Strong duality and boundedness in conic optimization

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Temitayo Ajayi* Akshay Gupte†, ‡ Amin Khademi§ Andrew J. Schaefer*

September 16, 2020

Abstract

Unlike linear programming, it is well-known that some conditions are required to achieve strong duality between a primal-dual pair of conic programs. The most common and well-known of these conditions is full-dimensionality of the cones and strict feasibility of one of the problems, also referred to as the Slater constraint qualification (CQ). Other sufficient conditions in literature for strong duality include a closedness condition for the full-dimensional case and a minimal facial property for general cones. We show that the closedness condition is also sufficient for strong duality when the cones are low-dimensional. The key step is to establish upper bounds on the duality gap through the separation of certain proximal points from the closure of an adjoint image of the cones. A consequence is a collection of specific sufficient conditions for strong duality, one of them being the generalized Slater CQ for low-dimensional cones, a few others being in terms of strict feasibility of the recession cones, and another being boundedness of the feasible region as a universal CQ. We also give various algebraic characterizations of the recession cone and its polar, thereby leading to many necessary and sufficient conditions for a bounded feasible region and also a theorem of the alternative in terms of approximate feasibility of the problem. Finally, we establish that under the generalized Slater CQ, finiteness of one problem and feasibility of the other problem are equivalent. This not only implies sufficiency of the Slater CQ for strong duality but it also allows us to characterize the projection of a conic set onto a linear subspace using extreme rays of a closed convex cone that generalizes the projection cone for polyhedral sets.

Keywords. Duality theory; Constraint qualification; Recession cone; Theorem of Alternative; Support function; Projection onto subspace

AMS 2020 subject classification. Primary 90C46, 90C25; Secondary 49N15; 90C22

1 Introduction

Let $\mathbf{E}$ and $\mathbf{E}'$ be two Euclidean spaces (finite-dimensional inner-product spaces over reals) with inner-products $\langle\cdot,\cdot\rangle_{\mathbf{E}}$ and $\langle\cdot,\cdot\rangle_{\mathbf{E}'}$. For a nonempty closed convex cone $K \subset \mathbf{E}'$ define the binary ordering relation $\preccurlyeq_K$ in $\mathbf{E}'$ as $u \preccurlyeq_K v$ if and only if $v - u \in K$. The $\succcurlyeq_K$ ordering is analogous. A linear map $\mathcal{A}\colon \mathbf{E} \to \mathbf{E}'$ and vector $b \in \mathbf{E}'$ yield the conic constraints $\mathcal{A}x \preccurlyeq_K b$ for $x \in \mathbf{E}$. The conic optimization problem for objective $c \in \mathbf{E}$ and nonempty closed convex cone $C \subseteq \mathbf{E}$ is

$$z_P^* = \sup\left\{\langle c, x\rangle_{\mathbf{E}} : \mathcal{A}x \preccurlyeq_K b,\ x \in C\right\}. \tag{1a}$$

*Department of Computational and Applied Mathematics, Rice University, Houston, TX, USA
†School of Mathematics, University of Edinburgh, Edinburgh, UK
‡Corresponding Author: `akshay.gupte@ed.ac.uk`
§Department of Industrial Engineering, Clemson University, Clemson, SC, USA

For nontriviality, we assume that $C \neq \{\mathbf{0}\}$ and that at least one of C or K is not equal to its ambient Euclidean space. Because K can be a Cartesian product of finitely many cones, the conic inequalities $\mathcal{A}x \preccurlyeq_K b$ can incorporate multiple constraints over different cones. Equality constraints can also be included as part of conic inequalities because $\mathcal{B}x = d$, for a linear map $\mathcal{B}$ and vector d, is equivalent to $\mathcal{B}x \preccurlyeq_{\{\mathbf{0}\}} d$. Writing equality constraints with the trivial cone $\{\mathbf{0}\}$ will be more useful to us than expressing them as $\preccurlyeq_K$ and $\succcurlyeq_K$ constraints. The conic constraint $x \in C$ is kept separate in (1a) because it involves a special linear map (identity) and constant vector being all zeros. When C is the trivial cone $\mathbf{E}$, we have only the conic inequalities. Our focus is on nonlinear conic programs and so for the sake of nontriviality, we assume that at least one of C or K is non-polyhedral.

Let $\mathcal{A}^\star \colon \mathbf{E}' \to \mathbf{E}$ be the adjoint of the linear map $\mathcal{A}$, and $K^\star \subset \mathbf{E}'$ and $C^\star \subseteq \mathbf{E}$ be the dual cones of K and C. The Lagrangian dual problem to the primal problem can be written in the dual space using the adjoint map and the ordering relation $\succcurlyeq_{C^\star}$ as the following conic program:

$$z_D^* = \inf \left\{ \langle b,\, y\rangle_{\mathbf{E}'} \,:\, \mathcal{A}^\star y \succcurlyeq_{C^\star} c,\ y \in K^\star \right\}. \tag{1b}$$

This problem is commonly also referred to as the conic dual of (1a) because it is also a conic program and Lagrangian duals of general convex programs do not always have their min-max simplified into an explicit minimization problem. Problems (1a) and (1b) form a primal-dual pair. They are symmetric in the sense that the conic dual of (1b) yields (1a), which means that the primal and dual problems can be referenced interchangeably.

The conic program (1a) was proposed many years ago as a generalization of the linear program [Duf57]. Since then, commonly studied conic optimization problems include second-order cone programs [AG03] and semidefinite programs [Tod01]. These have powerful modelling capabilities, thus appearing in many applications, and are also used as relaxations of nonconvex problems [Lob+98; WSV00; BN01; Nem06; AL12]. Other cones of interest are the cone of copositive matrices and the cone of completely positive matrices, which although cannot be optimized/separated over in polynomial-time, provide convex reformulations of nonconvex problems (cf. [AL12, chap. 8]).

This paper deals with duality theory for conic programs, where a fundamental question is to analyze conditions under which the dual problem is a strong dual to the primal problem. *Strong dual* means *zero duality gap* ($z_P^* = z_D^*$) and at least one of the two problems is *solvable* (has an optimal solution). The linear programming dual is a strong dual under the mild condition that either the primal or dual is feasible and finite-valued, which is implied if both problems are feasible. For general conic programs, some additional assumptions are required to achieve strong duality. The most common and well-known of these conditions is the Slater constraint qualification (Slater CQ). On the primal problem, this condition requires the cones C and K to be solid (i.e., be full-dimensional and hence have a nonempty interior) and that b belong to the interior of $\mathcal{A}(C) - K$, which is equivalent to existence of an x in the interior of C for which $\mathcal{A}x \preccurlyeq_K b$ is strictly feasible. Duality theory was developed in the 1990's for semidefinite programs [Ram97; RTW97]. The first reference that we could find on establishing strong duality under Slater CQ for general conic optimization with full-dimensional cones is [BN01, Theorem 1.7.1]. Around the same time, [Sha01, Proposition 2.6] showed the existence of strong duality when a certain set is closed, and this closedness condition is not a constraint qualification like the Slater CQ. Strong duality for cone programs was shown to exist in a more general setting of abstract convex programs by [BW81a, Theorem 4.2] using conditions on certain minimal faces of the cone K; see also [BW81b; Pat13] for related work. This allows for low-dimensional cones and implies that strong duality holds under the generalized Slater CQ which replaces interior with relative interior in the definition of Slater CQ (cf. [TW12, Corollary 4.8]). There have also been studies on the geometry of Slater CQ and degeneracy of solutions, see [WSV00; PT01; DW17]. The inexact duality theory for conic problems (inexact because it requires a closedness or strict feasibility or facial condition) yields approximate versions of the Farkas lemma (infeasibility certificates) for conic inequalities [BN01; PT09; DJ14; LP17].

1.1 Our Contributions and Organization of the Paper

The emphasis of this paper is two-fold. The first is to derive various conditions for strong duality in the most general case of low-dimensional (and possibly non-pointed) cones. The second is to give various characterizations for boundedness of the feasible region and for a bounded optimum.

On the first front, we establish in Theorem 3.1 that the closedness condition for strong duality with full-dimensional cones [Sha01, Proposition 2.6] in fact also holds in the most general case of low-dimensional cones, including non-pointed cones. Our proof technique relies on showing that a sufficient condition for ε-duality gap, for any $\varepsilon > 0$, is that certain ε-proximal points do not belong to the closure of an adjoint image of the cones. The linear map underlying this adjoint is closely related to the perspective image of the feasible set. Our arguments for bounding the duality gap are set-theoretic because we use the separation theorem for closed convex sets and polarity of cones. Hence, our approach is different than that of [Sha01] who used a functional analysis argument for value functions. Then we show in Corollary 5.3 that strong duality under generalized Slater CQ is a direct consequence, and also a special case, of the closedness condition. This gives an alternate proof for sufficiency of this CQ, as opposed to [TW12, Corollary 4.8] who used properties of minimal faces of K and [BN01, Theorem 1.7.1] who proved it directly for the full-dimensional case. Necessary and sufficient conditions for the existence of generalized Slater CQ are given in §4. Moreover, in §5, we derive several sufficient conditions for strong duality in Proposition 5.1 and Corollarys 5.4 and 5.5, where the last result implies that the boundedness of the feasible region is a universal CQ (i.e., CQ that is independent of b and c) for strong duality.

With regards to boundedness, we give various algebraic descriptions for the recession cone of the feasible region and the polar cone in §6.1. This leads to necessary and sufficient conditions for the boundedness of the feasible region in §6.2. We also generalize in Proposition 6.14 the theorem of alternative given by [BN01, Proposition 1.7.1] in terms of almost feasibility of the problem to the case of low-dimensional cones under some minor technical assumptions. A consequence noted in Corollary 6.17 is that feasibility and almost feasibility are equivalent when the recession cone is trivial. Theorem 7.1 is our final main result and it establishes that under Slater CQ, the finiteness of one problem and feasibility of the other problem are equivalent. We prove this result by explicitly characterizing the domain of support function of the affine preimage of a cone. There are two consequences to our theorem. One is that it leads to an alternate proof in §7.1 for strong duality with generalized Slater CQ. Another is Proposition 7.7 which describes the projection of a conic set onto a linear subspace in terms of extreme rays of a closed convex cone that is a generalization of the projection cone for polyhedral sets.

1.2 General Notation

Unless there is ambiguity, we drop the subscripts in the notation for inner-products. The vector of all zeros is written as $\mathbf{0}$. The Minkowski sum of two sets $S_1, S_2 \subset \mathbf{E}$ is $S_1 + S_2$, and we write the Minkowski difference of sets as $S_1 - S_2$ and define it as the Minkowski sum $S_1 + (-S_2)$. For a set $S \subset \mathbf{E}$, $\mathtt{ri}\, S$ is the relative interior, $\mathtt{int}\, S$ is the interior, $\mathtt{cl}\, S$ is the closure, $\partial\, S := \mathtt{cl}\, S \setminus \mathtt{ri}\, S$ is the relative boundary, $\mathtt{aff}\, S$ is the affine hull, $\mathtt{span}\, S$ is the linear hull (span). The orthogonal complement of a linear subspace $L \subset \mathbf{E}$, also called its annihilator, is denoted by $L^\perp$, and note that $(L^\perp)^\perp = L$. Two linear subspaces $L_1, L_2 \subset \mathbf{E}$ have $L_1 \subseteq L_2$ if and only if $L_1^\perp \supseteq L_2^\perp$. The recession cone of a nonempty convex set $S \subset \mathbf{E}$ is the convex cone $0^+S := \{r \in \mathbf{E}\colon x + \mu r \in S, \forall x \in S, \mu \geqslant 0\}$. The lineality space of S is the linear subspace $\mathtt{lin}\, S = 0^+S \cap -0^+S$. A set (e.g., a cone) is pointed if its lineality space is trivial (equal to $\{\mathbf{0}\}$).

For a map $\mathcal{L}\colon \mathbf{E} \to \mathbf{E}'$, the image of a set $S \subset \mathbf{E}$ is $\mathcal{L}(S) := \{\mathcal{L}(x)\colon x \in S\}$, the image of $\mathcal{L}$ is $\mathtt{Im}\,\mathcal{L} := \mathcal{L}(\mathbf{E})$, the preimage of a set $S \subset \mathbf{E}'$ is $\mathcal{L}^{-1}(S) := \{x \in \mathbf{E}\colon \mathcal{L}(x) \in S\}$. We work with linear and affine maps only. Affine maps are translates of linear maps. The kernel of a linear map $\mathcal{L}$ is $\mathtt{ker}\,\mathcal{L} := \mathcal{L}^{-1}\{\mathbf{0}\}$. The adjoint of a linear map $\mathcal{L}$ is a unique linear map $\mathcal{L}^\star\colon \mathbf{E}' \to \mathbf{E}$ which satisfies $\langle \mathcal{L}(x), y \rangle_{\mathbf{E}'} = \langle \mathcal{L}^\star(y), x \rangle_{\mathbf{E}}$ for all $x \in \mathbf{E}, y \in \mathbf{E}'$. We assume familiarity with fundamental properties of linear operators and their adjoints (cf. [Rom08]).

A cone $\mathcal{C} \subseteq \mathbf{E}$ is a set that is closed under positive scaling ($x \in \mathcal{C}, \lambda > 0$ implies $\lambda x \in \mathcal{C}$). We do not require a cone to contain $\mathbf{0}$, but when the cone is closed it necessarily contains $\mathbf{0}$. A cone is nontrivial if $\mathcal{C} \neq \{\mathbf{0}\}$ and $C \subsetneq \mathbf{E}$. The affine hull and span of a cone are equal and written as $\mathtt{aff}\,\mathcal{C}$, the orthogonal complement of $\mathtt{aff}\,\mathcal{C}$ is $\mathcal{C}^\perp$. The dual cone and polar cone are denoted by $\mathcal{C}^\star$ and $\mathcal{C}^\circ$, respectively, and defined as $\mathcal{C}^\star := \{y \in \mathbb{R}^n\colon \mathtt{inf}\{y^\top x\colon x \in \mathcal{C}\} = 0\}$ and $\mathcal{C}^\circ := \{y \in \mathbb{R}^n\colon \mathtt{sup}\{y^\top x\colon x \in \mathcal{C}\} = 0\} = -\mathcal{C}^\star$. These are closed convex cones. The binary relation $x \preccurlyeq_{\mathcal{C}} y$ means $y - x \in \mathcal{C}$. Strict inequality $x \prec_{\mathcal{C}} y$ means $y - x \in \mathtt{ri}\,\mathcal{C}$.

2 Preliminaries

Every nonempty convex set in $\mathbf{E}$ has a nonempty relative interior. The following basic results from convex analysis about closure and relative interior are useful in this paper.

Lemma 2.1 (cf. [Roc70])**.** *Let $S \subset \mathbf{E}$ be a nonempty convex set and $\mathcal{G}$ be an affine map.*

1. $\mathtt{ri}\, S = \mathtt{ri}\,(\mathtt{cl}\, S) \subseteq \mathtt{cl}\,(\mathtt{ri}\, S) = \mathtt{cl}\, S$.
2. $\mathtt{ri}\,\mathcal{G}(S) = \mathcal{G}(\mathtt{ri}\, S) \subseteq \mathcal{G}(\mathtt{cl}\, S) \subseteq \mathtt{cl}\,\mathcal{G}(S)$, *with* $\mathcal{G}(\mathtt{cl}\, S) = \mathtt{cl}\,\mathcal{G}(S)$ *when* $\mathcal{G}(\mathtt{cl}\, S)$ *is closed.*
3. $\mathcal{G}^{-1}(\mathtt{cl}\, S) \supseteq \mathtt{cl}\,\mathcal{G}^{-1}(S) \supseteq \mathtt{ri}\,\mathcal{G}^{-1}(S) \supseteq \mathcal{G}^{-1}(\mathtt{ri}\, S)$, *with* $\mathtt{ri}\,\mathcal{G}^{-1}(S) = \mathcal{G}^{-1}(\mathtt{ri}\, S)$ *and* $\mathcal{G}^{-1}(\mathtt{cl}\, S) = \mathtt{cl}\,\mathcal{G}^{-1}(S)$ *when* $\mathcal{G}^{-1}(\mathtt{ri}\, S) \neq \emptyset$.
4. $\mathtt{ri}$ *and* $\mathtt{cl}$ *distribute over a Cartesian product of nonempty convex sets and over intersection of convex sets when there is a point in the intersection of all the relative interiors.*

Commutativity of the $\mathtt{ri}$ operator with a linear map leads to its distributivity over the Minkowski sum and difference.

Lemma 2.2. $\mathtt{ri}\,(S_1 \pm S_2) = \mathtt{ri}\, S_1 \pm \mathtt{ri}\, S_2$ *for nonempty convex sets* $S_1, S_2 \subset \mathbf{E}$.

Proof. The Minkowski sum is the image of the linear map $\mathcal{L}\colon (u, v) \mapsto u+v$, i.e., $S_1 + S_2 = \mathcal{L}(S_1 \times S_2)$. Hence, $\mathtt{ri}\,(S_1 + S_2) = \mathtt{ri}\,\mathcal{L}(S_1 \times S_2) = \mathcal{L}(\mathtt{ri}\,(S_1 \times S_2)) = \mathcal{L}(\mathtt{ri}\, S_1 \times \mathtt{ri}\, S_2) = \mathtt{ri}\, S_1 + \mathtt{ri}\, S_2$. □

We use Lemmas 2.1 and 2.2 throughout this paper without necessarily referencing to them every time they are used.

2.1 Properties of Cones

For convex cones, duality and polarity are anti-inclusion preserving and distributive over the Cartesian product. A closed convex cone is both pointed and full-dimensional if and only if its dual cone is both pointed and full-dimensional. For a convex cone $\mathcal{C}$, we have $(\mathcal{C}^\star)^\star = \mathtt{cl}\,\mathcal{C}$, $(\mathtt{cl}\,\mathcal{C})^\star = (\mathtt{ri}\,\mathcal{C})^\star = \mathcal{C}^\star \supsetneq \mathtt{ri}\,\mathcal{C}^\star$, and $\mathcal{C}^\perp = \mathcal{C}^\star \cap \mathcal{C}^\circ$. The first relationship is called the Bipolar

Theorem for cones. Let us formally note the following fact that is useful in some arguments for the dual problem.

Lemma 2.3. *For a closed convex cone $\mathcal{C}$, $\operatorname{aff}\mathcal{C}^\star = \operatorname{aff}\mathcal{C}^\circ = (\operatorname{lin}\mathcal{C})^\perp$.*

Proof. Obviously, the dual and polar have the same span. For relation to the lineality space, consider the equality $\mathcal{C}^\perp = \mathcal{C}^\star \cap \mathcal{C}^\circ$ and replace $\mathcal{C}$ with $\mathcal{C}^\star$. This gives us $(\mathcal{C}^\star)^\perp = (\mathcal{C}^\star)^\star \cap (\mathcal{C}^\star)^\circ$. The Bipolar Theorem tells us $(\mathcal{C}^\star)^\star = \mathcal{C}$, and because the polar cone is the negative of the dual cone, we have $(\mathcal{C}^\star)^\circ = -\mathcal{C}$. Hence, $(\mathcal{C}^\star)^\perp = \mathcal{C} \cap -\mathcal{C} = \operatorname{lin}\mathcal{C}$, which leads to $\operatorname{aff}\mathcal{C}^\star = (\operatorname{lin}\mathcal{C})^\perp$. □

Linear subspaces are closed convex cones, but some of our results need to exclude such pathological cones. Non-subspace cones do not contain the origin in their relative interior or that of the dual cone.

