S \cdot H$ .

**Lemma A.11.** *The curve  $S$  is  $\bar{\Gamma}$ -invariant.*

*Proof.* Let  $Z$  be the  $\bar{\Gamma}$ -orbit of the curve  $S$ . Then  $Z = \mathcal{L}(V_{14}, D)$  because  $(V_{14}, D)$  is log canonical.

Suppose that  $S \neq Z$ . Let  $r$  be the number of irreducible components of  $Z$ . Then  $r \geq 11$  because there is no nontrivial homomorphism  $\bar{\Gamma} \rightarrow S_r$  if  $1 < r \leq 10$ .

Using (A.9) and the Riemann–Roch theorem, we see that

$$10 \geq 10 - h^0\left(\mathcal{O}_{V_{14}}(H) \otimes \mathcal{I}\right) = h^0\left(\mathcal{O}_{\mathcal{L}(V_{14}, D)} \otimes \mathcal{O}_{V_{14}}(H)\right) = r\left(S \cdot H - g + 1\right) \geq r$$

because  $\mathcal{L}(V_{14}, D) = Z$  and  $2g - 2 < S \cdot H$ . Thus, we see that  $r \leq 10$ .  $\square$

Therefore, there is a natural homomorphism  $\theta: \bar{\Gamma} \rightarrow \text{Aut}(S)$ .

**Lemma A.12.** *The homomorphism  $\theta$  is a monomorphism.*

*Proof.* Suppose that  $\theta$  is not a monomorphism. Then  $\ker(\theta) = \bar{\Gamma}$  because  $\bar{\Gamma}$  simple. Let  $P$  be a point in  $S$ . Then  $P$  is  $\bar{\Gamma}$ -invariant. The action of the group  $\bar{\Gamma}$  on the tangent space to the threefold  $V_{14}$  at the point  $P$  gives its faithful three-dimensional representation, which does not exist (see [15]).  $\square$

**Lemma A.13.** *The inequality  $g \geq 11$  holds.*

*Proof.* Suppose that  $g \leq 10$ . Then  $g \leq 1$  by Theorem 2.18 since  $|\bar{\Gamma}| = 660$ .

Moreover, it follows from the classification of finite subgroups of  $\text{PSL}_2(\mathbb{C})$  that the monomorphism  $\theta$  does not exist if  $g = 0$ . Thus, we see that  $g = 1$  and  $\text{Aut}(S)$  contains a simple non-abelian subgroup  $\theta(\bar{\Gamma}) \cong \text{PSL}_2(\mathbb{F}_{11})$ , which is impossible because  $\text{Aut}(S)$  is solvable.  $\square$

Using (A.9) and the Riemann–Roch theorem, we see that

$$10 \geq h^0(\mathcal{O}_{\mathcal{L}(V_{14}, D)} \otimes \mathcal{O}_{V_{14}}(H)) = S \cdot H - g + 1$$

because  $\mathcal{L}(V_{14}, D) = Z$  and  $2g - 2 < S \cdot H$ . Then  $g \geq S \cdot H - 9$ , but  $2g - 2 < S \cdot H$ . Hence  $2(S \cdot H) - 20 \leq 2g - 2 < S \cdot H$ , which implies that  $S \cdot H \leq 19$ . Therefore,  $g \leq 10$ , which is impossible by Lemma A.13.

The obtained contradiction shows that  $V_{14}$  is  $\bar{\Gamma}$ -birationally superrigid.

To complete the proof of Theorem A.5, we assume that the threefold  $V_3$  is not  $\bar{\Gamma}$ -birationally superrigid. Then there is a  $\bar{\Gamma}$ -invariant linear system  $\mathcal{M}$  without fixed components such that  $\lambda\mathcal{M} \sim_{\mathbb{Q}} -K_{V_3}$  for some  $\lambda \in \mathbb{Q}_{>0}$ . Then  $(V_3, \lambda\mathcal{M})$  is not canonical (see [7, Theorem 1.4.1]).

There is  $\mu \in \mathbb{Q}$  such that  $\mu < 2\lambda$  and  $(V_3, \mu\mathcal{M})$  is strictly log canonical.

Let  $S \subset V_3$  be a minimal center in  $\text{LCS}(V_3, \mu\mathcal{M})$ . Then  $\dim(S) \in \{0, 1\}$ .

