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# A Hyperbolic PDE with Parabolic Behavior\*

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**Abstract.** In this paper we present a hyperbolic partial differential equation (PDE) in one space and one time dimension. This equation arose in a study of numerical schemes for simulating evolving river topographies. The solution of this PDE, whose initial data are specified along a characteristic, is very similar to that of the canonical diffusion equation. This interesting example provides insight into the solution of hyperbolic PDEs when data is specified in this pathological way as well as illustrating some connections between the parabolic and hyperbolic classes of evolution equations.

**Key words.** partial differential equations, method of characteristics, evolution equations, Goursat's theorem

**AMS subject classification.** 35

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**I. Introduction.** This article is about the two partial differential equations (PDEs)

$$(1.1) \quad \begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2} & \text{for } x \in \mathfrak{R}, t \in \mathfrak{R}^+, \\ w(x, 0) &= f(x) & \text{for } x \in \mathfrak{R} \end{aligned}$$

and

$$(1.2) \quad \begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial t} & \text{for } x \in \mathfrak{R}, t \in \mathfrak{R}^+, \\ z(x, 0) &= f(x) & \text{for } x \in \mathfrak{R}, \\ \lim_{|x| \rightarrow \infty} |z(x, t)| &< \infty, & t \in \mathfrak{R}^+. \end{aligned}$$

We shall see that the second of these two equations is quite mysterious. It is mysterious because, although it does not seem to have enough side conditions to be well posed, it has a unique solution. It is also mysterious because, despite being a hyperbolic equation, its solution looks like that of the heat equation (1.1) in a sense that is made explicit subsequently.

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A first course in PDEs teaches the canonical classification of linear constant-coefficient second-order PDEs into parabolic, hyperbolic, and elliptic types. We then learn the different types of initial and boundary data that make these PDEs well posed. Here the physical intuition gained by interpreting parabolic equations as the heat equation, hyperbolic equations as the wave equation, and elliptic equations as Laplace's equation can help. We also study the method of characteristics, particularly in the context of hyperbolic equations, since it provides one of the few methods that survives the transition to nonlinear PDEs. The idea of characteristics helps, in the case of parabolic and hyperbolic (or "evolution") PDEs, to motivate an analogy with ordinary differential equations (ODEs). The idea is that the heat equation has only a first derivative in time and so requires only one piece of initial data, while the wave equation has a second derivative in time and so requires two pieces of initial data. The necessity for these conditions is also seen when we take the Fourier transform of the heat or wave equations with respect to the space variable. This intuition is, of course, only valid if the initial data is not specified along a characteristic. Even many mathematicians who use PDEs in their daily work are quite vague about what happens then. Goursat's theorem, which applies to hyperbolic PDEs in which one piece of initial data is given on a characteristic with another piece given on an intersecting curve, is sometimes invoked. This is analogous to solving a second-order ODE not with two pieces of initial data but with two pieces of boundary data. In fact, some linear second-order ODEs may also be solved with one piece of boundary data and another, weaker boundary condition at infinity. In this paper we exhibit a linear constant-coefficient hyperbolic PDE (1.2) on an infinite spatial domain with only one piece of initial data specified, along a characteristic. We show that specification of a weak boundary condition at infinity is enough to force a unique solution similar to the solution of the heat equation (1.1).

Equation (1.2) arose from a detailed study [1], [2] of a numerical model of braided rivers [3]. Braided rivers are characterized by a constantly changing channel structure in which branches recombine and split. Detailed braided river models require a combination of fluid and sediment transport equations in three spatial dimensions with moving boundary conditions. Because it is difficult to solve such systems, more tractable "cellular" models have recently been proposed [3]. A cellular model simulates the transport of water and sediment along a river bed represented by a grid of cells. In [1] and [2] we show that this model can be considered the finite difference solution of a nonlinear PDE with diffusive properties. A related cellular model in which water and sediment are followed in a "Lagrangian" approach results in a code representing the finite difference solution of a PDE having both diffusive and wavelike terms. Despite the difference in the equations, our results were similar to those reported in [3]. To find out what was going on, we simplified the equations by choosing a (nonphysical) parameter range that linearized them, by ignoring one of the spatial dimensions, and by choosing units for  $x$  and  $t$  in which the proportionality constant was one. The resulting pair of linear second-order constant-coefficient PDEs is (1.1) and (1.2). In section 4 of this paper we show that the paradox of seemingly very different equations yielding similar solutions persists. This is somewhat remarkable given the apparent differences between the two. However, it is appropriate to note that, unlike the canonical wave equation  $u_{tt} = u_{xx}$ , neither (1.1) nor (1.2) are invariant under time reversal.

