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DELTA INVARIANTS OF SINGULAR DEL PEZZO SURFACES

IVAN CHELTSOV, JIHUN PARK, CONSTANTIN SHRAMOV

Abstract. We use the methods introduced by Cheltsov–Rubinstein–Zhang in [CRZ18] to estimate δ-invariants of the seven singular del Pezzo surfaces with quotient singularities studied by Cheltsov–Park–Shramov in [CPS10] that have α-invariants less than $\frac{2}{3}$. As a result, we verify that each of these surfaces admits an orbifold Kähler–Einstein metric.

All varieties are assumed to be complex, projective and normal unless otherwise stated.

1. Introduction

Let S_d be a quasismooth and well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d, where $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_4$. Then S_d is given by a quasihomogeneous polynomial equation of degree d

$$f\big(x, y, z, t\big) = 0 \subset \mathbb{P}(a_0, a_1, a_2, a_3) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t\big]\Big),$$

where $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$. Here, being quasismooth simply means that the above equation defines a hypersurface that is singular only at the origin in $\mathbb{C}^4$, which implies that S_d has at most cyclic quotient singularities. On the other hand, being well-formed implies that

$$K_{S_d} \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}(a_0,a_1,a_2,a_3)}\big(d - a_0 - a_1 - a_2 - a_3\big),$$

see [Do82, Theorem 3.3.4], [IF00, 6.14].

Put $I = a_0 + a_1 + a_2 + a_3 - d$ and suppose that I is positive. Then S_d is a del Pezzo surfaces with at most quotient singularities. If S_d is smooth, then it always admits a Kähler–Einstein metric by [T90] (see also [C08, S10, T12, OSS16]). Singular del Pezzo surfaces with orbifold Kähler–Einstein metrics drew attention from Riemannian geometers because they may lift to Sasakian–Einstein 5-manifolds through S^1-bundle structures. Through this passage, Boyer, Galicki and Nakamaye yielded a significant amount of examples towards classification of simply-connected Sasakian–Einstein 5-manifolds (see [BGN03, BG08]).

For $I = 1$, Johnson and Kollár presented an algorithm in [JK01] that produces the (infinite) list of all possibilities for the quintuple (a_0, a_1, a_2, a_3, d) They also proved the following result:

Theorem 1.1 ([JK01, Theorem 8])**.** Suppose that S_d with $I = 1$ is singular and the quintuple (a_0, a_1, a_2, a_3, d) is not one of the following four quintuples:

$$(1, 2, 3, 5, 10), (1, 3, 5, 7, 15), (1, 3, 5, 8, 16), (2, 3, 5, 9, 18). \tag{1.2}$$

Then S_d admits an orbifold Kähler–Einstein metric.

Its proof uses the criterion given by the α-invariant (for the definition, see [CS08, Definition 1.2]) of the surface S_d, see [T87, N90, DK01]. It says that the surface S_d admits an (orbifold) Kähler–Einstein metric if the inequality

$$\alpha\big(S_d\big) > \frac{2}{3} \tag{1.3}$$

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holds, where $\alpha(S_d)$ is the α-invariant of the surface S_d. Indeed, Johnson and Kollár verified (1.3) in the case when $I = 1$, the surface S_d is singular, and the quintuple (a_0, a_1, a_2, a_3, d) is not one of the four exceptions (1.2). Two of the four remaining cases (1.2) have been treated in [A02] by Araujo, who proved the following result:

Theorem 1.4 ([A02, Theorem 4.1])**.** In the following two cases:

- $(a_0, a_1, a_2, a_3, d) = (1, 2, 3, 5, 10)$,
- $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the equation of S_d contains yzt,

the inequality $\alpha(S_d) > \frac{2}{3}$ holds. In particular, S_d admits an orbifold Kähler–Einstein metric.

The remaining two cases of (1.2) have been dealt with in the paper [CPS10]. In this paper, we succeeded in estimating their α-invariants from below by large enough numbers for the criterion (1.3). To be precise, we proved the following result:

Theorem 1.5 ([CPS10, Theorem 1.10])**.** Suppose that $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 8, 16)$ or $(2, 3, 5, 9, 18)$. Then $\alpha(S_d) > \frac{2}{3}$. In particular, S_d admits an orbifold Kähler–Einstein metric.

In particular, if $I = 1$, then S_d admits an orbifold Kähler–Einstein metric except possibly the case when $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the defining equation of the surface S_d does not contain yzt. Note that in the latter case one has

$$\alpha(S_d) = \frac{8}{15} < \frac{2}{3}$$

by [CPS10, Theorem 1.10], so that the criterion by the α-invariant could not be applied.

Meanwhile, since 2010 we have witnessed dramatic developments in the study of the Yau–Tian–Donaldson conjecture concerning the existence of Kähler–Einstein metrics on Fano manifolds and stability. The challenge to the conjecture has been heightened by Chen, Donaldson, Sun and Tian who have completed the proof for the case of Fano manifolds with anticanonical polarisations [CDS15, T15]. Following this celebrated achievement, useful technologies have been introduced to determine whether given Fano varieties are Kähler–Einstein or not, via the theorem of Chen–Donaldson–Sun and Tian.

Recently Fujita and Odaka introduced a new invariant of Fano varieties, which they called δ-invariant (for the definition, see [FO18, Definition 1.2]), that serves as a strong criterion for uniform K-stability (see [FO18]).

Theorem 1.6 ([FO18, BJ17])**.** Let X be a Fano variety with at most Kawamata log terminal singularities. Then X is uniformly K-stable if and only if $\delta(X) > 1$.

This powerful tool has been practiced for del Pezzo surfaces in [PW18, CRZ18, CZ18]. Around the same time, Li, Tian and Wang proved in [LTW17, LTW19] that the result of Chen, Donaldson, Sun and Tian also holds for some singular Fano varieties. In particular, it holds for del Pezzo surfaces with quotient singularities. Thus, if $\delta(S_d) > 1$, then the surface S_d admits an (orbifold) Kähler–Einstein metric. Note that $3\alpha(S_d) \geqslant \delta(S_d) \geqslant \frac{3}{2}\alpha(S_d)$ by [BJ17, Theorem A].

Now we are strongly reinforced by these new technologies, so that we could complete the study of existence of an (orbifold) Kähler–Einstein metric on the surface S_d in the case $I = 1$ started by Johnson and Kollár in [JK01]. In this paper, we prove the following result:

Theorem 1.7. Let S_d be a quasi-smooth hypersurface in $\mathbb{P}(1, 3, 5, 7)$ of degree 15 such that its defining equation does not contain yzt. Then $\delta(S_d) \geqslant \frac{6}{5}$. In particular, the surface S_d admits an orbifold Kähler–Einstein metric.

Corollary 1.8. If $I = 1$, then S_d admits an orbifold Kähler–Einstein metric.

For $I \geqslant 2$, the problem of existence of an orbifold Kähler–Einstein metric on the surface S_d was first studied by Boyer, Galicki and Nakamaye in [BGN03]. In this case, there is no reasonable classification similar to that obtained by Johnson and Kollár in [JK01]; note however that [P18] presents an algorithm that produces the (infinite) list of all possibilities for the quintuple (a_0, a_1, a_2, a_3, d) for every fixed $I \geqslant 2$. In [BGN03, CPS10, CS13], the existence of an orbifold Kähler–Einstein metric has been proved for (infinitely) many surfaces S_d with $I \geqslant 2$. However, in the following six cases their method did not work:

(1) $(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12)$ and the equation of S_d does not contain yzt;
(2) $(a_0, a_1, a_2, a_3, d) = (7, 10, 15, 19, 45)$;
(3) $(a_0, a_1, a_2, a_3, d) = (7, 18, 27, 37, 81)$;
(4) $(a_0, a_1, a_2, a_3, d) = (7, 15, 19, 32, 64)$;
(5) $(a_0, a_1, a_2, a_3, d) = (7, 19, 25, 41, 82)$;
(6) $(a_0, a_1, a_2, a_3, d) = (7, 26, 39, 55, 117)$.

In this paper, we use δ-invariants to show that S_d is Kähler–Einstein in these six cases as well:

Theorem 1.9. Suppose that (a_0, a_1, a_2, a_3, d) is one of the six quintuples listed above. Then $\delta(S_d) \geqslant \frac{65}{64}$. In particular, the surface S_d admits an orbifold Kähler–Einstein metric.

According to the similarity of the proofs, we handle the seven types of del Pezzo surfaces in Theorems 1.7 and 1.9 into three cases as follows:

Case A. $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the equation of S_d does not contain yzt;
$(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12)$ and the equation of S_d does not contain yzt;
Case B. $(a_0, a_1, a_2, a_3, d) = (7, 15, 19, 32, 64)$;
$(a_0, a_1, a_2, a_3, d) = (7, 19, 25, 41, 82)$;
Case C. $(a_0, a_1, a_2, a_3, d) = (7, 10, 15, 19, 45)$;
$(a_0, a_1, a_2, a_3, d) = (7, 18, 27, 37, 81)$;
$(a_0, a_1, a_2, a_3, d) = (7, 26, 39, 55, 117)$.

We will handle each of these cases separately in Sections 3, 4 and 5, respectively; see Corollaries 3.3, 4.3 and 5.6. In Section 2, we will present some results that will be used in the proofs of Theorems 1.7 and 1.9.

Let us briefly explain how we estimate $\delta(S_d)$ in the proofs of Theorems 1.7 and 1.9. In our old paper [CPS10], we developed a technique how to study possible singularities of log pairs (S_d, D), where D is an effective $\mathbb{Q}$-divisor on the surface S_d such that $D \sim_{\mathbb{Q}} -K_{S_d}$. This resulted in explicit values of $\alpha(S_d)$ in all considered cases. To estimate $\delta(S_d)$, one has to study singularities of similar log pairs with an additional condition: the $\mathbb{Q}$-divisor D has to be of *k-basis type* for $k \gg 1$ (for the definition, see [FO18, Definition 1.1]). By [FO18, Lemma 2.2] (see also Theorem 2.9 below), this extra condition provides strong upper bounds on multiplicities of the $\mathbb{Q}$-divisor D in various curves on S_d. We use these bounds (for some very particular curves in S_d) together with our original methods developed in [CPS10], to obtain the required estimates for $\delta(S_d)$. This approach was first used in [CRZ18] to estimate δ-invariants of the so-called asymptotically del Pezzo surfaces. Nevertheless, in our case we have an additional difficulty arising from the singularities of the surface S_d, while all surfaces considered in [CRZ18] are smooth.