By Lemma 2.13, there is a  $\bar{\Gamma}$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on the threefold  $V_3$  such that the set  $\text{LCS}(V_3, D)$  consists of irreducible components of the  $\bar{\Gamma}$ -orbit of  $S$  and that  $D \sim_{\mathbb{Q}} -\epsilon K_{V_3}$ , where  $\epsilon$  is a positive rational number such that  $\epsilon < 2$ .

Let  $\mathcal{I}$  be the multiplier ideal sheaf of the log pair  $(V_3, D)$ , and let  $\mathcal{L}(V_3, D)$  be the log canonical singularities subscheme of the log pair  $(V_3, D)$ . Then it follows from Theorem 2.4 that

$$\begin{aligned} \text{(A.14)} \quad h^0(\mathcal{O}_{\mathcal{L}(V_3, D)} \otimes \mathcal{O}_{V_3}(2H)) &= h^0(\mathcal{O}_{V_3}(2H)) - h^0(\mathcal{O}_{V_3}(2H) \otimes \mathcal{I}) \\ &= 15 - h^0(\mathcal{O}_{V_3}(2H) \otimes \mathcal{I}), \end{aligned}$$

where  $H$  is an ample generator of the group  $\text{Pic}(V_3)$ .

**Lemma A.15.** *The equality  $\dim(S) = 0$  is impossible.*

*Proof.* Suppose that  $\dim(S) = 0$ . Let  $\bar{F} \subset \bar{\Gamma}$  be a stabilizer of a point in  $\text{LCS}(V_3, D)$ . Then

$$|\bar{F}| = \frac{|\bar{\Gamma}|}{|\text{LCS}(V_3, D)|} \geq \frac{|\bar{\Gamma}|}{15} = 44$$

by (A.14). Thus, if  $\bar{F} \neq \bar{\Gamma}$ , then  $\bar{F}$  is isomorphic to either  $A_5$  or  $\mathbb{Z}_{11} \times \mathbb{Z}_5$  (see [15]).

We may identify  $\bar{\Gamma}$  with a subgroup in  $\text{Aut}(\mathbb{P}^4)$ . There is a subgroup  $G \subset \text{SL}_5(\mathbb{C})$  such that

$$\bar{\Gamma} = \phi(G) \subset \text{Aut}(\mathbb{P}^4) \cong \text{PGL}_5(\mathbb{C})$$

and  $G \cong \bar{\Gamma}$ , where  $\phi: \text{SL}_5(\mathbb{C}) \rightarrow \text{Aut}(\mathbb{P}^4)$  is the natural projection.

Put  $W = \mathbb{C}^5$ . Then  $W$  is an irreducible representation of the group  $G$  (see [1]) which implies that  $\bar{F} \neq \bar{\Gamma}$ . If  $F$  is a subgroup of the group  $G$  such that  $\phi(F) = \bar{F}$ , then one can show that  $W$  is an irreducible representation of the group  $F$ , which is a contradiction.  $\square$

It follows from Theorem 2.8 that  $S$  is a smooth curve of genus  $g$  such that  $g \leq S \cdot H$ .

**Lemma A.16.** *The curve  $S$  is  $\bar{\Gamma}$ -invariant.*

*Proof.* Let  $Z$  be the  $\bar{\Gamma}$ -orbit of the curve  $S$ . Then  $Z = \mathcal{L}(V_3, D)$  because  $(V_3, D)$  is log canonical.

Suppose that  $S \neq Z$ . Let  $r$  be the number of irreducible components of  $Z$ . Then  $r \geq 11$  because there is no non-trivial homomorphism  $\bar{\Gamma} \rightarrow S_r$  in the case when  $1 < r \leq 10$ .

Using (A.14) and the Riemann–Roch theorem, we see that

$$15 \geq 15 - h^0(\mathcal{O}_{V_3}(2H) \otimes \mathcal{I}) = h^0(\mathcal{O}_{\mathcal{L}(V_3, D)} \otimes \mathcal{O}_{V_3}(2H)) = r(2S \cdot H - g + 1)$$

because  $\mathcal{L}(V_3, D) = Z$  and  $g \leq S \cdot H$ . Since  $r \geq 11$ , one has  $2S \cdot H - g + 1 \leq 1$ , which contradicts the inequality  $g \leq S \cdot H$ .  $\square$

We see that there is a natural homomorphism  $\theta: \bar{\Gamma} \rightarrow \text{Aut}(S)$ . Arguing as in Lemma A.12, we see that  $\theta$  is a monomorphism. Arguing as in Lemma A.13, one obtains  $g \geq 11$ .