We present this example as an "experimental" result. In the final section of this paper we present an intuitive argument, based on the method of characteristics,

which explains this result. We think this example is fun because it brings together a lot of the threads, both theoretical and instrumental, from a first PDE course. In order to investigate this example we use techniques from Laplace transforms, we see a legitimate application of the method of variation of parameters, and we review some nice asymptotics. We also deepen our intuition about PDEs.

**2. The Two PDEs.** We consider two PDEs, one for  $w(x, t)$  and the other for  $z(x, t)$ . We take  $x \in \mathfrak{R}$  and  $t \in \mathfrak{R}^+$  and suppose both  $z$  and  $w$  to be sufficiently smooth. The first PDE has the solution

$$(2.1) \quad w(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x-u) e^{-\frac{u^2}{4t}} du,$$

the heat kernel convolved with  $f(x)$ . For bounded  $f$  we can therefore guarantee that

$$(2.2) \quad \lim_{|x| \rightarrow \infty} w(|x|, t) < \infty.$$

The second PDE for  $z(x, t)$  is, as we shall see, hyperbolic. We might therefore expect a unique solution to require two sets of initial data. In fact, because initial data is specified along a characteristic, we shall see that the addition of the weak condition analogous to (2.2),

$$\lim_{|x| \rightarrow \infty} z(|x|, t) < \infty,$$

suffices to make the equation well posed.

**3. Classification of the PDE.** We begin this section with a very brief and intuitive review of the ideas of characteristic curves and the classification of linear PDEs. More precise statements of the results and rigorous proofs may be found in many books on PDEs. The explanation in this section follows that of Carrier and Pearson [4, Chapter 5].

We begin by considering the diffusion equation (1.1). We may obtain a power series solution of this problem in the following way. We know  $w(x, 0)$  and hope to obtain  $w(x, t)$  via

$$w(x, t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\partial^k w}{\partial t^k}(x, 0).$$

To do this we need to find a way of evaluating all the partial derivatives with  $t$  of  $w$  at  $(x, 0)$ . The idea behind this is simple. For example, we use equation (1.1) to write  $\frac{\partial w}{\partial t}(x, 0) = \frac{\partial^2 w}{\partial x^2}(x, 0) = f''(x)$ . Repeated differentiation of this yields time derivatives of all orders. The Cauchy–Kowalewsky (CK) theorem ensures, under some rather weak conditions, both that this procedure is admissible and that the resulting series converges. A natural question to consider is whether the CK method still works with general “Cauchy data” which specifies  $w$  on a more general curve  $\zeta(x, y) = C$ . That the method does not work for all curves can be seen by considering

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2} && \text{for } x \in \mathfrak{R}, t \in \mathfrak{R}^+, \\ w(0, t) &= h(t) && \text{for } t \in \mathfrak{R}^+. \end{aligned}$$

Here the idea would be to find the Taylor series of the solution using spatial derivatives of all orders. But our earlier idea allows only even-order spatial partial derivatives of  $w$  to be computed. So the method fails.

If Cauchy data along a curve is not sufficient to solve a PDE using the CK method, the curve is said to be a *characteristic* of that PDE. Note that this does *not* mean that the PDE cannot be solved, just that the PDE cannot be solved using this method.

Turning our attention to the canonical wave equation

$$(3.1) \quad \begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} && \text{for } x \in \mathfrak{R}, t \in \mathfrak{R}^+, \\ u(x, 0) &= f(x) && \text{for } x \in \mathfrak{R}, \\ \frac{\partial u}{\partial t}(x, 0) &= g(x) && \text{for } x \in \mathfrak{R}, \end{aligned}$$

we see that the CK method will fail if  $u(x, 0)$  alone is specified but that it will work if the condition  $\frac{\partial u}{\partial t}(x, 0) = g(x)$  is added. It can also be shown that the families of curves  $x - t = C_1$  or  $x + t = C_2$  are characteristic curves for the wave equation.

Some more intuition about characteristics is obtained by setting  $g(x) = 0$  in equation (3.1). D'Alembert's solution of the simplified problem is

$$u(x, t) = \frac{1}{2} \left( f(x - t) + f(x + t) \right).$$

This shows that the solution of the wave equation at an arbitrary point  $P = (x, t)$  may be obtained by using the characteristic curves to “bring” initial data to the point. In particular, this shows that initial data outside the “cone” defined by the two characteristics which pass through  $P$  does not affect the solution at  $P$ . For a sketch of this, see the upper left-hand panel of Figure 6.1.