Lemma 2.4. *Let $\mathcal{C}$ be a nonempty closed convex cone that is not a linear subspace. We have*

1. $\operatorname{ri}\mathcal{C} \not\perp \mathcal{C}^\star \setminus \mathcal{C}^\perp$*; in particular, $\langle x, y\rangle > 0$ for $x \in \operatorname{ri}\mathcal{C}$ and $y \in \mathcal{C}^\star \setminus \mathcal{C}^\perp$,*
2. $\operatorname{ri}\mathcal{C}^\star \not\perp \mathcal{C} \setminus \operatorname{lin}\mathcal{C}$*; in particular, $\langle x, y\rangle > 0$ for $x \in \operatorname{ri}\mathcal{C}^\star$ and $y \in \mathcal{C} \setminus \operatorname{lin}\mathcal{C}$.*
3. $\mathcal{C}^\perp \subseteq \partial\mathcal{C}^\star$ *and* $\operatorname{lin}\mathcal{C} \subseteq \partial\mathcal{C}$*,*
4. $\mathbf{0} \notin \operatorname{ri}\mathcal{C} \cup \operatorname{ri}\mathcal{C}^\star$*.*

Proof. Observe that

$$\mathcal{C} \text{ is not a linear subspace} \iff \mathcal{C}\setminus\operatorname{lin}\mathcal{C} \neq \emptyset \iff C^\star \text{ is not a linear subspace} \iff \mathcal{C}^\star\setminus\mathcal{C}^\perp \neq \emptyset.$$

(1) Suppose the claim is not true and $\langle x, y\rangle = 0$. Because $y \notin \mathcal{C}^\perp$, there exists a $z \in \operatorname{aff}\mathcal{C}$ such that $\langle y, z\rangle \neq 0$. Because $x \in \operatorname{ri}\mathcal{C}$ and $z \in \operatorname{aff}\mathcal{C}$, we have $x + z \in \mathcal{C}$ and $x - z \in \mathcal{C}$. Now, $y \in \mathcal{C}^\star$ implies $\langle y, x+z\rangle \geqslant 0$ and $\langle y, x-z\rangle \geqslant 0$. After distributing the inner product and using $\langle x, y\rangle = 0$, we get $\langle y, z\rangle \geqslant 0$ and $-\langle y, z\rangle \geqslant 0$, leading to $\langle y, z\rangle = 0$. However, this is a contradiction to $\langle y, z\rangle \neq 0$.

(2) This is the dual version of the first claim where replacing $\mathcal{C}$ with $\mathcal{C}^\star$ leads to $\operatorname{ri}\mathcal{C}^\star \not\perp (\mathcal{C}^\star)^\star \setminus (\mathcal{C}^\star)^\perp$, and using the Bipolar Theorem and Lemma 2.3.

(3) The second claim implies that $\operatorname{ri}\mathcal{C}^\star \cap \mathcal{C}^\perp = \emptyset$. Because $\mathcal{C}^\perp \subseteq \mathcal{C}^\star$ and $\mathcal{C}^\star$ is the disjoint union $\operatorname{ri}\mathcal{C}^\star \cup \partial\mathcal{C}^\star$, it follows that $\mathcal{C}^\perp \subseteq \partial\mathcal{C}^\star$. Substituting $\mathcal{C}$ with $\mathcal{C}^\star$ and using the Bipolar Theorem and Lemma 2.3 transforms $\mathcal{C}^\perp \subseteq \partial\mathcal{C}^\star$ to $\operatorname{lin}\mathcal{C} \subseteq \partial\mathcal{C}$.

(4) Follows from the third claim due to $\mathbf{0} \in \operatorname{lin}\mathcal{C} \cap \mathcal{C}^\perp$, $\operatorname{ri}\mathcal{C} \cap \partial\mathcal{C} = \emptyset$ and $\operatorname{ri}\mathcal{C}^\star \cap \partial\mathcal{C}^\star = \emptyset$. □

Rockafellar [Roc70, Corollary 14.5.1] tells us that any closed convex set containing $\mathbf{0}$ is unbounded if and only if $\mathbf{0}$ is not in the interior of its polar set, and because a cone $\mathcal{C} \neq \{\mathbf{0}\}$ is unbounded, we obtain $\mathbf{0} \notin \operatorname{int}\mathcal{C}^\star$. However, this is applicable only when $\mathcal{C}^\star$ is full-dimensional, or equivalently when $\mathcal{C}$ is pointed. Lemma 2.4 subsumes the case of pointed cone and gives the more general result that the origin is not in the *relative* interior of the cone and its dual cone.

Another property of the relative interior is that it is invariant to addition with the cone.

Lemma 2.5. *For any nonempty closed convex cone $\mathcal{C}$, we have $\operatorname{ri}\mathcal{C} + \mathcal{C} = \operatorname{ri}\mathcal{C}$.*

Proof. The $\supseteq$-inclusion is trivial since $\mathcal{C}$ being closed implies that $\mathbf{0} \in \mathcal{C}$. Take any $x \in \operatorname{ri}\mathcal{C}$. There exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subset \mathcal{C}$ where $N_\varepsilon(\cdot) \subset \operatorname{aff}\mathcal{C}$ is the ε-neighbourhood around a point. It is straightforward to verify that $N_\varepsilon(v+w) = N_\varepsilon(v) + w$ for any $v, w \in \operatorname{aff}\mathcal{C}$. Therefore, for any $y \in \mathcal{C}$ we have $N_\varepsilon(x+y) = N_\varepsilon(x) + y$, and because $N_\varepsilon(x) \subset \mathcal{C}$ and $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$ for convex cones, we get $N_\varepsilon(x+y) \subset \mathcal{C}$, implying that $x + y \in \operatorname{ri}\mathcal{C}$. □

Existence of preimage of relative interior of a cone under a linear map implies that the preimage exists for all affine maps that are a suitable shift of the linear map.

Lemma 2.6. *Let $\mathcal{C}$ be a closed convex cone. If a linear map $\mathcal{L}$ has $\mathcal{L}^{-1}(\mathtt{ri}\,\mathcal{C}) \neq \emptyset$, then $\mathcal{G}_v^{-1}(\mathtt{ri}\,\mathcal{C}) \neq \emptyset$ for the affine map $\mathcal{G}_v(\cdot) = v + \mathcal{L}(\cdot)$ with $v \in \mathtt{aff}\,\mathcal{C}$. On the contrary, if an affine map $\mathcal{G}$ has $\mathcal{G}^{-1}(\mathtt{ri}\,\mathcal{C}) \neq \emptyset$, then $\mathcal{G}_\lambda^{-1}(\mathtt{ri}\,\mathcal{C}) \neq \emptyset$ for all affine maps $\mathcal{G}_\lambda(\cdot) = (1-\lambda)\mathcal{G}(\mathbf{0}) + \lambda\mathcal{G}(\cdot)$ where $\lambda \neq 0$.*

Proof. Take any $x \in \mathcal{L}^{-1}(\mathtt{ri}\,\mathcal{C})$ and $v \in \mathtt{aff}\,\mathcal{C}$. Then, $\mathcal{L}(x) \in \mathtt{ri}\,\mathcal{C}$ implies that there exists some $\varepsilon > 0$ for which $\mathcal{L}(x)+\varepsilon v \in \mathtt{ri}\,\mathcal{C}$. Scaling by ε and using the fact that $\mathtt{ri}\,\mathcal{C}$ is an open convex cone, gives us $\mathcal{L}(x/\varepsilon)+v \in \mathtt{ri}\,\mathcal{C}$. Hence, $\mathcal{G}_v(x/\varepsilon) \in \mathtt{ri}\,\mathcal{C}$, which implies that $x/\varepsilon \in \mathcal{G}_v^{-1}(\mathtt{ri}\,\mathcal{C})$. Now take $x \in \mathcal{G}^{-1}(\mathtt{ri}\,\mathcal{C})$. Denote $\mathcal{L}_{\mathcal{G}}(\cdot) = \mathcal{G}(\mathbf{0}) - \mathcal{G}(\cdot)$, which is a linear map. Then, $\mathcal{G}_\lambda(\cdot) = \mathcal{G}(\mathbf{0}) - \lambda\mathcal{L}_{\mathcal{G}}(\cdot)$. Because $\mathcal{G}(x) = \mathcal{G}(\mathbf{0}) - \mathcal{L}_{\mathcal{G}}(x) \in \mathtt{ri}\,\mathcal{C}$, linearity of $\mathcal{L}_{\mathcal{G}}$ implies that for any $\lambda \neq 0$, we have $\mathcal{G}(x) = \mathcal{G}(\mathbf{0}) - \lambda\mathcal{L}_{\mathcal{G}}(x/\lambda) = \mathcal{G}_\lambda(x/\lambda)$, and so $\mathcal{G}_\lambda^{-1}(\mathtt{ri}\,\mathcal{C}) \neq \emptyset$. □

For a convex cone $\mathcal{C}$ containing $\mathbf{0}$, the binary relation $\preccurlyeq_{\mathcal{C}}$ is reflexive, transitive, additive equivariant ($u \preccurlyeq_{\mathcal{C}} v, x \preccurlyeq_{\mathcal{C}} y$ implies $u + x \preccurlyeq_{\mathcal{C}} v + y$), and distributes over the Cartesian product of cones, i.e., $(x,u) \preccurlyeq_{\mathcal{C}\times\mathcal{C}'} (y,v)$ if and only if $x \preccurlyeq_{\mathcal{C}} y$ and $u \preccurlyeq_{\mathcal{C}'} v$. It is also antisymmetric when the cone is pointed, thereby making $\preccurlyeq_{\mathcal{C}}$ a partial order. The strict relation $\prec_{\mathcal{C}}$ is not reflexive because $\mathbf{0} \notin \partial\mathcal{C}$.

2.2 Basic Results on Conic Optimization

Denote the primal and dual feasible sets by

$$X = X(b) \coloneqq \{x \in C : \mathcal{A}x \preccurlyeq_K b\} = \{x \in C : \exists s \in K \text{ s.t. } s = b - \mathcal{A}x\}, \tag{2a}$$

$$Y = Y(c) \coloneqq \left\{y \in K^\star : \mathcal{A}^\star y \succcurlyeq_{C^\star} c\right\} = \left\{y \in K^\star : \exists w \in C^\star \text{ s.t. } w = \mathcal{A}^\star y - c\right\}. \tag{2b}$$

Note that $X(b) = C \cap G_p^{-1}(K)$ for $G_p \colon x \mapsto b - \mathcal{A}x$, and $Y(c) = K^\star \cap G_d^{-1}(C^\star)$ for $G_d \colon y \mapsto \mathcal{A}^\star y - c$. Let us also define the following two convex cones

$$\mathcal{C}_p \coloneqq \mathcal{A}(C) + K, \qquad \mathcal{C}_d \coloneqq \mathcal{A}^\star(K^\star) - C^\star. \tag{3}$$

These characterize the feasibility of the primal and dual.

Lemma 2.7. *$\mathcal{C}_p = \left\{b \in \mathbf{E}' : X(b) \neq \emptyset\right\}$ and $\mathcal{C}_d = \left\{c \in \mathbf{E} : Y(c) \neq \emptyset\right\}$. Consequently, we have the following necessary conditions for feasibility:*

1. *If $\mathcal{A}(C) \subseteq \mathtt{aff}\,K$, $X(b)$ is feasible only if $b \in \mathtt{aff}\,K$,*
2. *If $K \subseteq \mathcal{A}(\mathtt{aff}\,C)$, $X(b)$ is feasible only if $b \in \mathcal{A}(\mathtt{aff}\,C)$,*
3. *If $\mathcal{A}^\star(K^\star) \subseteq (\mathtt{lin}\,\mathcal{C})^\perp$, $Y(c)$ is feasible only if $c \in (\mathtt{lin}\,\mathcal{C})^\perp$,*
4. *If $C^\star \subseteq \mathcal{A}^\star((\mathtt{lin}\,K)^\perp)$, $Y(c)$ is feasible only if $c \in \mathcal{A}^\star((\mathtt{lin}\,K)^\perp)$.*

Proof. The equalities for $\mathcal{C}_p$ and $\mathcal{C}_d$ are straightforward from their definitions and equations (2). Because $\mathtt{aff}\,K + K = \mathtt{aff}\,K$, the assumption $\mathcal{A}(C) \subseteq \mathtt{aff}\,K$ implies that $\mathcal{C}_p \subset \mathtt{aff}\,K$. If $X(b)$ is feasible, then $b \in \mathcal{C}_p$ from the first part of this proof, and hence $b \in \mathtt{aff}\,K$. The second condition uses the fact that $\mathtt{aff}\,\mathcal{A}(C) = \mathcal{A}(\mathtt{aff}\,C)$. The third and fourth conditions for Y can be argued similarly as the first two conditions and using Lemma 2.3. □

By convention, we say that the primal optimum z_P^* from (1a) is equal to $-\infty$ if the primal is infeasible and $z_P^* = +\infty$ if the primal is unbounded, which necessarily means that the feasible set X is unbounded. For the dual, infeasibility is $z_D^* = +\infty$ and unboundedness is $z_D^* = -\infty$. Assuming feasibility, note that $z_P^* = +\infty$ if $c \notin (\mathtt{lin}\, X)^\perp$ and $z_D^* = -\infty$ if $b \notin (\mathtt{lin}\, Y)^\perp$. These conditions are trivially satisfied when the respective feasible sets are pointed, but they are not necessary for unboundedness. Because $(\mathtt{lin}\, X)^\perp = (\mathtt{lin}\, X)^\star \cap (\mathtt{lin}\, X)^\circ \subset (0^+X)^\circ$ and similarly $(\mathtt{lin}\, Y)^\perp \subset (0^+Y)^\circ$, another sufficient condition for unboundedness that is also not necessary in general is $c \notin (0^+X)^\circ$ and $b \notin (0^+Y)^\circ$, respectively. Trivial cases for computing z_P^* and z_D^* would be when the objective functions are constant-valued over the respective feasible sets. For the primal, $\langle c,\, x\rangle$ is constant-valued over X if and only if there exists some $u \in \mathbf{E}$ such that $u + c^\perp \supset X$, and for the dual, $\langle b,\, y\rangle$ is constant-valued over Y if and only if there exists some $u \in \mathbf{E}'$ such that $u + b^\perp \supset Y$. If the inner product is not constant-valued over a feasible set, then the optimum can be approximated arbitrarily well, i.e., for small $\varepsilon > 0$ there exists an $x \in X$ with $\langle c,\, x\rangle \geq z_P^* - \varepsilon$ (ε-suboptimal solution), and similarly for the dual.

Weak duality always holds between the primal and dual optima regardless of dimensionality or pointedness of the cones, because it is a consequence of elementary linear algebra: for any $(x, y) \in X \times Y$,

$$\langle c,\, x\rangle \;=\; \left\langle \mathcal{A}^\star(y) - w,\, x\right\rangle \;=\; \langle \mathcal{A}x,\, y\rangle - \langle w,\, x\rangle \;=\; \langle b,\, y\rangle - \langle s,\, y\rangle - \langle w,\, x\rangle \leqslant \langle b,\, y\rangle\,, \tag{4}$$

where the inequality is because of $(s, y) \in K \times K^\star$ and $(w, x) \in C^\star \times C$. This implies two things — (i) if one problem is unbounded, then the other problem is infeasible, (ii) if both problems are feasible, then they are both finite-valued. Unlike linear programming, feasibility and boundedness of one problem does not imply strong duality or even feasibility of the other problem. In particular, one problem can be finite-valued but the other problem is infeasible [BN01, Example 1.7.2], and it can also happen that both problems are finite-valued but their respective optima are not equal [DW17, Example 2.3.2]. In these examples, both the primal and dual lack strict feasibility, and we will see later in this paper that this condition guarantees equivalence between boundedness of one problem and feasibility of the other.

The duality gap is the difference $\langle c,\, x\rangle - \langle b,\, y\rangle$, and when this gap is zero and one of the two problems is solvable (has an optimum solution), then we say that *strong duality* holds. Finiteness of both z_P^* and z_D^* is a necessary condition for strong duality, whereas common sufficient conditions for strong duality are related to strict feasibility of the constraints and are referred to as constraint qualifications. Associated with strong duality is the notion of complementary slackness for a primal-dual pair of solutions. This does not depend on any specific sufficient condition for strong duality. In particular, if strong duality holds, regardless of the condition on the problem data under which strong duality holds, the following statements are equivalent for any $(x, y) \in X \times Y$:

1. x is optimal to the primal and y is optimal to the dual,
2. $\langle c,\, x\rangle = \langle b,\, y\rangle$,
3. $\langle y,\, b - \mathcal{A}x\rangle = 0$ and $\left\langle x,\, \mathcal{A}^\star(y) - c\right\rangle = 0$.

The equivalence of the first two statements is obvious and the third statement is straightforward to verify. It is well-known that unlike linear programming, strict complementary slackness does not hold in general (cf. [TW12, §5]).

The primal-dual pair of problems are symmetric. This can be seen in two ways. One is that taking the conic dual of the dual yields the primal. Another is that both these problems can be represented as maximization problems in a subspace form [BN01]. Primal-dual symmetry means that results proved for one problem can directly be extended to the other problem. Given this fact, henceforth, we generally prove our results only for the primal problem and omit analogous statements/proofs for the dual.

A final preliminary to note is that if we are given a basis for the span of C, then instead of writing the dual problem using the adjoint of $\mathcal{A}$, we can write it using a linear map that depends on the basis.

Proposition 2.8. *Let $B = \{v^1, \ldots, v^m\}$ be an orthonormal basis of $\mathtt{aff}\, C$ and denote the linear map $\mathcal{B}\colon y \in \mathbf{E}' \mapsto \sum_{j=1}^m \langle \mathcal{A}v^j,\, y\rangle\, v^j \in \mathtt{aff}\, C$. We have $z_D^* = \mathtt{inf}\{\langle b,\, y\rangle : \mathcal{B}y \succcurlyeq_{C^\star} c,\ y \in K^\star\}$.*

Proof. Any $x \in C$ can be written as $x = \sum_{i=1}^m \alpha_i v^i$ for some $\alpha \in \mathbb{R}^m$. We have

$$\begin{aligned}\langle x,\, \mathcal{B}y\rangle_{\mathbf{E}} = \left\langle \sum_{i=1}^m \alpha_i v^i,\, \sum_{j=1}^m \left\langle \mathcal{A}v^j,\, y\right\rangle_{\mathbf{E}'} v^j \right\rangle_{\mathbf{E}} &= \sum_{i=1}^m\sum_{j=1}^m \alpha_i \left\langle \mathcal{A}v^j,\, y\right\rangle_{\mathbf{E}'} \left\langle v^i,\, v^j\right\rangle_{\mathbf{E}} \\ &= \sum_{j=1}^m \alpha_j \left\langle \mathcal{A}v^j,\, y\right\rangle_{\mathbf{E}'} = \left\langle \mathcal{A}(\sum_{j=1}^m \alpha_j v^j),\, y\right\rangle_{\mathbf{E}'},\end{aligned}$$

where the penultimate equality is because of orthonormality of the basis. Thus,

$$\langle \mathcal{A}x,\, y\rangle_{\mathbf{E}'} = \langle x,\, \mathcal{B}y\rangle_{\mathbf{E}}, \quad x \in C, y \in \mathbf{E}'. \tag{5}$$

Since z_D^* is the Lagrangian dual of the primal, we have $z_D^* = \mathtt{inf}_{y\in K^\star}\,\mathtt{sup}_{x\in C}\,\langle c,\, x\rangle_{\mathbf{E}} + \langle y,\, b - \mathcal{A}x\rangle_{\mathbf{E}'}$. The inner maximization objective becomes $\langle b,\, y\rangle_{\mathbf{E}'} + \langle c,\, x\rangle_{\mathbf{E}} - \langle y,\, \mathcal{A}x\rangle_{\mathbf{E}'}$, and then equation (5) transforms this to $\langle b,\, y\rangle_{\mathbf{E}'} + \langle c - \mathcal{B}y,\, x\rangle_{\mathbf{E}}$. Hence, the inner maximum is finite (and equal to zero) if and only if $c - \mathcal{B}y \in C^\circ$, which is the dual constraint $\mathcal{B}y \succcurlyeq_{C^\star} c$. □

This implies that any dual conditions stated in terms of the adjoint $\mathcal{A}^\star$ can also be stated in terms of the linear map $\mathcal{B}$ using a orthonormal basis B. An analogous statement holds for the primal if we are given an orthonormal basis for $(\mathtt{lin}\, K)^\perp$.