By Theorem 2.18 we may assume that  $g = 14$ . Let  $a_i$  be the number of points on  $S$  whose stabilizers in  $\bar{\Gamma}$  are isomorphic to  $\mathbb{Z}_i$ . Using the Riemann–Hurwitz formula, we see that

$$2g - 2 = (2\bar{g} - 2) \cdot |\bar{\Gamma}| + 330a_2 + 440a_3 + 528a_5 + 550a_6 + 600a_{11},$$

where  $\bar{g}$  is the genus of the quotient curve  $S/\bar{\Gamma}$  (cf. the proof of Lemma 2.10). Then  $\bar{g} = 0$ . We have

$$1294 - 528a_5 = 330a_2 + 440a_3 + 550a_6 + 600a_{11},$$

which leads to a contradiction. The obtained contradiction completes the proof of Theorem A.5.

## APPENDIX B. DEL PEZZO FIBRATIONS

by YURI PROKHOROV\*

Let  $X$  be a threefold with at worst terminal singularities such that the group  $\text{Aut}(X)$  has a subgroup  $\bar{G} \cong A_6$ , and let  $\pi: X \rightarrow \mathbb{P}^1$  be a  $\bar{G}$ -Mori fibration.

The goal of this appendix is to prove the following result.

**Theorem B.1.** *The isomorphism  $X \cong \mathbb{P}^1 \times \mathbb{P}^2$  holds, and  $\pi$  is the projection to the first factor.*

---

\*This work was partially supported by the grants RFBR 08-01-00395-a, NSh-1983.2008.1 and NSh-1987.2008.1.

Recall that there exists no monomorphism  $\bar{G} \rightarrow \text{PGL}_2(\mathbb{C})$ .

**Lemma B.2** (cf. [38, Lemma 4.5]). *Let  $Y$  be a threefold with at worst terminal singularities such that  $\text{Aut}(Y)$  has a subgroup  $\bar{G} \cong A_6$ . Then  $Y$  contains no  $\bar{G}$ -invariant points.*

*Proof.* Suppose that  $Y$  contains a  $\bar{G}$ -invariant point  $P \in Y$ . Let us show that this assumption leads to a contradiction. Let  $T_{P,Y}$  be the Zariski tangent space to  $Y$  at the point  $P$ . Then  $\dim(T_{P,Y}) \geq 5$  because  $\bar{G}$  has no faithful representations of dimension less than 5. We see that  $Y$  is not Gorenstein at the point  $P$  (see [39, Section 3]). Let us regard  $(Y \ni P)$  as an analytic germ.

Let  $r$  be the Gorenstein index of the singularity  $(Y, P)$ , and let  $\pi: (Y^\sharp, P^\sharp) \rightarrow (Y, P)$  be the index one cover (see [39, Section 3.5]) where  $P^\sharp = \pi^{-1}(P)$ . Then

$$\pi: Y^\sharp \setminus \{P^\sharp\} \rightarrow Y \setminus \{P\}$$

is the topological universal cover (of degree  $r$ ). Thus, there is an exact sequence of groups

$$1 \longrightarrow \mathbb{Z}_r \xrightarrow{\alpha} \bar{G}^\sharp \xrightarrow{\beta} \bar{G} \longrightarrow 1,$$

where  $\bar{G}^\sharp$  is a finite subgroup in  $\text{Aut}(Y^\sharp, P^\sharp)$ . Since  $\bar{G}$  is simple, this is a central extension.

The group  $\bar{G}^\sharp$  naturally acts on the Zariski tangent space  $T_{P^\sharp, Y^\sharp}$  to  $Y^\sharp$  at the point  $P^\sharp$ .

Recall that  $(Y^\sharp, P^\sharp)$  is a hypersurface singularity. Hence, we have  $\dim(T_{P^\sharp, Y^\sharp}) \leq 4$ .

