# Infrared singularities and the high-energy limit

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We review recent results on the high-energy limit of gauge amplitudes, which can be derived from the universal properties of their infrared singularities. Using the dipole formula, a compact ansatz for infrared singularities of massless gauge amplitudes, and taking the high-energy limit, we provide a simple expression for the soft factor of a generic high-energy amplitude, valid to leading power in $t/s$ and to all logarithmic orders. This gives a direct and general proof of leading-logarithmic Reggeization for infrared divergent contributions to the amplitude, and it shows how Reggeization breaks down at NNLL level. We further show how the dipole formula constrains the high-energy limit of multi-particle amplitudes in multi-Regge kinematics, and how, on the other hand, Regge theory constrains possible corrections to the dipole formula.

*10th International Symposium on Radiative Corrections (Applications of Quantum Field Theory to Phenomenology)*
*September 26-30, 2011*
*Mamallapuram, India*

*Speaker.

 

## 1. Introduction

The high-energy limit of gauge theory scattering amplitudes and cross sections has been a source of interest for theory and phenomenology for more than half a century, starting with studies performed in the context of Regge theory, long before the construction of the Standard Model of particle physics [1]. In the context of Perturbative QCD, and using contemporary language, the problem may be formulated as follows. Consider the key example of a four-point scattering amplitude, characterized by the Mandelstam invariants $s$, $t$ and $u$, and by particle masses $m_i$, satisfying

$$s+t+u=\sum_{i=1}^{4} m_i^2 \,. \tag{1.1}$$

The high-energy limit is the limit in which the squared center-of-mass energy $s$ is much larger than other kinematic invariants, compatibly with the constraint in eq. (1.1). In the following, we concentrate exclusively on massless scattering, $m_i^2 = 0$, and one is then led to consider (in the center-of-mass frame) either forward scattering ($s \gg -t$, $u \sim -s$), or backward scattering ($s \gg -u$, $t \sim -s$). As expected in general for renormalizable field theories, in the presence of two disparate scales, say $s$ and $|t|$, amplitudes are dominated order by order by logarithmic contributions, which eventually spoil the (asymptotic) convergence of perturbation theory and need to be resummed. Regge theory suggests, under suitable assumptions, that this resummation can be achieved by replacing the $t$-channel propagator of the particle responsible for the leading power contribution to the high-energy limit at Born level with its 'Reggeized' counterpart, according to

$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)} , \tag{1.2}$$

where $\alpha(t)$ is the Regge trajectory for the chosen $t$-channel exchange. In Perturbative QCD, the Regge ansatz, eq. (1.2), can be studied explicitly, and 'Reggeization' can be proved, at least for selected amplitudes and to a given logarithmic accuracy [2, 3, 4]. Consider for example gluon-gluon scattering. To leading logarithmic (LL) accuracy [2], and to next-to-leading logarithmic (NLL) accuracy for the real part only [4], one can show that the scattering amplitude obeys a Regge factorization of the form

$$\mathcal{M}^{gg\to gg}_{a_1a_2a_3a_4}(s,t) \,=\, 2g_s^2\,\frac{s}{t}\left[(T^b)_{a_1a_3}C_{\lambda_1\lambda_3}(k_1,k_3)\right]\left(\frac{s}{-t}\right)^{\alpha(t)}\left[(T_b)_{a_2a_4}C_{\lambda_2\lambda_4}(k_2,k_4)\right], \tag{1.3}$$

where we take $s=(k_1+k_2)^2$ and $t=(k_1-k_3)^2$, $T^a$ are color generators in the adjoint representation, $g_s$ is the strong coupling, and the functions $C_{\lambda_i\lambda_j}(k_i,k_j)$ are universal 'impact factors' associated with gluons carrying momenta $k_i$ and helicities $\lambda_i$. Eq. (1.3) represents a universal factorization since impact factors depend only on the identity of the energetic partons interacting via $t$-channel exchange, while the $s$ dependence is confined to the Reggeized propagator, which in turn depends only on the identity of the particle being exchanged in the $t$ channel. Once this factorization is established, one may compute the various factors, and in particular the Regge trajectory $\alpha(t)$, using perturbation theory. A crucial fact is that the Regge trajectory is IR divergent order by order, since it requires integrations over virtual corrections which involve soft gluon exchanges. Using dimensional continuation (to $d \equiv 4 - 2\varepsilon > 4$) to regulate the infrared, it is practical to expand the Regge trajectory in powers of the $d$-dimensional running coupling [5], defined by the RG equation