3 Closedness Condition

This section establishes conic programming strong duality under a closedness condition. Consider the linear maps

$$\mathcal{L}_p\colon (\alpha, \alpha_0) \in \mathbf{E}\times\mathbb{R} \mapsto (\mathcal{A}\alpha + \alpha_0 b,\, -\alpha) \in \mathbf{E}'\times\mathbf{E}, \tag{6a}$$

$$\mathcal{L}_d\colon (\beta, \beta_0) \in \mathbf{E}'\times\mathbb{R} \mapsto (\mathcal{A}^\star\beta + \beta_0 c,\, \beta) \in \mathbf{E}\times\mathbf{E}'. \tag{6b}$$

The adjoints of these linear maps are

$$\mathcal{L}_p^\star\colon (y, w) \mapsto (\mathcal{A}^\star y - w,\, \langle b,\, y\rangle), \qquad \mathcal{L}_d^\star\colon (x, s) \mapsto (\mathcal{A}x + s,\, \langle c,\, x\rangle). \tag{6c}$$

Theorem 3.1. *Suppose both the primal and dual are feasible. Strong duality holds if either $\mathcal{L}_p^\star(K^\star\times C^\star)$ or $\mathcal{L}_d^\star(C\times K)$ is a closed set, with the dual being solvable in the first condition and the primal being solvable in the second condition.*

This theorem is proved in §3.2 after we have established a sufficient condition for upper bounding the duality gap in §3.1. Before getting to these, let us first discuss the significance of the linear maps in (6) and the assumptions on closedness.

Denote the perspective map by $\mathcal{P} : (\alpha, \alpha_0) \in \mathbf{E}\times\mathbb{R}\setminus\{0\} \mapsto \alpha/\alpha_0 \in \mathbf{E}$. The perspective image of the preimage of a cone under the linear map $\mathcal{L}_p$ is exactly the the preimage of a cone under the affine map $G\colon x \mapsto (b - \mathcal{A}x, x)$.

Lemma 3.2. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two nonempty convex cones and denote $S_1 = \{(\alpha, \alpha_0) \in \mathcal{L}_p^{-1}(\mathcal{C}' \times \mathcal{C}) : \alpha_0 > 0\}$ and $S_2 = G^{-1}(\mathcal{C}' \times \mathcal{C})$. We have $-\mathcal{P}(S_1) = S_2$.*

Proof. Take any $(\alpha, \alpha_0) \in S_1$. We have $-\alpha/\alpha_0 \in \mathcal{C}$ due to $-\alpha \in \mathcal{C}$ and $\alpha_0 > 0$. Also, $\alpha_0(b - (\mathcal{A}(-\alpha/\alpha_0))) = \mathcal{A}\alpha + \alpha_0 b \in \mathcal{C}'$ and $\alpha_0 > 0$ implies that $b - (\mathcal{A}(-\alpha/\alpha_0)) \in \mathcal{C}'$. Hence, $-\mathcal{P}(S_1) \subseteq S_2$. The reverse inclusion $-\mathcal{P}(S_1) \supseteq S_2$ follows from $G(x) = \mathcal{L}_p(-x, 1)$. □

Because $X = G^{-1}(K \times C)$, a consequence of this lemma is that X is the negative perspective image of the preimage of $K \times C$ under the map $\mathcal{L}_p$.

The convex cones $\mathcal{L}_p^\star(K^\star \times C^\star)$ and $\mathcal{L}_d^\star(C \times K)$ may not be closed sets in general because linear images of closed sets are not closed, see Lemma 5.2 for some sufficient conditions. Closedness of $\mathcal{L}_p^\star(K^\star \times C^\star)$ is referred to as the dual condition and closedness of $\mathcal{L}_d^\star(C \times K)$ is the primal condition. Note that if $(c, z) \in \mathcal{L}_p^\star(K^\star \times C^\star)$, then it must be that $c \in \mathcal{C}_d$ and $z \geqslant z_D^*$; this is because of the definition of $\mathcal{C}_d$ in (3) and $z_D^* = \inf\{\langle b, y\rangle : c \in \mathcal{C}_d\}$. The point (c, z_D^*) may not necessarily belong to $\mathcal{L}_p^\star(K^\star \times C^\star)$, meaning that the dual problem need not be solvable. However, taking closure includes this point.

Lemma 3.3. *If z_P^* is finite, then $(b, z_P^*) \in \operatorname{cl} \mathcal{L}_d^\star(C \times K)$, and if z_D^* is finite, then $(c, z_D^*) \in \operatorname{cl} \mathcal{L}_p^\star(K^\star \times C^\star)$.*

Proof. The dual value is $z_D^* = \inf_y\{\langle b, y\rangle : y \in Y(c)\}$. Adding an auxiliary variable t for the objective extends the dual problem to the (y,t)-space as $z_D^* = \inf_{y,t}\{t\colon y \in Y(c), \langle b, y\rangle - t = 0\}$. Because $y \in Y(c)$ is equivalent to $c \in \mathcal{C}_d$, we get that (y, t) is feasible if and only if $(c, t) \in \mathcal{L}_p^\star(K^\star \times C^\star)$. Therefore, the dual problem is $z_D^* = \inf_t\{t\colon (c, t) \in \mathcal{L}_p^\star(K^\star \times C^\star)\}$. The infimum of a linear function over a set is equal to the infimum over the closure of the set, and so $z_D^* = \inf\{t\colon (c, t) \in \operatorname{cl} \mathcal{L}_p^\star(K^\star \times C^\star)\}$. Let $S = \operatorname{cl} \mathcal{L}_p^\star(K^\star \times C^\star)$ and $I = \{t \in \mathbb{R}\colon (c, t) \in S\}$, so that $z_D^* = \inf\{t\colon t \in I\}$. Because S is a closed set, it is easy to see that I is a closed interval in $\mathbb{R}$. The finiteness of z_D^* implies that the infimum over I is achieved, which means that $z_D^* \in I$ and hence $(c, z_D^*) \in S$. □

A related question is knowing whether points with arbitrarily close values to the optimal objective belong to the adjoint images of the cones or their closures. These are ε-proximal points of the form $(b, z_P^* + \varepsilon)$ or $(c, z_D^* - \varepsilon)$. If the problems are solvable, then these points do not belong to the adjoint images.

Lemma 3.4. *If the primal is solvable, then $(b, z_P^* + \varepsilon) \notin \mathcal{L}_d^\star(C \times K)$, and if the dual is solvable, then $(c, z_D^* - \varepsilon) \notin \mathcal{L}_p^\star(K^\star \times C^\star)$, for any $\varepsilon > 0$.*

Proof. Solvability of the primal is $(b, z_P^*) \in \mathcal{L}_d^\star(C \times K)$ and solvability of the dual is $(c, z_D^*) \in \mathcal{L}_p^\star(K^\star \times C^\star)$, and so the claim is obvious because otherwise we would have a contradiction to the optimality of the solution. □

However, this does not mean that the proximal points can also be separated from the respective closures. In Proposition 3.5, we demonstrate this non-separability using an example that is a n-dimensional version of [BN01, Example 1.7.2], which showed infinite duality gap due to the lack of Slater CQ.

Proposition 3.5. *For every $n \geqslant 3$, there exists a primal-dual pair of conic problems for which $(c, z_D^* - \varepsilon) \in \operatorname{cl} \mathcal{L}_p^\star(K^\star \times C^\star)$ for every $\varepsilon > 0$.*

Proof. Let $A = \begin{bmatrix} I_{n-1} & e_1 \end{bmatrix}$ be an $(n-1) \times n$ matrix, where I_{n-1} is an identity matrix and e_1 is the column vector $(1, 0, \ldots, 0)^\top \in \mathbb{R}^{n-1}$, and $\mathcal{C}_n = \{x \in \mathbb{R}^n \colon x_n \geqslant \|(x_1, \ldots, x_{n-1})\|_2\}$ be the Lorentz cone in $\mathbb{R}^n$. Consider the primal-dual pair

$$z_P^* = \sup\{0 \colon Ax = \mathbf{1} - e_1,\ x \in \mathcal{C}_n\}, \quad z_D^* = \inf\left\{y_2 + \cdots + y_{n-1} \colon A^\top y \succcurlyeq_{\mathcal{C}_n} \mathbf{0},\ y \in \mathbb{R}^{n-1}\right\}.$$

With respect to the primal formulation in (1a) we have $K = \{\mathbf{0}\}$, $C = \mathcal{C}_n$, $b = \mathbf{1} - e_1$ and $c = \mathbf{0}$. The primal constraints $x_j = 1, j = 2, \ldots, n-1$ imply that for $x \in \mathcal{C}_n$, we must have $x_n \geqslant \sqrt{x_1^2 + n - 2}$, but then the first primal constraint $x_1 + x_n = 0$ makes the problem infeasible. Hence, $z_P^* = -\infty$. The dual is obviously feasible with $y = \mathbf{0}$. In fact, the dual optimum is $z_D^* = 0$ because $A^\top y = (y_1, \ldots, y_{n-1}, y_1)^\top$, and so every dual feasible solution has $y_1 \geqslant \|(y_1, \ldots, y_{n-1})\|_2$, which implies that $y_j = 0, j = 2, \ldots, n-1$.

The primal linear map and its adjoint from (6) are $\mathcal{L}_p(\alpha, \alpha_0) = (A\alpha + \alpha_0(\mathbf{1} - e_1), -\alpha)$ and $\mathcal{L}_p^\star(y, w) = (A^\top y - w, y_2 + \cdots + y_{n-1})$. Denote $S = \operatorname{cl} \mathcal{L}_p^\star(\{\mathbf{0}\}^\star \times \mathcal{C}_n^\star)$ and note that this is a closed convex cone. Because the Lorentz cone $\mathcal{C}_n$ is self-dual, we get that S is equal to $\operatorname{cl}\left\{(A^\top y - w, y_2 + \cdots + y_{n-1}) \colon y \in \mathbb{R}^{n-1}, w \in \mathcal{C}_n\right\}$. Take any $\varepsilon > 0$. By the separation theorem, the point $(\mathbf{0}, -\varepsilon)$, which is equal to $(c, z_D^* - \varepsilon)$, does not belong to S if and only if there exists $(\alpha, \alpha_0) \in S^\circ$ such that $\langle \mathbf{0}, \alpha \rangle - \varepsilon\alpha_0 > 0$. Suppose there exists $(\alpha, \alpha_0) \in S^\circ$ with $\varepsilon\alpha_0 < 0$. Lemma 3.6 tells us that $S^\circ = \mathcal{L}_p^{-1}(-\{\mathbf{0}\} \times -\mathcal{C}_n)$. Therefore, it must be that $\mathcal{L}_p(\alpha, \alpha_0) \in -\{\mathbf{0}\} \times -\mathcal{C}_n$, which implies that $A\alpha + \alpha_0(\mathbf{1} - e_1) = 0$ and $\alpha \in \mathcal{C}_n$. The linear equation can be scaled by $-\alpha_0$ to get $A\alpha' = \mathbf{1} - e_1$ for $\alpha' = -\alpha/\alpha_0$. Because $\alpha_0 < 0$ due to $\varepsilon > 0$ and $\varepsilon\alpha_0 < 0$, we have $\alpha' \in \mathcal{C}_n$ whenever $\alpha \in \mathcal{C}_n$. Hence, there must exist some $\alpha' \in \mathcal{C}_n$ for which $A\alpha' = \mathbf{1} - e_1$. However, this is exactly primal feasibility and we already argued that the primal is infeasible, therefore giving us a contradiction to the existence of α'. Hence, $(\mathbf{0}, -\varepsilon)$ cannot be separated from the closure of $\mathcal{L}_p^\star(\mathbb{R}^{n-1} \times \mathcal{C}_n)$ for any $\varepsilon > 0$. □

What we show in the next section is that being able to exclude from the closure such ε-proximal points guarantees a duality gap of at most ε. This further provides a recipe for arguing strong duality where we want the duality gap to be bounded by ε for arbitrarily small $\varepsilon > 0$.

3.1 Bounding the Duality Gap

We will need the following result about the polar of the preimage of a cone being equal to the closure of the adjoint image of the dual cone.

Lemma 3.6. *A linear map $\mathcal{L}$ and closed convex cone $\mathcal{C}$ satisfy $\big(\mathcal{L}^{-1}(-\mathcal{C})\big)^\circ = \operatorname{cl} \mathcal{L}^\star(\mathcal{C}^\star)$ and $\mathcal{L}^{-1}(-\mathcal{C}) = (\operatorname{cl} \mathcal{L}^\star(\mathcal{C}^\star))^\circ$.*

Proof. It is enough to argue the first identity, the second identity follows after applying the Bipolar Theorem.

($\supseteq$) It suffices to argue that $\big(\mathcal{L}^{-1}(-\mathcal{C})\big)^\circ \supseteq \mathcal{L}^\star(\mathcal{C}^\star)$; the containment of $\operatorname{cl} \mathcal{L}^\star(\mathcal{C}^\star)$ then follows from the polar being a closed set and the closure operator being inclusion-preserving. Take any $x \in \mathcal{L}^{-1}(-\mathcal{C})$ and $y = \mathcal{L}^\star(v)$ with $v \in \mathcal{C}^\star$. We have $\langle x, y \rangle = \big\langle x, \mathcal{L}^\star(v) \big\rangle = \big\langle \mathcal{L}(x), v \big\rangle \leqslant 0$, where the inequality is due to $v \in \mathcal{C}^\star$ and $\mathcal{L}(x) \in -\mathcal{C}$.

($\subseteq$) Take any $y \in \big(\mathcal{L}^{-1}(-\mathcal{C})\big)^\circ$ and suppose $y \notin \operatorname{cl} \mathcal{L}^\star(\mathcal{C}^\star)$. Because $\operatorname{cl} \mathcal{L}^\star(\mathcal{C}^\star)$ is a closed convex cone, the separation theorem tells us that there exists some $\alpha \in \big(\operatorname{cl} \mathcal{L}^\star(\mathcal{C}^\star)\big)^\circ$ such that $\langle \alpha, y \rangle > 0$. The anti-inclusion preserving property of polarity and $\operatorname{cl} \mathcal{L}^\star(\mathcal{C}^\star) \supseteq \mathcal{L}^\star(\mathcal{C}^\star)$ implies $\big(\operatorname{cl} \mathcal{L}^\star(\mathcal{C}^\star)\big)^\circ \subseteq \big(\mathcal{L}^\star(\mathcal{C}^\star)\big)^\circ$. Therefore, $\alpha \in \big(\mathcal{L}^\star(\mathcal{C}^\star)\big)^\circ$. The definition of polarity gives us $\langle \alpha, \mathcal{L}^\star(z)\rangle \leqslant 0$ for all $z \in \mathcal{C}^\star$. By the definition of the adjoint map, this is equivalent to $\langle -\mathcal{L}(\alpha), z\rangle \geqslant 0$ for all $z \in \mathcal{C}^\star$, which is equivalent to $-\mathcal{L}(\alpha) \in (\mathcal{C}^\star)^\star$. The Bipolar theorem tells us that $(\mathcal{C}^\star)^\star = \mathcal{C}$, and so we have $-\mathcal{L}(\alpha) \in \mathcal{C}$. But this means that $\alpha \in \mathcal{L}^{-1}(-\mathcal{C})$, a contradiction to $y \in \left(\mathcal{L}^{-1}(-\mathcal{C})\right)^\circ$ and $\langle \alpha, y\rangle > 0$, thereby completing our proof. □

The polar cone of $\mathcal{L}^{-1}(-\mathcal{C})$ can also be obtained from the Farkas lemma for conic linear systems (cf. [DJ14, Theorem 2.1]) which states that for any $c \in \mathbf{E}$, exactly one of the following holds: either $c \in \operatorname{cl} \mathcal{L}^\star(\mathcal{C}^\star)$ or there exists $y \in \mathbf{E}$ such that $\langle c, y\rangle > 0$ and $\mathcal{L}(y) \preccurlyeq_{\mathcal{C}} \mathbf{0}$. This conic Farkas lemma can be seen as a consequence of the separation theorem, and so the proof provided for Lemma 3.6, which also relies on the separation theorem, is more direct and also standalone.

Lemma 3.7. *For finite z_P^* and z_D^*, we have $z_D^* - z_P^* \leqslant \varepsilon$ for some $\varepsilon > 0$ if either $(c, z_D^* - \varepsilon) \notin \operatorname{cl} \mathcal{L}_p^\star(K^\star \times C^\star)$ or $(b, z_P^* + \varepsilon) \notin \operatorname{cl} \mathcal{L}_d^\star(C \times K)$.*

Proof. We give arguments for the point $(c, z_D^* - \varepsilon)$ and remark that the proof for $(b, z_P^* + \varepsilon)$ is analogous from primal-dual symmetry. For convenience, denote $S := \operatorname{cl} \mathcal{L}_p^\star(K^\star \times C^\star)$. Because the conic and convexity properties of a set are preserved under taking topological closure and linear image, S is a closed convex cone. By the separation theorem for cones, the point $(c, z_D^* - \varepsilon)$ not belonging to S means that there exists an $(\alpha, \alpha_0) \in S^\circ$ such that $\langle (c, z_D^* - \varepsilon), (\alpha, \alpha_0)\rangle_{\mathbf{E}\times\mathbb{R}} > 0$. The inner product distributes in an additive fashion over a direct product of Hilbert spaces, and so we get $\langle c, \alpha\rangle + (z_D^* - \varepsilon)\alpha_0 > 0$, which after rearranging terms becomes

$$\langle c, \alpha\rangle_{\mathbf{E}} + \alpha_0 z_D^* > \alpha_0 \varepsilon. \tag{7}$$

Lemma 3.6 gives us $S^\circ = \mathcal{L}_p^{-1}(-K \times -C)$, and so $\mathcal{L}_p(\alpha, \alpha_0) \in -K \times -C$.

