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# Reduction in a Linear Lambda-calculus with Applications to Operational Semantics

Alex Simpson

LFCS, School of Informatics  
University of Edinburgh, UK

**Abstract.** We study beta-reduction in a linear lambda-calculus derived from Abramsky’s linear combinatory algebras. Reductions are classified depending on whether the redex is in the computationally active part of a term (“surface” reductions) or whether it is suspended within the body of a thunk (“internal” reductions). If surface reduction is considered on its own then any normalizing term is strongly normalizing. More generally, if a term can be reduced to surface normal form by a combined sequence of surface and internal reductions then every combined reduction sequence from the term contains only finitely many surface reductions. We apply these results to the operational semantics of `Lily`, a second-order linear lambda-calculus with recursion, introduced by Bierman, Pitts and Russo, for which we give simple proofs that call-by-value, call-by-name and call-by-need contextual equivalences coincide.

## 1 Introduction

The language `Lily` was introduced by Bierman, Pitts and Russo in [3]. It is a typed lambda-calculus based on a second-order intuitionistic linear type theory with recursion. What makes it interesting from a programming language perspective is that, following ideas of Plotkin [10], the language is able to encode a remarkably rich range of datatype constructs (eager products, lazy products, coproducts, polymorphism, abstract types, recursive types, etc.). Furthermore, its linearity makes it potentially useful for modelling single-threadedness and other state and resource-related concepts, cf. [7].

The main achievement of [3] was to establish direct operational techniques for reasoning about `Lily` up to contextual equivalence. Such techniques include useful extensionality properties, and a powerful framework for establishing program equalities using an adaptation (based on [8]) of Reynolds’ relational parametricity (first introduced in [11]). In order to get this machinery to work, the authors of [3] need to first establish one key result about `Lily`, a result which pervades all further developments in their paper. This result, the so-called Strictness Theorem, asserts the (surprising at first sight) fact that call-by-name and call-by-value operational semantics for `Lily` both give rise to the same notion of contextual equivalence.

The outline proof of the Strictness Theorem in [3] makes rather heavy use of the well-stocked armoury of known operational techniques. In particular it uses Howe’s method [4] to obtain a version of Mason and Talcott’s *ciu theorem* [6]. The starting point for the research in this paper was the realisation that basic techniques from rewriting

could be applied to obtain an alternative, self-contained and essentially simple proof of the Strictness Theorem.

In Sec. 2, we review `Lily` and its operational semantics. Then, in Secs. 3 and 4, we present our alternative proof of the Strictness Theorem. We translate `Lily` into a simple untyped linear lambda-calculus containing: *linear* lambda abstractions,  $\lambda x.M$ ; *non-linear* lambda abstractions,  $\lambda!x.M$ , which require their arguments to be suspended as “thunks”; and *thunks* themselves,  $!M$ . We study beta-reduction in this untyped calculus, making the restriction that, as thunks are considered suspended, reductions should not take place within a thunk. This restricted relation, which we call *surface reduction*, turns out to be extremely well behaved: as well as the expected confluence property, it holds that every normalizing term is strongly normalizing. The Strictness Theorem for `Lily` follows easily from this latter fact, using straightforward simulations under surface reduction of call-by-name and call-by-value evaluation for `Lily`.

In Sec. 5, we (temporarily) turn attention away from `Lily` and take a deeper look at our untyped linear lambda-calculus and its reduction properties. In order to obtain a conversion relation between terms that is a congruence, it is necessary to consider also reductions inside thunks. We call such reductions *internal reductions*, and we call arbitrary reductions (either surface or internal) *combined reductions*. As well as the expected confluence properties (for both internal and combined reductions), we show that internal reductions can always be postponed until after surface reductions. Further, we show that if a term reduces (under combined reduction) to a surface normal form then any sequence of combined reductions contains only finitely many surface reductions.

Next, in Sec. 6, we return to `Lily` and show that the results of Sec. 5 again have applications to operational semantics. We use them to establish the equivalence of the call-by-name operational semantics of `Lily` with an implementation-oriented call-by-need semantics. Once again, the equivalence of these two semantics had previously been established by the authors of [3], but with an intricate and lengthy proof (private communication). Our proof turns out to be relatively straightforward.