$$\mu \frac{\partial \alpha_s}{\partial \mu} = \beta(\varepsilon, \alpha_s) = -2\varepsilon \alpha_s - \frac{\alpha_s^2}{2\pi} \sum_{n=0}^{\infty} b_n \left( \frac{\alpha_s}{\pi} \right)^n , \tag{1.4}$$

where $\varepsilon < 0$ and $b_0 = (11 C_A - 2 n_f)/3$. One then writes

$$\alpha(t) = \frac{\alpha_s(-t, \varepsilon)}{4\pi} \, \alpha^{(1)} + \left( \frac{\alpha_s(-t, \varepsilon)}{4\pi} \right)^2 \alpha^{(2)} + \mathcal{O}\left(\alpha_s^3\right) , \tag{1.5}$$

where the coupling is naturally evaluated at the scale $\mu = -t$,

$$\alpha_s(-t, \varepsilon) = \left( \frac{\mu^2}{-t} \right)^{\varepsilon} \alpha_s(\mu^2) + \mathcal{O}\left(\alpha_s^2\right) . \tag{1.6}$$

The first two perturbative coefficients of the gluon Regge trajectory are known (see for example [6, 7, 8, 9, 10]), and in the $\overline{\rm MS}$ scheme they are given by

$$\alpha^{(1)} = C_A \frac{\widehat{\gamma}_K^{(1)}}{\varepsilon} , \qquad \alpha^{(2)} = C_A \left[ - \frac{b_0}{\varepsilon^2} + \widehat{\gamma}_K^{(2)} \frac{2}{\varepsilon} + C_A \left( \frac{404}{27} - 2 \zeta_3 \right) + n_f \left( - \frac{56}{27} \right) \right] . \tag{1.7}$$

In eq. (1.7) we introduced the cusp anomalous dimension [11], $\gamma_K^{(i)}(\alpha_s)$, where $i$ denotes the color representation (the adjoint representation in the present case), and we extracted the corresponding quadratic Casimir eigenvalue, writing $\gamma_K^{(i)}(\alpha_s) \equiv C_i \, \widehat{\gamma}_K(\alpha_s)$. The first coefficients of the universal function $\widehat{\gamma}_K(\alpha_s)$, which appear in eq. (1.7), are well known and are given by

$$\widehat{\gamma}_K^{(1)} = 2 , \qquad \widehat{\gamma}_K^{(2)} = \left( \frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} n_f . \tag{1.8}$$

The fact that the Regge trajectory is IR divergent suggests that studies of the high-energy limit could be pursued by using the detailed knowledge of IR and collinear divergences that has been developed over the last decades. Notably, considerable progress was achieved in the last few years, which in particular led to an all-order ansatz [12, 13] for the anomalous dimension governing the exponentiation of IR and collinear poles in generic multi-particle massless scattering amplitudes. In the following, we will first describe this ansatz (the 'dipole formula'), briefly discussing its properties and limitations, and then we will show how it can be used to improve our understanding of the high-energy limit [14]. We will show that, at leading power in $|t|/s$ and to arbitrary logarithmic accuracy, IR divergences are generated by a soft operator with a simple color and kinematic structure, and we will use this result to prove Reggeization of IR divergent contributions to the amplitude, at LL level, for the scattering of particles belonging to generic color representations. Our soft operator, however, is accurate to all logarithmic orders, so we can go beyond LL: we find that standard Regge factorization breaks down at NNLL, and we give a general expression for the leading Reggeization-breaking operator. We then show how our method applies to multi-particle scattering in Multi-Regge kinematics, and we point out how the Regge limit constrains potential corrections to the dipole ansatz.