For some $\delta \geqslant 0$ take any $y_\delta \in Y$ with $\langle b, y_\delta\rangle = z_D^* + \delta$. Such a point y_δ exists because z_D^* is finite and either y_δ is an optimal solution, so that $\delta = 0$, or y_δ is a δ-suboptimal solution. We have $\mathcal{A}^\star y_\delta - w = c$ for some $w \in C^\star$. Hence,

$$\begin{aligned}
\langle c, \alpha\rangle + \alpha_0 z_D^* &= \langle \mathcal{A}^\star y_\delta - w, \alpha\rangle + \alpha_0(\langle b, y_\delta\rangle - \delta) \\
&= \langle \mathcal{A}^\star y_\delta, \alpha\rangle - \langle w, \alpha\rangle + \alpha_0 \langle b, y_\delta\rangle - \alpha_0\delta \\
&= \langle y_\delta, \mathcal{A}\alpha + \alpha_0 b\rangle + \langle w, -\alpha\rangle - \alpha_0\delta \\
&= \langle (\mathcal{A}\alpha + \alpha_0 b, -\alpha), (y_\delta, w)\rangle - \alpha_0\delta \\
&= \langle \mathcal{L}_p(\alpha, \alpha_0), (y_\delta, w)\rangle - \alpha_0\delta \\
&\leqslant -\alpha_0\delta,
\end{aligned}$$

where the last inequality is due to $\mathcal{L}_p(\alpha, \alpha_0) \in -K \times -C = -(K \times C)$ and $(y_\delta, w) \in K^\star \times C^\star = (K \times C)^\star$. Therefore, (7) implies $\alpha_0\varepsilon < -\alpha_0\delta$, and because $\varepsilon > 0$ and $\delta \geqslant 0$, it follows that $\alpha_0 < 0$. Now, $\mathcal{L}_p(\alpha, \alpha_0) = (\mathcal{A}\alpha + \alpha_0 b, -\alpha) \in -K \times -C$ and dividing by $\alpha_0 < 0$ implies that $\frac{1}{\alpha_0}\mathcal{L}_p(\alpha, \alpha_0) \in K \times C$, which leads to $-\alpha/\alpha_0 \in X$. Dividing by $\alpha_0 < 0$ in (7) gives us $-\left\langle c, \frac{-\alpha}{\alpha_0}\right\rangle_{\mathbf{E}} + z_D^* < \varepsilon$, and then the primal feasibility of $-\alpha/\alpha_0$ implies that $-z_P^* + z_D^* < \varepsilon$. □

3.2 Proof of Theorem 3.1

By weak duality, feasibility of both problems implies finiteness of their optimal values. We argue sufficiency of the dual condition — closedness of $\mathcal{L}_p^\star(K^\star \times C^\star)$, and remark that the primal condition is analogous by primal-dual symmetry. Lemma 3.3 and the dual condition imply $(c, z_D^*) \in \mathcal{L}_p^\star(K^\star \times C^\star)$. This means that the dual is solvable. Then, Lemma 3.4 gives us $(c, z_D^* - \varepsilon) \notin \mathcal{L}_p^\star(K^\star \times C^\star)$ for all $\varepsilon > 0$. Lemma 3.7 implies $z_D^* - z_P^* \leqslant \varepsilon$ and because this is for all $\varepsilon > 0$, we have $z_D^* - z_P^* \leqslant 0$. Weak duality gives us $z_D^* - z_P^* \geqslant 0$. Hence, $z_D^* - z_P^* = 0$. □

4 Strict Feasibility

The set of *relative* strictly feasible solutions to the primal problem is

$$\mathtt{strict}\, X \;=\; \mathtt{strict}\, X(b) \;\coloneqq\; \{x \in \mathtt{ri}\, C : \; \mathcal{A}x \prec_K b\}\,, \tag{8a}$$

and the set of relative strictly feasible solutions to the dual problem is

$$\mathtt{strict}\, Y \;=\; \mathtt{strict}\, Y(c) \;\coloneqq\; \left\{y \in \mathtt{ri}\, K^\star : \; \mathcal{A}^\star y \succ_{C^\star} c\right\}. \tag{8b}$$

These definitions are exactly the same as those for the full-dimensional case up to replacing $\mathtt{int}$ with $\mathtt{ri}$ in the definitions for $\prec_K$ and $\succ_K$. The ordinary Slater's condition for conic programs is the existence of $x \in \mathtt{int}\, C$ such that $b - \mathcal{A}x \in \mathtt{int}\, K$, and so the existence of relative strictly feasible solutions is a *generalized Slater's condition.* Henceforth, we refer to it simply as strict feasibility and do not distinguish between the full-dimensional and low-dimensional cases.

The notion of strictly feasible solutions is algebraic in its nature because it depends on the algebraic representation of the set, and so the same set represented in two different ways could have strictly feasible solutions for one representation but not the other. For example, $X = \{x \in \mathbb{R}^n_+ : a^\top x \leqslant 1, -a^\top x \leqslant -1\}$, for some vector $a > \mathbf{0}$, has $\mathtt{strict}\, X = \emptyset$, but writing the same set as $X = \{x \in \mathbb{R}^n_+ : a^\top x = 1\}$ gives $\mathtt{strict}\, X = \{x > \mathbf{0} : a^\top x = 1\}$. The relative interior of a set is a topological concept, thus differing from the algebraic concept of strict feasibility, and is nonempty for a nonempty convex set. When strict feasible solutions do exist, they are indeed exactly the points in the relative interior of the set, as seen in the above example of X.

Proposition 4.1. *For $X \neq \emptyset$, $\mathtt{ri}\, X = \mathtt{strict}\, X$ if and only if $\mathtt{strict}\, X \neq \emptyset$.*

Proof. The only if direction is due to the feasibility of a convex set being equivalent to the feasibility of its relative interior. Now suppose $\mathtt{strict}\, X \neq \emptyset$. The strict conic inequality $\mathcal{A}x \prec_K b$ is defined as $b - \mathcal{A}x \in \mathtt{ri}\, K$, which is equivalent to $x \in \mathcal{A}^{-1}(b - \mathtt{ri}\, K)$. We will use the properties from Lemma 2.1. The map $K \mapsto b - K$ is affine and then commutativity of an affine map with $\mathtt{ri}$ implies that $b - \mathtt{ri}\, K = \mathtt{ri}\,(b - K)$, leading to $\mathcal{A}x \prec_K b$ if and only if $x \in \mathcal{A}^{-1}(\mathtt{ri}\,(b - K))$. Thus, $\mathtt{strict}\, X = \mathcal{A}^{-1}(\mathtt{ri}\,(b - K)) \cap \mathtt{ri}\, C$, and so $\mathcal{A}^{-1}(\mathtt{ri}\,(b - K))$ is nonempty. A nonempty convex set S and affine map $\mathcal{G}$ have $\mathcal{G}^{-1}(\mathtt{ri}\, S) \subseteq \mathtt{ri}\, \mathcal{G}^{-1}(S)$, and the two sets are equal when $\mathcal{G}^{-1}(\mathtt{ri}\, S) \neq \emptyset$. Hence, $\mathtt{strict}\, X = \mathtt{ri}\, \mathcal{A}^{-1}(b - K) \cap \mathtt{ri}\, C$. Because $X = \mathcal{A}^{-1}(b - K) \cap C$, the distributivity of $\mathtt{ri}$ over nonempty intersection implies that $\mathtt{ri}\, X = \mathtt{ri}\, \mathcal{A}^{-1}(b - K) \cap \mathtt{ri}\, C$, giving us the desired equality $\mathtt{ri}\, X = \mathtt{strict}\, X$. □

Analogous to the feasibility cones characterizing feasibility with respect to the right-hand side (cf. Lemma 2.7), we have that the relative interior of each feasibility cone characterizes strict feasibility of the corresponding problem.

Proposition 4.2. $\mathtt{ri}\, \mathcal{C}_p = \{b \in \mathbf{E}' : \; \mathtt{strict}\, X(b) \neq \emptyset\}$.

Proof. This follows from basic properties of $\mathtt{ri}$. Distributivity from Lemma 2.2 gives us $\mathtt{ri}\, \mathcal{C}_p = \mathtt{ri}\, \mathcal{A}(C) + \mathtt{ri}\, K$, and commutativity with a linear map from Lemma 2.1 leads to $\mathtt{ri}\, \mathcal{C}_p = \mathcal{A}(\mathtt{ri}\, C) + \mathtt{ri}\, K$. The definition of strict feasibility makes it easy to see that $\mathtt{strict}\, X(b) \neq \emptyset$ if and only if $b \in \mathcal{A}(\mathtt{ri}\, C) + \mathtt{ri}\, K$. □

The definition of strict feasibility tells us that strictly feasible solutions exist only when $X \cap \mathtt{ri}\, C \neq \emptyset$ and so $X \not\subset \partial C$ is necessary for their existence. But this is far from being a sufficient condition. The example $X = \{x \in \mathbb{R}^n_+ \colon a^\top x \leqslant 1,\, -a^\top x \leqslant -1\}$ mentioned earlier satisfies $X \not\subset \cup_{j=1}^n \{x \colon x_j = 0\} = \partial \mathbb{R}^n_+$, but it does not have any strictly feasible solutions because the slacks of the two inequality constraints are negative of each other, implying that X is contained in the subspace formed by the slack variables equal to zero. We show that the slacks of conic constraints forming a full-dimensional set is a sufficient condition for strict feasibility, and a necessary condition, under a technicality, is that X have the same dimension as C. Define the set of slack values for X as $\mathtt{slack}\, X := b - \mathcal{A}(X) = \{b - \mathcal{A}x : x \in X\}$ and let $\mathtt{dim} \cdot$ denote the affine dimension of a set.

Proposition 4.3. *1.* $\mathtt{strict}\, X \neq \emptyset$ *if* $\mathtt{aff}(\mathtt{slack}\, X) = \mathtt{aff}\, K$ *and* $\emptyset \neq X \not\subset \partial C$.

2. When $\mathcal{A}(\mathtt{aff}\, C) \subseteq \mathtt{aff}\, K$, $\mathtt{strict}\, X \neq \emptyset$ *only if* $\mathtt{dim}\, X = \mathtt{dim}\, C$.

Proof. (1) For a nonempty convex set X, we have that $\mathtt{slack}\, X$, which is the affine image $b - \mathcal{A}(X)$, is also a nonempty convex set, and therefore $\mathtt{ri}\,(\mathtt{slack}\, X) \neq \emptyset$. Because $\mathtt{slack}\, X \subset K$, the assumption of equal affine hulls for K and $\mathtt{slack}\, X$ implies that $\mathtt{ri}\,(\mathtt{slack}\, X) \subseteq \mathtt{ri}\, K$. The commutativity of $\mathtt{ri}$ and affine images of convex sets gives us $\mathtt{ri}\,(\mathtt{slack}\, X) = b - \mathcal{A}(\mathtt{ri}\, X)$. Hence, $b - \mathcal{A}(\mathtt{ri}\, X) \subseteq \mathtt{ri}\, K$. When $X \not\subset \partial C$, we have $\mathtt{ri}\, X \cap \mathtt{ri}\, C \neq \emptyset$ because otherwise $X \subset C = \mathtt{ri}\, C \cup \partial C$, $\mathtt{cl}\,(\mathtt{ri}\, X) = X$, and ∂C being a closed set implies the contradiction $X \subset \partial C$. Therefore, there exists some $x \in \mathtt{ri}\, C$ with $b - \mathcal{A}x \in \mathtt{ri}\, K$, and hence $\mathtt{strict}\, X \neq \emptyset$.

(2) Clearly, $\mathtt{dim}\, X \leqslant \mathtt{dim}\, C$ because $X \subseteq C$. Suppose $\mathtt{strict}\, X \neq \emptyset$. Take any $x \in \mathtt{strict}\, X$ and $y \in \mathtt{aff}\, C$. We have $\mathcal{A}y \in \mathtt{aff}\, K$ due to the assumption that $\mathcal{A}(\mathtt{aff}\, C) \subseteq \mathtt{aff}\, K$. Because $\mathtt{strict}\, X = \mathcal{A}^{-1}(b - \mathtt{ri}\, K) \cap \mathtt{ri}\, C$, there exists some $\varepsilon > 0$ such that $x + \varepsilon y \in C$ and $b - \mathcal{A}x + \varepsilon \mathcal{A}y \in K$. The linearity of $\mathcal{A}$ makes $b - \mathcal{A}x + \varepsilon \mathcal{A}y \in K$ equivalent to $b - \mathcal{A}(x + \varepsilon y) \in K$, and so $x + \varepsilon y \in X$. Therefore, for any basis $\{y^1, \ldots, y^k\}$ of the subspace $\mathtt{aff}\, C$, the points $\{x, x + \varepsilon y^1, \ldots, x + \varepsilon y^k\}$ are affinely independent in X, implying that $\mathtt{dim}\, X \geqslant \mathtt{dim}\, C$, as desired. □

This leads to a case where full-dimensionality of X is necessary and sufficient for the existence of strictly feasible solutions. We will need the basic fact that taking a linear image is a dimension-reducing operation.

Lemma 4.4. *A linear map* $\mathcal{L}$ *and set* S *have* $\mathtt{dim}\, \mathcal{L}(S) \leqslant \mathtt{dim}\, S$, *with equality holding when* $\mathcal{L}$ *is injective.*

Corollary 4.5. *Suppose* C *is full-dimensional and* $\mathcal{A}$ *is an injective map with* $\mathtt{Im}\, \mathcal{A} \subseteq \mathtt{aff}\, K$. *Then* $\mathtt{strict}\, X \neq \emptyset$ *if and only if* $X \not\subset \partial C$ *and* X *is a full-dimensional set.*

Proof. The only if direction is directly from the second claim in Proposition 4.3, whereas the if direction is from the first claim in the proposition. To see the if direction, suppose X is a full-dimensional set. We have $\mathtt{aff}(\mathtt{slack}\, X) = \mathtt{aff}(b - \mathcal{A}(X)) = b - \mathtt{aff}\, \mathcal{A}(X) = b - \mathcal{A}(\mathtt{aff}\, X)$, which implies $\mathtt{dim}(\mathtt{slack}\, X) = \mathtt{dim}\, \mathcal{A}(\mathtt{aff}\, X)$, and so full-dimensionality of X and Lemma 4.4 imply that $\mathtt{slack}\, X$ is also full-dimensional. It follows from $\mathtt{slack}\, X \subset K$ that $\mathtt{aff}(\mathtt{slack}\, X) = \mathtt{aff}\, K = \mathbf{E}'$. □

A sufficient condition for $X(b)$ to be feasible, in fact strictly feasible, for arbitrary b is that the set $X(\mathbf{0}) := \{x \in C \colon \mathcal{A}x \preccurlyeq_K \mathbf{0}\}$ be strictly feasible.

Proposition 4.6. *If* $\mathtt{strict}\, X(\mathbf{0}) \neq \emptyset$, *then* $\mathtt{strict}\, X(b) \neq \emptyset$ *for every* $b \in \mathtt{aff}\, K$.

Proof. $X(\mathbf{0})$ is the preimage of the cone $K \times C$ under the linear map $\mathcal{L}\colon x \mapsto (-\mathcal{A}x, x)$, and $X(b)$ is the preimage of $K \times C$ under the affine map $(b, \mathbf{0}) + \mathcal{L}(x)$. The definition of strict feasibility is that $\mathtt{strict}\, X(\mathbf{0}) = \mathcal{L}^{-1}(\mathtt{ri}\, K \times \mathtt{ri}\, C)$. The claim $\mathtt{strict}\, X(b) \neq \emptyset$ follows from Lemma 2.6 due to $(b, \mathbf{0}) \in \mathtt{aff}\, K \times \mathtt{aff}\, C$. □

5 Specific Conditions for Strong Duality

We begin by noting a special case that does not require any constraint qualification, if the objective vectors belong to specific parts of feasibility cones of the other problem.

Proposition 5.1. *If the primal (resp. dual) is feasible, then strong duality holds when $c \in \mathcal{A}^\star(K^\perp)$ (resp. $b \in \mathcal{A}(\mathtt{lin}\, C)$).*

Proof. Let $c = \mathcal{A}^\star(y)$ for some $y \in K^\perp$. Because $\mathcal{A}^\star(K^\perp) \subseteq \mathcal{A}^\star(K^\star) \subseteq \mathcal{C}_d$, we have $c \in \mathcal{C}_d$ and hence the dual is feasible with $y \in Y$. Every $x \in X$ has $\mathcal{A}x + s = b$ for $s \in K$. Then, $\langle c, x\rangle = \langle \mathcal{A}^\star(y), x\rangle = \langle y, \mathcal{A}x\rangle = \langle y, b - s\rangle = \langle y, b\rangle - \langle y, s\rangle = \langle y, b\rangle$, where the last equality is due to $y \in K^\perp$ and $s \in K$. Thus, there is zero duality gap and solvability of the dual. The arguments for the other condition $b \in \mathcal{A}(\mathtt{lin}\, C)$ are similar after using Lemma 2.3. □

Because $\mathbf{0} = \mathcal{A}^\star\mathbf{0} \in \mathcal{A}^\star(K^\perp)$ and $\mathbf{0} = \mathcal{A}\mathbf{0} \in \mathcal{A}(\mathtt{lin}\, C)$, it follows that a particular case of strong duality occurs if either $c = \mathbf{0}$ or $b = \mathbf{0}$,

Now we derive consequences of Theorem 3.1 and show how specific sufficient conditions emerge from the general closedness condition. Some of these conditions are related to imposing a constraint qualification in the problem. The basic ingredient of our derivation is known sufficient conditions for the closedness of the linear image of a cone. Because linear images of polyhedral cones are closed, we consider only the non-polyhedral case.

Lemma 5.2 ([Roc70; Pat07])**.** *Given a linear map $\mathcal{L}$ and a non-polyhedral closed convex cone $\mathcal{C}$, the linear image $\mathcal{L}^\star(\mathcal{C}^\star)$ is a closed set if any of the following conditions hold:*

1. $\mathtt{Im}\, \mathcal{L} \cap \mathtt{ri}\, \mathcal{C} \neq \emptyset$,
2. $\mathtt{ker}\, \mathcal{L}^\star \cap \mathtt{ri}\, \mathcal{C}^\star \neq \emptyset$,
3. $x \in \mathtt{lin}\, \mathcal{C}$ *for every* $x \in \mathcal{C} \cap \mathtt{Im}\, \mathcal{L}$,
4. $x \in \mathcal{C}^\perp$ *for every* $x \in \mathcal{C}^\star \cap \mathtt{ker}\, \mathcal{L}^\star$.

These four conditions on closedness imply four sufficient conditions for strong duality. First, we have the well-known condition of strict feasibility (cf. (8a) and (8b)), also referred to as generalized Slater CQ, which was directly proven by [BN01, Theorem 1.7.1] in the full-dimensional case, but now we see it as an immediate consequence of sufficiency of the closedness condition in the general case.

Corollary 5.3. *If both the primal and dual are feasible with one problem being strictly feasible, then strong duality holds and the other problem is solvable.*

Proof. Suppose that $\mathtt{strict}\, X \neq \emptyset$ and $Y \neq \emptyset$. Because $\mathtt{strict}\, X = G^{-1}(\mathtt{ri}\, K \times \mathtt{ri}\, C)$, applying Lemma 3.2 with $\mathcal{C}' = \mathtt{ri}\, K$ and $\mathcal{C} = \mathtt{ri}\, C$ gives us $\mathcal{L}_p^{-1}(\mathtt{ri}\, K \times \mathtt{ri}\, C) \neq \emptyset$. Polarity and $\mathtt{ri}$ operators distribute over the Cartesian product, and so $(K \times C)^\star = K^\star \times C^\star$ and $\mathtt{ri}\,(K \times C) = \mathtt{ri}\, K \times \mathtt{ri}\, C$. Lemma 5.2 with $\mathcal{C} = K \times C$ implies that $\mathcal{L}_p^\star(K^\star \times C^\star)$ is a closed set, and then Theorem 3.1 completes the proof. □

Next we give two sets of conditions that use strict feasibility of the recession cone instead of strict feasibility of the feasible set as in the previous corollary. This means generalized Slater CQ for the set defined by conic inequalities with right-hand side equal to zero (either $X(\mathbf{0})$ or $Y(\mathbf{0})$) because it is well-known, and formally established later in §6.1 along with other results on the recession cone, that such a homogenous system defines the recession cone of the corresponding feasible set.