The general second-order constant-coefficient partial differential equation in  $x$  and  $t$  may be classified by the number of independent families of characteristics. If, as for the wave equation, there are two independent families, the PDE is said to be hyperbolic. A parabolic PDE is one for which there is just one independent family of characteristic curves (it turns out that the diffusion equation falls into this category). If there are no characteristic curves, the PDE is said to be elliptic. Determining these curves is a simple problem in multivariate calculus covered in, for example, [5, Chapter 3].

It is easy to see that (1.2) is everywhere hyperbolic. The characteristics of this equation are  $t = \zeta$  and  $x - t = \eta$ . This means our initial conditions are specified along a characteristic. That sets off warning bells. The fact that the lines of constant time are characteristics provides the first hint that this equation may have diffusive behavior, since the diffusion equation (1.1) shares the characteristic  $t = \zeta$ . This tells us that, while (1.2) is hyperbolic, it shares the “infinite propagation speed of initial data” property of parabolic equations. The characteristics may be used to write the PDE in the first canonical form

$$\begin{aligned} \frac{\partial^2 z}{\partial \zeta \partial \eta} &= \frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \eta}, \\ z(\eta, 0) &= f(\eta). \end{aligned}$$

The change of variables  $\alpha = x$ ,  $\beta = 2t - x$  may be used to write (1.2) in the second canonical form:

$$\frac{1}{2} \left( \frac{\partial^2 z}{\partial^2 \alpha} - \frac{\partial^2 z}{\partial^2 \beta} \right) = \frac{\partial z}{\partial \beta},$$

$$z(-\beta, \beta) = f(-\beta).$$

These forms showcase the wavelike nature of the equation, albeit with first derivative terms where the canonical wave equation has no such terms. However, neither is easier to solve than the initial equation. As we shall see, the behavior of the solutions here is more diffusive than wavelike. It is interesting that changing the partial differential operator with respect to the first derivative terms, which do not even come into the classification scheme, leads to the qualitative change in solution properties shown in section 5 of this paper.

A hyperbolic equation can be solved by using two characteristics to bring data from different points along the curve to any point in the solution region. If only one function is specified, this procedure breaks down. The solution of linear hyperbolic PDEs with data given on a characteristic has been studied and is called the Goursat problem [6, pp. 117–119].

The Goursat problem is the problem of finding the solution of a linear hyperbolic PDE satisfying a prescribed condition on one characteristic and another prescribed condition on a monotonic increasing curve which intersects that characteristic. Its solution still requires two sets of initial data. The underlying physical problem does not supply us with the second set of initial data here. In the next section we shall see that we can nonetheless solve (1.2).

**4. Solutions.** We solve (1.2) using the Laplace transform. We show that a unique solution which uses one initial condition and a weak boundary condition at infinity can be obtained using Laplace transforms with respect to the variable  $t$ . Let  $Z(x, s)$ ,  $s > 0$ , be the Laplace transform of  $z(x, t)$ . Then  $Z(x, s)$  satisfies

$$(4.1) \quad \frac{\partial^2 Z}{\partial x^2} + s \frac{\partial Z}{\partial x} - sZ = -g(x) \quad \text{for } x \in \mathfrak{R}, s \in \mathfrak{R}^+,$$

$$\lim_{|x| \rightarrow \infty} |Z(x, s)| < \infty \quad \text{for } s \in \mathfrak{R}^+,$$

where

$$g(x) = f(x) - f'(x).$$

The solution to problem (1.2) is the inverse Laplace transform of the solution  $Z(x, s)$ . Below we prove that the solution  $z(x, t)$  is

$$(4.2) \quad z(x, t) = \int_{-\infty}^t g(x-u) e^{u-2t} I_0 \left( 2\sqrt{t(t-u)} \right) du,$$

where  $I_0$  is the modified Bessel function of the first kind of order 0. In particular, since  $z(x, t)$  is continuous and  $Z(x, s)$  is the unique solution of system (4.1), Lerch's theorem [7, p. 119] guarantees that  $z(x, t)$  is the unique solution of the original problem (1.2). Now we supply the details.