## 2. Dipoles

Our understanding of IR and collinear divergences in gauge theory amplitudes rests on a powerful factorization theorem, which is the final result of many years of theoretical work [15]. In order to state the theorem, let us first note that a scattering amplitude in a non-abelian gauge theory is a vector in the space of the available color configurations, which is a subspace of the tensor product of the color representations of the $n$ particles participating in the scattering process. Choosing a basis of color tensors $c_J$ in this space we may write

$$\mathscr{M}_{a_1 \ldots a_L}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \varepsilon\right) = \sum_J \mathscr{M}_J\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \varepsilon\right) (c_J)_{a_1 \ldots a_L}\,, \tag{2.1}$$

In the following we will consider massless fixed-angle scattering amplitudes, so that $p_i^2 = 0$, and all invariants $p_i \cdot p_j \gg \Lambda_{\rm QCD}$ are taken to be of the same parametric size. The factorization theorem then states [16] that each component of the vector $\mathscr{M}$ can be written as a product

$$\mathscr{M}_J\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \varepsilon\right) = \sum_K \mathscr{S}_{JK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \varepsilon\right) H_K\left(\frac{2 p_i \cdot p_j}{\mu^2}, \frac{(2 p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \varepsilon\right) \times \prod_{i=1}^{L} \frac{J_i\left(\frac{(2 p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \varepsilon\right)}{\mathscr{J}_i\left(\frac{2 (\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \varepsilon\right)}\,. \tag{2.2}$$

Here we have introduced the light-like four velocities $\beta_i^\mu \equiv p_i^\mu / Q$, with $Q \sim \mu$ an arbitrary scale, and the auxiliary four-vectors $n_i^\mu$, $n_i^2 \neq 0$, to be discussed below. The matrix $\mathscr{S}$ is responsible for soft divergences, and it acts as a color operator on the vector of finite functions $H$. It is defined by the expectation value of a product of Wilson lines, as

$$\left(c^J\right)_{\{a_k\}} \mathscr{S}_{JK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \varepsilon\right) \equiv \sum_{\{b_k\}} \left\langle 0 \left| \prod_{k=1}^{L} \Big[\Phi_{\beta_k}(\infty, 0)\Big]_{a_k b_k} \right| 0 \right\rangle_{\rm ren.} \left(c^K\right)_{\{b_k\}}\,, \tag{2.3}$$

where

$$\Big[\Phi_{\beta_l}(\infty, 0)\Big]_{a_l b_l} \equiv \left[\mathscr{P} \exp\left({\rm i} g_s \int_0^\infty dt\, \beta_l \cdot A(t \beta_l)\right)\right]_{a_l b_l}\,. \tag{2.4}$$

Collinear divergences associated with each external hard particle are collected in the jet functions $J_i$, which are defined in a gauge-invariant way with the help of the auxiliary vectors $n_i$. As an example, for quark jets one writes

$$\overline{u}(p_l)\, J_l\left(\frac{(2 p_l \cdot n_l)^2}{n_l^2 \mu^2}, \alpha_s(\mu^2), \varepsilon\right) = \langle p_l \,|\, \overline{\psi}(0)\, \Phi_{n_l}(0, -\infty)\, |0\rangle\,. \tag{2.5}$$

In taking the product of soft and collinear factors, one double-counts the soft-collinear region. The problem is easily fixed by dividing each particle jet by its own eikonal approximation $\mathscr{J}_i$, which is obtained simply by replacing the field $\psi$ in eq. (2.5) with a Wilson line in the direction of the corresponding momentum $p_i$. Note that the dependence on the 'factorization vectors' $n_i$ must cancel between the jet factors and the hard functions. Upon modifying the hard function in order to effect this cancellation, the factorization theorem in eq. (2.2) can be summarized [13] in a more compact and elegant form as

$$\mathscr{M}\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\varepsilon\right) = Z\left(\frac{p_i}{\mu_f},\alpha_s(\mu_f^2),\varepsilon\right)\,\mathscr{H}\left(\frac{p_i}{\mu},\frac{\mu_f}{\mu},\alpha_s(\mu^2),\varepsilon\right), \tag{2.6}$$