Corollary 5.4. *Suppose that both the primal and dual are feasible. Strong duality holds when any of the following conditions hold:*

1. $\mathtt{strict}(0^+X) \neq \emptyset$ *and* $b \in \mathtt{aff}\, K$,
2. $\mathtt{strict}(0^+Y) \neq \emptyset$ *and* $c \in (\mathtt{lin}\, C)^\perp$,
3. $b^\perp \cap \mathtt{strict}(0^+Y) \neq \emptyset$,
4. $c^\perp \cap \mathtt{strict}(0^+X) \neq \emptyset$,

with the dual being solvable in the first and third conditions, and the primal being solvable in the second and fourth conditions.

Proof. It suffices to argue the first and third conditions, because the other two are symmetric analogues. Also note that $\mathtt{aff}\, C^\star = (\mathtt{lin}\, C)^\perp$ from Lemma 2.3.

(1) Follows from Proposition 4.6 and Corollary 5.3.

(3) By Theorem 3.1, it suffices to argue the closedness of $\mathcal{L}_p^\star(K^\star \times C^\star)$. The second condition in Lemma 5.2 tells us that $\mathcal{L}_p^\star(K^\star \times C^\star)$ is a closed set when $\mathtt{ker}\, \mathcal{L}_p^\star \cap (\mathtt{ri}\, K^\star \times \mathtt{ri}\, C^\star) \neq \emptyset$. This means that we have to argue there exists $y \in \mathtt{ri}\, K^\star$ and $w \in \mathtt{ri}\, C^\star$ such that $\mathcal{A}^\star y = w$ and $\langle b, y \rangle = 0$. Thus, we want some $y \in \mathtt{ri}\, K^\star$ with $\mathcal{A}^\star y \in \mathtt{ri}\, C^\star$ and $y \in b^\perp$. This is exactly what the condition $b^\perp \cap \mathtt{strict}(0^+Y) \neq \emptyset$ guarantees, because the dual analogue of equation (9) tells us that $\mathtt{strict}(0^+Y) = \{y \in \mathtt{ri}\, K^\star \colon \mathcal{A}^\star y \succ_{C^\star} \mathbf{0}\}$. □

Proposition 5.1 established strong duality for $c \in \mathcal{A}^\star(K^\perp)$ without requiring any constraint qualification. When $\mathcal{A}(\mathtt{lin}\, C) \subseteq \mathtt{aff}\, K$, the condition $c \in \mathcal{A}^\star(K^\perp)$ implies the condition $c \in (\mathtt{lin}\, C)^\perp$ in Corollary 5.4 because $\mathcal{A}^\star(K^\perp) = (\mathcal{A}^{-1}(\mathtt{aff}\, K))^\perp$ due to Lemma 7.6, and $\mathcal{A}(\mathtt{lin}\, C) \subseteq \mathtt{aff}\, K$ is equivalent to $\mathtt{lin}\, C \subseteq \mathcal{A}^{-1}(\mathtt{aff}\, K)$, which is equivalent to $(\mathtt{lin}\, C)^\perp \supseteq (\mathcal{A}^{-1}(\mathtt{aff}\, K))^\perp$.

A consequence is that the boundedness of the feasible region guarantees strong duality for certain objective functions. We prove this claim here using a necessary condition for boundedness that is established later in §6.2.

Corollary 5.5. *If the primal (resp. dual) has a nonempty and bounded feasible region and* $c \in (\mathtt{lin}\, C)^\perp$ *(resp.* $b \in \mathtt{aff}\, K$*), then strong duality holds and the primal (resp. dual) is solvable.*

Proof. Solvability of the primal is from the extreme value theorem for lower semi-continuous functions over compact sets. Lemma 6.8 gives us $\mathtt{strict}(0^+Y) \neq \emptyset$ when the primal has a bounded feasible set. The dual analogue of Proposition 4.6 makes the dual also feasible, and then the claim follows from Corollary 5.4. □

We will show later in Corollary 6.5 that when one of the problems has a bounded set then the other problem has an unbounded set, assuming at least one of the cones is not a subspace.

A pointed cone C has $\mathtt{lin}\, C = \{\mathbf{0}\}$, and so in this case, the previous corollary holds for arbitrary $c \in \mathbf{E}$. This makes boundedness of the feasible region a *universal CQ* for strong duality, i.e., a constraint qualification that is independent of the data b, c, when C is pointed.

Corollary 5.6. *If C is pointed (resp. K is full-dimensional), then boundedness of the primal (resp. dual) feasible region is a universal CQ for strong duality.*

A different universal CQ is provided in [TW12, Theorem 4.12].

Lastly, we mention that some more sufficient conditions for strong duality can be derived using the last two sets of conditions for closedness in Lemma 5.2. However, because these conditions end up being rather specific and are not applicable when the cones C and K are pointed, we do not describe them explicitly here and leave their derivation to the reader.

6 Bounded Feasible Region

6.1 Recession Cone and its Polar

It is well-known (cf. [Roc70]) that the recession cone of a nonempty closed convex set $S \subset \mathbf{E}$ is equal to the set of all $r \in \mathbf{E}$ for which there exists some $x \in X$ such that $x + \mu r \in S$ for all $\mu \geqslant 0$, and this leads to $0^+(\mathcal{L}^{-1}(S)) = \mathcal{L}^{-1}(0^+S)$ for a linear map $\mathcal{L}$ with $\mathcal{L}^{-1}(S) \neq \emptyset$. For an affine map $\mathcal{G}$, the 0^+ operator does not commute with the preimage $\mathcal{G}^{-1}$ but instead it commutes with the linear map associated with $\mathcal{G}$.

Lemma 6.1. $0^+\mathcal{G}^{-1}(S) = -\mathcal{L}_{\mathcal{G}}^{-1}(0^+S)$*, where* $\mathcal{G}$ *is an affine map with* $\mathcal{G}^{-1}(S) \neq \emptyset$ *and* $\mathcal{L}_{\mathcal{G}}\colon x \mapsto \mathcal{G}(\mathbf{0}) - \mathcal{G}(x)$*.*

Proof. Because $\mathcal{G}(x) = \mathcal{G}(\mathbf{0}) - \mathcal{L}_{\mathcal{G}}(x)$, we have $\mathcal{G}^{-1}(S) = \mathcal{L}_{\mathcal{G}}^{-1}(\mathcal{G}(\mathbf{0}) - S)$. Commutativity of 0^+ with $\mathcal{L}_{\mathcal{G}}^{-1}$ gives us $0^+\mathcal{G}^{-1}(S) = \mathcal{L}_{\mathcal{G}}^{-1}(0^+(\mathcal{G}(\mathbf{0}) - S))$. Translation invariance of 0^+ means that $0^+(\mathcal{G}(\mathbf{0}) - S) = 0^+(-S) = -\,0^+S$, and so we have our claim. □

Recall the dual feasibility cone $\mathcal{C}_d$ from (3). We make the following observation.

Observation 6.2. $(0^+X)^\circ \supseteq \mathcal{C}_d$ *when* $X \neq \emptyset$*.*

Proof. If there exists some $c \in \mathcal{C}_d \setminus (0^+X)^\circ$, we have $\langle c, r\rangle > 0$ for some $r \in 0^+X$, which leads to $z_P^* = \infty$, and then weak duality and Lemma 2.7 imply the contradiction $c \notin \mathcal{C}_d$. □

In fact, the same argument tells us the stronger statement that $\mathcal{C}_d$ is a subset of the domain of the support function of X, we will state this explicitly in §7. The inclusion-preserving property of $\mathtt{cl}$ and the fact that polar cones are closed gives us $\mathtt{cl}\,\mathcal{C}_d \subseteq (0^+X)^\circ$. In our next result describing the recession cone and its polar, we show that the equality holds, thereby making the recession cone equal to the polar of $\mathcal{C}_d$. We also show that almost feasibility is a sufficient condition for membership in the polar cone of the other problem, where the set $X(b)$ is said to be *almost feasible* if for any $\varepsilon > 0$ and any norm $\|\cdot\|$ in $\mathbf{E}'$, there exists $b^\varepsilon \in \mathbf{E}'$ such that $\|b^\varepsilon\| \leqslant \varepsilon$ and $X(b + b^\varepsilon) \neq \emptyset$. The definition for $Y(c)$ is analogous. Note that feasibility implies almost feasibility because we can take $b^\varepsilon = \mathbf{0}$.

Proposition 6.3. *For* $X \neq \emptyset$*, we have* $0^+X = -\mathcal{A}^{-1}(K) \cap C = \mathcal{C}_d{}^\circ$ *and* $(0^+X)^\circ = \mathtt{cl}\,\mathcal{C}_d$*. Furthermore,*

$$(0^+X)^\circ \supseteq \{c \in \mathbf{E}\colon Y(c) \text{ is almost feasible}\} \supseteq \mathtt{ri}\,(0^+X)^\circ = \mathtt{ri}\,\mathcal{C}_d = \{c \in \mathbf{E}\colon\ \mathtt{strict}\,Y(c) \neq \emptyset\}.$$

Proof. The affine map $G(x) = (b - \mathcal{A}x, x)$ has $X = G^{-1}(K \times C)$ and the associated linear map being $\mathcal{L}_G = (\mathcal{A}x, -x)$. Because $0^+(K \times C) = 0^+K \times 0^+C = K \times C$ and $\mathcal{L}_G^{-1}(K \times C) = \{x\colon \mathcal{A}x \in K, x \in -C\} = \mathcal{A}^{-1}(K) \cap -C$, Lemma 6.1 tells us that $0^+X = -\mathcal{L}_G^{-1}(K \times C)$ and hence the equality $0^+X = -\mathcal{A}^{-1}(K) \cap C$ follows immediately. Lemma 3.6 implies that the polar of 0^+X is equal to the closure of $\mathcal{L}_G^\star(K^\star \times C^\star)$. Because the adjoint of $\mathcal{L}_G$ is the linear map $(y, w) \mapsto \mathcal{A}^\star y - w$, we obtain that $(0^+X)^\circ = \mathcal{A}^\star(K^\star) - C^\star$, and then the definition of $\mathcal{C}_d$ in (3) gives us the equality of the polar cone to $\mathtt{cl}\,\mathcal{C}_d$. Applying the Bipolar Theorem to this equality and using the fact that the polars of $\mathcal{C}_d$ and $\mathtt{cl}\,\mathcal{C}_d$ are equal leads us to 0^+X being equal to the polar of $\mathcal{C}_d$.

Now we argue the relative interior. A convex set and its closure have the same relative interior, implying that $\mathtt{ri}\,\mathcal{C}_d = \mathtt{ri}\,(\mathtt{cl}\,\mathcal{C}_d)$, and then the first equality follows immediately from the first part of this proof. Distributivity from Lemma 2.2 and commutativity of $\mathtt{ri}$ with a linear map implies that $\mathtt{ri}\,\mathcal{C}_d = \mathcal{A}^\star(\mathtt{ri}\,K^\star) - \mathtt{ri}\,C^\star$. The definition of dual strict feasibility in (8b) means that $\mathtt{strict}\,Y(c) \neq \emptyset$ if and only if $c \in \mathcal{A}^\star(\mathtt{ri}\,K^\star) - \mathtt{ri}\,C^\star = \mathtt{ri}\,\mathcal{C}_d$.

Because strict feasibility implies feasibility which implies almost feasibility, we obtain the second inclusion. Let us prove the first inclusion by contraposition. Let $c \notin (0^+X)^\circ$. Then there exists some nonzero $r \in 0^+X$ for which $\langle c, r\rangle > 0$. Pick any norm $\|\cdot\|$ in $\mathbf{E}'$ and set $\delta = \langle c, r\rangle$ and $\varepsilon = \frac{\delta}{2\|r\|}$. For any $c^\varepsilon \in \mathbf{E}'$ such that $\|c^\varepsilon\| \leqslant \varepsilon$, we have

$$\langle c + c^\varepsilon, r\rangle \geqslant \langle c, r\rangle - \left|\langle c^\varepsilon, r\rangle\right| \geqslant \langle c, r\rangle - \|c^\varepsilon\|\|r\| \geqslant \langle c, r\rangle - \varepsilon\|r\| = \delta - \frac{\delta}{2} = \frac{\delta}{2} > 0,$$

where the second inequality is by the Cauchy-Schwarz inequality. Therefore, $c + c^\varepsilon \notin (0^+X)^\circ$, and then Observation 6.2 and Lemma 2.7 imply $Y(c + c^\varepsilon) = \emptyset$. Because c^ε was arbitrary up to $\|c^\varepsilon\| \leqslant \varepsilon$, it follows that $Y(c)$ is not almost feasible. □

The recession cone being exactly the set of feasible solutions to the homogenous system $\{x \in C\colon \mathcal{A}x \preccurlyeq_K \mathbf{0}\}$ means that it is invariant to the right-hand side vector b, and so the parametric conic set $X(\beta) := \{x \in C\colon \mathcal{A}x \preccurlyeq_K \beta\}$ has $0^+X(\beta) = X(\mathbf{0})$ whenever $X(\beta) \neq \emptyset$.

Remark 1. Regardless of whether X is feasible or not, we write 0^+X to mean the set $X(\mathbf{0})$. Similarly, 0^+Y means the set $Y(\mathbf{0}) = \{y \in K^\star\colon \mathcal{A}^\star y \succcurlyeq_{C^\star} \mathbf{0}\}$.

We have showed in Proposition 6.3 that almost feasibility is a sufficient condition for membership in the polar cone of the other problem. We will show later in §6.3 that under some minor technical assumptions, the almost feasibility condition is also necessary, thereby giving a characterization of the polar cone in terms of almost feasibility of the other problem.

6.1.1 Strict Feasibility of the Recession Cone

Applying equation (8a) with $b = \mathbf{0}$ implies that the strictly feasible solutions in the recession cone can be described as

$$\mathtt{strict}(0^+X) = -\mathcal{A}^{-1}(\mathtt{ri}\,K) \cap \mathtt{ri}\,C = \{x \in \mathtt{ri}\,C : \mathcal{A}x \prec_K \mathbf{0}\}. \tag{9}$$

These solutions may not always exist but when they do exist, Proposition 4.6 tells us that every parametric conic set with a right-hand side in the span of K is strictly feasible and unbounded. We can apply the conditions of §4 to the set $X(\mathbf{0}) = 0^+X$ to certify when it has strictly feasible solutions. Another necessary condition is the unboundedness of X.

Proposition 6.4. *If either C or K is not a linear subspace and* $\mathtt{strict}(0^+X) \neq \emptyset$, *then $X(b)$ is unbounded for every $b \in \mathtt{aff}\,K$.*

Proof. Proposition 4.6 gives us the strict feasibility of $X(b)$. Hence, boundedness is equivalent to $X(\mathbf{0}) = \{\mathbf{0}\}$. If $X(\mathbf{0}) = \{\mathbf{0}\}$, then $\mathtt{strict}(0^+X) \subseteq X(\mathbf{0})$ implies that $\mathtt{strict}(0^+X) = \{\mathbf{0}\}$. This means that $\mathbf{0} \in \mathtt{ri}\,C$ and $\mathbf{0} \in \mathtt{ri}\,K$. However, we have reached a contradiction due to Lemma 2.4 telling us that the origin cannot be in both the relative interiors when at least one of C or K is not a linear subspace. □

A consequence of this is that at least one of the two problems must have an unbounded feasible set.

Corollary 6.5. *Suppose that at least one of C or K is not a linear subspace. If X is nonempty and bounded, then $Y(c)$ is unbounded for every $c \in (\operatorname{lin} C)^{\perp}$.*

Proof. Lemma 6.8 gives us $\mathtt{strict}(0^+Y) \neq \emptyset$ when the primal has a bounded feasible set. Then unboundedness of $Y(c)$ is from the dual analogue of Proposition 6.4 and using the equivalence between a convex cone not being a subspace and its dual cone not being a subspace. ☐

If the recession cone does have strictly feasible solutions, then the closure operator in Proposition 6.3 can be dropped so that we have equality in Observation 6.2. To prove this claim, we use known sufficient conditions for the sum of two closed convex cones, such as polar cones, to be closed.

Lemma 6.6 ([Roc70; Pat07])**.** *For two nonempty closed convex cones $\mathcal{C}$ and $\mathcal{C}'$, $\left(\mathcal{C} \cap \mathcal{C}'\right)^{\circ} = \operatorname{cl}\left(\mathcal{C}^{\circ} + \mathcal{C}'^{\circ}\right)$, and the sum $\mathcal{C}^{\circ} + \mathcal{C}'^{\circ}$ is a closed set when either $\operatorname{ri}\mathcal{C} \cap \operatorname{ri}\mathcal{C}' \neq \emptyset$ or $\operatorname{ri}\mathcal{C}^{\star} \cap \operatorname{ri}\mathcal{C}'^{\circ} \neq \emptyset$.*

Proposition 6.7. $\left(0^+X\right)^{\circ} = \mathcal{C}_d$ *when* $\mathtt{strict}(0^+X) \neq \emptyset$.

Proof. By Proposition 6.3, it suffices to argue that $\mathcal{C}_d$ is a closed set. The existence of a strict solution in the recession cone means that $\mathcal{A}^{-1}(\operatorname{ri} K) \neq \emptyset$, and then Lemma 5.2 tells us that $\mathcal{A}^{\star}(K^{\star})$ is a closed cone. Therefore, $\mathcal{C}_d$ is a Minkowski difference of two closed cones. Lemma 3.6 tells us that $\mathcal{C}_d$ is the Minkowski sum of the polars of $\mathcal{A}^{-1}(-K)$ and C. For this sum to be closed, a sufficient condition from Lemma 6.6 is that the relative interiors of $\mathcal{A}^{-1}(-K)$ and C intersect. From Lemma 2.1 we have $\operatorname{ri}\mathcal{A}^{-1}(-K) \supseteq \mathcal{A}^{-1}(-\operatorname{ri} K)$. Strict feasibility implies $\mathcal{A}^{-1}(-\operatorname{ri} K) \cap \operatorname{ri} C \neq \emptyset$, and therefore, $\mathcal{C}_d$ is a closed set. ☐

6.2 Conditions for Boundedness

The descriptions of 0^+X and $(0^+X)^{\circ}$ from §6.1 lead to primal and dual characterizations for boundedness of X. The closed convex set X is bounded if and only if $0^+X = \{\mathbf{0}\}$, and so it is obvious from Proposition 6.3 that a characterization of unboundedness of X is that there exists $\mathbf{0} \neq x \in C$ such that $\mathcal{A}x \preccurlyeq_K \mathbf{0}$. Checking this condition is solving the convex maximization problem of maximizing $\|x\|$, for any norm $\|\cdot\|$, over $x \in C$ and $\mathcal{A}x \preccurlyeq_K \mathbf{0}$.

An unbounded set may or may not have strictly feasible solutions to its recession cone. We show that a pointed feasible set is unbounded if and only if the recession cone of the other problem does not have any strictly feasible solutions. The necessity part does not require pointedness of the feasible set.