It would be interesting to study the problem of existence of an orbifold Kähler–Einstein metric on S_d in the remaining cases. In some of these cases, the del Pezzo surface S_d is indeed not Kähler–Einstein. For instance, the surface S_d does not admit an orbifold Kähler–Einstein metric in the case when $I > 3a_0$. This follows from the obstruction found by Gauntlett, Martelli, Sparks, and Yau [GMSY07]. On the other hand, we expect the following to be true:

Conjecture 1.10. If $I = 2$ or $I = 3$, then S_d admits an orbifold Kähler–Einstein metric.

We believe that this conjecture can be proved using a similar approach to the one we use in the proofs of Theorems 1.7 and 1.9.

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2. Basic tools

Let S be a surface with at most cyclic quotient singularities, let C be an irreducible reduced curve on S, let P be a point of the curve C, and let D be an effective $\mathbb{R}$-divisor on the surface S. In this section, we present a few of well-known (local and global) results that will be used in the proof of Theorem 1.9. We start with

Lemma 2.1 ([K97])**.** Suppose that P is a smooth point of the surface S, and the singularities of the log pair (S, D) are not log canonical at P. Then $\mathrm{mult}_P(D) > 1$.

This immediately implies

Corollary 2.2. If P is a smooth point of the surface S, the log pair (S, D) is not log canonical at P, and C is not contained in the support of the divisor D, then $D \cdot C > 1$.

To state an analogue of this result in the case when S is singular at P, recall that S has a cyclic quotient singularity of type $\frac{1}{n}(a, b)$ at the point P, where a and b are coprime positive integers that are also coprime to n. Thus, if $n = 1$, then P is a smooth point of the surface S. For $n > 1$, Corollary 2.2 can be generalized as follows:

Lemma 2.3. Suppose that the log pair (S, D) is not log canonical at P, and C is not contained in the support of the divisor D. Then $D \cdot C > \frac{1}{n}$.

Proof. This follows from [CPS10, Lemma 2.2] and [CPS10, Lemma 2.3], cf. [BMO]. □

In general, the curve C may be contained in the support of the divisor D. Thus, we write

$$D = aC + \Delta,$$

where a is a non-negative real number, and Δ is an effective $\mathbb{R}$-divisor on S whose support does not contain the curve C. Then we have the following useful result:

Lemma 2.4. Suppose that $a \leqslant 1$, the surface S is smooth at the point P, the curve C is also smooth at P, and the log pair (S, D) is not log canonical at P. Then

$$C \cdot \Delta \geqslant \big(C \cdot \Delta\big)_P > 1,$$

where $\big(C \cdot \Delta\big)_P$ is the local intersection number of C and Δ at P.

Proof. This is a special case of a much more general result, known as the inversion of adjunction (see [S93, P01]). □

The inversion of adjunction also holds for singular varieties. In our two-dimensional case, it can be stated as follows:

Lemma 2.5. Suppose that $a \leqslant 1$, the log pair (S, C) is purely log terminal at P, and the log pair (S, D) is not log canonical at P. Then

$$C \cdot \Delta > \frac{1}{n}.$$

Proof. The required inequality follows from a more general version of the inverse of adjunction (see [S93, P01]). See also the proof of [CPS10, Lemma 2.5]. □

By our assumption, the surface S has a cyclic quotient singularity of type $\frac{1}{n}(a, b)$ at the point P. Thus, locally near P, the surface S is a quotient of $\mathbb{C}^2$ by the group $\mathbb{Z}_n$ that acts on $\mathbb{C}^2$ as

$$\big(x, y\big) \mapsto \big(\omega^a x, \omega^b y\big),$$

where ω is a primitive nth root of unity. We can consider x and y as weighted coordinates around the point P.

Remark 2.6. The pair (S, C) has purely log terminal singularity at P if and only if C is given by $x = 0$ or $y = 0$ for an appropriate choice of weighted coordinates x and y. This follows from [P01, Theorem 2.1.2], see also [K97, § 9.6]. Geometrically, this means that C is smooth at P, and its proper transform on the minimal resolution of singularities of the singular point P intersects the tail curve in the chain of exceptional curves. If (S, C) has purely log terminal singularities, then

$$\big(K_S + C\big) \cdot C = -2 + \sum_{O \in C} \frac{n_O - 1}{n_O},$$

where we assume that S has a cyclic quotient singularity of index n_O at the point O.

Let $f\colon \widetilde{S} \to S$ be the weighted blow up of the point P with $\mathrm{wt}(x) = a$ and $\mathrm{wt}(y) = b$, and let E be the exceptional curve of the morphism f. Then $\widetilde{S}$ has at most cyclic quotient singularities, one has $E \cong \mathbb{P}^1$, and the log pair $(\widetilde{S}, E)$ has purely log terminal singularities. Moreover, the curve E has at most two singular points of the surface $\widetilde{S}$. One of then is a singular point of type $\frac{1}{a}(n, -b)$, and another is a singular point of type $\frac{1}{b}(-a, n)$. Furthermore, we have

$$K_{\widetilde{S}} \sim_{\mathbb{Q}} f^*\big(K_S\big) + \frac{a + b - n}{n} E.$$

If the curve C is locally given by $x = 0$ near the point P, then

$$\widetilde{C} \sim_{\mathbb{Q}} f^*\big(C\big) - \frac{a}{n} E,$$

where $\widetilde{C}$ is the proper transform of the curve C on the surface $\widetilde{S}$. For more properties of weighted blow ups and their defining equations, see [P01, Section 3] or [BMO].

Denote by $\widetilde{D}$ the proper transform of the divisor D via f. Then

$$\widetilde{D} \sim_{\mathbb{R}} f^*\big(D\big) - mE$$

for some non-negative rational number m. If C is not contained in the support of the divisor D, we can estimate m using

$$0 \leqslant \widetilde{D} \cdot \widetilde{C} = \big(f^*\big(D\big) - mE\big) \cdot \widetilde{C} = D \cdot C - mE \cdot \widetilde{C},$$

where $D \cdot C$ and $E \cdot \widetilde{C}$ can be computed in every case. Note that

$$K_{\widetilde{S}} + \widetilde{D} + \Big(m - \frac{a + b - n}{n}\Big) E \sim_{\mathbb{R}} f^*\big(K_S + D\big).$$

This implies

Proposition 2.7. The log pair (S, D) is log canonical at P if and only if the log pair

$$\left(\widetilde{S}, \widetilde{D} + \left(m - \frac{a+b-n}{n}\right)E\right)$$

is log canonical along the curve E.

So far, we considered only local properties of the divisor D on the surface S. These properties will be used later to prove Theorem 1.9. However, the nature of this theorem is global, so that we will need one global result that is due to Fujita and Odaka. To state it, we remind the reader of what the volume $\mathrm{vol}(D)$ of the $\mathbb{R}$-divisor D is. If D is a Cartier divisor, then its volume is simply the number

$$\mathrm{vol}(D) = \limsup_{k\in\mathbb{N}} \frac{h^0(\mathcal{O}_S(kD)}{k^2/2!},$$

where the lim sup can be replaced by limit (see [L04, Example 11.4.7]). Likewise, if D is a $\mathbb{Q}$-divisor, we can define its volume using the identity

$$\mathrm{vol}(D) = \frac{\mathrm{vol}\big(\lambda D\big)}{\lambda^2}$$

for an appropriate positive rational number λ. One can show that the volume $\mathrm{vol}(D)$ only depends on the numerical equivalence class of the divisor D. Moreover, the volume function can be continuously extended to $\mathbb{R}$-divisors (see [L04] for details).

If D is not pseudoeffective, then $\mathrm{vol}(D) = 0$. If D is pseudoeffective, its volume can be computed using its Zariski decomposition [P03, BKS04]. Namely, if D is pseudoeffective, then there exists a nef $\mathbb{R}$-divisor N on the surface S such that

$$D \sim_{\mathbb{R}} N + \sum_{i=1}^{r} a_i C_i,$$

where each C_i is an irreducible curve on S with $N \cdot C_i = 0$, each a_i is a non-negative real number, and the intersection form of the curves $C_1, \ldots, C_r$ is negative definite. Such decomposition is unique, and it follows from [BKS04, Corollary 3.2] that

$$\mathrm{vol}\big(D\big) = \mathrm{vol}\big(N\big) = N^2. \tag{2.8}$$

Recall that $D = aC + \Delta$, where a is a non-negative real number, and Δ is an effective divisor whose support does not contain the curve C. Let

$$\tau = \sup\Big\{x \in \mathbb{R}_{>0} \ \Big|\ D - xC \text{ is pseudoeffective}\Big\}.$$

Then $a \leqslant \tau$. However, to prove Theorem 1.9, we have to find a better bound for a under an additional assumption that D is an ample $\mathbb{Q}$-divisor of k-basis type for $k \gg 1$ (for the definition, see [FO18, Definition 1.1] and the proof of Theorem 2.9 below). One such estimate is given by the following very special case of [FO18, Lemma 2.2].

Theorem 2.9. Suppose that D is a big $\mathbb{Q}$-divisor of k-basis type for $k \gg 1$. Then

$$a \leqslant \frac{1}{D^2}\int_0^{\tau} \mathrm{vol}\big(D - xC\big)dx + \epsilon_k,$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$.

Proof. Let us give a sketch of the proof that shows the nature of the required bound. First, recall from [FO18] that being k-basis type simply means that

$$D = \frac{1}{kd_k}\sum_{i=1}^{d_k}\{s_i = 0\},$$

where $d_k = h^0(S, \mathcal{O}_S(kD))$ and $s_1, \ldots, s_{d_k}$ are linearly independent sections in $H^0(S, \mathcal{O}_S(kD))$. Here, we assume that kD is a Cartier divisor and $k \gg 0$.