where we introduced a factorization scale $\mu_f$, and where $Z$ acts as a color operator on the vector of modified hard functions $\mathscr{H}$. The operator $Z$, which is built out of fields and Wilson lines, is multiplicatively renormalizable, and thus obeys a (matrix) RG equation of the form

$$\mu\frac{d}{d\mu}Z\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\varepsilon\right) = -Z\left(\frac{p_i}{\mu},\alpha_s(\mu^2),\varepsilon\right)\Gamma\left(\frac{p_i}{\mu},\alpha_s(\mu^2)\right), \tag{2.7}$$

with a finite anomalous dimension matrix $\Gamma$. In dimensional regularization, eq. (2.7) can be easily solved to yield

$$Z\left(\frac{p_l}{\mu},\alpha_s(\mu^2),\varepsilon\right) = \mathscr{P}\exp\left[\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\Gamma\left(\frac{p_i}{\lambda},\alpha_s(\lambda^2)\right)\right]. \tag{2.8}$$

Clearly, our understanding of IR and collinear divergences to all orders hinges upon the knowledge of the anomalous dimension matrix $\Gamma$. The dipole formula is an ansatz for this matrix, which represents the simplest solution to a set of exact equations satisfied by $\Gamma$ as a consequence of factorization and of conformal invariance of light-like Wilson lines [12, 13]. It reads

$$\Gamma_{\rm dip}\left(\frac{p_i}{\lambda},\alpha_s(\lambda^2)\right) = \frac{1}{4}\widehat{\gamma}_K\left(\alpha_s(\lambda^2)\right)\sum_{(i,j)}\ln\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \sum_{i=1}^{L}\gamma_{J_i}\left(\alpha_s(\lambda^2)\right), \tag{2.9}$$

where $-s_{ij} = 2|p_i\cdot p_j|{\rm e}^{-{\rm i}\pi\lambda_{ij}}$, with $\lambda_{ij}=1$ if particles $i$ and $j$ both belong to either the initial or the final state, and $\lambda_{ij}=0$ otherwise. The color structure is expressed in a basis-independent way by the color generators $\mathbf{T}_i$, which act on hard parton $i$ as gluon insertion operators, and $\gamma_{J_i}$ are color singlet anomalous dimensions for the jets. The crucial feature of eq. (2.9) is that it involves only pairwise correlations between hard partons, an extremely drastic simplification with respect to the expected level of complexity. A side effect of this simplification is the fact that the matrix structure of $\Gamma_{\rm dip}$ is fixed at one loop, and radiative corrections enter only through the anomalous dimensions $\widehat{\gamma}_K$ and $\gamma_{J_i}$: as a consequence, the path ordering in eq. (2.8) becomes immaterial. The dipole formula reproduces all known results for IR divergences of massless gauge theory amplitudes, and in principle it could be the definitive solution of the problem. We know however that eq. (2.9) may receive, at sufficiently high orders, precisely two classes of corrections [12, 13]. First of all, eq. (2.9) was derived assuming that the cusp anomalous dimensions in representation $i$ remains proportional to the quadratic Casimir eigenvalue $C_i$ to all orders in perturbation theory. This proportionality ("Casimir scaling") is established through explicit calculations only up to three loops, and in principle corrections proportional to higher-order Casimir operators might arise starting at four loops (although arguments were given [13] that at four loops this does not actually happen). The second class of possible corrections to eq. (2.9) arises because the conformal invariance of light-like Wilson lines cannot constrain the dependence of $\Gamma$ on conformally invariant cross-ratios of momenta such as $\rho_{ijkl} \equiv (p_i \cdot p_j p_k \cdot p_l)/(p_i \cdot p_k p_j \cdot p_l)$. If such corrections are present, they modify the anomalous dimension matrix according to

$$\Gamma\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) = \Gamma_{\rm dip}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) + \Delta\left(\rho_{ijkl}, \alpha_s(\lambda^2)\right) . \tag{2.10}$$

The 'quadrupole' correction $\Delta$ can be non-vanishing starting at three loops and when at least four hard particles are present. It has been shown to be tightly constrained not only by factorization, but also by collinear limits, Bose symmetry and transcendentality requirements. A few examples of viable functions at three loops have however been constructed [17], and the general structure of possible corrections at four loops has been studied [18]. As we will see shortly, the high-energy limit provides further nontrivial constraints, which come tantalizingly close to showing that $\Delta$ should vanish, at least at three loops.