Lemma 6.8. *When X is nonempty and bounded, then $\mathcal{A}(C) \cap -K = \{\mathbf{0}\}$ and $\mathtt{strict}(0^+Y) \neq \emptyset$.*

Proof. The first necessary condition follows from Proposition 6.3 and $\mathcal{A}(0^+X) = -K \cap \mathcal{A}(C)$. To derive the dual condition, we have $(0^+X)^{\circ} = \mathbf{E}$, and so Proposition 6.3 gives us $\operatorname{cl}\mathcal{C}_d = \mathbf{E}$, which implies that $\mathbf{0} \in \operatorname{ri}(\operatorname{cl}\mathcal{C}_d) = \operatorname{ri}\mathcal{C}_d$. The dual analogue of Proposition 4.2 and $0^+Y = Y(\mathbf{0})$ yields $\mathtt{strict}(0^+Y) \neq \emptyset$. ☐

To prove sufficiency, we use a characterization for strict feasibility. Equation (9) and primal-dual symmetry give us

$$\mathtt{strict}(0^+Y) \;=\; \left\{y \in \operatorname{ri} K^{\star} \colon \mathcal{A}^{\star}y \succ_{C^{\star}} \mathbf{0}\right\},$$

and so $\mathtt{strict}(0^+Y) \neq \emptyset$ if and only if $\mathcal{A}^{\star}(\operatorname{ri} K^{\star}) \cap \operatorname{ri} C^{\star} \neq \emptyset$. There are other equivalent statements.

Lemma 6.9. *The following statements are equivalent:*

1. $\mathtt{strict}(0^+Y) = \emptyset$,
2. $\mathbf{0} \notin \mathtt{ri}\,\mathcal{C}_d$,
3. *there exists* $\mathbf{0} \neq \lambda \in \mathbf{E}$ *such that* $\langle \lambda, y \rangle \leqslant 0 \leqslant \langle \lambda, y' \rangle$ *for all* $y \in \mathtt{cl}\,\mathcal{A}^\star(K^\star)$ *and* $y' \in C^\star$.

Proof. (1 $\iff$ 2) $\mathtt{strict}(0^+Y) = \emptyset$ if and only if $\mathcal{A}^\star(\mathtt{ri}\,K^\star) \cap \mathtt{ri}\,C^\star = \emptyset$, which is equivalent to $\mathbf{0} \notin \mathcal{A}^\star(\mathtt{ri}\,K^\star) - \mathtt{ri}\,C^\star$. Lemmas 2.1 and 2.2 mean that $\mathcal{A}^\star(\mathtt{ri}\,K^\star) - \mathtt{ri}\,C^\star = \mathtt{ri}\,\mathcal{C}_d$.

(1 $\iff$ 3) Because $\mathcal{C}^\star = (-\mathcal{C})^\circ$ for any convex cone $\mathcal{C}$, Lemma 3.6 gives us $(\mathcal{A}^{-1}(K))^\star = \mathtt{cl}\,\mathcal{A}^\star(K^\star)$, and so we have $\mathtt{ri}\,(\mathcal{A}^{-1}(K))^\star = \mathtt{ri}\,(\mathtt{cl}\,\mathcal{A}^\star(K^\star)) = \mathtt{ri}\,\mathcal{A}^\star(K^\star) = \mathcal{A}^\star(\mathtt{ri}\,K^\star)$. Therefore, statement (1) is equivalent to $\mathtt{ri}\,(\mathcal{A}^{-1}(K))^\star \cap \mathtt{ri}\,C^\star = \emptyset$. The separation theorem states that an empty intersection of relative interiors is equivalent to the existence of a separator λ satisfying the third statement. □

Proposition 6.10. *Suppose X is a nonempty pointed set.*

1. *X is bounded if and only if* $\mathtt{strict}(0^+Y) \neq \emptyset$.
2. *Exactly one of the following statements hold: either there exists* $\mathbf{0} \neq x \in C$ *such that* $\mathcal{A}x \preccurlyeq_K \mathbf{0}$, *or there exists* $y \in \mathtt{ri}\,K^\star$ *such that* $\mathcal{A}^\star y \in \mathtt{ri}\,C^\star$.

Proof. Necessity in the first claim is from Lemma 6.8. To argue sufficiency, suppose $\mathtt{strict}(0^+Y) \neq \emptyset$. Lemma 6.9 implies $\mathbf{0} \in \mathtt{ri}\,\mathcal{C}_d$, and then Proposition 6.3 gives us $\mathbf{0} \in \mathtt{ri}\,(0^+X)^\circ$. Pointedness of X means that $0^+X \cap -0^+X = \{\mathbf{0}\}$. Hence if X is unbounded, then 0^+X cannot be a linear subspace. Lemma 2.4 applied to 0^+X gives us the contradiction $\mathbf{0} \notin \mathtt{ri}\,(0^+X)^\circ$. Therefore, X must be a bounded set. The second claim is due to the two systems being $X(\mathbf{0}) \neq \{\mathbf{0}\}$ or $\mathtt{strict}(0^+Y) \neq \emptyset$, and the latter was argued in this proof to be equivalent to $X(\mathbf{0}) = \{\mathbf{0}\}$. □

The second statement in the above proposition is a conic version of Gordan's theorem of the alternative.

Lemma 6.8 observed that $\mathcal{A}(C) \cap -K = \{\mathbf{0}\}$ is a necessary condition for boundedness. This condition is also sufficient when $\mathcal{A}$ is injective and K is pointed. It is not sufficient when only C is pointed, which is the other condition that can make X a pointed set for Proposition 6.10, because we could have $\mathtt{ker}\,\mathcal{A} \cap C \neq \emptyset$ which would imply existence of a nonzero point in 0^+X and therefore, unboundedness of X.

Corollary 6.11. *Suppose X is feasible and $\mathcal{A}$ is injective. Then, X is bounded if and only if* $\mathcal{A}(C) \cap -K = \{\mathbf{0}\}$. *In particular, X is bounded if* $\mathcal{A}(C) \subseteq K$ *and K is pointed, or* $\mathcal{A}(C) \subseteq K^\star$.

Proof. The only if part is from Lemma 6.8. For the if part, Proposition 6.3 tells us that it suffices to show $\mathcal{A}(C \setminus \{\mathbf{0}\}) \cap -K = \emptyset$. Suppose there exists a nonzero $x \in C$ for which $\mathcal{A}x \in -K$. Set $y = \mathcal{A}x$. Then $y \in \mathcal{A}(C) \cap -K$. Because $\mathcal{A}(C) \cap -K = \{\mathbf{0}\}$, it must be that $y = \mathbf{0}$ and therefore $x \in \mathtt{ker}\,\mathcal{A}$. The assumption that $\mathcal{A}$ is injective is equivalent to $\mathtt{ker}\,\mathcal{A} = \{\mathbf{0}\}$, and so $x = \mathbf{0}$, which is a contradiction. Therefore, X' must be bounded. The particular conditions each imply $\mathcal{A}(C) \cap -K = \{\mathbf{0}\}$ because $K \cap -K = \{\mathbf{0}\}$ for a pointed K and any closed convex cone $\mathcal{C}$ satisfies $\mathcal{C}^\star \cap -\mathcal{C} = \{\mathbf{0}\}$. □

We also derive a sufficient condition for boundedness in terms of a basis for the span of C.

Corollary 6.12. *A nonempty pointed set X is bounded if there exists an orthonormal basis* $B = \{v^1, \ldots, v^m\}$ *of* $\mathtt{aff}\,C$ *such that* $B \subseteq C^\star$ *and* $\mathcal{A}(B) \subseteq K$ *with* $v^j \in \mathtt{ri}\,C^\star$ *and* $\mathcal{A}v^j \in K \setminus \mathtt{lin}\,K$ *for some j.*

Proof. Proposition 2.8 tells us that we can replace the adjoint map in the dual with the linear map $\mathcal{B}\colon y \mapsto \sum_{j=1}^{m} \langle \mathcal{A}v^j, y \rangle v^j$. Therefore, the recession cone of the dual is $\{y \in K^\star \colon \mathcal{B}y \succcurlyeq_{C^\star} \mathbf{0}\}$. Proposition 6.10 applied to this recession cone tells us that to show that X is bounded, it suffices to show that there exists $y \in \operatorname{ri} K^\star$ such that $\mathcal{B}y \in \operatorname{ri} C^\star$. The assumption $\mathcal{A}v^j \in K \setminus \operatorname{lin} K$ implies that $K \setminus \operatorname{lin} K \neq \emptyset$, which is equivalent to K not being a linear subspace. Then the last claim in Lemma 2.4 implies that there exists a nonzero $y \in \operatorname{ri} K^\star$. The second claim in this lemma combined with the assumption $\mathcal{A}v^j \in K \setminus \operatorname{lin} K$ gives us $\langle \mathcal{A}v^j, y \rangle > 0$. This leads to $\langle \mathcal{A}v^j, y \rangle v^j \in \operatorname{ri} C^\star$ due to $v^j \in \operatorname{ri} C^\star$ and the fact that $\operatorname{ri} C^\star$ is a cone. The assumptions $v^i \in C^\star$ and $\mathcal{A}v^i \in K$ for $i \neq j$ give us $\langle \mathcal{A}v^i, y \rangle \geqslant 0$ and $\langle \mathcal{A}v^i, y \rangle v^i \in C^\star$, thereby leading to $\sum_{i \neq j} \langle \mathcal{A}v^i, y \rangle v^i \in C^\star$. Since $\mathcal{B}y = \langle \mathcal{A}v^j, y \rangle v^j + \sum_{i \neq j} \langle \mathcal{A}v^i, y \rangle v^i$, Lemma 2.5 gives us the desired claim $\mathcal{B}y \in \operatorname{ri} C^\star$. □

6.2.1 Packing Sets

We say that the primal problem is a *conic packing* problem if it has $\mathcal{A}(C) \subseteq K$. This definition is motivated by the polyhedral case where the set $\{x \in \mathbb{R}^n_+ \colon Ax \leqslant b\}$ is called a packing polyhedron when the matrix A and vector b have nonnegative entries and A has full column-rank. Such a polyhedron is trivially nonempty and bounded. We extend this fact to conic sets.

Proposition 6.13. *A conic packing set X is nonempty if and only if $b \in K$. Furthermore, a nonempty conic packing set is bounded when K is not a subspace and either*

1. *K is pointed and $\mathcal{A}$ is injective, or*
2. *$C = \mathbb{R}^n_+$ and $A_j \in \operatorname{ri} K$ for $j = 1, \ldots, n$, where $A_j = \mathcal{A}\mathbf{e}_j$ for the unit coordinate vector $\mathbf{e}_j$.*

Proof. The if direction is obvious because $b \in K$ allows $x = \mathbf{0}$ to be feasible. The arguments for the only if direction are similar to those for the first condition in Lemma 2.7 after replacing $\operatorname{aff} K$ with K. Boundedness under the first condition is directly from Corollary 6.11. The arguments for the second condition are similar to those for Corollary 6.12. The orthant $\mathbb{R}^n_+$ is a pointed cone, which makes X pointed, and note that an orthonormal span of $\mathbb{R}^n_+$, which is self-dual, is given by its extreme rays $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Since K is not a subspace, we have that $K^\star \setminus K^\perp \neq \emptyset$ and then there exists some $y \in \operatorname{ri} K^\star \setminus K^\perp$ (because otherwise $\operatorname{cl}(\operatorname{ri} K^\star) = \operatorname{cl} K^\star$ would give a contradiction). Hence, by Lemma 2.4, $\langle A_j, y \rangle > 0$ for all j. Observe that a property of the nonnegative orthant is that a positive linear combination of all its extreme rays produces a point in the interior of the orthant. Therefore, we get $\mathcal{B}y \in \operatorname{ri} C^\star$, which is a sufficient condition for boundedness as argued in Corollary 6.12. □

6.3 Almost Feasibility and a Theorem of the Alternative

Recall that Proposition 6.3 related almost feasibility of the dual to the polar of the recession cone of the primal. Now we show this to be an exact relationship under some conditions.

Proposition 6.14. $(0^+X)^\circ = \{c \in \mathbf{E} \colon Y(c)$ *is almost feasible*$\}$ *when either K is a subspace or $\mathcal{A}^\star(\operatorname{ri} K^\star) \cap (\operatorname{lin} C)^\perp \neq \emptyset$. Similarly,* $(0^+Y)^\circ = \{b \in \mathbf{E}' \colon X(b)$ *is almost feasible*$\}$ *when either C is a subspace or $\mathcal{A}(\operatorname{ri} C) \cap \operatorname{aff} K \neq \emptyset$.*

Note that if C is pointed, then the first equality holds, and if K is pointed then the second one holds. One could also interpret the above result as a theorem of the alternative because it tells us that either $Y(c)$ is almost feasible or $\langle c, x \rangle > 0$ for some $x \in C$ with $\mathcal{A}x \preccurlyeq_K \mathbf{0}$, but both statements cannot be true. Note that the theorem of the alternative in [BN01, Proposition

1.7.1], which can be restated as $b \in (0^+Y)^\circ$ if and only if $X(b)$ is almost feasible, deals with $C = \mathbf{E}$ and a full-dimensional K and therefore, is a special case of our result because it satisfies both the conditions.

We will need some technical lemmata for proving Proposition 6.14. First, we have that elements of the polar cone that do not permit feasibility of an auxiliary dual problem do permit almost feasibility of the original dual.

Lemma 6.15. *We have* $(0^+X)^\circ \cap \Omega = \{c \in \Omega \colon Y(c) \text{ is almost feasible}\}$ *for*

$$\Omega := \left\{c \in \mathbf{E} \colon \nexists y \in K^\star, w \in K^\star \setminus \mathtt{lin}\, K^\star \text{ s.t. } \mathcal{A}^\star(y - w) \succcurlyeq_{C^\star} c\right\}.$$

Proof. The $\supseteq$ inclusion is from Proposition 6.3. Take $c \in \Omega$ and suppose $Y(c)$ is not almost feasible. We have to prove that $c \notin (0^+X)^\circ$. Choose any $\xi \in \mathtt{ri}\, C^\star$ and $\gamma \in \mathtt{ri}\, K$ and consider the conic problem

$$z^* = \inf\left\{t_1 + t_2 + \langle \gamma, w\rangle \;:\; \mathcal{A}^\star y - \mathcal{A}^\star w + t_1 c + t_2 \xi \succcurlyeq_{C^\star} c,\; y \in K^\star, w \in K^\star, t_1, t_2 \geqslant 0\right\}.$$

Denote the feasible set by Y_ξ. Observe that $(\bar y, \bar y, 1, 1) \in \mathtt{strict}\, Y_\xi$ for any $\bar y \in \mathtt{ri}\, K^\star$. The dual problem to z^* is

$$\sup\left\{\langle c, x\rangle \;:\; \mathcal{A}x \preccurlyeq_K \mathbf{0}, -\mathcal{A}x \preccurlyeq_K \gamma, \langle c, x\rangle \leqslant 1, \langle \xi, x\rangle \leqslant 1, x \in C\right\}.$$

This has a feasible solution $x = 0$ due to $\gamma \in \mathtt{ri}\, K$. Then strong duality from Corollary 5.3 implies that there exists a feasible x^* to the dual problem with $\langle c, x^*\rangle = z^*$. It is clear that $z^* \geqslant 0$. We claim that $z^* > 0$. Because the feasible set of the dual problem is a subset of $\{x \in C \colon \mathcal{A}x \preccurlyeq_K \mathbf{0}\} = 0^+X$, we get $x^* \in 0^+X$, and so $z^* > 0$ implies $c \notin (0^+X)^\circ$, thereby finishing our proof.

Suppose for the sake of contradiction that $z^* = 0$. By Lemma 2.4, the term $\langle \gamma, w\rangle$ is equal to zero for any $w \in K^\star$ if and only if $w \in K^\perp$. Lemma 2.3 with $\mathcal{C} = K^\star$ and the Bipolar Theorem gives us $K^\perp = \mathtt{lin}\, K^\star$. Because $(\bar y, \bar y, 1, 1) \in \mathtt{strict}\, Y_\xi$ for any $\bar y \in \mathtt{ri}\, K^\star$, we know that the objective function is not identically equal to zero over Y_ξ. Because the infimum is equal to zero, for arbitrarily small $\varepsilon > 0$ there exist feasible solutions $(y, w, t) \in Y_\xi$ with $t_1 + t_2 + \langle \gamma, w\rangle = \varepsilon$. Then one of the following two things must happen: (i) at least one of t_1 or t_2 is strictly positive and $w \in \mathtt{lin}\, K^\star$, or (ii) $t_1 = t_2 = 0$ and $w \in K^\star \setminus \mathtt{lin}\, K^\star$. However, $c \in \Omega$ means that the second case is not possible. In the first case, set $\delta = \varepsilon(\|c\| + \|\xi\|)$ and $c^\delta = -t_1 c - t_2 \xi$. Because $t_1, t_2 \leqslant \varepsilon$, we have $\|c^\delta\| \leqslant \delta$. Because ε is arbitrary close to zero, δ is also arbitrary close to zero. Then $\mathcal{A}^\star(y - w) \succcurlyeq_{C^\star} c + c^\delta$ and $y - w \in K^\star$ due to $y \in K^\star, w \in \mathtt{lin}\, K^\star$ implies that for every $\delta > 0$, we have $y - w \in Y(c + c^\delta)$, thereby making $Y(c)$ almost feasible, which is a contradiction. □

One of our conditions will need the following geometric properties of cones.

Lemma 6.16. *Let $\mathcal{C}$ be a nonempty closed convex cone and $x \in \mathcal{C}$ and $y \in \mathtt{aff}\, \mathcal{C}$.*

1. *If $y \notin \mathcal{C}$, there exists a finite $t' \geqslant 0$ such that $x + ty \in \mathcal{C}$ for all $0 \leqslant t \leqslant t'$ and $x + ty \notin \mathcal{C}$ for all $t > t'$. Also, $t' = 0$ only if $x \in \partial\mathcal{C}$.*

Now suppose $x \in \mathtt{ri}\, \mathcal{C}$.

2. *There exists a finite $\delta^\star > 0$ such that $y + \delta x \in \mathtt{ri}\, \mathcal{C}$ for all $\delta > \delta^\star$.*
3. *If $y \notin \mathcal{C}$, then $y + \delta^\star x \in \partial\mathcal{C}$ and $y + \delta x \notin \mathcal{C}$ for all $0 \leqslant \delta < \delta^\star$.*

Proof. Suppose $y \notin \mathcal{C}$ and $x + ty \in \mathcal{C}$ for all $t \geqslant 0$. Because $\mathcal{C}$ is a cone, for $t > 0$, we have $\frac{1}{t}(x+ty) = \frac{x}{t}+y \in C$. Because $\mathcal{C}$ is closed, $\lim_{t\to\infty} \frac{x}{t}+y \in \mathcal{C}$. But this limit is equal to y, giving a contradiction to $y \notin \mathcal{C}$. Hence, there exists some $\tilde{t} \geqslant 0$ for which $x + \tilde{t}y \notin \mathcal{C}$. Convexity of $\mathcal{C}$ and $x \in \mathcal{C}$ implies that $x + ty \in \mathcal{C}$ is not possible for any $t \geqslant \tilde{t}$. Taking $t' := \inf\{\tilde{t} : x + \tilde{t}y \notin \mathcal{C}\}$, which is equal to $\sup\{t : x + ty \in \mathcal{C}\}$, finishes our first claim on existence of a t'. The second claim on $t' = 0$ is obvious from $\operatorname{aff} \mathcal{C}$ being a subspace and $x + \varepsilon y \in \mathcal{C}$ for some small $\varepsilon > 0$ when $x \in \operatorname{ri} \mathcal{C}$ and $y \in \operatorname{aff} \mathcal{C}$.