Let M be a positive rational number such that $M \geqslant \tau$. We may assume that kM is an integer. Then there is a filtration of vector spaces

$$\begin{aligned} 0 = H^0\Big(S, \mathcal{O}_S(kD-(kM+1)C)\Big) \subseteq H^0\Big(S, \mathcal{O}_S(kD-kMC)\Big) \subseteq \\ \subseteq H^0\Big(S, \mathcal{O}_S(kD-(kM-1)C)\Big) \subseteq \ldots \subseteq H^0\Big(S, \mathcal{O}_S(kD-3C)\Big) \subseteq \\ \subseteq H^0\Big(S, \mathcal{O}_S(kD-2C)\Big) \subseteq H^0\Big(S, \mathcal{O}_S(kD-C)\Big) \subseteq H^0\Big(S, \mathcal{O}_S(kD)\Big). \end{aligned}$$

Let $r_i = h^0(S, \mathcal{O}_S(kD - iC))$. Then

$$0 = r_{kM+1} \leqslant r_{kM} \leqslant r_{kM-1} \leqslant \ldots \leqslant r_3 \leqslant r_2 \leqslant r_1 \leqslant r_0 = d_k.$$

Since the sections $s_1, \ldots, s_{d_k}$ are linearly independent, we see that at most r_1 of them are contained in

$$H^0\Big(S, \mathcal{O}_S(kD - C\Big).$$

Among them at most r_2 are contained in $H^0(S, \mathcal{O}_S(kD - 2C))$. Among them at most r_3 are contained in $H^0(S, \mathcal{O}_S(kD - 3C))$ etc. Finally, at most r_{kM} sections among $s_1, \ldots, s_{d_k}$ are contained in

$$H^0\Big(S, \mathcal{O}_S(kD - kMC\Big),$$

and there are no sections in $H^0(\mathcal{O}_S(kD - (kM + 1)C) = 0$. Then

- at most r_1 sections among $s_1, \ldots, s_{d_k}$ vanish at C;
- at most r_2 sections among $s_1, \ldots, s_{d_k}$ vanish at C with order $\geqslant 2$;
- at most r_3 sections among $s_1, \ldots, s_{d_k}$ vanish at C with order $\geqslant 3$;
- $\ldots$
- at most r_{kM-1} sections among $s_1, \ldots, s_{d_k}$ vanish at C with order $\geqslant kM - 1$;
- at most r_{kM} sections among $s_1, \ldots, s_{d_k}$ vanish at C with order kM;
- no sections among $s_1, \ldots, s_{d_k}$ vanish at C with order $\geqslant kM + 1$.

This immediately implies that the the order of vanishing of the product $s_1 \cdot s_2 \cdot s_3 \cdot \ldots \cdot s_{d_n}$ at the curve C is at most

$$\begin{aligned} kMr_{kM} + (kM-1)\big(r_{kM-1} - r_{kM}\big) + (kM-2)\big(r_{kM-2} - r_{kM-1}\big) + \ldots \\ \ldots + 4\big(r_4 - r_5\big) + 3\big(r_3 - r_4\big) + 2\big(r_2 - r_3\big) + \big(r_1 - r_2\big) = \sum_{i=1}^{kM} r_i. \end{aligned}$$

Then we have

$$a \leqslant \frac{r_1 + r_2 + \ldots + r_{kM-1} + r_{kM}}{kr_0}.$$

As $k \to \infty$, the right hand side in this inequality converges to

$$\frac{1}{D^2}\int_0^\tau \mathrm{vol}\big(D - xC\big)dx,$$

which gives the upper bound on a. For a detailed proof, we refer the reader to [FO18]. $\square$

Corollary 2.10. Suppose that D is a big $\mathbb{Q}$-divisor of k-basis type for $k \gg 1$, and

$$C \sim_{\mathbb{Q}} \mu D$$

for some positive rational number μ. Then

$$a \leqslant \frac{1}{3\mu} + \epsilon_k,$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$.

Proof. Using Theorem 2.9, we get

$$a \leqslant \frac{1}{D^2}\int_0^{\infty} \mathrm{vol}\big(D - \lambda C\big)d\lambda + \epsilon_k,$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$. But

$$\int_0^{\infty} \mathrm{vol}\big(D - \lambda C\big)d\lambda = \int_0^{\infty} \mathrm{vol}\big((1 - \lambda\mu)D\big)d\lambda = D^2\int_0^{\frac{1}{\mu}} (1 - \lambda\mu)^2 d\lambda = \frac{D^2}{3\mu}.$$

This implies the assertion. $\square$

3. Case A

In this section, we consider two types of quasismooth hypersurfaces as follows:

- S_{15} : a quasismooth hypersurface in $\mathbb{P}(1, 3, 5, 7)$ of degree 15;
- S_{12} : a quasismooth hypersurface in $\mathbb{P}(2, 3, 4, 5)$ of degree 12.

By suitable coordinate changes, S_{15} may be assumed to be given by

$$\begin{aligned} z^3 + y^5 + xt^2 + b_1yzt + b_2xy^3z + b_3x^2yz^2 + b_4x^2y^2t + \\ + b_5x^3zt + b_6x^3y^4 + b_7x^4y^2z + b_8x^5z^2 + b_9x^5yt + b_{10}x^6y^3 + \\ + b_{11}x^7yz + b_{12}x^8t + b_{13}x^9y^2 + b_{14}x^{10}z + b_{15}x^{12}y + b_{16}x^{15} = 0 \end{aligned}$$

and S_{12} by

$$z(z - x^2)(z - \epsilon x^2) + y^4 + xt^2 + b_1yzt + b_2xy^2z + b_3x^2yt + b_4x^3y^2 = 0,$$

where ϵ ($\epsilon \neq 0$ and $\epsilon \neq 1$), b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}, b_{15} and b_{16} are constants. Note that the surface S_{15} has the only singular point at $O_t = [0 : 0 : 0 : 1]$. Meanwhile, S_{12} has exactly four singular points at $O_x = [1 : 0 : 0 : 0]$, $O_t = [0 : 0 : 0 : 1]$, $Q_1 = [1 : 0 : 1 : 0]$ and $Q_2 = [1 : 0 : \epsilon : 0]$.

In the sequel, we use S for the surfaces S_{15} and S_{12} if properties or conditions are satisfies by both the surfaces.

Denote by C_x the curve in S cut out by the equation $x = 0$. Then the curve C_x is reduced and irreducible in both the cases. It is easy to check

$$\mathrm{lct}\big(S_{15}, C_x\big) = \begin{cases} 1 & \text{if } a_1 \neq 0, \\ \dfrac{8}{15} & \text{if } a_1 = 0, \end{cases}$$

$$\mathrm{lct}\big(S_{12}, C_x\big) = \begin{cases} 1 & \text{if } a_1 \neq 0, \\ \dfrac{7}{12} & \text{if } a_1 = 0, \end{cases}$$

where $\mathrm{lct}\big(S, C_x\big)$ is the log canonical threshold of C_x on S. Moreover, one has $\alpha(S) = \mathrm{lct}(S, C_x)$ by [CPS10, Theorem 1.10]. Thus, since $\delta(S) \geqslant \frac{3}{2}\alpha(S)$, we obtain

Corollary 3.1. If $b_1 \neq 0$, then $\delta(S) \geqslant \frac{3}{2}$.

From now on, we suppose that $b_1 = 0$.

Proposition 3.2. Let D be an effective $\mathbb{Q}$-divisor on S such that

$$D \sim_{\mathbb{Q}} -K_S.$$

Write $D = aC_x + \Delta$, where a is a non-negative number, and Δ is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curve C_x. Suppose also that $a \leqslant \frac{8}{21}$. Then the log pair $(S, \frac{6}{5}D)$ is log canonical.

Corollary 3.3. One has $\delta(S) \geqslant \frac{6}{5}$.

Proof. Let D be a $\mathbb{Q}$-divisor of k-basis type divisor on S with $k \gg 0$. Write $D = aC_x + \Delta$, where a is a non-negative number, and Δ is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curve C_x. By Corollary 2.10, we have $a \leqslant \frac{8}{21}$ for $k \gg 0$. Thus, the log pair $(S, \frac{6}{5}D)$ is log canonical for $k \gg 0$ by Proposition 3.2. This implies that $\delta(S) \geqslant \frac{6}{5}$ by Corollary 3.1. □

To prove Proposition 3.2, we fix an effective $\mathbb{Q}$-divisor D on the surface S such that

$$D \sim_{\mathbb{Q}} -K_S.$$

Write $D = aC_x + \Delta$, where a is a non-negative number, and Δ is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curve C_x. Suppose also that $a \leqslant \frac{8}{21}$. Let us show that the log pair $(S, \frac{6}{5}D)$ is log canonical.

Lemma 3.4. The log pair $(S, \frac{6}{5}D)$ is log canonical outside C_x.

Proof. The required assertion follows from [CPS10, Lemma 2.7]. For convenience of the reader, let us give the detailed proof here. Let P be a point in $S \setminus C_x$. Since $P \notin C_x$, there are complex numbers c_1 and c_2 such that P satisfies the following system of equations:

$$\begin{cases} z + c_1x^5 = 0 \\ y + c_2x^3 = 0 \quad \text{for } S_{15}; \end{cases}$$

$$\begin{cases} y^2 + c_1x^3 = 0 \\ z + c_2x^2 = 0 \quad \text{for } S_{12}. \end{cases}$$

Let $\mathcal{P}$ be the pencil of curves that is given by

$$\nu\big(z + c_1x^5\big) + \mu\big(yx^2 + c_2x^5\big) = 0 \quad \text{on } S_{15},$$

$$\nu\big(y^2 + c_1x^3\big) + \mu\big(zx + c_2x^3\big) = 0 \quad \text{on } S_{12}$$

for $[\nu : \mu] \in \mathbb{P}^1$. Then the base locus of the pencil $\mathcal{P}$ consists of finitely many points. Moreover, by construction, the point P is one of them. Let C be a general curve in $\mathcal{P}$. Then

$$C \cdot D \leqslant \frac{5}{6},$$

so that $(S, \frac{6}{5}D)$ is log canonical at P by Corollary 2.2 if P is a smooth point of the surface S. This verifies the statement for S_{15}.