**4.1. Solution of (4.1).** We solve the system (4.1), where  $g(x)$  is a continuous, bounded, integrable function.

This is a second-order linear differential equation in the variable  $x$ . The solution has the form

$$Z(x, s) = A(s)Z_1(x, s) + B(s)Z_2(x, s) + Z_p(x, s),$$

where

$$\begin{aligned} Z_1(x, s) &= e^{\frac{\sqrt{s^2+4s}-s}{2}x}, \\ Z_2(x, s) &= e^{-\frac{s+\sqrt{s^2+4s}}{2}x} \end{aligned}$$

are solutions to the homogeneous version of (4.1). The general solution  $Z(x, s)$  can be obtained from the homogeneous solutions by the method of variation of parameters. The general solution has the form

$$(4.3) \quad \begin{aligned} Z(x, s) &= \left[ A(s) - \frac{1}{\sqrt{s^2+4s}} \int_{-\infty}^x g(u) e^{-r_2(s)u} du \right] e^{r_2(s)x} \\ &+ \left[ B(s) + \frac{1}{\sqrt{s^2+4s}} \int_{-\infty}^x g(u) e^{-r_1(s)u} du \right] e^{r_1(s)x}, \end{aligned}$$

where

$$\begin{aligned} r_1(s) &= \frac{1}{2}(s - \sqrt{s^2+4s}), \\ r_2(s) &= \frac{1}{2}(s + \sqrt{s^2+4s}). \end{aligned}$$

The boundary conditions of (4.1) imply

$$\begin{aligned} A(s) &= \frac{1}{\sqrt{s^2+4s}} \int_{-\infty}^{\infty} g(u) e^{-r_2(s)u} du, \\ B(s) &= 0. \end{aligned}$$

Substituting the equations for  $A(s)$  and  $B(s)$  into the above equation for  $Z(x, s)$  yields the unique solution

$$(4.4) \quad Z(x, s) = \frac{1}{\sqrt{s^2+4s}} \int_{-\infty}^{\infty} g(u) e^{-\frac{s}{2}(x-u)} e^{-\frac{1}{2}\sqrt{s^2+4s}|x-u|} du.$$

The solution  $z(x, t)$  is the inverse Laplace transform of  $Z(x, s)$  in equation (4.4). We find this inverse Laplace transform next.

**4.2. Inverting the Laplace Transform.** Let  $F(s)$  be the Laplace transform of  $f(t)$ . Then the following are true:

$$(4.5) \quad \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t),$$

$$(4.6) \quad \mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)H(t-a),$$

$$(4.7) \quad \mathcal{L}^{-1}\left\{(s^2-a^2)^{-\frac{1}{2}}e^{-b\sqrt{s^2-a^2}}\right\} = H(t-b)I_0(a\sqrt{t^2-b^2}),$$

where the function  $I_0$  in (4.7) is the modified Bessel function of the first kind and zeroth order and  $H(x)$  is the Heaviside step function: 1 for positive and 0 for negative inputs.

The proofs of the first two relations can be found in any standard Laplace transform textbook (e.g., [7, p. 127]). The third equation follows from (4.6) and [8, formula 42, p. 250].

We need to find the inverse Laplace transform of the function

$$U(s, y) = \frac{1}{\sqrt{s^2 + 4s}} e^{-\frac{y}{2}s} e^{-\frac{|y|}{2}\sqrt{s^2+4s}},$$

with variable  $s$  and parameter  $y$ , since then

$$(4.8) \quad z(x, t) = \int_{-\infty}^{\infty} g(u) \mathcal{L}^{-1} \{U(s, x - u)\} du.$$

Rewriting  $U(s, y)$  as

$$U(s, y) = \frac{1}{\sqrt{(s + 2)^2 - 4}} e^{-\frac{y}{2}(s+2)} e^{y} e^{-\frac{1}{2}\sqrt{(s+2)^2-4}|y|}$$

and applying the above Laplace transform formulas yields

$$\mathcal{L}^{-1}(U(s, y)) = e^{y-2t} H\left(t - \frac{y}{2}\right) H\left(\left(t - \frac{y}{2}\right) - \frac{|y|}{2}\right) I_0\left(2\sqrt{\left(t - \frac{y}{2}\right)^2 - \frac{y^2}{4}}\right).$$

This can be simplified to

$$\mathcal{L}^{-1}\{U(s, y)\} = H(t - \max(0, y)) e^{y-2t} I_0\left(2\sqrt{t(t-y)}\right),$$

which, using the substitution  $u = x - y$  and (4.8), yields the solution

$$z(x, t) = \int_{-\infty}^t g(x - y) e^{y-2t} I_0\left(2\sqrt{t(t-y)}\right) dy.$$

In order to recast the solution of this problem to a form that is easily compared to (2.1), we require the following lemma.