We consider two cases: $y \in \mathcal{C}$ and $y \in \operatorname{aff} \mathcal{C} \setminus \mathcal{C}$. For the first case we claim $\delta^\star$ is any positive scalar by arguing that

$$\operatorname{ri} \mathcal{C} + \mathcal{C} \subseteq \operatorname{ri} \mathcal{C}. \tag{10}$$

Take any $u \in \operatorname{ri} \mathcal{C}$. There exists $\varepsilon > 0$ such that $N_\varepsilon(u) \subset \mathcal{C}$ where $N_\varepsilon(\cdot) \subset \operatorname{aff} \mathcal{C}$ is the ε-neighbourhood around a point. It is straightforward to verify that $N_\varepsilon(v + w) = N_\varepsilon(v) + w$ for any $v, w \in \operatorname{aff} \mathcal{C}$. Therefore, for any $y \in \mathcal{C}$ we have $N_\varepsilon(u + y) = N_\varepsilon(u) + y$, and because $N_\varepsilon(u) \subset \mathcal{C}$ and $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$ for convex cones, we get $N_\varepsilon(u + y) \subset \mathcal{C}$, implying that $u + y \in \operatorname{ri} \mathcal{C}$.

Now consider the second case $y \in \operatorname{aff} \mathcal{C} \setminus \mathcal{C}$. First we argue the existence of $\delta^\star > 0$ such that $y + \delta x \in \mathcal{C}$ for all $\delta \geqslant \delta^\star$. Consider t' from the first claim in this proof. The value of t' is equal to 0 if and only if $x + ty \notin \mathcal{C}$ for all $t > 0$. When $x \in \operatorname{ri} \mathcal{C}$, there exists some small enough $\varepsilon > 0$ such that $x + \varepsilon d \in \mathcal{C}$ for all $d \in \operatorname{aff} \mathcal{C}$. Taking $d = y$ implies $x + \varepsilon y \in \mathcal{C}$, and therefore $t' > 0$. Set $\delta^\star := 1/t'$; we have $\delta^\star \in (0, \infty)$ due to $t' \in (0, \infty)$. Take any $\delta \geqslant \delta^\star$. Because $1/\delta \in (0, t']$, we have $x + \frac{1}{\delta} y \in \mathcal{C}$. This leads to $\delta(x + \frac{1}{\delta} y) = y + \delta x \in \mathcal{C}$ because $\mathcal{C}$ is a cone. If $y + \delta x \in \mathcal{C}$ for some $0 \leqslant \delta < \delta^\star$, then $\frac{1}{\delta} y + x \in \mathcal{C}$, a contradiction to the first claim for $t = 1/\delta$ which is larger than $t' := 1/\delta^\star$. Hence, we have $y + \delta x \notin \mathcal{C}$ for all $0 \leqslant \delta < \delta^\star$. It follows then that $y + \delta^\star x \in \partial \mathcal{C}$ because $y \notin \mathcal{C}$. Finally, consider $y + \delta x$ for $\delta > \delta^\star$. Because $y + \delta x = y + \delta^\star x + (\delta - \delta^\star)x$, and $y + \delta^\star x \in \partial \mathcal{C}$ and $(\delta - \delta^\star)x \in \operatorname{ri} \mathcal{C}$ due to $\delta > \delta^\star$, equation (10) gives us $y + \delta x \in \operatorname{ri} \mathcal{C}$. □

Note that $x \in \partial \mathcal{C}$ is not a sufficient condition for $t' = 0$, e.g., for $\mathcal{C} = \mathbb{R}^2_+$ and $y = (y_1, y_2)$ for some $y_1 < 0, y_2 > 0$, we have $t' = -1/y_1$ for $x = (1, 0)$ whereas $t' = 0$ for $x = (0, 1)$.

***Proof of Proposition* 6.14.** A dual cone of a subspace is also a subspace and so K being a subspace implies $K^\star = \operatorname{lin} K^\star$, which makes $\Omega = \mathbf{E}$ in Lemma 6.15. The equality for the polar cone follows from the lemma.

Now let $\mathcal{A}^\star(\operatorname{ri} K^\star) \cap \operatorname{aff} C^\star \neq \emptyset$. The $\supseteq$ inclusion is from Proposition 6.3. We argue the $\subseteq$ inclusion by contraposition. Suppose $Y(c)$ is not almost feasible. Choose any $\xi \in \operatorname{ri} C^\star$ and consider the conic problem

$$z^* = \inf \left\{ t_1 + t_2 : \mathcal{A}^\star y + t_1 c + t_2 \xi \succcurlyeq_{C^\star} c,\ y \in K^\star, t \geqslant \mathbf{0} \right\}.$$

Denote the feasible set Y_ξ. The dual problem to z^* is

$$\sup \left\{ \langle c,\, x \rangle \ : \ \mathcal{A}x \preccurlyeq_K \mathbf{0}, \langle c,\, x \rangle \leqslant 1, \langle \xi,\, x \rangle \leqslant 1, x \in C \right\}.$$

This has a feasible solution $x = 0$. By assumption, there exists some $\bar{y} \in \operatorname{ri} K^\star$ with $\mathcal{A}^\star \bar{y} \in \operatorname{aff} C^\star$. Lemma 6.16 with $\mathcal{C} = C^\star$ and $\xi \in \operatorname{ri} C^\star$ imply that $(\bar{y}, 1, t_2) \in \operatorname{strict} Y_\xi$ for some $t_2 > 0$. Then strong duality from Corollary 5.3 implies that there exists a feasible x^* to the dual problem with $\langle c,\, x^* \rangle = z^*$. It is clear that $z^* \geqslant 0$. We claim that $z^* > 0$. Because the feasible set of the dual problem is a subset of $\{x \in C \colon \mathcal{A}x \preccurlyeq_K \mathbf{0}\} = 0^+ X$, we get $x^* \in 0^+ X$, and so $z^* > 0$ implies that $c \notin (0^+ X)^\circ$, which finishes our proof by contraposition for the $\subseteq$ inclusion. To argue the claim $z^* > 0$, suppose that $z^* = 0$. This means that there exist feasible solutions to Y_ξ with both t_1 and t_2 arbitrarily close to zero. Then for any $\varepsilon > 0$, we can choose $c^\varepsilon = -t_1 c - t_2 \xi$ for small values of t_1 and t_2 to get $\|c^\varepsilon\| \leqslant \varepsilon$, and we would have some $y \in K^\star$ such that $\mathcal{A}^\star y \succcurlyeq_{C^\star} c + c^\varepsilon$. But this would make $Y(c)$ almost feasible, which is a contradiction. □

A consequence is the equivalence of feasibility and almost feasibility for bounded sets.

Corollary 6.17. *If $0^+X = \{\mathbf{0}\}$ and either C is a subspace or $\mathcal{A}(\mathtt{ri}\, C) \cap \mathtt{aff}\, K \neq \emptyset$, then for any $b \in \mathbf{E}'$, $X(b)$ is almost feasible if and only if $X(b)$ is feasible.*

Proof. The if direction is trivially true without any assumptions. Assume $0^+X = \{\mathbf{0}\}$ and suppose $b \in \mathbf{E}'$ is such that $X(b)$ is almost feasible. Lemma 6.8 tells us that $0^+X = \{\mathbf{0}\}$ implies $\mathtt{strict}(0^+Y) \neq \emptyset$. The dual analogue of Proposition 6.7 gives us $(0^+Y)^\circ = \mathcal{C}_p$, and then Proposition 6.14 implies that $b \in \mathcal{C}_p$, and so $X(b)$ is feasible by Lemma 2.7. □

7 Domain of the Support Function

Weak duality tells us that a sufficient condition for one problem (primal or dual) to have a bounded optimum is that the other problem is feasible. Our main result here is that under the assumption of strict feasibility, this sufficient condition also becomes necessary.

Theorem 7.1. *If one problem (primal or dual) is feasible with either the recession cone or feasible region being strictly feasible, then that problem has finite optimum if and only if the other problem is feasible.*

Proposition 6.7 gives us the only if direction when the recession cone is strictly feasible. Therefore, the main thing to prove is the only if direction under the assumption of strict feasibility of the feasible region. The two conditions are indeed distinct because strict feasibility of the recession cone does not imply strict feasibility of the feasible set. This is for two reasons. First is that the recession cone of a closed convex set S is contained in S if and only if S contains the origin. Second is that the relative interior operator is not inclusion-preserving unless the two sets have the same affine span, and so even if the recession cone were contained in the feasible set, the strictly feasible solution to the former may not be strictly feasible to the latter. Although the two strict feasibility conditions are not related in general, note that Proposition 4.6 tells us that if the right-hand side belongs to the span of the corresponding cone, then strict feasibility of the recession cone is a stronger condition.

To prove Theorem 7.1 under strict feasibility of the feasible region, we establish a general result about the domain of support function of the affine preimage of a cone. The support function of a nonempty closed convex set S is the optimal value functional $\sigma_S\colon \mathbf{E} \to \mathbb{R} \cup \{\infty\}$ defined as

$$\sigma_S(c) \coloneqq \sup\left\{\langle c, x\rangle : x \in S\right\}, \quad \text{with} \quad \mathtt{dom}\, \sigma_S = \left\{c \in \mathbf{E} :\ \sigma_S(c) < \infty\right\}.$$

The domain $\mathtt{dom}\, \sigma_S$ is a convex cone, also referred to as the *barrier cone* of S. This cone, which may be neither open or closed in general, closely approximates the polar of the recession cone.

Lemma 7.2. *Any nonempty closed convex set S satisfies*

$$\mathtt{ri}\,(\mathtt{dom}\, \sigma_S) = \mathtt{ri}\,(0^+S)^\circ \subseteq \mathtt{dom}\, \sigma_S \subseteq (0^+S)^\circ = (\mathtt{ri}\, 0^+S)^\circ = \mathtt{cl}\,(\mathtt{dom}\, \sigma_S).$$

Proof. A convex cone $\mathcal{C}$ obeys $\mathcal{C}^\circ = (\mathtt{ri}\, \mathcal{C})^\circ = (\mathtt{cl}\, \mathcal{C})^\circ$. Because 0^+S is a closed convex cone, we obtain $(0^+S)^\circ = (\mathtt{ri}\, 0^+S)^\circ$. The inclusion of the barrier cone in the polar cone is by the definition of these cones. We have $0^+S = (\mathtt{dom}\, \sigma_S)^\circ$ from [Roc70, Corollary 14.2.1], and then the Bipolar Theorem and $\mathcal{C}^\circ = (\mathtt{cl}\, \mathcal{C})^\circ$ gives us $(0^+S)^\circ = \mathtt{cl}\,(\mathtt{dom}\, \sigma_S)$. This implies $\mathtt{ri}\,(0^+S)^\circ = \mathtt{ri}\,(\mathtt{cl}\,(\mathtt{dom}\, \sigma_S)) = \mathtt{ri}\,(\mathtt{dom}\, \sigma_S) \subseteq \mathtt{dom}\, \sigma_S$. □

The containment of the relative interior of the polar cone in the barrier cone can be shown independently using Lemma 2.4, and this leads to the other relationships using basic facts about $\mathtt{ri}$ and $\mathtt{cl}$ from Lemma 2.1. In general, $\mathtt{dom}\,\sigma_S$ could be equal to $\mathtt{ri}\,(0^+S)^\circ$, as in the case of the epigraph of a parabola in $\mathbb{R}^2$, or be equal to $(0^+S)^\circ$ in which case the set is said to be a thin convex set, or be neither open nor closed as with the epigraph of a parabola cut in half.

Let $\mathcal{C} \subset \mathbf{E}'$ be a nonempty closed convex cone, $\mathcal{G}\colon \mathbf{E} \to \mathbf{E}'$ be an affine map whose associated linear map is $\mathcal{L}_\mathcal{G}\colon x \mapsto \mathcal{G}(\mathbf{0}) - \mathcal{G}(x)$, and assume that $\mathcal{G}^{-1}(\mathcal{C}) \neq \emptyset$. Our result on the domain of support function is the following.

Proposition 7.3. $\mathtt{dom}\,\sigma_{\mathcal{G}^{-1}(\mathcal{C})} = \mathcal{L}_\mathcal{G}^\star(\mathcal{C}^\star)$ *if any of the following conditions hold:*

1. $\mathcal{G}^{-1}(\mathtt{ri}\,\mathcal{C}) \neq \emptyset$,
2. $\mathcal{L}_\mathcal{G}^{-1}(\mathtt{ri}\,\mathcal{C}) \neq \emptyset$,
3. $\mathtt{ker}\,\mathcal{L}_\mathcal{G}^\star \cap \mathtt{ri}\,\mathcal{C}^\star \neq \emptyset$,
4. $x \in \mathtt{lin}\,\mathcal{C}$ *for every* $x \in \mathcal{C} \cap \mathtt{Im}\,\mathcal{L}_\mathcal{G}$,
5. $x \in \mathcal{C}^\perp$ *for every* $x \in \mathcal{C}^\star \cap \mathtt{ker}\,\mathcal{L}_\mathcal{G}^\star$.

Note that due to Lemma 2.6, the second condition implies the first condition when $\mathcal{G}(\mathbf{0}) \in \mathtt{aff}\,\mathcal{C}$, but not otherwise. Before proving our proposition, we show how a direct consequence of it is our main theorem on the equivalence of the finiteness of the primal and feasibility of the dual.

***Proof of Theorem** 7.1.* The if direction is trivial from weak duality and does not require strict feasibility. For the only if direction, we have to show that $\mathtt{dom}\,\sigma_X \subseteq \mathcal{C}_d$, because by Lemma 2.7 the dual feasibility is characterized by $\mathcal{C}_d$. This is immediate when $\mathtt{strict}(0^+X) \neq \emptyset$, because Proposition 6.7 gives us $\left(0^+X\right)^\circ = \mathcal{C}_d$ and Lemma 7.2 gives us $\left(0^+X\right)^\circ \supseteq \mathtt{dom}\,\sigma_X$.

Now suppose $\mathtt{strict}\,X \neq \emptyset$. Consider the affine map $\mathcal{G}(x) = (b - \mathcal{A}x, x)$, whose associated linear map is $\mathcal{L}_\mathcal{G}(x) = (b, \mathbf{0}) - \mathcal{G}(x) = (\mathcal{A}x, -x)$. The primal feasible set is $X = \mathcal{G}^{-1}(K \times C)$, and strict feasibility (cf. (8a) and (9)) is $\mathtt{strict}\,X = \mathcal{G}^{-1}(\mathtt{ri}\,K \times \mathtt{ri}\,C) = \mathcal{G}^{-1}(\mathtt{ri}\,(K \times C))$ and $\mathtt{strict}(0^+X) = \mathcal{L}_\mathcal{G}^{-1}(-\mathtt{ri}\,K \times -\mathtt{ri}\,C) = -\mathcal{L}_\mathcal{G}^{-1}(\mathtt{ri}\,(K \times C))$. Because the primal is finite-valued if and only if $c \in \mathtt{dom}\,\sigma_X$ and the dual is feasible if and only if $c \in \mathcal{C}_d$, we have to show that $\mathtt{dom}\,\sigma_X = \mathcal{C}_d$ when $\mathcal{G}^{-1}(\mathtt{ri}\,\mathcal{C})$ is nonempty for $\mathcal{C} = K \times C$. Proposition 7.3 implies $\mathtt{dom}\,\sigma_X = \mathcal{L}_\mathcal{G}^\star(\mathcal{C}^\star)$ and then distributivity of the dual cone operator over the Cartesian product means that $\mathtt{dom}\,\sigma_X = \mathcal{L}_\mathcal{G}^\star(K^\star \times C^\star)$. It is straightforward to verify that $\mathcal{L}_\mathcal{G}^\star(y, w) = \mathcal{A}^\star y - w$, and so $\mathtt{dom}\,\sigma_X = \mathcal{A}^\star(K^\star) - C^\star$. The definition of $\mathcal{C}_d$ in (3) gives us the desired equality $\mathtt{dom}\,\sigma_X = \mathcal{C}_d$. □

The requirement of strict feasibility in Theorem 7.1 is necessary in general to obtain the desired equivalence because it is possible for one problem to be feasible, but not strictly feasible, with bounded optimum and the other problem to be infeasible. Any of the conditions for strict feasibility derived in §4 could be used for applying this theorem.

The remainder of this section is devoted to proving Proposition 7.3. We begin by noting that a weaker statement than this proposition, obtained by taking closure on both sides, is always true without any assumptions.

Lemma 7.4. $\mathcal{L}_\mathcal{G}^\star(\mathcal{C}^\star) \subseteq \mathtt{dom}\,\sigma_{\mathcal{G}^{-1}(\mathcal{C})} \subseteq \mathtt{cl}\,(\mathtt{dom}\,\sigma_{\mathcal{G}^{-1}(\mathcal{C})}) = \left(0^+\mathcal{G}^{-1}(\mathcal{C})\right)^\circ = \mathtt{cl}\,\mathcal{L}_\mathcal{G}^\star(\mathcal{C}^\star)$.

Proof. Taking $S = \mathcal{G}^{-1}(\mathcal{C})$ in Lemma 7.2 gives us $\mathtt{cl}\,(\mathtt{dom}\,\sigma_{\mathcal{G}^{-1}(\mathcal{C})}) = \left(0^+\mathcal{G}^{-1}(\mathcal{C})\right)^\circ$. Lemma 6.1 and $0^+\mathcal{C} = \mathcal{C}$ gives us $0^+\mathcal{G}^{-1}(\mathcal{C}) = \mathcal{L}_\mathcal{G}^{-1}(-\mathcal{C})$, and then Lemma 3.6 yields the second equality.

The second inclusion is obvious. For the first inclusion, take any $c \in \mathcal{L}_{\mathcal{G}}^{\star}(\mathcal{C}^{\star})$. Therefore, $c = \mathcal{L}_{\mathcal{G}}^{\star}(\bar{x})$ for some $\bar{x} \in \mathcal{C}^{\star}$. We have to argue that $\sigma_{\mathcal{G}^{-1}(\mathcal{C})}(c) < \infty$. By definition, $\sigma_{\mathcal{G}^{-1}(\mathcal{C})}(c) = \sup_x\{\left\langle \mathcal{L}_{\mathcal{G}}^{\star}(\bar{x}),\, x \right\rangle : \mathcal{G}(x) \in \mathcal{C}\}$. The property of adjoints means that the objective is $\langle \bar{x},\, \mathcal{L}_{\mathcal{G}}(x)\rangle$, and because $\mathcal{L}_{\mathcal{G}}(x) = \mathcal{G}(\mathbf{0}) - \mathcal{G}(x)$, we get that $\sigma_{\mathcal{G}^{-1}(\mathcal{C})}(c) = \langle \bar{x},\, \mathcal{G}(\mathbf{0})\rangle - \inf_x\{\langle \bar{x},\, \mathcal{G}(x)\rangle : \mathcal{G}(x) \in \mathcal{C}\}$. Because $\bar{x} \in \mathcal{C}^{\star}$, the infimum is equal to zero, and hence $\sigma_{\mathcal{G}^{-1}(\mathcal{C})}(c) < \infty$. □

Thus, it follows that Proposition 7.3 holds when $\mathcal{L}_{\mathcal{G}}^{\star}(\mathcal{C}^{\star})$ is a closed set. The second to fifth conditions in this proposition are indeed conditions that guarantee this closedness, and so our main task is to prove the $\subseteq$ inclusion under the first condition. The first step in doing this is the following technical result that does not require any assumptions.