For S_{12}, we suppose that $(S_{12}, \frac{6}{5}D)$ is not log canonical at P. Then P must be one of the points O_x, Q_1,Q_2. Observe that the point P belongs to the curve C_y cut by $y = 0$. Moreover, the curve C_y is irreducible and the log pair $(S_{12}, \frac{6}{5}\cdot\frac{2}{3}C_y)$ is log canonical. Thus, it follows from [CS08, Remark 2.22] that there exists an effective $\mathbb{Q}$-divisor D' on the surface S_{12} such that

$$D' \sim_{\mathbb{Q}} -K_{S_{12}},$$

the log pair $(S_{12}, \frac{6}{5}D')$ is not log canonical at the point P, and the support of the divisor D' does not contain the curve C_y. However,

$$D' \cdot C_y = \frac{6}{10},$$

which is impossible by Lemma 2.3 since $(S_{12}, \frac{6}{5}D')$ is not log canonical at the point P. This completes the proof for S_{12}. □

Lemma 3.5. The log pair $(S, \frac{6}{5}D)$ is log canonical at a point in $C_x \setminus \{O_t\}$.

Proof. Let P be a point in $C_x \setminus \{O_t\}$. Observe that P is a smooth point of the surface S, and C_x is smooth at the point P. Note also that $\frac{6}{5}a < 1$. Thus, we can apply Lemma 2.4 to $(S, \frac{6}{5}D)$ and the curve C_x at the point P. Indeed, since

$$\Big(C_x \cdot \Delta\Big)_P \leqslant C_x \cdot \Delta = \frac{1-a}{7} \leqslant \frac{5}{6} \quad \text{on } S_{15},$$
$$\Big(C_x \cdot \Delta\Big)_P \leqslant C_x \cdot \Delta = \frac{1-2a}{5} \leqslant \frac{5}{6} \quad \text{on } S_{12},$$

the log pair $(S, \frac{6}{5}D)$ must be log canonical at P. □

Note that S_{15} (resp. S_{12}) has singularity of type $\frac{1}{7}(3,5)$ (resp. $\frac{1}{5}(3,4)$) at the point O_t. In the chart defined by $t=1$, the surface S_{15} is given by

$$\begin{aligned} z^3 + y^5 + x + b_2xy^3z + b_3x^2yz^2 + b_4x^2y^2 + \\ + b_5x^3z + b_6x^3y^4 + b_7x^4y^2z + b_8x^5z^2 + b_9x^5y + b_{10}x^6y^3 + \\ + b_{11}x^7yz + b_{12}x^8 + b_{13}x^9y^2 + b_{14}x^{10}z + b_{15}x^{12}y + b_{16}x^{15} = 0, \end{aligned}$$

and S_{12} by

$$z(z-x^2)(z-\epsilon x^2) + y^4 + x + a_1yz + a_2xy^2z + a_3x^2y + a_4x^3y^2 = 0.$$

Thus, in a neighborhood of the point O_t, we may regard y and z as local weighted coordinates with $\mathrm{wt}(y) = 3$ and $\mathrm{wt}(z) = 5$ for S_{15} and with $\mathrm{wt}(y) = 3$ and $\mathrm{wt}(z) = 4$ for S_{12}.

Let $f\colon \widetilde{S} \to S$ be the weighted blow up at the singular point O_t with weights $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 5$ for S_{15} and with weights $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 4$ for S_{12}. Denote by E the exceptional curve of the blow up f. Then

$$K_{\widetilde{S}_{15}} \sim_{\mathbb{Q}} f^*\big(K_{S_{15}}\big) + \frac{1}{7}E;$$
$$K_{\widetilde{S}_{12}} \sim_{\mathbb{Q}} f^*\big(K_{S_{12}}\big) + \frac{2}{5}E.$$

The surface S has two singular points in E. One is a point of type $\frac{1}{3}(1,1)$, and the other is a singular point of type $\frac{1}{5}(1,1)$ on $\widetilde{S}_{15}$ ($\frac{1}{4}(1,1)$ on $\widetilde{S}_{12}$). Denote the former by O_3 and the latter by O. Observe that

$$E^2 = -\frac{7}{15} \quad \text{on } \widetilde{S}_{15};$$
$$E^2 = -\frac{5}{12} \quad \text{on } \widetilde{S}_{12};$$

and $E \cong \mathbb{P}^1$.

Let $\widetilde{C}_x$ be the proper transform of the curve C_x on the surface $\widetilde{S}$. Then

$$\widetilde{C}_x \sim_{\mathbb{Q}} f^*\big(C_x\big) - cE \quad \text{for } S_{15},$$

where $c = \frac{15}{7}$ for S_{15} and $c = \frac{12}{5}$ for S_{12}, and the intersection $E \cap \widetilde{C}_x$ consists of a single point, which is different from O_3 and O. Note that the curves E and $\widetilde{C}_x$ intersect transversally at the point $E \cap \widetilde{C}_x$.

Denote by $\widetilde{\Delta}$ be the proper transform of the $\mathbb{Q}$-divisor Δ on the surface $\widetilde{S}$. Then

$$\widetilde{\Delta} \sim_{\mathbb{Q}} f^*\big(\Delta\big) - mE$$

for some non-negative rational number m. To estimate it, observe that

$$0 \leqslant \widetilde{C}_x \cdot \widetilde{\Delta} = \Big(f^*\big(C_x\big) - cE\Big) \cdot \Big(f^*\big(\Delta\big) - mE\Big) = C_x \cdot \Delta - m = C_x \cdot (D - aC_x) - m,$$

so that $m \leqslant \frac{1-a}{7}$ for S_{15} and $m \leqslant \frac{1-2a}{5}$ for S_{12}. Now we are ready to prove

Lemma 3.6. The log pair $(S, \frac{6}{5}D)$ is log canonical at O_t.

Proof. Suppose that the log pair $(S, \frac{6}{5}D)$ is not log canonical at O_t. Let us seek for a contradiction. Let $\lambda = \frac{6}{5}$. Then

$$K_{\widetilde{S}} + \lambda a \widetilde{C}_x + \lambda \widetilde{\Delta} + \mu E \sim_{\mathbb{Q}} f^*\big(K_S + \lambda D\big),$$

where

$$\mu = \frac{15\lambda a}{7} + \lambda m - \frac{1}{7} \quad \text{for } S_{15},$$

$$\mu = \frac{12\lambda a}{5} + \lambda m - \frac{2}{5} \quad \text{for } S_{12}.$$

Thus, the log pair

$$\Big(\widetilde{S}, \lambda a \widetilde{C}_x + \lambda \widetilde{\Delta} + \mu E\Big) \tag{3.7}$$

is not log canonical at some point $Q \in E$. Note that $\mu \leqslant 1$ because $m \leqslant \frac{1-a}{7}$ (or $m \leqslant \frac{1-2a}{5}$) and $a \leqslant \frac{8}{21}$.

We first apply Lemmas 2.4 or 2.5 to (3.7) and the curve E at the point Q. Indeed,

$$E \cdot \widetilde{\Delta} = E \cdot (f^*\big(\Delta\big) - mE) = -mE^2 = \begin{cases} \dfrac{7m}{15} \leqslant \dfrac{1-a}{15} \leqslant \dfrac{1}{6} & \text{on } \widetilde{S}_{15}, \\ \dfrac{5m}{12} \leqslant \dfrac{1-2a}{12} \leqslant \dfrac{5}{24} & \text{on } \widetilde{S}_{12}. \end{cases}$$

This shows that Q must be the intersection point of E and $\widetilde{C}_x$.

Applying Lemma 2.4 again, we see that

$$\frac{5}{6} = \frac{1}{\lambda} < \Big(a\widetilde{C}_x + \widetilde{\Delta}\Big) \cdot E = a + \widetilde{\Delta} \cdot E = \begin{cases} a + \dfrac{7m}{15} \leqslant a + \dfrac{1-a}{15} & \text{on } \widetilde{S}_{15}, \\ a + \dfrac{5m}{12} \leqslant a + \dfrac{1-2a}{12} & \text{on } \widetilde{S}_{12}. \end{cases}$$

However, these inequalities contradict our assumption $a \leqslant \frac{8}{21}$. Therefore, the log pair $(S, \frac{6}{5}D)$ is log canonical at O_t. □

Proposition 3.2 is completely verified.

4. Case B

The way to evaluate δ-invariants for Case B is almost same as that of Case A. In spite of this, we write the proof for the readers' convenience.

In this section, we consider the following two types of quasismooth hypersurfaces:

- S_{64} : a quasismooth hypersurface in $\mathbb{P}(7, 15, 19, 32)$ of degree 64;
- S_{82} : a quasismooth hypersurface in $\mathbb{P}(7, 19, 25, 41)$ of degree 82.

As in the previous section, we use S for the surfaces S_{64} and S_{82} if properties or conditions are satisfies by both the surfaces.

We may assume that the surface S_{64} is given by the equation

$$t^2 + y^3z + xz^3 + x^7y = 0$$

in $\mathbb{P}(7, 15, 19, 32)$ and S_{82} by the equation

$$t^2 + y^3z + xz^3 + x^9y = 0$$

in $\mathbb{P}(7, 19, 25, 41)$. The surface S is singular at the points $O_x = [1 : 0 : 0 : 0]$, $O_y = [0 : 1 : 0 : 0]$ and $O_z = [0 : 0 : 1 : 0]$, and is smooth away from them. Moreover, the surface S_{64} (resp. S_{82}) has quotient singularity of types $\frac{1}{7}(5, 4)$, $\frac{1}{15}(7, 2)$, $\frac{1}{19}(2, 3)$ (resp. $\frac{1}{7}(2, 3)$, $\frac{1}{19}(7, 3)$, $\frac{1}{25}(2, 3)$) at the points O_x, O_y, O_z, respectively.