Finally, in Sec. 7, we observe that our untyped linear lambda-calculus is exactly the lambda-calculus counterpart of Abramsky’s *linear combinatory algebras*, presented in [1]. This connection makes us believe that the linear lambda-calculus introduced in this paper is rather natural. Accordingly, it is plausible that the properties of reduction established in Secs. 3 and 5 may turn out to have other applications, perhaps again in the area of operational semantics, but possibly more widely.

## 2 `Lily` and its Operational Semantics

In this section, we review the language `Lily`, a typed  $\lambda$ -calculus, based on second-order intuitionistic linear type theory with recursion, introduced in [3].

The language of types for `Lily` contains just three type constructors: linear function space  $\sigma \multimap \tau$ ; linear “exponentials”  $!\sigma$ , used to type thunks; and universally quantified types  $\forall\alpha.\sigma$ , used for polymorphism. Types  $\sigma, \tau, \dots$  are thus built up from type variables  $\alpha, \beta, \dots$ , according to the grammar:

$$\sigma ::= \alpha \mid \sigma \multimap \tau \mid !\tau \mid \forall\alpha.\tau .$$

$$\begin{array}{c}
\frac{}{\Gamma; x: \sigma \vdash x: \sigma} \qquad \frac{}{\Gamma, x: \sigma; - \vdash x: \sigma} \\
\\
\frac{\Gamma; \Delta, x: \sigma \vdash t: \tau}{\Gamma; \Delta \vdash \lambda x: \sigma. t: \sigma \multimap \tau} \qquad \frac{\Gamma; \Delta \vdash s: \sigma \multimap \tau \quad \Gamma; \Delta' \vdash t: \sigma}{\Gamma; \Delta, \Delta' \vdash s(t): \tau} \\
\\
\frac{\Gamma; - \vdash t: \tau}{\Gamma; - \vdash !t: !\tau} \qquad \frac{\Gamma; \Delta \vdash s: !\sigma \quad \Gamma, x: \sigma; \Delta' \vdash t: \tau}{\Gamma; \Delta, \Delta' \vdash \text{let } !x = s \text{ in } t: \tau} \Delta \# x: \sigma \\
\\
\frac{\Gamma; \Delta \vdash t: \tau}{\Gamma; \Delta \vdash \Lambda \alpha. t: \forall \alpha. \tau} \alpha \notin \text{ftv}(\Gamma, \Delta) \qquad \frac{\Gamma; \Delta \vdash t: \forall \alpha. \tau}{\Gamma; \Delta \vdash t(\sigma): \tau[\sigma/\alpha]} \qquad \frac{\Gamma, x: \sigma; - \vdash t: \sigma}{\Gamma; - \vdash \text{rec } x: \sigma. t: \sigma}
\end{array}$$

**Fig. 1.** Typing rules for `Lily`.

As usual,  $\alpha$  is bound in  $\forall \alpha. \tau$ . We write  $\text{ftv}(\sigma)$  for the set of free type variables in  $\sigma$  (and below apply the same notation to terms and contexts in the evident way). If  $\text{ftv}(\sigma) = \emptyset$  then  $\sigma$  is said to be *closed*.

Although simple, the above language of types is remarkably rich. For example, the other type constructors of intuitionistic linear logic can all be encoded: non-linear (intuitionistic) function space,  $\sigma \rightarrow \tau$ , using Girard’s  $!\sigma \multimap \tau$ ; tensor,  $\otimes$ , product,  $\&$  and sum,  $\oplus$ . One can also encode basic ground types (booleans, natural numbers, etc.), and existentially quantified types  $\exists \alpha. \sigma$ , and, due to the recursion operator in `Lily`, arbitrary recursive types. These encodings are due to Plotkin [10], see [3] for details.