By the classification of three-dimensional terminal singularities (see [39, Section 6.1]) the action of the group  $\alpha(\mathbb{Z}_r)$  on  $T_{P^\sharp, Y^\sharp}$  in some coordinate system has one of the following forms:

- either  $\dim(T_{P^\sharp, Y^\sharp}) = 3$  and  $(x_1, x_2, x_3) \mapsto (\varepsilon x_1, \varepsilon^{-1} x_2, \varepsilon^a x_3)$ ,
- or  $\dim(T_{P^\sharp, Y^\sharp}) = 4$  and  $(x_1, x_2, x_3, x_4) \mapsto (\varepsilon x_1, \varepsilon^{-1} x_2, \varepsilon^a x_3, x_4)$ ,
- or  $\dim(T_{P^\sharp, Y^\sharp}) = r = 4$  and  $(x_1, x_2, x_3, x_4) \mapsto (\sqrt{-1}x_1, -\sqrt{-1}x_2, \pm\sqrt{-1}x_3, -x_4)$ ,

where  $\varepsilon$  is a primitive  $r$ -th root of unity and  $\gcd(r, a) = 1$ .

The case  $\dim(T_{P^\sharp, Y^\sharp}) = 3$  and  $r = 2$  is impossible because the group  $2.A_6$  does not have faithful three-dimensional representations. Therefore, the central subgroup  $\alpha(\mathbb{Z}_r) \subset \bar{G}^\sharp$  has at least 2 different eigenvalues. Then  $T_{P^\sharp, Y^\sharp}$  is a reducible representation of the group  $\bar{G}^\sharp$ .

If  $\dim(T_{P^\sharp, Y^\sharp}) = 4$ , then the subgroup  $\alpha(\mathbb{Z}_r)$  has at least 3 different eigenvalues.

Hence, in every possible case, the group  $\bar{G}^\sharp$  has a subrepresentation of dimension at least 2, which is impossible, because the group  $\bar{G} \cong A_6$  does not admit any embedding to  $\text{PGL}_2(\mathbb{C})$ . □

**Lemma B.3** (cf. [16]). *Let  $S$  be a smooth del Pezzo surface such that  $S$  admits a non-trivial action of the group  $\bar{G}$ . Then  $S \cong \mathbb{P}^2$ .*

*Proof.* Note that  $S \not\cong \mathbb{P}^1 \times \mathbb{P}^1$  because there exists no monomorphism  $\bar{G} \rightarrow \text{PGL}_2(\mathbb{C})$ .

Suppose that  $S \not\cong \mathbb{P}^2$ . Let us derive a contradiction.

If  $K_S^2 \geq 5$ , then the action of  $\bar{G}$  on  $\text{Pic}(S)$  is trivial because  $\text{rk Pic}(S) \leq 5$  and the canonical class  $K_S$  is  $\bar{G}$ -invariant. Then any  $(-1)$ -curve on  $S$  must be invariant, a contradiction.

Let  $C$  be a  $\bar{G}$ -invariant curve in  $|-K_S|$ . Then every component of the curve  $C$  is either a rational or an elliptic curve. Moreover, the curve  $C$  consists of at most 4 components, which immediately implies that  $C$  is not  $\bar{G}$ -invariant because  $\bar{G}$  is simple.

Put  $V = H^0(\mathcal{O}_S(-K_S))$ . By the above, the group  $\bar{G}$  acts non-trivially on  $V$ . Then

$$4 \geq K_S^2 = h^0(\mathcal{O}_S(-K_S)) - 1 \geq 4,$$

which implies that  $K_S^2 = 4$ , and the space  $V$  is an irreducible five-dimensional representation of the group  $\bar{G}$  because  $\bar{G}$  has no non-trivial representations of dimension less than 5 (see [15]).

We see that  $S = Q_1 \cap Q_2 \subset \mathbb{P}^4$ , where  $Q_1$  and  $Q_2$  are irreducible quadric hypersurfaces.

The action of the group  $\bar{G}$  on the space  $V$  induces its action on  $\mathbb{P}^4$ .