Let C_x be the curve in S cut out by $x = 0$ and C_y by $y = 0$. Then both the curves C_x and C_y are irreducible. We have

$$\frac{35}{54} = \mathrm{lct}\Big(S_{64}, \frac{9}{7}C_x\Big) < \mathrm{lct}\Big(S_{64}, \frac{9}{15}C_y\Big) = \frac{25}{18},$$
$$\frac{7}{12} = \mathrm{lct}\Big(S_{82}, \frac{10}{7}C_x\Big) < \mathrm{lct}\Big(S_{82}, \frac{10}{19}C_y\Big) = \frac{19}{12},$$

which imply $\alpha(S_{64}) \leqslant \frac{35}{54}$ and $\alpha(S_{82}) \leqslant \frac{7}{12}$. In fact, we have $\alpha(S_{64}) = \frac{35}{54}$ and $\alpha(S_{82}) = \frac{7}{12}$ by [CPS10, Theorem 1.10].

Proposition 4.1. Let D be an effective $\mathbb{Q}$-divisor on S such that

$$D \sim_{\mathbb{Q}} -K_S.$$

Write $D = aC_x + \Delta$, where a is a non-negative number, and Δ is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curve C_x. Suppose also that $a \leqslant \frac{1}{2}$. Then the log pair $(S, \frac{19}{18}D)$ is log canonical.

Proof. Suppose also that $a \leqslant \frac{1}{2}$.

We first consider a point P that lies neither on C_x nor on C_y. Observe that P is a smooth point of the surface S. Since $P \notin C_x$, there are complex numbers c_1 and c_2 such that P satisfies the following system of equations:

$$\begin{cases} y^7 + c_1x^{15} = 0 \\ y^2z + c_2x^7 = 0 \end{cases} \quad \text{for } S_{64};$$

$$\begin{cases} y^4 + c_1x^5t = 0 \\ y^3 + c_2xz^2 = 0 \end{cases} \quad \text{for } S_{82}.$$

Moreover, since $P \notin C_y$, we have $c_1 \neq 0$. Let $\mathcal{P}$ be the pencil given by

$$\nu\big(y^7 + c_1x^{15}\big) + \mu x^8\big(y^2z + c_2x^7\big) = 0 \quad \text{on } S_{64};$$
$$\nu\big(y^4 + c_1x^5t\big) + \mu y\big(y^3 + c_2xz^2\big) = 0 \quad \text{on } S_{82}$$

for $[\nu:\mu]\in\mathbb{P}^1$. The base locus of the pencil $\mathcal{P}$ consists of finitely many points. Furthermore, by construction, the point P is one of them. Let C be a general curve in $\mathcal{P}$. Then

$$\mathrm{mult}_P(D)\leqslant C\cdot D\leqslant\frac{18}{19}.$$

It immediately follows from Corollary 2.2 that the log pair $(S,\frac{19}{18}D)$ is log canonical outside C_x and C_y.

We next consider a point P on C_x different from O_z. Since $a\leqslant\frac{1}{2}$, we apply Lemmas 2.4 and 2.5 to the log pair $(S,\frac{18}{19}aC_x+\frac{18}{19}\Delta)$. Indeed, since

$$\Big(C_x\cdot\Delta\Big)_P\leqslant C_x\cdot\Delta=\frac{18-14a}{285}\leqslant\frac{6}{95}\quad\text{ on } S_{64},$$

$$\Big(C_x\cdot\Delta\Big)_P\leqslant C_x\cdot\Delta=\frac{20-14a}{475}\leqslant\frac{18}{19\cdot 19}\quad\text{ on } S_{82},$$

the log pair $(S,\frac{19}{18}D)$ must be log canonical at P.

We now let P be a point on C_y different from O_z. Suppose that the log pair $(S,\frac{19}{18}D)$ is not log canonical at the point P. Recall that $(S_{64},\frac{19}{18}\cdot\frac{9}{15}C_y)$ and $(S_{82},\frac{19}{18}\cdot\frac{10}{19}C_y)$ are log canonical, and the curve C_y is irreducible. Thus, it follows from [CS08, Remark 2.22] that there exists an effective $\mathbb{Q}$-divisor D' on the surface S such that

$$D'\sim_{\mathbb{Q}}-K_S,$$

the log pair $(S,\frac{19}{18}D')$ is not log canonical at the point P and the support of the divisor D' does not contain the curve C_y. Observe

$$C_y\cdot D'=\left\{\begin{matrix}\dfrac{18}{19\cdot 7} & \text{on } S_{64}\\ \dfrac{4}{35} & \text{on } S_{82}\end{matrix}\right\}\leqslant\frac{18}{19\cdot 7}.$$

This implies that the log pair $(S,\frac{19}{18}D')$ is log canonical at the point P. This contradicts our assumption. Thus, we see that $(S,\frac{19}{18}D)$ is log canonical away from O_z. Hence, to complete the proof of Proposition 4.1, we have to show that $(S,\frac{19}{18}D)$ is log canonical at the point O_z.

Recall that S_{64} (resp. S_{82}) has singularity of type $\frac{1}{19}(2,3)$ (resp. $\frac{1}{25}(2,3)$) at the point O_z. In the chart $z=1$, the surface S_{64} is given by

$$t^2+y^3+x+x^7y=0$$

and S_{82} by

$$t^2+y^3+x+x^9y=0.$$

In a neighborhoods of the point O_z, we can consider y and t as local weighted coordinates such that $\mathrm{wt}(y)=2$ and $\mathrm{wt}(t)=3$.

Let $f\colon\widetilde{S}\to S$ be the weighted blow up at the singular point O_z with weights $\mathrm{wt}(y)=2$ and $\mathrm{wt}(t)=3$. Denote by E the exceptional curve of the blow up f. Then

$$K_{\widetilde{S}_{64}}\sim_{\mathbb{Q}}f^*\big(K_{S_{64}}\big)-\frac{14}{19}E;$$

$$K_{\widetilde{S}_{82}}\sim_{\mathbb{Q}}f^*\big(K_{S_{82}}\big)-\frac{20}{25}E.$$

The surface S has two singular points in E. One is a point of type $\frac{1}{2}(1,1)$ and the other is of type $\frac{1}{3}(1,1)$. Denote the former by O_2 and the latter by O_3. Observe

$$E^2 = -\frac{19}{6} \quad \text{on } \widetilde{S}_{64};$$
$$E^2 = -\frac{25}{6} \quad \text{on } \widetilde{S}_{82}$$

and $E \cong \mathbb{P}^1$.

Let $\widetilde{C}_x$ be the proper transform of the curve C_x on the surface $\widetilde{S}$. Then

$$\widetilde{C}_x \sim_{\mathbb{Q}} f^*\big(C_x\big) - cE,$$

where $c = \frac{6}{19}$ for S_{64} and $c = \frac{6}{25}$ for S_{82}, and the intersection $E \cap \widetilde{C}_x$ consists of a single point different from O_2 and O_3. Note that the curves E and $\widetilde{C}_x$ intersect transversally.

Denote by $\widetilde{\Delta}$ be the proper transform of the $\mathbb{Q}$-divisor Δ on the surface $\widetilde{S}$. Then

$$\widetilde{\Delta} \sim_{\mathbb{Q}} f^*\big(\Delta\big) - mE$$

for some non-negative rational number m. To estimate it, observe

$$0 \leqslant \widetilde{C}_x \cdot \widetilde{\Delta} = \Big(f^*\big(C_x\big) - cE\Big) \cdot \Big(f^*\big(\Delta\big) - mE\Big) = C_x \cdot \Delta - m = C_x \cdot (D - aC_x) - m.$$

This implies $m \leqslant \frac{18-14a}{285}$ for S_{64} and $m \leqslant \frac{20-14a}{19 \cdot 25}$ for S_{82}.

We finally suppose that the log pair $(S, \frac{19}{18}D)$ is not log canonical at O_z. Let $\lambda = \frac{19}{18}$. Then

$$K_{\widetilde{S}} + \lambda a\widetilde{C}_x + \lambda\widetilde{\Delta} + \mu E \sim_{\mathbb{Q}} f^*\big(K_S + \lambda D\big),$$

where

$$\mu = \frac{6\lambda a}{19} + \lambda m + \frac{14}{19} \quad \text{for } S_{64};$$
$$\mu = \frac{6\lambda a}{25} + \lambda m + \frac{20}{25} \quad \text{for } S_{82}.$$

Thus, the log pair

$$\Big(\widetilde{S}, \lambda a\widetilde{C}_x + \lambda\widetilde{\Delta} + \mu E\Big) \tag{4.2}$$

is not log canonical at some point $Q \in E$.

Using $m \leqslant \frac{18-14a}{15 \cdot 19}$ for S_{64}, $m \leqslant \frac{20-14a}{19 \cdot 25}$ for S_{82} and $a \leqslant \frac{1}{2}$, we get

$$\frac{6\lambda a}{19} + \lambda m + \frac{14}{19} \leqslant \frac{4\lambda a}{15} + \frac{6\lambda}{95} + \frac{14}{19} \leqslant \frac{56\lambda}{285} + \frac{14}{19} = \frac{2422}{2565} < 1,$$
$$\frac{6\lambda a}{25} + \lambda m + \frac{20}{25} \leqslant \frac{4\lambda a}{19} + \frac{4\lambda}{95} + \frac{4}{5} \leqslant \frac{14\lambda}{95} + \frac{4}{5} = \frac{817}{855} < 1.$$

Since

$$E \cdot \widetilde{\Delta} = E \cdot (f^*\big(\Delta\big) - mE) = -mE^2 = \begin{cases} \dfrac{19m}{6} \leqslant \dfrac{9-7a}{45} \leqslant \dfrac{6}{19} & \text{on } \widetilde{S}_{64}, \\[2ex] \dfrac{25m}{6} \leqslant \dfrac{20-14a}{6 \cdot 19} \leqslant \dfrac{6}{19} & \text{on } \widetilde{S}_{82}. \end{cases}$$

Lemmas 2.4 and 2.5 imply that Q must be the intersection point of E and $\widetilde{C}_x$. It then follows from Lemma 2.4 that

$$\frac{18}{19} = \frac{1}{\lambda} < \Big(a\widetilde{C}_x + \widetilde{\Delta}\Big) \cdot E = a + \widetilde{\Delta} \cdot E = \begin{cases} a + \dfrac{19m}{6} \leqslant a + \dfrac{9-7a}{45} & \text{on } \widetilde{S}_{64}, \\[2ex] a + \dfrac{25m}{6} \leqslant a + \dfrac{20-14a}{6 \cdot 19} & \text{on } \widetilde{S}_{82}. \end{cases}$$

This contradicts our assumption $a \leqslant \frac{1}{2}$. The obtained contradiction completes the proof. $\square$

Corollary 4.3. One has $\delta(S) \geqslant \frac{19}{18}$.