## 3. High-energy

Since the dipole formula is an ansatz applicable to all orders and for any number of hard particles, it is natural to study its high-energy limit. Note that, strictly speaking, the fixed-angle assumption under which the dipole formula was derived breaks down as $|t|/s \to 0$. The ensuing corrections however must take the form of logarithms of $s/(-t)$, with coefficients that remain finite as $\varepsilon \to 0$, since the high-energy limit cannot generate new divergent contributions. In order to write the result in a compact and appealing form, it is useful to introduce color operators associated with each Mandelstam channel [19]. Explicitly

$$\mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_2 = -(\mathbf{T}_3 + \mathbf{T}_4) , \qquad \mathbf{T}_t = \mathbf{T}_1 + \mathbf{T}_3 = -(\mathbf{T}_2 + \mathbf{T}_4) \tag{3.1}$$

and similarly for $\mathbf{T}_u$. In eq. (3.1), we enforced color conservation, expressed in this context by $\sum_i \mathbf{T}_i = 0$. For scattering between particles in generic color representations, the color operators in eq. (3.1) satisfy a constraint reminiscent of eq. (1.1),

$$\mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = \sum_{i=1}^{4} C_i . \tag{3.2}$$

We consider now the limit $|t| \ll s$, which implies $u \sim -s$. The energy dependence of the four-point amplitude simplifies drastically in this limit, and in particular the IR operator $Z$ factorizes, to leading *power* in $|t|/s$, as

$$Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \varepsilon\right) = \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \varepsilon\right) Z_{\mathbf{1}}\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) . \tag{3.3}$$

The crucial feature of eq. (3.3) is that the factor $Z_{\mathbf{1}}$, which is independent of $s$, is proportional to the unit matrix in color space, while the non-trivial color dependence, contained in the factor $\widetilde{Z}$, is remarkably simple. One finds

$$\widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \varepsilon\right) = \exp\left\{K\Big(\alpha_s(\mu^2), \varepsilon\Big)\left[\ln\left(\frac{s}{-t}\right)\mathbf{T}_t^2 + {\rm i}\pi\,\mathbf{T}_s^2\right]\right\} , \tag{3.4}$$

where the function $K(\alpha_s, \varepsilon)$ is a simple integral of the cusp anomalous dimension, which plays a role in several different QCD applications, given by

$$K\Big(\alpha_s(\mu^2), \varepsilon\Big) \equiv -\frac{1}{4} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \, \widehat{\gamma}_K \left(\alpha_s(\lambda^2, \varepsilon)\right) . \tag{3.5}$$

We note that the special role played by the cusp anomalous dimension in the high-energy limit was previously demonstrated in [9, 10]; in particular, Ref. [10] derived an expression analogous to eq. (3.5), and used it to compute the gluon Regge trajectory at NLL. One readily sees, in fact, that the simple dependence of eq. (3.4) on the high-energy logarithm $\ln(s/(-t))$ has immediate consequences on Reggeization, at least for divergent contributions to the amplitude. At LL level, one can obviously discard the imaginary part of the exponent of eq. (3.4), and the matrix element in eq. (2.6) takes the form

$$\mathscr{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \varepsilon\right) = \exp\left\{K\Big(\alpha_s(\mu^2), \varepsilon\Big) \ln\left(\frac{s}{-t}\right) \mathbf{T}_t^2\right\} Z_{\mathbf{1}} \, \mathscr{H}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \varepsilon\right) , \tag{3.6}$$