The term language of `Lily` is the expected typed  $\lambda$ -calculus associated with the above types, together with a recursion operator.<sup>1</sup> Raw terms  $s, t, \dots$  are built from term variables  $x, y, \dots$  according to the grammar:

$$t ::= x \mid \lambda x: \sigma. t \mid s(t) \mid !t \mid \text{let } !x = s \text{ in } t \mid \Lambda \alpha. t \mid t(\sigma) \mid \text{rec } x: \sigma. t .$$

Here,  $x$  is bound in  $\lambda x: \sigma. t$ , in  $\text{let } !x = s \text{ in } t^2$  and in  $\text{rec } x: \sigma. t$ , and  $\alpha$  is bound in  $\Lambda \alpha. t$ . We write  $\text{fv}(t)$  for the set of free variables in a term  $t$ . We identify terms up to  $\alpha$ -equivalence.

The typing rules for `Lily` are based on Barber and Plotkin’s DILL [2]. We use  $\Gamma, \Delta, \dots$  to range over “contexts”, which are finite functions from term variables to types. We write  $\Gamma \# \Delta$  to say that the domains of  $\Gamma$  and  $\Delta$  are disjoint. The typing rules manipulate sequents  $\Gamma; \Delta \vdash t: \sigma$  where  $\Gamma \# \Delta$ . Here,  $\Gamma$  types the “intuitionistic” variables appearing in the term  $t$ , which have no restriction on how they occur, and  $\Delta$  types the “linear” variables, each of which occurs exactly once in  $t$ , not within the

<sup>1</sup> We depart from [3] by building an explicit recursion operator into `Lily`, instead of incorporating recursion within `thunks`. This is an inessential difference.

<sup>2</sup> For simplicity, we place an inessential restriction in the typing rules ensuring that the term  $\text{let } !x = s \text{ in } t$  is well typed only when  $x$  does not occur free in  $s$ .

$$\begin{array}{c}
\frac{s \rightarrow s'}{s(t) \rightarrow s'(t)} \quad \frac{t \rightarrow t'}{t(\sigma) \rightarrow t'(\sigma)} \quad \frac{}{(\Lambda\alpha. t)(\sigma) \rightarrow t[\sigma/\alpha]} \\
\\
\frac{s \rightarrow s'}{\text{let } !x = s \text{ in } t \rightarrow \text{let } !x = s' \text{ in } t} \quad \frac{}{\text{let } !x = !s \text{ in } t \rightarrow t[s/x]} \quad \frac{}{\text{rec } x : \sigma. t \rightarrow t[\text{rec } x : \sigma. t/x]} \\
\\
\frac{t \rightarrow_{v1} t'}{(\lambda x : \sigma. s)(t) \rightarrow_{v1} (\lambda x : \sigma. s)(t')} \quad \frac{}{(\lambda x : \sigma. s)(v) \rightarrow_{v1} s[v/x]} \quad \frac{}{(\lambda x : \sigma. s)(t) \rightarrow_{nm} s[t/x]}
\end{array}$$

**Fig. 2.** Call-by-value and Call-by-name Evaluation for `Lily`

scope of a `!` or `rec` operator. The typing rules are presented in Fig. 1. In them, a comma always denotes a disjoint union of contexts and a dash denotes the empty context. We write  $t : \tau$  to mean that the sequent  $\vdash t : \tau$  is derivable, where  $\tau$  is a closed type ( $t$  is necessarily a closed term).

Following [3], we define two operational semantics for `Lily`, one using a call-by-value evaluation of function application, and one using call-by-name. In both cases, the operational semantics reduces terms to *values*  $v, \dots$ , which are terms of the form:

$$v ::= \lambda x : \sigma. t \mid !t \mid \Lambda\alpha. t .$$

In contrast to [3], we give the operational semantics in a small-step style. This facilitates our proofs, but only in an inessential way, the equivalence of big-step and small-step definitions being anyway easy to establish.

Figure 2 defines two small-step evaluation relations  $t \rightarrow_{v1} t'$  and  $t \rightarrow_{nm} t'$  between `Lily` terms. The call-by-value (or strict) relation  $t \rightarrow_{v1} t'$  is inductively defined by the two specific  $\rightarrow_{v1}$  rules for application together with all rules written using the neutral  $\rightarrow$  notation. Similarly, the call-by-name (or non-strict) relation  $t \rightarrow_{nm} t'$  is defined by the specific  $\rightarrow_{nm}$  rule for application together with the neutral rules. Note that both operational semantics are deterministic.