Proof. See the proof of Corollary 3.3. $\square$

5. Case C

In this section, we consider the following three types of quasismooth hypersurfaces:

- S_{45} : a quasismooth hypersurface in $\mathbb{P}(7,10,15,19)$ of degree 45;
- S_{81} : a quasismooth hypersurface in $\mathbb{P}(7,18,27,37)$ of degree 81;
- S_{117} : a quasismooth hypersurface in $\mathbb{P}(7,26,39,55)$ of degree 117.

As in the previous sections, we use S for all the surfaces S_{45}, S_{81}, and S_{117} if properties or conditions are satisfies by all the surfaces.

By appropriate coordinate changes, we may assume that the surface S_{45} is defined by the equation

$$z^3-y^3z+xt^2+x^5y=0$$

in $\mathbb{P}(7,10,15,19)$, the surface S_{81} by

$$z^3-y^3z+xt^2+x^9y=0$$

in $\mathbb{P}(7,18,27,37)$, and the surface S_{117} by

$$z^3-y^3z+xt^2+x^{13}y=0$$

in $\mathbb{P}(7,26,39,55)$.

The surface S is singular at the points

$$O_x=[1:0:0:0], O_y=[0:1:0:0], O_t=[0:0:0:1], Q=[0:1:1:0],$$

and is smooth away from them. Moreover, the surface S_{45} (resp. S_{81} and S_{117}) has quotient singularity of types $\frac{1}{7}(1,5)$, $\frac{1}{10}(7,9)$, $\frac{1}{19}(2,3)$, $\frac{1}{5}(1,2)$ (resp. $\frac{1}{7}(3,1)$, $\frac{1}{18}(7,1)$, $\frac{1}{37}(2,3)$, $\frac{1}{9}(7,1)$ and $\frac{1}{7}(2,3)$, $\frac{1}{26}(7,3)$, $\frac{1}{55}(2,3)$, $\frac{1}{13}(7,3)$) at the points O_x, O_y, O_t, Q, respectively.

Let C_x be the curve in S that is cut out by $x=0$. Then

$$C_x=L_{xz}+R_x,$$

where L_{xz} is the curve given by $x=z=0$ and R_x by $x=z^2-y^3=0$ in the ambient weighted projective space. These two curves L_{xz} and R_x meets each other at the point O_t. Also, we have

$$\begin{aligned}
&L_{xz}^2=-\frac{23}{10\cdot 19},\quad R_x^2=-\frac{8}{5\cdot 19},\quad L_{xz}\cdot R_x=\frac{3}{19}\quad \text{on } S_{45};\\
&L_{xz}^2=-\frac{47}{18\cdot 37},\quad R_x^2=-\frac{20}{9\cdot 37},\quad L_{xz}\cdot R_x=\frac{3}{37}\quad \text{on } S_{81};\\
&L_{xz}^2=-\frac{71}{26\cdot 55},\quad R_x^2=-\frac{32}{13\cdot 55},\quad L_{xz}\cdot R_x=\frac{3}{55}\quad \text{on } S_{117}.
\end{aligned}\tag{5.1}$$

Note also that the curve R_x is singular at the point O_t.

Let C_y be the curve in S cut out by $y=0$. Then C_y is irreducible and

$$\begin{aligned}
\frac{35}{54}&=\mathrm{lct}\left(S_{45},\frac{6}{7}C_x\right)<\mathrm{lct}\left(S_{45},\frac{6}{10}C_y\right)=\frac{25}{18};\\
\frac{35}{72}&=\mathrm{lct}\left(S_{81},\frac{8}{7}C_x\right)<\mathrm{lct}\left(S_{81},\frac{8}{18}C_y\right)=\frac{15}{8};\\
\frac{7}{18}&=\mathrm{lct}\left(S_{117},\frac{10}{7}C_x\right)<\mathrm{lct}\left(S_{117},\frac{10}{26}C_y\right)=\frac{13}{6}.
\end{aligned}$$

In fact, in these three cases $\alpha(S)$ is given by the numbers $\frac{35}{54}$, $\frac{35}{72}$, and $\frac{7}{18}$ on the left-hand sides by [CPS10, Theorem 1.10].

To estimate $\delta(S)$, we fix an effective $\mathbb{Q}$-divisor D on the surface S such that

$$D \sim_{\mathbb{Q}} -K_S$$

and write $D = aL_{xz} + bR_x + \Delta$, where a and b are non-negative numbers, and Δ is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curves L_{xz} and R_x.

Lemma 5.2. If the $\mathbb{Q}$-divisor D is of k-basis type with $k \gg 0$, then

$$a \leqslant \left\{\begin{array}{c} \frac{2}{5} \\ \frac{1}{2} \\ \frac{11}{20} \end{array}\right\}, \quad b \leqslant \left\{\begin{array}{ll} \frac{1}{3} & \text{on } S_{45} \\ \frac{1}{5} & \text{on } S_{81} \\ \frac{12}{25} & \text{on } S_{117} \end{array}\right\}.$$

Proof. Suppose that D is of k-basis type with $k \gg 0$. Theorem 2.9 implies that

$$a \leqslant \frac{1}{(-K_S)^2}\int_0^{\infty} \mathrm{vol}\big(-K_S - \lambda L_{xz}\big)d\lambda + \epsilon_k,$$

where ϵ_k is a small constant depending on k such that $\epsilon_k \to 0$ as $k \to \infty$. Since

$$-K_S - \lambda L_{xz} \sim_{\mathbb{Q}} \begin{cases} \left(\frac{6}{7} - \lambda\right) L_{xz} + \frac{6}{7}R_x & \text{on } S_{45} \\ \left(\frac{8}{7} - \lambda\right) L_{xz} + \frac{8}{7}R_x & \text{on } S_{81} \\ \left(\frac{10}{7} - \lambda\right) L_{xz} + \frac{10}{7}R_x & \text{on } S_{117} \end{cases}$$

and $R_x^2 < 0$, we have $\mathrm{vol}(-K_S - \lambda L_{xz}) = 0$ for $\lambda \geqslant \frac{6}{7}$ on S_{45}, $\lambda \geqslant \frac{8}{7}$ on S_{81} and $\lambda \geqslant \frac{10}{7}$ on S_{117}. Similarly, using (5.1), we see that

$$(-K_S - \lambda L_{xz}) \cdot R_x = \begin{cases} \left(\left(\frac{6}{7} - \lambda\right) L_{xz} + \frac{6}{7}R_x\right) \cdot R_x = \frac{6-15\lambda}{19\cdot 5} & \text{on } S_{45} \\ \left(\left(\frac{8}{7} - \lambda\right) L_{xz} + \frac{8}{7}R_x\right) \cdot R_x = \frac{8-27\lambda}{37\cdot 9} & \text{on } S_{81} \\ \left(\left(\frac{10}{7} - \lambda\right) L_{xz} + \frac{10}{7}R_x\right) \cdot R_x = \frac{10-39\lambda}{13\cdot 55} & \text{on } S_{117}. \end{cases}$$

This shows that the divisor $-K_S - \lambda L_{xz}$ is nef for $\lambda \leqslant \frac{2}{5}$ on S_{45}, $\lambda \leqslant \frac{8}{27}$ on S_{81} and $\lambda \leqslant \frac{10}{39}$ on S_{117}. Thus, we have

$$\mathrm{vol}\big(-K_S - \lambda L_{xz}\big) = \big(-K_S - \lambda L_{xz}\big)^2 = \begin{cases} \frac{54}{665} - \frac{6\lambda}{95} - \frac{23\lambda^2}{190} & \text{for } \lambda \leqslant \frac{2}{5} \text{ on } S_{45} \\ \frac{32}{777} - \frac{8\lambda}{333} - \frac{47\lambda^2}{666} & \text{for } \lambda \leqslant \frac{8}{27} \text{ on } S_{81} \\ \frac{200}{7007} - \frac{12\lambda}{1001} - \frac{36}{715}\lambda^2 & \text{for } \lambda \leqslant \frac{10}{39} \text{ on } S_{117}. \end{cases}$$

To compute $\mathrm{vol}(-K_S-\lambda L_{xz})$ for $\frac{2}{5}<\lambda<\frac{6}{7}$ on S_{45}, $\frac{8}{27}<\lambda<\frac{8}{7}$ on S_{81} and $\frac{10}{39}<\lambda<\frac{10}{7}$ on S_{117}, we let

$$N=\begin{cases}\left(\dfrac{6}{7}-\lambda\right)L_{xz}+\left(\dfrac{6}{7}-\dfrac{15\lambda-6}{8}\right)R_x & \text{for } S_{45}\\ \left(\dfrac{8}{7}-\lambda\right)L_{xz}+\left(\dfrac{8}{7}-\dfrac{27\lambda-8}{20}\right)R_x & \text{for } S_{81}\\ \left(\dfrac{10}{7}-\lambda\right)L_{xz}+\left(\dfrac{10}{7}-\dfrac{39\lambda-10}{32}\right)R_x & \text{for } S_{117}.\end{cases}$$

Then, using (5.1) again, we see that $N\cdot R_x=0$ and $N\cdot L_{xz}\geqslant 0$. Thus, we conclude that the divisor N is nef on the respective interval for λ. This shows that

$$-K_S-\lambda L_{xz}\sim_{\mathbb{Q}}\begin{cases}N+\dfrac{15\lambda-6}{8}R_x & \text{on } S_{45}\\ N+\dfrac{27\lambda-8}{20}R_x & \text{on } S_{81}\\ N+\dfrac{39\lambda-10}{32}R_x & \text{on } S_{117}\end{cases}$$

is the Zariski decomposition of the divisor $-K_S-\lambda L_{xz}$. Hence, we have

$$\mathrm{vol}\big(-K_S-\lambda L_{xz}\big)=N^2=\begin{cases}\dfrac{1}{280}(6-7\lambda)^2 & \text{on } S_{45}\\ \dfrac{1}{1260}(8-7\lambda)^2 & \text{on } S_{81}\\ \dfrac{369}{1121120}(10-7\lambda)^2 & \text{on } S_{117}\end{cases}$$

by (2.8). Thus, integrating, we get

$$a\leqslant\frac{1}{(-K_S)^2}\int_0^\infty\mathrm{vol}\big(-K_S-\lambda L_{xz}\big)d\lambda+\epsilon_k=\begin{cases}\dfrac{118}{315}+\epsilon_k & \text{for } S_{45}\\ \dfrac{760}{1701}+\epsilon_k & \text{for } S_{81}\\ \dfrac{8780}{17199}+\epsilon_k & \text{for } S_{117}.\end{cases}$$

This gives us the asserted bounds for a.