LEMMA 4.1. *Let*

$$z(x, t) = \int_{-\infty}^t \left[ f(x - u) - f'(x - u) \right] e^{u-2t} I_0(2\sqrt{t(t-u)}) du,$$

where  $x$  and  $t$  are real numbers satisfying  $0 < t < \infty$  and  $-\infty < x < \infty$  and where  $f(x)$  is a smooth bounded function. Then

$$z(x, t) = e^{-t} f(x - t) + \int_{-\infty}^t f(x - u) e^{u-2t} \sqrt{\frac{t}{t-u}} I_1\left(2\sqrt{t(t-u)}\right) du.$$

*Proof.* We recall some facts about modified Bessel functions. The following are taken from [9]:

$$(4.9) \quad \begin{aligned} I_0'(x) &= I_1(x), \\ I_0(0) &= 1, \\ I_1(0) &= 0. \end{aligned}$$

When  $\nu$  is fixed and  $x$  is small,

$$(4.10) \quad I_\nu(x) \sim \frac{x^\nu}{2^\nu \Gamma(\nu + 1)},$$

and when  $\nu$  is fixed and  $x$  is large,

$$(4.11) \quad I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left( 1 - \frac{4\nu^2 - 1}{8x} + \dots \right).$$

To prove the lemma we write  $z(x, t) = A_1 - A_2$ , where

$$A_1 = \int_{-\infty}^t f(x-u) e^{u-2t} I_0(2\sqrt{t(t-u)}) du,$$

$$A_2 = \int_{-\infty}^t f'(x-u) e^{u-2t} I_0(2\sqrt{t(t-u)}) du.$$

Integrate  $A_2$  by parts to obtain

$$A_2 = -f(x-u) e^{u-2t} I_0(2\sqrt{t(t-u)}) \Big|_{-\infty}^t + \int_{-\infty}^t f(x-u) \frac{d}{du} \left[ e^{u-2t} I_0(2\sqrt{t(t-u)}) \right] du.$$

Now

$$\lim_{u \rightarrow -\infty} f(x-u) e^{u-2t} I_0(2\sqrt{t(t-u)}) = \lim_{u \rightarrow \infty} e^{-u} f(x+u) e^{-2t} I_0(2\sqrt{t(t+u)}).$$

Using (4.11) this becomes

$$\begin{aligned} \lim_{u \rightarrow \infty} e^{-u} f(x+u) e^{-2t} \frac{e^{2\sqrt{t(t+u)}}}{2\sqrt{\pi} \sqrt{t(t+u)}} &= \lim_{u \rightarrow \infty} \frac{e^{-(t+u-2\sqrt{t(t+u)}+t)}}{2\sqrt{\pi} \sqrt{t} \sqrt{\frac{u}{t}}} f(x+u) \\ &= \lim_{u \rightarrow \infty} \frac{e^{-(\sqrt{t+u}-\sqrt{t})^2}}{2\sqrt{\pi} \sqrt{t} \sqrt{\frac{u}{t}}} f(x+u) = 0, \end{aligned}$$

since  $f(x+u)$  is bounded as  $u \rightarrow \infty$ .

Thus

$$A_2 = -e^{-t} I_0(0) f(x-t) + \int_{-\infty}^t f(x-u) \left[ e^{u-2t} I_0(2\sqrt{t(t-u)}) + e^{u-2t} \frac{d}{du} I_0(2\sqrt{t(t-u)}) \right] du.$$

But  $I_0(0) = 1$ , so

$$\begin{aligned} A_2 &= -e^{-t} f(x-t) + \int_{-\infty}^t f(x-u) e^{u-2t} I_0(2\sqrt{t(t-u)}) du \\ &\quad + \int_{-\infty}^t e^{u-2t} I_0'(2\sqrt{t(t-u)}) \frac{d}{du} (2\sqrt{t(t-u)}) du. \end{aligned}$$

The second summand of this is  $A_1$  and  $z = A_1 - A_2$ , so we can put the two pieces together to obtain

$$z(x, t) = e^{-t}f(x - t) + \int_{-\infty}^t f(x - u)e^{u-2t}I_0'\left(2\sqrt{t(t - u)}\right)\sqrt{\frac{t}{t - u}}du.$$

We finish the proof using (4.9):

$$z(x, t) = e^{-t}f(x - t) + \int_{-\infty}^t f(x - u)e^{u-2t}I_1(2\sqrt{t(t - u)})\sqrt{\frac{t}{t - u}}du. \quad \square$$

**5. Comparing the Two Solutions.** With this result in hand we are able to progress towards our goal of comparing the solution (2.1) of the diffusion equation,

$$w(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x - u)e^{-\frac{u^2}{4t}} du,$$

with the solution of the hyperbolic equation (1.2):

$$(5.1) \quad z(x, t) = e^{-t}f(x - t) + \int_{-\infty}^t f(x - u)e^{u-2t}\sqrt{\frac{t}{t - u}}I_1(2\sqrt{t(t - u)})du.$$

At first glance the two solutions do not appear similar. For example, if the initial function  $f(x)$  is even, (2.1) must also be even in  $x$  at any fixed time  $t > 0$ . In contrast, (5.1) involves integration over the interval  $(-\infty, t]$ , indicating an asymmetric time evolution of an even initial function. Despite these premonitions, numerical results (see Figure 5.1) indicate that the two solutions become very similar.