Let  $\mathcal{P}$  be the pencil generated by  $Q_1$  and  $Q_2$  is  $\bar{G}$ -invariant. Then  $\mathcal{P}$  is  $\bar{G}$ -invariant. Since there is no monomorphism  $\bar{G} \rightarrow \text{PGL}_2(\mathbb{C})$ , both quadrics  $Q_1$  and  $Q_2$  are  $\bar{G}$ -invariant, which is impossible, because otherwise the vertex of a degenerate quadric in pencil is a fixed point.  $\square$

**Corollary B.4.** *Let  $F_\pi$  be a general fiber of the morphism  $\pi$ . Then  $K_{F_\pi}^2 = 9$*

Let  $F$  be any scheme fiber of the morphism  $\pi: X \rightarrow \mathbb{P}^1$ . Then  $F$  is  $\bar{G}$ -invariant.

**Lemma B.5.** *The threefold  $X$  is smooth and  $F \cong \mathbb{P}^2$ .*

*Proof.* First we show that  $\text{Supp}(F)$  is irreducible, normal and has at worst Kawamata log terminal singularities. This step is similar to the proof of [34, Proposition 2.6].

Take  $\mu \in \mathbb{Q}$  such that  $(X, \mu F)$  is strictly log-canonical. Then there are a  $\bar{G}$ -invariant  $\pi$ -ample divisor  $H$  and a small positive  $\delta_1$  and  $\delta_2 \in \mathbb{Q}$  such that  $(X, (\mu - \delta_1)F + \delta_2 H)$  is log canonical and

$$\text{LCS}(X, (\mu - \delta_1)F + \delta_2 H) = \bigcup_{g \in \bar{G}} g(S),$$

where  $S$  is a minimal center in  $\text{LCS}(X, \mu F)$  (cf. Lemma 2.13 or the proof of [34, Proposition 2.6]).

Put  $D = (\mu - \delta_1)F + \delta_2 H$ . Then  $\text{LCS}(X, D)$  is a  $\bar{G}$ -orbit of the center  $S$ .

By Theorem 2.5, the locus  $\text{LCS}(X, D)$  is connected. It follows from Lemma 2.7 that  $\text{LCS}(X, D) = S$ , which implies that  $S$  is not a point by Lemma B.2.

If  $S$  is a curve, then  $S \cong \mathbb{P}^1$  by Theorem 2.8, which is again impossible by Lemma B.2.

We see that  $S$  is a  $\bar{G}$ -invariant surface. Then  $S = \text{Supp}(F)$  because  $\pi$  is a  $\bar{G}$ -Mori fibred space.

By Theorem 2.8, the surface  $S$  is normal and has Kawamata log terminal singularities.

Let  $\Sigma \subset S$  be a subset consisting of points where  $S$  is not Cartier. If  $F$  is not reduced, then  $1 \leq |\Sigma| \leq 4$  by [34, Theorem 1.1]. Since  $\Sigma$  is  $\bar{G}$ -invariant, we see that  $\Sigma = \emptyset$  by Lemma B.2.



We see that  $F$  is a reduced normal surface having only quotient singularities, which implies that  $F$  is a degeneration of  $\mathbb{P}^2$  by Corollary B.4. Hence, the inequality  $|\text{Sing}(F)| \leq 3$  holds (see [32, Main Theorem], [20, Corollary 1.2]). So, the surface  $F$  is smooth by Lemma B.2, which immediately implies that  $F \cong \mathbb{P}^2$ .  $\square$

*Proof of Theorem B.1.* It follows from Lemma B.5 that every scheme fiber of the morphism  $\pi$  is isomorphic to  $\mathbb{P}^2$ . By [3, Proposition V.4.1], there are integers  $b \geq a \geq 0$  such that

$$X \cong \text{Proj}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)\right).$$

Recall that the action of the group  $\bar{G}$  on the base  $\mathbb{P}^1$  is trivial.