Meanwhile, we have

$$\mathrm{vol}\big(-K_S-\lambda R_x\big)=\big(-K_S-\lambda R_x\big)^2=\begin{cases}\dfrac{54}{665}-\dfrac{12\lambda}{95}-\dfrac{8\lambda^2}{95} & \text{for } 0\leqslant\lambda\leqslant\dfrac{1}{5} \text{ on } S_{45}\\ \dfrac{32\cdot 21}{9\cdot 37\cdot 49}-\dfrac{16\lambda}{9\cdot 37}-\dfrac{20\lambda^2}{9\cdot 37} & \text{for } 0\leqslant\lambda\leqslant\dfrac{4}{27} \text{ on } S_{81}\\ \dfrac{30}{1001}-\dfrac{4\lambda}{143}-\dfrac{32\lambda^2}{715} & \text{for } 0\leqslant\lambda\leqslant\dfrac{5}{39} \text{ on } S_{117}.\end{cases}$$

since the divisor $-K_S - \lambda R_x$ is nef for the values λ in the respective interval. The Zariski decomposition of the divisor $-K_S - \lambda R_x$ is given by

$$\begin{cases} \underbrace{\Big(\frac{6}{7} - \frac{30\lambda-6}{23}\Big)L_{xz} + \Big(\frac{6}{7}-\lambda\Big)R_x}_{\text{nef } \mathbb{R}\text{-divisor}} + \frac{30\lambda-6}{23}L_{xz} & \text{for } \frac{1}{5} < \lambda \leqslant \frac{6}{7} \text{ on } S_{45} \\ \underbrace{\Big(\frac{8}{7} - \frac{54\lambda-8}{47}\Big)L_{xz} + \Big(\frac{8}{7}-\lambda\Big)R_x}_{\text{nef } \mathbb{R}\text{-divisor}} + \frac{54\lambda-8}{47}L_{xz} & \text{for } \frac{4}{27} < \lambda \leqslant \frac{8}{7} \text{ on } S_{81} \\ \underbrace{\Big(\frac{10}{7} - \frac{78\lambda-10}{71}\Big)L_{xz} + \Big(\frac{10}{7}-\lambda\Big)R_x}_{\text{nef } \mathbb{R}\text{-divisor}} + \frac{78\lambda-10}{71}L_{xz} & \text{for } \frac{5}{39} < \lambda \leqslant \frac{10}{7} \text{ on } S_{117}, \end{cases}$$

so that we could obtain

$$\mathrm{vol}\big(-K_S - \lambda R_x\big) = \begin{cases} \bigg(\Big(\frac{6}{7} - \frac{30\lambda-6}{23}\Big)L_{xz} + \Big(\frac{6}{7}-\lambda\Big)R_x\bigg)^2 = \frac{2}{5\cdot 7\cdot 23}(6-7\lambda)^2 \\ \bigg(\Big(\frac{8}{7} - \frac{54\lambda-8}{47}\Big)L_{xz} + \Big(\frac{8}{7}-\lambda\Big)R_x\bigg)^2 = \frac{2}{7\cdot 9\cdot 47}(8-7\lambda)^2 \\ \bigg(\Big(\frac{10}{7} - \frac{78\lambda-10}{71}\Big)L_{xz} + \Big(\frac{10}{7}-\lambda\Big)R_x\bigg)^2 = \frac{2}{7\cdot 13\cdot 71}(10-7\lambda)^2. \end{cases}$$

Finally, $\mathrm{vol}(-K_S - \lambda R_x) = 0$ for $\lambda > \frac{6}{7}$ on S_{45}, for $\lambda > \frac{8}{7}$ on S_{81}, and for $\lambda > \frac{10}{7}$ on S_{117} since $-K_S - \lambda R_x$ is not pseudoeffective for these values λ. Thus, by Theorem 2.9, we have

$$b \leqslant \frac{1}{(-K_S)^2}\int_0^\infty \mathrm{vol}\big(-K_S - \lambda R_x\big)d\lambda + \varepsilon_k = \begin{cases} \frac{97}{315} + \varepsilon_k & \text{for } S_{45} \\ \frac{10709068}{58281363} + \varepsilon_k & \text{for } S_{81} \\ \frac{1205}{2457} + \varepsilon_k & \text{for } S_{117}. \end{cases}$$

This yields the required bounds for b. ☐

Now we prove the main assertion in this section.

Proposition 5.3. If a and b satisfies the bounds in Lemma 5.2 then the log pair $(S, \frac{65}{64}D)$ is log canonical.

Proof. We suppose that a and b satisfies the bounds in Lemma 5.2.

We fist claim that the log pair $(S, \frac{65}{64}D)$ is log canonical outside of C_x and C_y. This immediately follows from the same argument as in the beginning of the proof of Proposition 4.1 with the pencil $\mathcal{P}$ given by

$$\begin{aligned} \nu\big(x^{10} + c_1y^7\big) + \mu y^4\big(z^2 + c_2y^3\big) = 0 \quad & \text{on } S_{45}, \\ \nu\big(x^{18} + c_1y^7\big) + \mu y^4\big(z^2 + c_2y^3\big) = 0 \quad & \text{on } S_{81}, \\ \nu\big(x^{26} + c_1y^7\big) + \mu y^4\big(z^2 + c_2y^3\big) = 0 \quad & \text{on } S_{117}, \end{aligned}$$

where c_1 and c_2 are appropriate constants, for $[\nu : \mu] \in \mathbb{P}^1$. For a general member C in $\mathcal{P}$ we obtain

$$C \cdot D \leqslant \frac{64}{65},$$

which verifies the claim. Notice that the surface S is smooth outside C_x and C_y.

We now consider a point P on C_y different from O_t. Suppose that the log pair $(S, \frac{65}{64}D)$ is not log canonical at the point P. Recall that $(S, \frac{65e}{64}C_y)$ is log canonical, where e is the positive rational number such that $-K_S \sim_{\mathbb{Q}} eC_y$, and that the curve C_y is irreducible. Thus, it follows from [CS08, Remark 2.22] that there exists an effective $\mathbb{Q}$-divisor D' on the surface S such that

$$D' \sim_{\mathbb{Q}} -K_S,$$

the log pair $(S, \frac{65}{64}D')$ is not log canonical at the point P, and the support of the divisor D' does not contain the curve C_y. Observe that

$$C_y \cdot D' \leqslant \frac{64}{7 \cdot 65}.$$

This implies that the log pair $(S, \frac{65}{64}D')$ is log canonical at the point P. This contradiction shows that the log pair $(S, \frac{65}{64}D)$ is log canonical outside C_x.

Let P be a point on C_x other than O_t. We have two cases for the location of P, i.e., when P lies on L_{xz} and when it lies on R_x. Note that we always have $\frac{65a}{64} < 1$ and $\frac{65b}{64} < 1$.

We first consider the case where P belongs to L_{xz}. Then the log pair $(S, L_{xz} + \frac{65b}{64}R_x + \frac{65}{64}\Delta)$ is log canonical at P. Indeed,

$$(bR_x + \Delta) \cdot L_{xz} = \big(D - aL_{xz}\big) \cdot L_{xz} = \begin{cases} \dfrac{6+23a}{190} \leqslant \dfrac{64}{65 \cdot 10} & \text{for } S_{45} \\ \dfrac{8+47a}{37 \cdot 18} \leqslant \dfrac{64}{65 \cdot 18} & \text{for } S_{81} \\ \dfrac{10+71a}{55 \cdot 26} \leqslant \dfrac{64}{65 \cdot 26} & \text{for } S_{117}. \end{cases}$$

Lemmas 2.4 or 2.5 imply that $(S, \frac{65}{64}D)$ is log canonical at the point P. If the point P must lie on R_x, then we consider

$$(aL_{xz} + \Delta) \cdot R_x = \big(D - bR_x\big) \cdot R_x = \begin{cases} \dfrac{3+8b}{95} \leqslant \dfrac{64}{65 \cdot 5} & \text{for } S_{45} \\ \dfrac{8+20b}{9 \cdot 37} \leqslant \dfrac{64}{65 \cdot 9} & \text{for } S_{81} \\ \dfrac{10+32b}{13 \cdot 55} \leqslant \dfrac{64}{65 \cdot 13} & \text{for } S_{117}. \end{cases}$$

Lemmas 2.4 or 2.5 then show that $(S, \frac{65}{64}D)$ is log canonical at the point P.

Now it is enough to show that $(S, \frac{65}{64}D)$ is log canonical at O_t.