Now we provide an analytic result that supports these numerical results. We begin with the Green’s function case in which the initial data is a Dirac delta function  $f(x) = \delta(x)$ . Then

$$w_\delta(x, t) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t}}, \quad t > 0,$$

and

$$z_\delta(x, t) = e^{x-2t}\sqrt{\frac{t}{t - x}}I_1(2\sqrt{t(t - x)}), \quad 0 < x < t, \quad t > 0.$$

Note that the restriction on  $x$  is a direct consequence of the characteristic curve  $x = t$  bounding the rate at which information specified at  $x = 0$  can travel to the right.

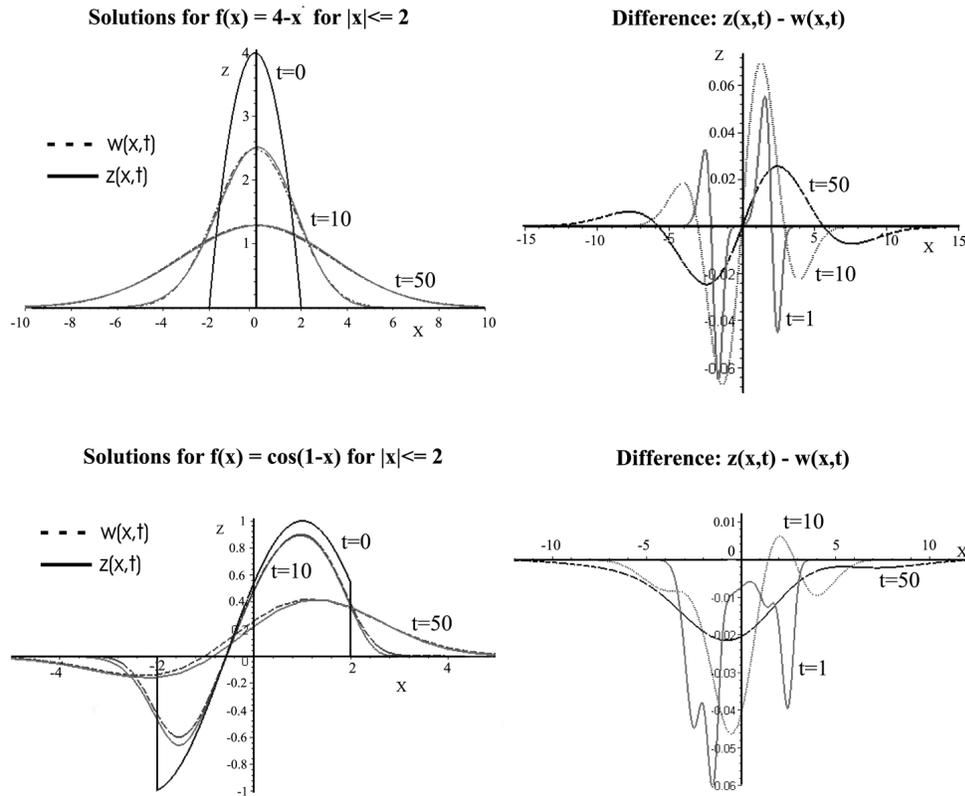
LEMMA 5.1. *Define the relative error between the two solutions as*

$$(5.2) \quad r_\delta(x, t) = \frac{|z_\delta(x, t) - w_\delta(x, t)|}{w_\delta(x, t)}.$$

For  $x \ll t$ ,

$$\lim_{t \rightarrow \infty; x \sim t^\gamma} r_\delta(x, t) \rightarrow 0 \quad \text{like} \quad \frac{3x}{4t},$$

where  $0 \leq \gamma < \frac{1}{2}$ . This, of course, includes the case in which we fix  $x$  and allow  $t$  to grow without bound.