This automatically implies (LL) Reggeization, for any representation content, under the sole assumption that the amplitude be dominated at lowest order and at leading power in $|t|/s$ by the exchange of a specific color state in the $t$ channel. If this is the case the color operator $\mathbf{T}_t^2$ can be replaced by its Casimir eigenvalue $C_t$, and the color state exchanged in the $t$ channel is found to Reggeize, with a Regge trajectory given by the function $C_t K(\alpha_s, \varepsilon)$. For example, for gluon-gluon scattering one finds

$$\mathbf{T}_t^2 \, \mathscr{H}^{gg \to gg} = C_A \, \mathscr{H}_t^{gg \to gg} + \mathscr{O}(|t|/s) , \tag{3.7}$$

which implies

$$\mathscr{M}^{gg \to gg} = \left(\frac{s}{-t}\right)^{C_A K\left(\alpha_s(\mu^2), \varepsilon\right)} Z_{\mathbf{1}} \, \mathscr{H}_t^{gg \to gg} . \tag{3.8}$$

Computing the integral in eq. (3.5), one easily recovers the divergent terms in eq. (1.7). It is easy to see, however, that the same reasoning applies in full generality to different $t$-channel exchanges.

Since our master formula, eq. (3.4), is valid to all logarithmic orders for divergent contributions, it is straightforward to go beyond LL accuracy. One may start by applying the Baker-Campbell-Hausdorff formula to write the operator $\widetilde{Z}$ as a product of exponentials of decreasing logarithmic weight, as

$$\begin{aligned} \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \varepsilon\right) = & \left(\frac{s}{-t}\right)^{K \, \mathbf{T}_t^2} \exp\left\{\mathrm{i}\, \pi K \, \mathbf{T}_s^2\right\} \exp\left\{-\,\mathrm{i}\, \frac{\pi}{2} K^2 \ln\left(\frac{s}{-t}\right) \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right\} \\ & \times \exp\left\{\frac{K^3}{6}\left(-2\pi^2 \ln\left(\frac{s}{-t}\right) \left[\mathbf{T}_s^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right] + \mathrm{i}\pi \ln^2\left(\frac{s}{-t}\right) \left[\mathbf{T}_t^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right]\right)\right\} \exp\left\{\mathscr{O}\left(K^4\right)\right\} , \end{aligned} \tag{3.9}$$

where $K = K(\alpha_s, \varepsilon)$. From eq. (3.9), it is apparent that color exchanges that are not diagonal in the $t$-channel basis become relevant at NLL accuracy for the imaginary part of the amplitude (as exemplified by the presence of the color operator $\mathbf{T}_s^2$ in the second factor on the *r.h.s.* of eq. (3.9)), and for the real part of the amplitude at NNLL accuracy. This implies a breakdown of Regge factorization, as given in eq. (1.3). A general form of the leading Reggeization-breaking operator at NNLL for the real part of the amplitude is easily read off from eq. (3.9) and is given by

$$\mathscr{E}\left(\frac{s}{t}, \alpha_s, \varepsilon\right) \equiv -\frac{\pi^2}{3} K^3(\alpha_s, \varepsilon) \ln\left(\frac{s}{-t}\right) \left[\mathbf{T}_s^2, \left[\mathbf{T}_t^2, \mathbf{T}_s^2\right]\right] , \tag{3.10}$$

where we note that the expansion of $K^3(\alpha_s, \varepsilon)$ starts out at $\mathscr{O}\left(\alpha_s^3/\varepsilon^3\right)$.

The remarkable simplicity of eqs. (3.3) and (3.4) suggests that IR divergences of multi-particle amplitudes might similarly simplify in a suitably defined high-energy limit. The relevant limit is well-known [20], and is characterized by strongly ordered rapidities $y_i$ of the emitted particles, while transverse momenta $k_i^\perp$ are of the same parametric size. This configuration is called 'Multi-Regge' kinematical regime, characterized by

$$y_3 \gg y_4 \gg \ldots \gg y_L \qquad |k_i^\perp| \simeq |k_j^\perp| \quad \forall i, j \, . \tag{3.11}$$