Lemma 7.5. *For any $c \in \operatorname{dom} \sigma_{\mathcal{G}^{-1}(\mathcal{C})}$ with $c^{\perp} \not\supseteq \mathcal{L}_{\mathcal{G}}^{-1}(\operatorname{aff} \mathcal{C})$, there exists a $y \in \mathcal{C}^{\star}$ such that $\langle \mathcal{G}(x),\, y\rangle \leqslant 0$ for all $x \in \mathbf{E}$ with $\langle c,\, x\rangle \geqslant \sigma_{\mathcal{G}^{-1}(\mathcal{C})}(c)$.*

Proof. Let $H = \{x\colon\, \langle c,\, x\rangle \geqslant \sigma_{\mathcal{G}^{-1}(\mathcal{C})}(c)\}$. This halfspace is nonempty due to $c \in \operatorname{dom} \sigma_{\mathcal{G}^{-1}(\mathcal{C})}$. We argue that $\mathcal{G}(H) \cap \operatorname{ri} \mathcal{C} = \emptyset$. Then the separation theorem tells us that there exists a $\mathbf{0} \neq y \in \mathbf{E}'$ such that $\sup_{x \in \mathcal{G}(H)} \langle y,\, \mathcal{G}(x)\rangle \leqslant \inf_{w \in \mathcal{C}} \langle y,\, w\rangle$, and because $\mathcal{C}$ is a cone and $H \neq \emptyset$, the infimum must be zero so that $y \in \mathcal{C}^{\star}$.

Suppose for the sake of contradiction there exists some $x \in H$ such that $\mathcal{G}(x) \in \operatorname{ri} \mathcal{C}$. By the assumption $c^{\perp} \not\supseteq \mathcal{L}_{\mathcal{G}}^{-1}(\operatorname{aff} \mathcal{C})$ on the linear subspaces $c^{\perp}$ and $\mathcal{L}_{\mathcal{G}}^{-1}(\operatorname{aff} \mathcal{C})$, we know there exists some $d \in \mathcal{L}_{\mathcal{G}}^{-1}(\operatorname{aff} \mathcal{C})$ for which $\langle c,\, d\rangle > 0$. Because $\mathcal{G}(x) \in \operatorname{ri} \mathcal{C}$ and $\mathcal{L}_{\mathcal{G}}(d) \in \operatorname{aff} \mathcal{C}$, we can choose a small $\varepsilon > 0$ such that $\mathcal{G}(x) - \varepsilon \mathcal{L}_{\mathcal{G}}(d) \in \mathcal{C}$. Linearity of $\mathcal{L}_{\mathcal{G}}$ and $\mathcal{G}(x) = \mathcal{G}(\mathbf{0}) - \mathcal{L}_{\mathcal{G}}(x)$ gives us $\mathcal{G}(\mathbf{0}) - \mathcal{L}_{\mathcal{G}}(x + \varepsilon d) \in \mathcal{C}$, which means that $\mathcal{G}(x + \varepsilon d) \in \mathcal{C}$. Hence, $x + \varepsilon d \in \mathcal{G}^{-1}(\mathcal{C})$. The objective value is $\langle c,\, x + \varepsilon d\rangle = \langle c,\, x\rangle + \varepsilon \langle c,\, d\rangle > \langle c,\, x\rangle \geqslant \sigma_{\mathcal{G}^{-1}(\mathcal{C})}(c)$, where the strict inequality is by $\varepsilon > 0$ and $\langle c,\, d\rangle > 0$, and the inequality is from $x \in H$. The feasibility of $x + \varepsilon d$ to $\mathcal{G}^{-1}(\mathcal{C})$ implies that we have reached a contradiction to the optimality of the value $\sigma_S(c)$. Therefore, $\mathcal{G}(H)$ and $\operatorname{ri} \mathcal{C}$ do not intersect. □

We will also need an identity for the adjoint of a linear map and orthogonal complement of a linear subspace, which is a generalization of the fundamental fact from linear algebra that the orthogonal complement to the kernel of a linear map is exactly the image of the adjoint map.

Lemma 7.6. $(\mathcal{L}^{-1}(L))^{\perp} = \mathcal{L}^{\star}(L^{\perp})$ *for a linear map $\mathcal{L}\colon \mathbf{E} \to \mathbf{E}'$ and linear subspace $L \subseteq \mathbf{E}'$.*

Proof. For any $x \in \mathcal{L}^{-1}(L)$ and $y \in L^{\perp}$, we have $\langle x,\, \mathcal{L}^{\star}(y)\rangle_{\mathbf{E}} = \langle \mathcal{L}(x),\, y\rangle_{\mathbf{E}'} = 0$, implying that $\mathcal{L}^{\star}(y) \in (\mathcal{L}^{-1}(L))^{\perp}$ and hence the $\supseteq$-inclusion. Arguing the $\subseteq$-inclusion is equivalent to showing that $\mathcal{L}^{-1}(L) \supseteq (\mathcal{L}^{\star}(L^{\perp}))^{\perp}$. For any $x \in (\mathcal{L}^{\star}(L^{\perp}))^{\perp}$ and $y \in L^{\perp}$, we have $0 = \langle x,\, \mathcal{L}^{\star}(y)\rangle_{\mathbf{E}} = \langle \mathcal{L}(x),\, y\rangle_{\mathbf{E}'}$, and so $y \in L^{\perp}$ implies $\mathcal{L}(x) \in (L^{\perp})^{\perp} = L$, which means that $x \in \mathcal{L}^{-1}(L)$. □

Finally, we are ready to prove our result on support functions.

Proof of Proposition 7.3. The $\supseteq$ inclusion is from Lemma 7.4. This lemma also tells us that equality holds when $\mathcal{L}_{\mathcal{G}}^{\star}(\mathcal{C}^{\star})$ is a closed set. The second to fifth conditions are exactly the conditions for closedness obtained from Lemma 5.2. It remains to prove the $\subseteq$ inclusion under the first condition, $\mathcal{G}^{-1}(\operatorname{ri} \mathcal{C}) \neq \emptyset$.

For $c \in \operatorname{dom} \sigma_{\mathcal{G}^{-1}(\mathcal{C})} \cap \mathcal{L}_{\mathcal{G}}^{\star}(\mathcal{C}^{\perp})$, it is obvious that $c \in \mathcal{L}_{\mathcal{G}}^{\star}(\mathcal{C}^{\star})$ due to $\mathcal{C}^{\perp} \subseteq \mathcal{C}^{\star}$. Now take $c \in \operatorname{dom} \sigma_{\mathcal{G}^{-1}(\mathcal{C})} \setminus \mathcal{L}_{\mathcal{G}}^{\star}(\mathcal{C}^{\perp})$. By Lemma 7.6, $\mathcal{L}_{\mathcal{G}}^{\star}(\mathcal{C}^{\perp}) = (\mathcal{L}_{\mathcal{G}}^{-1}(\operatorname{aff} \mathcal{C}))^{\perp}$, and so $c \notin (\mathcal{L}_{\mathcal{G}}^{-1}(\operatorname{aff} \mathcal{C}))^{\perp}$. This implies $\operatorname{span} c \cap (\mathcal{L}_{\mathcal{G}}^{-1}(\operatorname{aff} \mathcal{C}))^{\perp} = \{\mathbf{0}\}$ and $c^{\perp} \not\supseteq \mathcal{L}_{\mathcal{G}}^{-1}(\operatorname{aff} \mathcal{C})$. Lemma 7.5 gives us $\langle \mathcal{G}(x),\, y\rangle \leqslant 0$

for all $x \in \mathbf{E}$ with $\langle c,\, x\rangle \geqslant \sigma_{\mathcal{G}^{-1}(\mathcal{C})}(c)$. Because $\mathcal{G}(x) = \mathcal{G}(\mathbf{0}) - \mathcal{L}_{\mathcal{G}}(x)$, we have the consequence that for some $y \in \mathcal{C}^\star$,

$$\begin{aligned} \langle c,\, x\rangle_{\mathbf{E}} \geqslant \sigma_{\mathcal{G}^{-1}(\mathcal{C})}(c) &\implies \left\langle \mathcal{L}_{\mathcal{G}}(x),\, y\right\rangle_{\mathbf{E}'} \geqslant \left\langle \mathcal{G}(\mathbf{0}),\, y\right\rangle_{\mathbf{E}'} \\ &\implies \left\langle \mathcal{L}^\star_{\mathcal{G}}(y),\, x\right\rangle_{\mathbf{E}} \geqslant \left\langle \mathcal{G}(\mathbf{0}),\, y\right\rangle_{\mathbf{E}'}. \end{aligned} \tag{11}$$

Farkas' lemma for linear inequalities gives us that there exists $\mu \geqslant 0$ for which $\mathcal{L}^\star_{\mathcal{G}}(y) = \mu c$. Now we argue that such a μ is positive. This implies that $c = \frac{1}{\mu}\mathcal{L}^\star_{\mathcal{G}}(y) = \mathcal{L}^\star_{\mathcal{G}}(y/\mu)$, and because $y/\mu \in \mathcal{C}^\star$ due to y belonging to the cone $\mathcal{C}^\star$, it follows that $c \in \mathcal{L}^\star_{\mathcal{G}}(\mathcal{C}^\star)$.

Suppose for the sake of contradiction that $\mu = 0$. Then, $\mathcal{L}^\star_{\mathcal{G}}(y) = \mathbf{0}$, and so for any $x \in \mathbf{E}$,

$$\left\langle y,\, \mathcal{G}(x)\right\rangle_{\mathbf{E}'} = \left\langle y,\, \mathcal{G}(\mathbf{0}) - \mathcal{L}_{\mathcal{G}}(x)\right\rangle_{\mathbf{E}'} = \left\langle y,\, \mathcal{G}(\mathbf{0})\right\rangle_{\mathbf{E}'} - \left\langle \mathcal{L}^\star_{\mathcal{G}}(y),\, x\right\rangle_{\mathbf{E}} = \left\langle y,\, \mathcal{G}(\mathbf{0})\right\rangle_{\mathbf{E}'} \leqslant \mathbf{0}, \tag{12}$$

where the inequality is from (11). By the assumption $\mathcal{G}^{-1}(\operatorname{ri}\mathcal{C}) \neq \emptyset$, there exists some $x \in \mathbf{E}$ for which $\mathcal{G}(x) \in \operatorname{ri}\mathcal{C}$. We consider two cases to argue contradiction. First suppose $y \notin \mathcal{C}^\perp$. We know $y \in \mathcal{C}^\star$, and so then the first claim in Lemma 2.4 gives us the strict inequality $\left\langle y,\, \mathcal{G}(x)\right\rangle > 0$, which is a contradiction to (12). The second case is $y \in \mathcal{C}^\perp$. This gives us $\left\langle y,\, \mathcal{G}(x)\right\rangle = 0$, and then (12) implies $\left\langle y,\, \mathcal{G}(\mathbf{0})\right\rangle = 0$. Recall that the nonzero vector $y \in \mathbf{E}'$ obtained from Lemma 7.5 defined the hyperplane $y^\perp$ separating the image $\mathcal{G}(H)$ of the halfspace $H = \{x\colon\, \langle c,\, x\rangle \geqslant \sigma_S(c)\}$ from $\mathcal{C}$. Proper separation tells us that either $\mathcal{G}(H) \not\subset y^\perp$ or $\mathcal{C} \not\subset y^\perp$ or both. Hence, the case $y \in \mathcal{C}^\perp$, which is equivalent to $y^\perp \supset \mathcal{C}$, implies that $\left\langle y,\, \mathcal{G}(\bar{x})\right\rangle \neq 0$ for some $\bar{x} \in H$. Equation (12) makes this $\left\langle y,\, \mathcal{G}(\mathbf{0})\right\rangle \neq 0$, which is a contradiction because we argued earlier in this case that $\left\langle y,\, \mathcal{G}(\mathbf{0})\right\rangle = 0$. Thus, we are done proving our claim $\mu > 0$. ☐

Now we discuss two implications of the main results of this section.

7.1 Alternate proof for Strong Duality with Strict Feasibility

Recall that Corollary 5.3, which establishes strong duality under the strict feasibility condition, was proved as a special case of the more general closedness condition of Theorem 3.1. We mention an alternate proof here. This resembles the final step in the strong duality proof of [BN01, Theorem 1.7.1].

Take $\mathcal{G}(x) = (b - \mathcal{A}x, x)$, $\mathcal{L}_{\mathcal{G}}(x) = (b, \mathbf{0}) - \mathcal{G}(x) = (\mathcal{A}x, -x)$, and $\mathcal{C} = K \times C$ as in the proof of Theorem 7.1. We have $\operatorname{strict} X = \mathcal{G}^{-1}(\operatorname{ri}(K \times C))$. For $c \in \operatorname{dom}\sigma_{\mathcal{G}^{-1}(\mathcal{C})}$, Proposition 7.3 tells us that $c \in \mathcal{C}_d$, i.e., $c = \mathcal{A}^\star y - w$ for some $(y, w) \in K^\star \times C^\star$. Equation (11) with $x \in \mathbf{E}$ such that $\langle c,\, x\rangle_{\mathbf{E}} = \sigma_{\mathcal{G}^{-1}(\mathcal{C})}(c)$ yields $\langle c,\, x\rangle_{\mathbf{E}} \geqslant \left\langle (b, \mathbf{0}),\, (y, w)\right\rangle_{\mathbf{E}'\times\mathbf{E}} = \langle b,\, y\rangle_{\mathbf{E}'}$. Because $z^*_P = \langle c,\, x\rangle$ by construction of x, $\langle b,\, y\rangle \geqslant z^*_D$ due to $y \in Y(c)$, and $z^*_P \leqslant z^*_D$ due to weak duality, we obtain zero duality gap.

7.2 Projecting onto a Subspace

A well-known result for linear programming is that an orthogonal projection of a polyhedron onto a subspace of variables can be obtained by multiplying the algebraic linear description of the polyhedron by every extreme ray of a specific cone called the projection cone (cf. [Bal05, Theorem 1.1]). This can be proved using LP strong duality and it is very useful for deriving valid inequalities to mixed-integer programs. It is natural to expect that a similar result holds for sets described using conic inequalities because conic programs also exhibit strong duality, albeit under stronger assumptions than those required for linear programming. We describe the projection of conic sets onto arbitrary linear subspaces by giving a straightforward proof that is motivated by the polyhedral case.

Consider the conic set $X = \{x \in C\colon \mathcal{A}x \preccurlyeq_K b\}$ and let $L \subset \mathbf{E}$ be a linear subspace. The projection of X onto L is defined as $\mathtt{proj}_L X := \{x \in L : \exists u \in L^\perp \text{ s.t. } x + u \in X\}$. Denote

$$\mathscr{C} := \left\{(y, w) \in K^\star \times C^\star : \mathcal{A}^\star y - w \in L\right\}.$$

Clearly, this is a convex cone. It is also a closed set because it is equal to the intersection of two closed sets — (i) $K^\star \times C^\star$, and (ii) the preimage of the subspace L under the linear map $(y, w) \mapsto \mathcal{A}^\star y - w$. The latter is a closed set because the preimage of any closed set under a continuous map is a closed set. Thus, $\mathscr{C}$ is a closed convex cone and so it is generated by its extreme rays.

Proposition 7.7. *Suppose there exists $\hat{y} \in \mathtt{ri}\, K^\star$ and $\hat{w} \in \mathtt{ri}\, C^\star$ such that $\mathcal{A}^\star \hat{y} - \hat{w} \in L$. Then,*

$$\mathtt{proj}_L X = \left\{x \in L : \left\langle \mathcal{A}^\star y - w,\, x\right\rangle_{\mathbf{E}} \leqslant \langle b,\, y\rangle_{\mathbf{E}'},\ \forall \text{ extreme rays } (y, w) \in \mathscr{C}\right\}.$$

Proof. The definition of projection tells us that

$$\mathtt{proj}_L X = \left\{x \in L : \exists u \in L^\perp \text{ s.t. } \mathcal{A}u \preccurlyeq_K b - \mathcal{A}x,\ u \in C - x\right\}. \tag{13}$$

Consider the conic program

$$\sup_u \left\{\langle \mathbf{0},\, u\rangle \ : \ \mathcal{A}u \preccurlyeq_K b - \mathcal{A}x,\ u \in C - x,\ u \in L^\perp\right\}.$$

Using the fact that $(L^\perp)^\perp = L$, the dual of the above primal can be easily derived to be

$$\inf_{y,w} \left\{\langle b - \mathcal{A}x,\, y\rangle + \langle x,\, w\rangle \ : \ (y, w) \in \mathscr{C}\right\}.$$

The existence of $(\hat{y}, \hat{w})$ guarantees strict feasibility of $\mathscr{C}$ because $\mathtt{ri}\, L = L$ for a subspace L. Because the dual optimizes over a cone, its value is either 0 or $-\infty$. Theorem 7.1 gives us that the dual value is 0 if and only if the primal is feasible. Hence, the existence of a u in equation (13) can be equivalently replaced with the infimum of the dual being zero, which is equivalent to $\langle b - \mathcal{A}x,\, y\rangle + \langle x,\, w\rangle \geqslant 0$ for all $(y, w) \in \mathscr{C}$. Because $\mathscr{C}$ is a cone, it suffices to enforce this nonnegativity for only its extreme rays. Substituting this in equation (13) yields

$$\mathtt{proj}_L X = \left\{x \in L : \langle b - \mathcal{A}x,\, y\rangle + \langle x,\, w\rangle \geqslant 0,\ \forall \text{ extreme rays } (y, w) \in \mathscr{C}\right\}.$$

The claimed expression follows after rearranging terms and using the definition of the adjoint. □

The cone $\mathscr{C}$ could have uncountably many extreme rays, in which case the projection is defined by an infinite system of linear inequalities. Applying Proposition 7.7 to a high-dimensional set gives us its orthogonal projection.

Corollary 7.8. *The orthogonal projection of $S = \{(x, x') \in C \times C'\colon \mathcal{A}x + \mathcal{B}x' \preccurlyeq_K b\}$ onto the x-space is equal to*

$$\left\{x \in C\colon \left\langle \mathcal{A}^\star y,\, x\right\rangle \leqslant \langle b,\, y\rangle,\ \forall \text{ extreme rays } y \in \mathscr{C}\right\},$$

for the cone $\mathscr{C} = \{y \in K^\star\colon \mathcal{B}^\star y \succcurlyeq_{C'^\star} \mathbf{0}\}$, when there exists a $y \in \mathtt{ri}\, K^\star$ with $\mathcal{B}^\star y \in \mathtt{ri}\, C'^\star$.

8 Conclusions

This paper establishes various conditions for strong duality to hold between a primal-dual pair of conic programs. When the cones are low-dimensional, closedness of a particular adjoint image is sufficient for strong duality. This leads to many specific sufficient conditions that are more tractable to verify, such as generalized Slater constraint qualification, boundedness of the feasible region, and some that are in terms of the recession cones of primal or dual feasible regions. The paper also gives many necessary and sufficient conditions for having a bounded feasible region and then provides a conic theorem of the alternative in terms of approximate feasibility. Finally, it shows that under generalized Slater constraint qualifications, finiteness of one problem and feasibility of the other are equivalent.

An interesting direction of future research is to see whether the closedness condition and its many implications also hold for abstract convex programs that have been studied in literature with regards to a minimal facial property of cones. Another open question is to investigate which of the results in this paper extend to infinite-dimensional conic programs over general Hilbert spaces.

9 Acknolwedgments

Ajayi and Schaefer were partially supported by National Science Foundation grant CMMI-1933373. Gupte was partially supported by National Science Foundation grant DMS-1913294.

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