Our interest lies in the operational semantics of `Lily programs`, i.e. of closed terms of closed type. It is easily seen that if  $t : \sigma$  and  $t \rightarrow_{v1} t'$  then  $t' : \sigma$  (and similar if  $t \rightarrow_{nm} t'$ ). Also, by induction on the structure of  $t$ , one sees that if  $t : \sigma$  then  $t$  does not reduce under  $\rightarrow_{v1}$  if and only if  $t$  is a value (and similar for  $\rightarrow_{nm}$ ). We write  $t \downarrow_{v1}$  (resp.  $t \downarrow_{nm}$ ) for the “termination” property: there exists a value  $v$  such that  $t \rightarrow_{v1}^* v$  (resp.  $t \rightarrow_{nm}^* v$ ), where, as usual,  $R^*$  (resp.  $R^+$ ) denotes the reflexive-transitive (resp. transitive) closure of the relation  $R$ .

The program below shows that sometimes call-by-name evaluation terminates when call-by-value does not (cf. [3, Example 2.2]).

$$(\lambda f : \forall\alpha. \alpha \multimap \forall\alpha. \alpha. \lambda x : \forall\alpha. \alpha. f(x))(\text{rec } g : \forall\alpha. \alpha \multimap \forall\alpha. \alpha. g) \quad (1)$$

This program has type  $\forall\alpha. \alpha \multimap \forall\alpha. \alpha$ . An important insight of [3], is that the most useful notion of contextual equivalence for `Lily` is obtained by only observing termi-

nation for programs of exponential type  $!\tau$ . The restriction to such observations corresponds to observing termination at ground types (such as booleans, naturals, etc.), it yields desirable extensionality properties for contextual equivalence, and it is crucial to the correctness of Plotkin’s [10] encodings of datatype constructions in `Lily`.

The key result of [3] that underpins its entire study of contextual equivalence for `Lily` is the “Strictness Theorem”.

**Theorem 2.1 (Strictness Theorem [3]<sup>3</sup>).** *If  $t : !\tau$  then  $t \downarrow_{v1}$  if and only if  $t \downarrow_{nm}$ .*

When termination observations are restricted to exponential types, it follows immediately from the theorem that both call-by-value and call-by-name operational semantics induce the same contextual equivalence.

We remark that the Strictness Theorem is stated in the most general form possible: the result holds for no types other than exponential types, as simple adaptations of (1) readily show. This suggests that any proof of Theorem 2.1 has to uncover some crucial property of exponential types. The machinery used in [3] to this end has already been mentioned in Section 1. In this paper, we shall instead prove Theorem 2.1 using surprisingly elementary techniques from rewriting, translating `Lily` into a very simple untyped linear  $\lambda$ -calculus in which (the appropriate notion of)  $\beta$ -reduction simulates both call-by-value and call-by-name operational semantics. This untyped linear  $\lambda$ -calculus includes explicit thunks, and it is the treatment of these thunks that will reflect the all-important behaviour of `Lily` at exponential type.

### 3 A Linear Lambda-calculus and Surface Reduction

In this section, we introduce our untyped linear  $\lambda$ -calculus. Its main ingredients are: applications  $MN$ ; linear lambda abstractions,  $\lambda x.M$ ; non-linear lambda abstractions,  $\lambda!x.M$ , which require their arguments to be suspended as thunks; and thunks themselves,  $!M$ . Formally, raw terms  $M, N, \dots$  are built up from variables  $x, y, \dots$  according to the grammar:

$$M ::= x \mid MN \mid \lambda x.M \mid \lambda!x.M \mid !M .$$

The variable  $x$  is bound in both  $\lambda x.M$  and  $\lambda!x.M$ . We write  $\equiv$  for syntactic equality of terms modulo  $\alpha$ -equivalence.