In this limit, the relevant Mandelstam invariants can be written as

$$\begin{aligned} -s \equiv -s_{12} \simeq |k_3^\perp||k_L^\perp| \, e^{-\mathrm{i}\pi} \, \mathrm{e}^{y_3 - y_L} \, , \quad -s_{1i} \simeq |k_3^\perp||k_i^\perp| \, \mathrm{e}^{y_3 - y_i} \, , \quad -s_{2i} \simeq |k_L^\perp||k_i^\perp| \, \mathrm{e}^{y_i - y_L} \, , \\ -s_{ij} \simeq |k_i^\perp||k_j^\perp| \, \mathrm{e}^{y_i - y_j} \, e^{-\mathrm{i}\pi} \qquad (3 \leq i < j \leq L) \, , \end{aligned} \tag{3.12}$$

so that logarithms of ratios of kinematical invariants are dominated by rapidity differences. Taking the Multi-Regge limit in eq. (2.9) yields once again a factorized expression, in which the dominant kinematic dependence is characterized by $t$-channel color exchanges. One finds [14]

$$Z\left(\frac{p_l}{\mu}, \alpha_s(\mu^2), \varepsilon\right) = \widetilde{Z}^{\rm MR}\Big(\Delta y_k, \alpha_s(\mu^2), \varepsilon\Big) \, Z_{\mathbf{1}}^{\rm MR}\left(\frac{|k_i^\perp|}{\mu}, \alpha_s(\mu^2), \varepsilon\right) , \tag{3.13}$$

where the transverse-momentum dependent factor $Z_{\mathbf{1}}^{\rm MR}$ is again proportional to the unit matrix in color space. Non-trivial color exchanges are encoded into the operator

$$\widetilde{Z}^{\rm MR}\Big(\Delta y_k, \alpha_s(\mu^2), \varepsilon\Big) = \exp\left\{K\Big(\alpha_s(\mu^2), \varepsilon\Big)\left[\sum_{k=3}^{L-1} \mathbf{T}_{t_{k-2}}^2 \, \Delta y_k + \mathrm{i}\pi \mathbf{T}_s^2\right]\right\} , \tag{3.14}$$

where we introduced the $t$-channel color matrices

$$\mathbf{T}_{t_k} = \mathbf{T}_1 + \sum_{p=1}^{k} \mathbf{T}_{p+2} \, . \tag{3.15}$$

It is important to note that the operators $\mathbf{T}_{t_k}^2$ commute with each other, so that it is always possible to choose a basis in color space in which they are all simultaneously diagonal. Given eq. (3.14), it is clear that the patterns of Reggeization, and Reggeization breaking, which have been discussed in the case of the four-point amplitude generalize to the Multi-Regge kinematics in a straightforward manner. Notice that it is not necessary that the tree-level hard amplitude be dominated by the exchange of a single color state across all the $t_k$ subchannels: if multiple representations contribute at leading power in the high-energy limit, they will separately Reggeize (at LL), with the expected Regge trajectories, as a consequence of eq. (3.14).

## 4. Quadrupoles?

So far we have discussed the implications of the dipole formula on the high-energy limit of scattering amplitudes. It is important to note that known results concerning the high-energy limit can, in turn, be used to constrain possible corrections to the dipole formula. As was outlined in

Sec. 2, we already know that corrections to the dipole formula can only arise from violations of Casimir scaling for the cusp anomalous dimension, or in the form of functions of the conformal cross ratios $\rho_{ijkl}$, denoted by $\Delta$ in eq. (2.10). In order to explore the existence of possible constraints on these functions arising from the high-energy limit, let us focus on four-point amplitudes. In this case three different cross ratios can be constructed: $\rho_{1234}$, $\rho_{1342}$, $\rho_{1423}$, subject to the constraint $\rho_{1234}\rho_{1324}\rho_{1423} = 1$. They are given by

$$\rho_{1234} = \left(\frac{s}{-t}\right)^2 \mathrm{e}^{-2\mathrm{i}\pi}, \qquad \rho_{1342} = \left(\frac{-t}{s+t}\right)^2, \qquad \rho_{1423} = \left(\frac{s+t}{s}\right)^2 \mathrm{e}^{2\mathrm{i}\pi}, \tag{4.1}$$

where care was taken to retain the correct phases. In the high-energy limit, the logarithms of the cross ratios, $L_{ijkl} \equiv \log \rho_{ijkl}$ can be expressed, up to corrections suppressed by powers of $|t|/s \to 0$, in terms of the high-energy logarithm $L \equiv \log(s/(-t))$ as

$$L_{1234} = 2(L - \mathrm{i}\pi), \qquad L_{1342} = -2L, \qquad L_{1423} = 2\mathrm{i}\pi. \tag{4.2}$$

