Vortices versus monopoles in color confinement *  
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We construct the creation operator of a vortex for \textit{SU}(2) pure gauge theory using the methods developed for monopoles. We interpret its vacuum expectation value as a disorder parameter for the deconfinement phase transition and find that it behaves in the vacuum in a similar way to monopoles. Results are extrapolated to the thermodynamical limit using finite-size scaling.

1. Introduction  

Two kinds of topological defects were proposed by 't Hooft as possible configurations condensing in the disordered phase of \textit{SU}(N) Yang-Mills theories to produce confinement: \textit{Z}_N vortices \cite{1} and monopoles \cite{2}.  

Monopoles have the natural topology to three dimensional space, corresponding to a mapping: \textit{S}^2 \to \textit{SU}(2). Their condensation implies dual superconductivity of the vacuum, and gives an appealing physical picture of confinement in terms of dual Abrikosov tubes produced by dual Meissner effect \cite{3}. Indeed, condensation of magnetic charge has been unambiguously demonstrated by numerical simulations on the lattice \cite{4}. An operator \( \mu \) carrying non-zero magnetic charge is constructed, and its vev \( \langle \mu \rangle \) is measured in the confined and deconfined phases, as a candidate disorder parameter. The extrapolation to the thermodynamical limit is done by finite size scaling techniques and the result is:

\begin{align*}
\langle \mu \rangle &\neq 0, \quad \text{for } T < T_c, \quad (1) \\
\langle \mu \rangle & = 0, \quad \text{for } T > T_c \quad (2) \\
\langle \mu \rangle & \propto (1 - T/T_c)\delta, \quad \text{for } T \simeq T_c \quad (3)
\end{align*}

Both \( T_c \) and the critical index of the correlation length \( \nu \) are obtained, and they agree with determinations by other techniques \cite{5}. The values of \( \delta \) are:

\begin{align*}
\delta &= 0.25(10) \quad \text{for } \textit{SU}(2) \quad (4) \\
\delta &= 0.54(4) \quad \text{for } \textit{SU}(3)
\end{align*}

\( \langle \mu \rangle \neq 0 \) signals dual superconductivity.  

The same disorder parameter can be defined in full \textit{QCD}, and shows a similar behaviour at the deconfinement phase transition \cite{6}. This phenomenon seems to be independent of the Abelian projection chosen to define the monopoles. The meaning of these results is that the (yet unknown) excitations which condense in the confined phase and are weakly interacting in the dual picture, must be magnetically charged with respect to all abelian projections. Attempts to identify these excitations with the monopoles of the maximal abelian projection do not seem to work as expected \cite{7,8}.  

In order to understand better the properties of the dual excitations, we investigate by similar techniques the role of vortices \cite{9}. In 3+1 dimensional vortices are string-like topological defects, associated to closed curves \( C \). The operator \( B(C) \) which creates such a vortex obeys the following algebraic relation with the Wilson loop \( W(C') \), defined as the parallel transport along the curve \( C' \):

\begin{equation}
B(C)W(C') = W(C')B(C) \exp \left( \frac{i2\pi n_{CC'}}{N} \right) \quad (5)
\end{equation}

with \( n_{CC'} \) being the linking number of the curves \( C \) and \( C' \).  

It follows from Eq. \( \ref{5} \) that whenever \( \langle W(C') \rangle \) obeys the area law, \( \langle B(C) \rangle \) obeys the perimeter law and viceversa \cite{5}. A consequence of the area law for \( \langle W(C') \rangle \) is that the Polyakov line, defined as the parallel transport along a line in the time direction closed by periodic boundary con-
ditions (pbc), has to vanish. Instead, it can be that \( P(x) \neq 0 \) when the Wilson loop obeys the perimeter law. We argue that the same happens for the dual loops \( B(C) \). We define a disorder parameter \( \langle \mu \rangle \) as the “dual Polyakov line”, corresponding to the operator \( B(C) \) for a curve \( C \) going through the lattice, e.g. parallel to the \( z \) axis, at fixed time and closed by pbc. In the phase where \( \langle B(C) \rangle \) obeys the area law (and therefore \( \langle W(C') \rangle \) obeys a perimeter law), \( \langle \mu \rangle = 0 \); whenever \( \langle \mu \rangle \neq 0 \) the perimeter law is possible for \( \langle B(C) \rangle \). This observation helps in defining a disorder parameter in 3+1 dimension.

In 2+1 dimension the vortex is point-like, it can be described by a local field, and a conserved quantum number can be associated to it, which is broken in the disordered phase \( \Phi \). Instead, no conserved quantum number can be associated to vortices in 3+1 dimension.

2. Creation operator of a vortex

For the sake of definiteness, let us take for the curve \( C \) a rectangle \( R \) in the \( xy \) plane:
\[
R = \{(x, y, z) : x_0 < x < x_1, y = y_0, z_0 < z < z_1 \}
\]
(6)
The definition given below can be extended trivially to any curve \( C \).