Recall that S_{45} (resp. S_{81} snd S_{117}) has singularity of type $\frac{1}{19}(2,3)$ (resp. $\frac{1}{37}(2,3)$ and $\frac{1}{55}(2,3)$) at the point O_t. In the chart given by $t = 1$, the surface S_{45} is given by

$$z^3 - y^3z + x + x^5y = 0,$$

the surface S_{81} by

$$z^3 - y^3z + x + x^9y = 0,$$

and the surface S_{117} by

$$z^3 - y^3z + x + x^{13}y = 0.$$

In a neighborhood of the point O_t, we can consider y and z as local weighted coordinates such that $\mathrm{wt}(y) = 2$ and $\mathrm{wt}(z) = 3$.

Let $f\colon \widetilde{S}\to S$ be the weighted blow up at the singular point O_t such that $\mathrm{wt}(y)=2$ and $\mathrm{wt}(z)=3$. Denote by E the exceptional curve of the blow up f. Then

$$K_{\widetilde{S}_{45}}\sim_{\mathbb{Q}} f^*\big(K_{S_{45}}\big)-\frac{14}{19}E;$$

$$K_{\widetilde{S}_{81}}\sim_{\mathbb{Q}} f^*\big(K_{S_{81}}\big)-\frac{32}{37}E;$$

$$K_{\widetilde{S}_{117}}\sim_{\mathbb{Q}} f^*\big(K_{S_{117}}\big)-\frac{10}{11}E.$$

The surface S has two singular points in E. One is of type $\frac{1}{2}(1,1)$ and the other is of type $\frac{1}{3}(1,1)$. Denote the former one by O_2 and the latter one by O_3. Observe

$$E^2=\begin{cases} -\dfrac{19}{6} & \text{on } \widetilde{S}_{45},\\ -\dfrac{37}{6} & \text{on } \widetilde{S}_{81},\\ -\dfrac{55}{6} & \text{on } \widetilde{S}_{117},\end{cases}$$

and $E\cong\mathbb{P}^1$.

Let $\widetilde{L}_{xz}$ and $\widetilde{R}_x$ be the proper transforms of the curve L_{xz} and R_x to the surface $\widetilde{S}$, respectively. Then

$$\widetilde{L}_{xz}\sim_{\mathbb{Q}} f^*\big(L_{xz}\big)-\frac{3}{c}E,\quad \widetilde{R}_x\sim_{\mathbb{Q}} f^*\big(R_x\big)-\frac{6}{c}E,$$

where c is the index of singularity O_t. The intersection $E\cap\widetilde{L}_{xz}$ consists of the point O_2 and the intersection $E\cap\widetilde{R}_x$ consists of a single smooth point. Note that $\widetilde{L}_{xz}\cdot E=\frac{1}{2}$ and the curves E and $\widetilde{R}_x$ intersect transversally.

Recall that $D=aL_{xz}+bR_x+\Delta$. Denote by $\widetilde{\Delta}$ be the proper transform of the $\mathbb{Q}$-divisor Δ on the surface $\widetilde{S}$. Then

$$\widetilde{\Delta}\sim_{\mathbb{Q}} f^*\big(\Delta\big)-mE$$

for some non-negative rational number m. To estimate m, consider the intersection

$$0\leqslant \widetilde{R}_x\cdot\widetilde{\Delta}=\widetilde{R}_x\cdot\Big(f^*\big(\Delta\big)-mE\Big)=R_x\cdot\Delta-m.$$

Applying (5.1), we are able to obtain

$$(5.4)\qquad m\leqslant\begin{cases} \dfrac{6}{5\cdot 19}-\dfrac{3a}{19}+\dfrac{8b}{5\cdot 19}\leqslant\dfrac{6}{5\cdot 19}+\dfrac{8b}{5\cdot 19}\leqslant\dfrac{26}{285} & \text{for } S_{45},\\ \dfrac{8}{9\cdot 37}-\dfrac{3a}{37}+\dfrac{20b}{9\cdot 37}\leqslant\dfrac{8}{9\cdot 37}+\dfrac{20b}{9\cdot 37}\leqslant\dfrac{4}{111} & \text{for } S_{81},\\ \dfrac{2}{11\cdot 13}-\dfrac{3a}{55}+\dfrac{32b}{13\cdot 55}\leqslant\dfrac{2}{11\cdot 13}+\dfrac{32b}{13\cdot 55}\leqslant\dfrac{634}{17875} & \text{for } S_{117}.\end{cases}$$

We now suppose that the log pair $(S,\frac{65}{64}D)$ is not log canonical at O_t. Put $\lambda=\frac{65}{64}$. Then

$$K_{\widetilde{S}}+\lambda a\widetilde{L}_{xz}+\lambda b\widetilde{R}_x+\lambda\widetilde{\Delta}+\mu E\sim_{\mathbb{Q}} f^*\big(K_S+\lambda D\big),$$

where

$$\mu = \begin{cases} \dfrac{3\lambda a}{19} + \dfrac{6\lambda b}{19} + \lambda m + \dfrac{14}{19} & \text{for } S_{45}, \\ \dfrac{3\lambda a}{37} + \dfrac{6\lambda b}{37} + \lambda m + \dfrac{32}{37} & \text{for } S_{81}, \\ \dfrac{3\lambda a}{55} + \dfrac{6\lambda b}{55} + \lambda m + \dfrac{10}{11} & \text{for } S_{117}. \end{cases}$$

Thus, the log pair

$$\left(\widetilde{S}, \lambda a \widetilde{L}_{xz} + \lambda b \widetilde{R}_x + \lambda \widetilde{\Delta} + \mu E\right) \tag{5.5}$$

is not log canonical at some point $O \in E$. Using (5.4) and bounds for b, we can easily check

$$\mu \leqslant \begin{cases} \dfrac{3\lambda a}{19} + \dfrac{6\lambda b}{19} + \dfrac{6\lambda}{95} - \dfrac{3a\lambda}{19} + \dfrac{8\lambda b}{95} + \dfrac{14}{19} = \dfrac{2\lambda b}{5} + \dfrac{6\lambda}{95} + \dfrac{14}{19} \leqslant 1 & \text{for } S_{45}, \\ \dfrac{3\lambda a}{37} + \dfrac{6\lambda b}{37} + \dfrac{8\lambda}{9\cdot 37} - \dfrac{3\lambda a}{37} + \dfrac{20\lambda b}{9\cdot 37} + \dfrac{32}{37} = \dfrac{2\cdot 29\lambda b}{3\cdot 37} + \dfrac{8\lambda}{9\cdot 37} + \dfrac{32}{37} \leqslant 1 & \text{for } S_{81}, \\ \dfrac{3\lambda a}{55} + \dfrac{6\lambda b}{55} + \dfrac{2\lambda}{11\cdot 13} - \dfrac{3\lambda a}{55} + \dfrac{32\lambda b}{13\cdot 55} + \dfrac{10}{11} = \dfrac{2\lambda b}{13} + \dfrac{2\lambda}{11\cdot 13} + \dfrac{10}{11} \leqslant 1 & \text{for } S_{117}. \end{cases}$$

If $O = E \cap \widetilde{R}_x$, then we apply Lemma 2.4 to (5.5) and E. This yields

$$\lambda b + \lambda \widetilde{\Delta} \cdot E = \left(\lambda b \widetilde{R}_x + \lambda \widetilde{\Delta}\right) \cdot E > 1,$$

so that we could obtain absurd inequalities

$$\frac{64}{65} = \frac{1}{\lambda} < b + \widetilde{\Delta} \cdot E = b + \frac{cm}{6} \leqslant \begin{cases} \dfrac{1}{3} + \dfrac{19}{60} = \dfrac{13}{20} & \text{for } S_{45}, \\ \dfrac{1}{5} + \dfrac{37}{6\cdot 25} = \dfrac{67}{150} & \text{for } S_{81}, \\ \dfrac{12}{25} + \dfrac{1}{3} = \dfrac{61}{75} & \text{for } S_{117}, \end{cases}$$

where c is the index of the singularity O_t. The inequality

$$\widetilde{\Delta} \cdot E = \frac{cm}{6} \leqslant \left\{ \begin{array}{ll} \dfrac{13}{45} & \text{for } S_{45}, \\ \dfrac{2}{9} & \text{for } S_{81}, \\ \dfrac{317}{975} & \text{for } S_{117}. \end{array} \right\} \leqslant \frac{1}{3\lambda} = \frac{64}{3\cdot 65}$$

implies that $O = O_2$. However, using (5.4) and Lemma 2.5 (applied to (5.5) and E), we conclude that the log pair (5.5) is log canonical everywhere since

$$\left(a\widetilde{L}_{xz} + \widetilde{\Delta}\right) \cdot E = \frac{a}{2} + \widetilde{\Delta} \cdot E = \frac{a}{2} + \frac{cm}{6} \leqslant \left\{ \begin{array}{ll} \dfrac{1}{5} + \dfrac{4b}{15} \leqslant \dfrac{13}{45} & \text{for } S_{45}, \\ \dfrac{4}{27} + \dfrac{10b}{27} \leqslant \dfrac{2}{9} & \text{for } S_{81}, \\ \dfrac{5}{39} + \dfrac{16b}{39} \leqslant \dfrac{317}{975} & \text{for } S_{117}, \end{array} \right\} \leqslant \frac{1}{2\lambda} = \frac{32}{65}$$

This completes the proof. □

Corollary 5.6. The δ-invariant of S is at least $\frac{65}{64}$.

Proof. This immediately follows from Proposition 5.3 and Lemma 5.2. □

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Ivan Cheltsov
School of Mathematics, The University of Edinburgh
Edinburgh EH9 3JZ, UK

National Research University Higher School of Economics, Laboratory of Algebraic Geometry,
6 Usacheva street, Moscow, 117312, Russia
`I.Cheltsov@ed.ac.uk`

Jihun Park
Center for Geometry and Physics, Institute for Basic Science
77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk, 37673, Korea

Department of Mathematics, POSTECH
77 Cheongam-ro, Nam-gu, Pohang, Gyeongbuk, 37673, Korea
`wlog@postech.ac.kr`

Constantin Shramov
Steklov Mathematical Institute of Russian Academy of Sciences
8 Gubkina street, Moscow, 119991, Russia

National Research University Higher School of Economics, Laboratory of Algebraic Geometry,
6 Usacheva street, Moscow, 117312, Russia
`costya.shramov@gmail.com`