One may now examine the existing examples of functions $\Delta$ which satisfy all previously examined constraints [17]. The simplest example, a symmetric polynomial in the logarithms $L_{ijkl}$, is given by

$$\begin{aligned} \Delta^{(212)}(\rho_{ijkl}, \alpha_s) = \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \Big[ & f^{ade} f^{cbe} L_{1234}^2 \left(L_{1423} L_{1342}^2 + L_{1423}^2 L_{1342}\right) \\ + & f^{cae} f^{dbe} L_{1423}^2 \left(L_{1234} L_{1342}^2 + L_{1234}^2 L_{1342}\right) \\ + & f^{bae} f^{cde} L_{1342}^2 \left(L_{1423} L_{1234}^2 + L_{1423}^2 L_{1234}\right) \Big]. \end{aligned} \tag{4.3}$$

In the high-energy limit (and using Jacobi identities for the structure constants $f^{abc}$) one finds

$$\begin{aligned} \Delta^{(212)}(\rho_{ijkl}, \alpha_s) = \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \, 32\,\mathrm{i}\,\pi \Big[ & \left(-L^4 - \mathrm{i}\pi L^3 - \pi^2 L^2 - \mathrm{i}\pi^3 L\right) f^{ade} f^{cbe} \\ & + \left(2\mathrm{i}\pi L^3 - 3\pi^2 L^2 - \mathrm{i}\pi^3 L\right) f^{cae} f^{dbe} \Big] + \mathscr{O}\left(|t/s|\right). \end{aligned} \tag{4.4}$$

Such a function cannot contribute to the anomalous dimension matrix $\Gamma$, since it would generate super-leading high-energy logarithms, $\alpha_s^p L^q$, with $q > p$, starting at three loops. One may easily show that all explicit examples constructed in Ref. [17] suffer from this problem: indeed, the problem appears to be rather generic, since the function $\Delta$ is constrained by collinear limits and Bose symmetry to contain a minimum number of factors behaving logarithmically at large $s$. In order to put together a viable example of a three-loop correction to the dipole formula, one would have to construct a linear combination of functions such as eq. (4.3), each with the ‘wrong’ high-energy behavior, arranging for the cancellation of the unwanted logarithms. Since gluon Reggeization is proven in general to NLL accuracy, in eq. (4.4) one would need to cancel not only the super-leading $L^4$ contribution, but the leading ($L^3$) and next-to-leading ($L^2$) terms as well, since those are correctly predicted by the dipole formula, and would have a color structure incompatible with gluon Reggeization. So far, no explicit example of such a function has been constructed.

## 5. Perspective

We have summarized some recent developments concerning the IR structure of scattering amplitudes in massless gauge theories, with applications to the widely studied high-energy regime. The dipole formula, if correct, would be the first case in which the structure of a multi-particle non-abelian anomalous dimension is understood exactly to all orders in perturbation theory: note that it applies to any massless gauge theory and it is exact in the $1/N_c$ expansion. In the high-energy limit, the dipole formula implies Reggeization of IR-singular contributions to the amplitude at LL accuracy, and at NLL accuracy for the real part of the amplitude. Beyond NLL, it predicts the existence of specific Reggeization-breaking contributions, which can be explicitly computed in practical cases once the relevant hard sub-amplitudes are known. In turn, known results on the high-energy limit of scattering amplitudes constrain possible corrections to the dipole formula in a significant way, ruling out individually all viable examples which had been constructed so far, and it is conceivable that a definitive answer on the IR structure of multiparticle massless gauge theory amplitudes will become available in a not-too-far future.

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