**Fig. 5.1**  $w(x, t)$  and  $z(x, t)$ .

*Proof.* Write

$$r_{\delta}^x(t) = \left| 2\sqrt{\pi t} e^{x-2t} e^{\frac{x^2}{4t}} \left(1 - \frac{x}{t}\right)^{-\frac{1}{2}} I_1 \left(2t\sqrt{1 - \frac{x}{t}}\right) - 1 \right|.$$

Under the assumptions of the lemma, the argument of  $I_1$  grows without bound. We may therefore use (4.11) and some cancellation to rewrite this as

$$r_{\delta}(x, t) \sim \left| e^{x-2t + \frac{x^2}{4t} + 2t\sqrt{1 - \frac{x}{t}}} \left[ \left(1 - \frac{x}{t}\right)^{-\frac{3}{4}} - \frac{3}{16t} \left(1 - \frac{x}{t}\right)^{-\frac{5}{4}} \right] - 1 \right|.$$

To leading order, the second part of this expression may be neglected, yielding

$$r_{\delta}(x, t) \sim \left| e^{x-2t + \frac{x^2}{4t} + 2t\sqrt{1 - \frac{x}{t}}} \left(1 - \frac{x}{t}\right)^{-\frac{3}{4}} - 1 \right|.$$

Now expand  $\sqrt{1 - \frac{x}{t}}$  in a Taylor series to write

$$\sqrt{1 - \frac{x}{t}} \sim 1 - \frac{x}{2t} - \frac{x^2}{8t^2} - \frac{x^3}{16t^3}.$$

Inserting this into the argument of the exponential and cancelling yields

$$r_{\delta}(x, t) \sim \left| e^{-\frac{x^3}{8t^2}} \left(1 - \frac{x}{t}\right)^{-\frac{3}{4}} - 1 \right|.$$

Provided that  $x \sim t^\gamma$  when  $0 \leq \gamma < \frac{2}{3}$ , this will vanish in the limit as  $t \rightarrow \infty$ . Here  $0 < \gamma < \frac{1}{2}$ , so we may expand in one more Taylor series and keep terms of leading order to write

$$z_\delta(x, t) \sim w_\delta(x, t) \left( 1 + \frac{3x}{4t} + \cdots \right),$$

and so

$$r_\delta(x, t) \sim \frac{3x}{4t}.$$

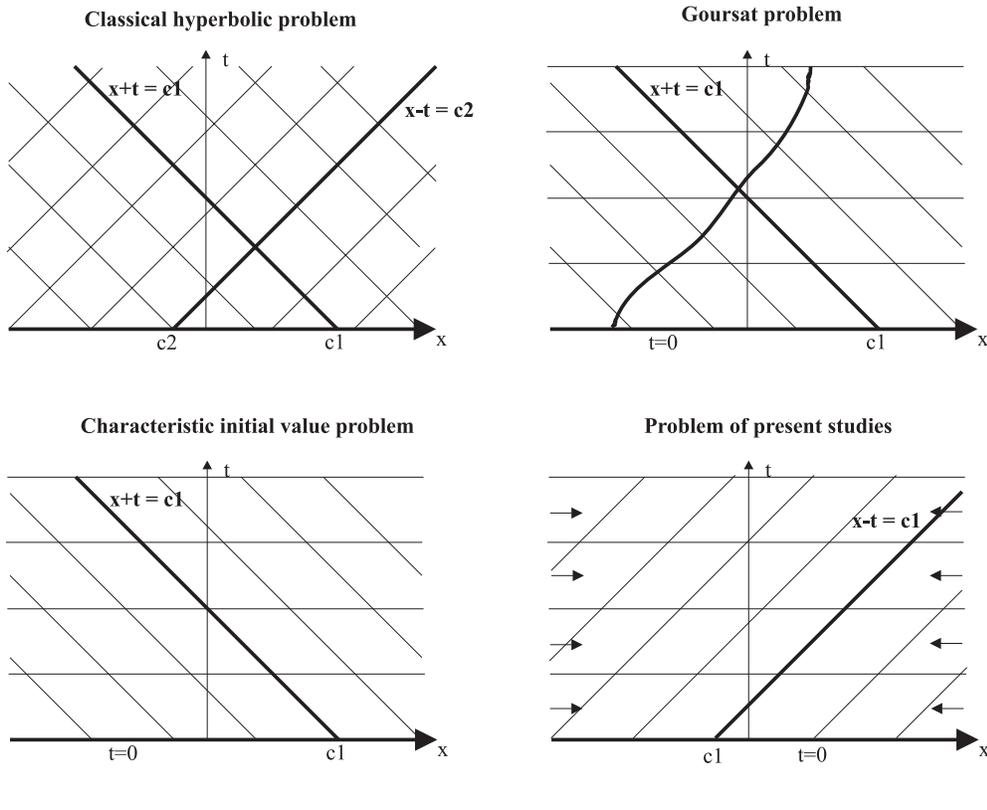
In the case  $x \sim \sqrt{At}$ , which is motivated by the random walk solution of the heat equation, this must be modified to read