We say that  $x$  is *linear in  $M$*  if  $x$  occurs free exactly once in  $M$  and, moreover, this free occurrence of  $x$  does not lie within the scope of a  $!$  operator in  $M$ . A term  $M$  is said to be *linear* if, in every subterm of  $M$  the form  $\lambda x.M'$ , it holds that  $x$  is linear in  $M'$ . *Henceforth, we consider linear terms only.*

In Fig. 3, we define a version of  $\beta$ -reduction for our calculus. The important points are the two types of redex, and that no reduction occurs under the scope of a  $!$  operator. The latter restriction reflects the idea that thunks are suspended computations. We call the reduction defined in Fig. 3 *surface reduction*. It is easily shown that when  $M$  is linear and  $M \rightarrow N$  then  $N$  is linear. From now on, all similar observations about linearity will be omitted. All operations we consider will respect the linearity of terms.

<sup>3</sup> The theorem as stated here is easily shown to be equivalent to the original [3, Theorem 2.3].

$$\begin{array}{c}
\frac{}{(\lambda x.M)(N) \rightarrow M[N/x]} \qquad \frac{}{(\lambda !x.M)(!N) \rightarrow M[N/x]} \\
\\
\frac{M \rightarrow M'}{MN \rightarrow M'N} \qquad \frac{N \rightarrow N'}{MN \rightarrow MN'} \qquad \frac{M \rightarrow M'}{\lambda x.M \rightarrow \lambda x.M'} \qquad \frac{M \rightarrow M'}{\lambda !x.M \rightarrow \lambda !x.M'}
\end{array}$$

**Fig. 3.** Surface Reduction

A term is said to be in *surface normal form* if there is no surface reduction from the term. Trivially, any term  $!M$  is in surface normal form. A *reduction sequence* from  $M$  is a finite or infinite sequence  $M \equiv M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \dots$ . A *completed* reduction sequence is a reduction sequence that is either infinite or is finite with the last term in the sequence in surface normal form.

The linearity restriction on terms combines with the disallowance of reduction within thunks to ensure that the basic well-behavedness properties of surface reduction are almost trivial to establish. The main, though very simple, results of this section are Corollaries 3.3 and 3.4 below. (Only the latter is used in the proof of Theorem 2.1.)

**Lemma 3.1.**

1. If  $M \rightarrow M'$  then  $M[N/x] \rightarrow M'[N/x]$ .
2. If  $N \rightarrow N'$  and  $x$  is linear in  $M$  then  $M[N/x] \rightarrow M[N'/x]$ .

**Proposition 3.2.** If  $M \rightarrow L$  and  $M \rightarrow L'$  then either  $L \equiv L'$  or there exists  $N$  such that  $L \rightarrow N$  and  $L' \rightarrow N$ .

*Proof.* By induction on the structure of  $M$ , considering all possible cases for  $M \rightarrow L$  and  $M \rightarrow L'$ . We consider only the two redex cases.

If  $M \equiv (\lambda x.M_1)(M_2) \rightarrow M_1[M_2/x] \equiv L$  and  $L \not\equiv L'$  then either  $L' \equiv (\lambda x.L'_1)(M_2)$  where  $M_1 \rightarrow L'_1$  or  $L' \equiv (\lambda x.M_1)(L'_2)$  where  $M_2 \rightarrow L'_2$ . In the first case, we have  $L \rightarrow L'_1[M_2/x]$ , by Lemma 3.1.1, and also  $L' \rightarrow L'_1[M_2/x]$ . In the second, we have  $L \rightarrow M_1[L'_2/x]$ , by Lemma 3.1.2, and also  $L' \rightarrow M_1[L'_2/x]$ .

If  $M \equiv (\lambda !x.M_1)(!M_2) \rightarrow M_1[M_2/x] \equiv L$  and  $L \not\equiv L'$  then  $L' \equiv (\lambda !x.L'_1)(!M_2)$  where  $M_1 \rightarrow L'_1$ . Thus  $L \rightarrow L'_1[M_2/x]$ , by Lemma 3.1.1, and also  $L' \rightarrow L'_1[M_2/x]$ .  $\square$

**Corollary 3.3 (Confluence).** If  $M \rightarrow^* M_1$  and  $M \rightarrow^* M_2$  then there exists  $N$  such that  $M_1 \rightarrow^* N$  and  $M_2 \rightarrow^* N$ .