We define the vacuum expectation value (vev) \( \langle B(C) \rangle \equiv \bar{Z}/Z \), where \( Z \) is the ordinary partition function for pure Yang-Mills theories on the lattice defined by the Wilson action:
\[
S[U] = \sum_{x, \mu, \nu} \text{Re} \text{Tr} [1 - P_{\mu\nu}]
\]
(7)
with \( P_{\mu\nu} \) is the usual plaquette. \( \bar{Z} \) is the partition function corresponding to the action \( \bar{S} \) obtained from \( S \) by the change:
\[
P_{\bar{y}y}(t_0, x_0 < x < x_1, y = y_0, z_0 < z < z_1) \mapsto e^{i2\pi/N} P_{\bar{y}y}(t_0, x_0 < x < x_1, y = y_0, z_0 < z < z_1)
\]

We study by numerical simulations, the behaviour of the “dual Polyakov line”, which corresponds to the above definition with:
\[
z_0 \to -\infty
\]
\[
z_1 \to +\infty
\]
in that case the change of variables becomes:
\[
P_{\bar{y}y}(t_0, x_0 < x < x_1, y = y_0, z) \mapsto e^{i2\pi/N} P_{\bar{y}y}(t_0, x_0 < x < x_1, y = y_0, z), \quad \text{for all } z
\]
(8)
For the proof that Eq. \( \bar{S} \) really corresponds to the definition of \( B(C) \) as defined by Eq. \( \bar{S} \) and for comparison with alternative definitions in the literature, we refer to [9].

We shall also compute the correlator:
\[
\Gamma(t) = \langle \mu(t_0, x_0, y_0) \mu(t_0 + t, x_0, y_0) \rangle \quad \text{for } t \to \infty,
\]
(9)
whence \( \langle \mu \rangle \) can be extracted.

At finite \( T \) there is no propagation in time and \( \langle \mu \rangle \) is computed directly, provided \( C^* \) boundary conditions are adopted [9].

3. Numerical results

We compute \( \langle \mu \rangle \) by the techniques explained above, or better we compute:
\[
\rho = \frac{d}{d\beta} \log \langle \mu(t_0, x_0, y_0) \rangle = \langle S \rangle \bar{S} - \langle \bar{S} \rangle \bar{S} \quad \text{(10)}
\]
which contains the same information and is less noisy [10, 11]. The behaviour of \( \rho \) vs \( \beta \) for a \( N_t \times N_s^3 \) lattice with \( N_t = 4 \) and \( N_s = 12, 16, 20, 24, 32 \) is plotted in Fig. [1]: the corresponding \( \rho \) for the creation of monopoles is plotted in the same figure for comparison. As explained in the introduction, \( \rho \) is independent within errors of the Abelian projection used to define the magnetic charges [4].

Similarly to the case of monopoles, the behaviour of \( \rho \) at large \( \beta \) is proportional to \( N_s \)
\[
\rho \sim -4N_s + C
\]

implying that \( \langle \mu \rangle = \exp \left[ -\int_0^\beta d\beta' \rho(\beta') \right] \) vanishes for \( \beta > \beta_c \) as \( N_s \to \infty \). At low \( \beta \), \( \rho \) is bounded from below when the volume is increased, which implies \( \langle \mu \rangle \neq 0 \) for \( \beta < \beta_c \).

Around \( \beta_c \), \( \langle \mu \rangle \sim (\beta_c - \beta)^{\delta} \). In the scaling region one expects:
\[
\langle \mu \rangle \sim (\beta_c - \beta)^{\delta} \Phi \left( \frac{N_s}{\xi} \right)
\]
where $\xi \sim (\beta_c - \beta)^{-\nu}$ is the correlation length.

This yields the scaling law:

$$\frac{\rho}{L^{1/\nu}} = f \left( L^{1/\nu} (\beta_c - \beta) \right)$$

(11)

We shall assume $f(x) = -\delta/x + c$. Requiring scaling fixes $\beta_c, \nu$ and $\delta$. The quality of the scaling for $SU(2)$ is shown in Fig. 2.

The critical exponents that we obtain are:

$\beta_c = 2.30(1), \delta = 0.5(1), \nu = 0.7(1)$

$\beta_c$ and $\nu$ are compatible with the standard determinations [3]. $\delta$ is equal within errors to the corresponding index for the monopole disorder parameter. Preliminary data for $SU(3)$ present a similar behaviour [12].

We conclude that the “dual Polyakov line” is a good disorder parameter for confinement, in the same way as the ordinary Polyakov line is a good order parameter. An analogous analysis for $SU(3)$ is in preparation and gives similar results [12]. A measurement of the critical indices using a monopole operator in full QCD is in progress [6]. We are also trying to extend to full QCD the study of the dual Polyakov line. The dependence on $N_c$ of both approaches is an interesting question, and deserves numerical investigation, in order to check the basic ideas of the large-$N_c$ limit.

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