$$r_\delta(x, t) \sim \frac{(6-A)x}{8t}. \quad \square$$

A similar result should also hold for general bounded initial conditions  $w(x, 0) = z(x, 0) = f(x)$ . Let  $f \star g = \int_{-\infty}^{\infty} f(u)g(x-u)du$  denote the convolution of  $f$  with  $g$ . Then  $w(x, t) = (w_\delta \star f)(x, t)$  and  $z(x, t) = (z_\delta \star f)(x, t)$ . Therefore,  $z - w = (z_\delta - w_\delta) \star f$ . Now, from the above work we see that  $z_\delta - w_\delta \sim \frac{3}{4} \sqrt{\frac{1}{t}} w_\delta$  so  $z - w \sim \frac{3}{4} \sqrt{\frac{1}{t}} w_\delta \star f$ . So our result for Green's functions carries over, at least approximately, to the more general case.

**6. Discussion and Conclusions.** This example illustrates some technicalities of the method of characteristics. Figure 6.1 is a schematic diagram of how the method of characteristics works for hyperbolic PDEs in one space and one time variable. The upper left-hand panel represents the typical case in which the two pieces of initial data specified along  $t = 0$  are brought by the two characteristics to each point for which a solution is possible. The upper right-hand panel represents the generic Goursat problem in which the line  $t = 0$  is a characteristic. The information from this line may still be distributed within the solution region by the second family of characteristics. The required second piece of data at each point within the solution region is collected from another curve which intersects the line  $t = 0$  by the other family of characteristics. The bottom left-hand panel represents the "characteristic initial value problem" [6, p. 118] in which both pieces of data are specified on a characteristic. The characteristic initial value problem may be solved in the same way as the more general Goursat problem. The bottom right-hand panel represents the case we meet here, in which a weak "finiteness" condition is specified at infinity. This condition, while weak, is enough in this setting to reduce to one piece the amount of data required to force uniqueness at each point.

In the typical hyperbolic case, the two characteristics that intersect a point in the solution region define a "light cone" which eventually intersects a finite subset of the  $t = 0$  line. Initial data taken from this subset is then used to determine the solution value at that point. In this case data specified at  $|x| = \infty$  cannot impact the solution at any finite space and time. If one of the families of characteristics is  $t = \zeta$ , making initial data specified along the characteristic  $t = 0$ , however, data specified at  $|x| = \infty$  can be brought to bear at any point within the solution region. This "finiteness" information can be combined with the initial data carried to the point by the other characteristic to determine the solution at that point. Whether or not such a set of information is enough to fully specify the solution will depend on the



**Fig. 6.1** Characteristics for hyperbolic PDE.

problem, but in our case (and in the following second-order linear ODE), it is. In such cases it is the pathological nature of the problem which *allows* unique solutions to be found.

Along the same lines, we may write down a pair of ODEs which are, in some sense, analogous to the problem treated here. They are

$$(6.1) \quad \begin{aligned} \frac{dx}{dt} &= -x \quad \text{for } t > 0, \\ x(0) &= A, \end{aligned}$$

and

$$(6.2) \quad \begin{aligned} \frac{d^2y}{dt^2} + (1 - \alpha) \frac{dy}{dt} &= \alpha y \quad \text{for } t > 0, \\ y(0) &= A, \\ \lim_{t \rightarrow \infty} |y(t)| &< \infty. \end{aligned}$$

These seemingly different equations have the same solution:  $x(t) = y(t) = Ae^{-t}$ . If  $\alpha = 0$ , the similarity between the two equations becomes clear. The pathological nature of the example is clearly revealed by rewriting the second ODE as  $(y' + y)' - \alpha(y' + y) = 0$ . Note that in this ODE example the two solutions are identical for all

time, while in the PDE example they have the same initial and final values, deviating at intermediate times.

Our interest in these equations came from numerical analysis. Numerical analysts prefer parabolic equations to hyperbolic ones because it is much harder to cope with numerical instability in the hyperbolic case. Here we show an example of a hyperbolic equation that is “close” to a superficially quite different parabolic equation. This is somewhat reminiscent of the viscosity method technique [10] for solving hyperbolic equations by adding a small diffusion term to smooth sharp wavefronts and shocks.

This result also ties in with ongoing research on the similarities and differences between the diffusion and wave equations. It connects with the physically motivated papers of [11], [12], and [13] which postulate a class of fractional PDEs that smoothly interpolate between the diffusion and wave equations. This class of problems, however, still requires two initial data functions for the hyperbolic while requiring just a single initial function for the parabolic case. Another equation with wavelike and diffusive properties is the telegrapher’s equation (see [14] for an illuminating analysis). However, this also requires two pieces of initial data. The example produced here could lead to a smooth transition between a hyperbolic and a parabolic equation without requiring an abrupt jump in the number of initial data required.

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