By Lemma B.2, every fiber of the morphism  $\pi$  contains no  $\bar{G}$ -fixed points or  $\bar{G}$ -invariant lines, which implies that  $a = b = 0$ , and so  $X \cong \mathbb{P}^1 \times \mathbb{P}^2$ .  $\square$

#### REFERENCES

- [1] A. Adler, *On the automorphism group of a certain cubic threefold*, American Journal of Mathematics **100** (1978), 1275–1280 MR522700 (80d:14026)
- [2] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, *Geometry of algebraic curves. Volume I*, Grundlehren der Mathematischen Wissenschaften **267** (1985) Springer-Verlag, New York-Heidelberg MR770932 (86h:14019)
- [3] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Springer-Verlag, Berlin, 1984 MR749574 (86c:32026)
- [4] A. Beauville, *Non-rationality of the symmetric sextic Fano threefold*, arXiv:math/1102.1255 (2011).
- [5] W. Bosma, J. Cannon, C. Playoust, *The Magma algebra system. I. The user language*, Journal of Symbolic Computation, **24** (1997), 235–265 MR1484478
- [6] T. Breuer, *Characters and automorphism groups of compact Riemann surfaces*, London Mathematical Society Lecture Note Series **280**, Cambridge University Press, 2000 MR1796706 (2002i:14034)
- [7] I. Cheltsov, *Birationally rigid Fano varieties*, Russian Mathematical Surveys **60** (2005), 875–965 MR2195677 (2007d:14028)
- [8] I. Cheltsov, *Log canonical thresholds of del Pezzo surfaces*, Geometric and Functional Analysis, **18** (2008), 1118–1144 MR2465686 (2009m:14054)
- [9] I. Cheltsov, C. Shramov, *Log canonical thresholds of smooth Fano threefolds*, Russian Mathematical Surveys **63** (2008), 73–180 MR2484031 (2010b:14082)
- [10] I. Cheltsov, C. Shramov, *On exceptional quotient singularities*, Geometry and Topology, **15** (2011), 1843–1882
- [11] C. Clemens, P. Griffiths, *The intermediate Jacobian of the cubic threefold*, Annals of Mathematics **95** (1972), 73–100 MR0302652 (46:1796)
- [12] A. Corti, *Factorizing birational maps of threefolds after Sarkisov*, Journal of Algebraic Geometry **4** (1995), 223–254 MR1311348 (96c:14013)
- [13] A. Corti, *Singularities of linear systems and 3-fold birational geometry*, L.M.S. Lecture Note Series **281** (2000), 259–312 MR1798984 (2001k:14041)
- [14] A. Corti, K. E. Smith, J. Kollár, *Rational and nearly rational varieties*, Cambridge Studies in Advanced Mathematics **92** (2004), Cambridge University Press MR2062787 (2005i:14063)
- [15] J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985 MR827219 (88g:20025)
- [16] I. Dolgachev, V. Iskovskikh, *Finite subgroups of the plane Cremona group*, Progress in Mathematics, Birkhauser, Boston **269** (2009), 443–548 MR2641179 (2011c:14032)
- [17] W. Feit, *The current situation in the theory of finite simple groups*, Actes du Congrès International des Mathématiciens, Gauthier–Villars, Paris (1971), 55–93 MR0427449 (55:481)

- [18] H. Finkelberg, *Small resolutions of the Segre cubic*, *Indagationes Mathematicae* **90** (1987), 261–277 MR914085 (88j:14050)
- [19] H. Finkelberg, J. Werner, *Small resolutions of nodal cubic threefolds*, *Indagationes Mathematicae* **92** (1989), 185–198 MR1005050 (90i:14010)
- [20] P. Hacking, Yu. Prokhorov, *Smoothable del Pezzo surfaces with quotient singularities*, *Compositio Mathematica* **146** (2010), 169–192 MR2581246 (2011f:14062)
- [21] C. Hacon, J. McKernan, Ch. Xu, *On the birational automorphisms of varieties of general type*, arXiv:math/1011.1464 (2010)
- [22] R. Hartshorne, *Algebraic geometry*, *Graduate Texts in Mathematics* **52** (1977) Springer-Verlag, New York-Heidelberg MR0463157 (57:3116)
- [23] B. Hunt, *The geometry of some special arithmetic quotients*, *Lecture Notes in Mathematics* **1637** (Springer-Verlag, New York, 1996) MR1438547 (98c:14033)
- [24] V. Iskovskikh, Yu. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, *Matematiskii Sbornik* **86** (1971), 140–166 MR0291172 (45:266)
- [25] V. Iskovskikh, Yu. Prokhorov, *Fano varieties*, *Encyclopaedia of Mathematical Sciences* **47** (1999) Springer, Berlin MR1668579 (2000b:14051b)
- [26] V. Iskovskikh, A. Pukhlikov, *Birational automorphisms of multidimensional algebraic manifolds*, *Journal of Mathematical Sciences* **82** (1996), 3528–3613 MR1426395 (98e:14009)
- [27] Y. Kawamata, *On Fujita’s freeness conjecture for 3-folds and 4-folds*, *Mathematische Annalen* **308** (1997), 491–505 MR1457742 (99c:14008)
- [28] Y. Kawamata, *Subadjunction of log canonical divisors II*, *American Journal of Mathematics* **120** (1998), 893–899 MR1646046 (2000d:14020)
- [29] J. Kollár, *Singularities of pairs*, *Proceedings of Symposia in Pure Mathematics* **62** (1997), 221–287 MR1492525 (99m:14033)
- [30] R. Lazarsfeld, *Positivity in algebraic geometry II*, Springer-Verlag, Berlin, 2004 MR2095472 (2005k:14001b)
- [31] N. Lemire, V. Popov, Z. Reichstein, *Cayley groups*, *Journal of the American Mathematical Society* **19** (2006), 921–967 MR2219306 (2007e:14074)
- [32] M. Manetti, *Normal degenerations of the complex projective plane*, *Journal für die Reine und Angewandte Mathematik* **419** (1991), 89–11 MR1116920 (92f:14028)
- [33] M. Mella, *Birational geometry of quartic 3-folds II: the importance of being  $\mathbb{Q}$ -factorial*, *Mathematische Annalen* **330** (2004), 107–126 MR2091681 (2005h:14030)
- [34] S. Mori, Yu. Prokhorov, *Multiple fibers of del Pezzo fibrations*, *Proceedings of the Steklov Institute of Mathematics* **264** (2009), 131–145 MR2590844 (2011b:14037)
- [35] S. Mukai, *Curves and symmetric spaces*, *Proceedings of the Japan Academy, Series A, Mathematical Sciences* **68** (1992), 7–10 MR1158012 (93d:14042)
- [36] I. Nieto, *The normalizer of the level (2, 2)-Heisenberg group*, *Manuscripta Mathematica* **76** (1992), 257–267 MR1185019 (93j:14062)
- [37] K. Pettersen, *On nodal determinantal quartic hypersurfaces in  $\mathbb{P}^4$* , Thesis, University of Oslo (1998)
- [38] Yu. Prokhorov, *Simple finite subgroups of the Cremona group of rank 3*, *Journal of Algebraic Geometry*, to appear
- [39] M. Reid, *Young person’s guide to canonical singularities*, *Proceedings of Symposia in Pure Mathematics* **46** (1987), 345–414 MR927963 (89b:14016)
- [40] J.-P. Serre, *A Minkowski-style bound for the order of the finite subgroups of the Cremona group of rank 2, over an arbitrary field*, *Moscow Mathematical Journal* **9** (2009), 183–198 MR2567402 (2011e:14025)
- [41] V. Shokurov, *Three-fold log flips*, *Russian Academy of Sciences, Izvestiya Mathematics* **40** (1993), 95–202 MR1162635 (93j:14012)
- [42] C. Shramov, *Birational automorphisms of nodal quartic threefolds*, arXiv:math/0803.4348 (2008)
- [43] G. Tian, *On Kähler–Einstein metrics on certain Kähler manifolds with  $c_1(M) > 0$* , *Inventiones Mathematicae* **89** (1987), 225–246 MR894378 (88e:53069)
- [44] G. Tian, S.-T. Yau, *Kähler–Einstein metrics on complex surfaces with  $C_1 > 0$* , *Communications in Mathematical Physics* **112** (1987), 175–203 MR904143 (88k:32070)
- [45] J. A. Todd, *Configurations defined by six lines in space of three dimensions*, *Mathematical Proceedings of the Cambridge Philosophical Society* **29** (1933), 52–68

- [46] D. Vazzana, *Invariants and projections of six lines in projective space*, Transactions of the American Mathematical Society **353** (2001), 2673–2688 MR1828467 (2002e:14092)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF EDINBURGH, EDINBURGH EH9 3JZ, UNITED KINGDOM

*E-mail address:* `I.Cheltsov@ed.ac.uk`

STEKLOV MATHEMATICAL INSTITUTE, GUBKINA STR. 8, 119991, MOSCOW, RUSSIA

*E-mail address:* `shramov@mccme.ru`