**Corollary 3.4 (Uniform normalization).** If  $M \rightarrow^* V$  is a  $k$ -step reduction sequence, where  $V$  is in surface normal form, then every reduction sequence from  $M$  has at most  $k$  steps, and every completed reduction sequence has exactly  $k$  steps and terminates with  $V$ . In particular, if a term is normalizing under surface reduction then it is strongly normalizing.

## 4 Proof of the Strictness Theorem

The proof is based on a simple translation of `Lily` into the untyped linear  $\lambda$ -calculus of Sec. 3. The translation uses an untyped recursion construct, defined by:

$$\mu x.M =_{\text{def}} (\lambda!x.M[x(!x)/x])(! \lambda!x.M[x(!x)/x]) .$$

Observe that  $\mu x.M \rightarrow M[(\mu x.M)/x]$ .

To every raw term  $t$  of `Lily`, we define a raw term  $t^*$ , in the grammar from Sec. 3, by induction on the structure of  $t$ . In the definition, we make use of a distinguished variable  $u$ , used as a dummy translation for types.

$$\begin{aligned} x^* &=_{\text{def}} x & (\text{let } !x = s \text{ in } t)^* &=_{\text{def}} (\lambda!x.t^*)(s^*) \\ (\lambda x:\sigma. t)^* &=_{\text{def}} \lambda x.t^* & (\Lambda\alpha. t)^* &=_{\text{def}} \lambda!w.t^* & w \notin \text{fv}(t^*) \\ (s(t))^* &=_{\text{def}} s^* t^* & (t(\sigma))^* &=_{\text{def}} t^*(!u) \\ (!t)^* &=_{\text{def}} !t^* & (\text{rec } x:\sigma. t)^* &=_{\text{def}} \mu x.t^* . \end{aligned}$$

The four lemmas below are straightforward.

**Lemma 4.1.**  $(s[t/x])^* \equiv s^*[t^*/x]$ .

**Lemma 4.2.** If  $\Gamma; \Delta \vdash t : \sigma$  then the raw term  $t^*$  is linear.

**Lemma 4.3.** If  $t_1 \rightarrow_{v1} t_2$  then  $t_1^* \rightarrow t_2^*$ .

**Lemma 4.4.** If  $t_1 \rightarrow_{nm} t_2$  then  $t_1^* \rightarrow t_2^*$ .

**Corollary 4.5.** If  $t : !\tau$  then the following are equivalent:

1.  $t \downarrow_{v1}$ ,
2.  $t \downarrow_{nm}$ ,
3.  $t^*$  is surface normalizing.

*Proof.* To show that 1 implies 3, suppose that  $t \downarrow_{v1}$ . Then there exists  $v$  with  $v : !\tau$  such that  $t \rightarrow_{v1}^* v$ . As  $v : !\tau$ , it holds that  $v \equiv !t'$ . By Lemma 4.3,  $t^* \rightarrow^* (!t')^* \equiv !(t'^*)$ . But  $!(t'^*)$  is in surface normal form, hence  $t^*$  is surface normalizing.

For the converse, suppose  $t \not\downarrow_{v1}$ . Then there exists an infinite sequence of call-by-value evaluation steps  $t \equiv t_0 \rightarrow_{v1} t_1 \rightarrow_{v1} t_2 \rightarrow_{v1} \dots$ . Whence, by Lemma 4.3,  $t^*$  has an infinite surface reduction sequence. Thus, by Corollary 3.4,  $t^*$  is not normalizing under surface reduction.

The equivalence of 2 and 3 is shown in the same way, using Lemma 4.4.  $\square$

Theorem 2.1 is immediate from the corollary. Note that the point that fails for `Lily` programs  $t : \sigma$  of arbitrary type is that it is not in general the case that  $t \downarrow_{v1}$  (or  $t \downarrow_{nm}$ ) implies that  $t^*$  is surface normalizing, because, apart from at exponential type, `Lily` values do not necessarily translate to surface normal forms, indeed not even to surface normalizing terms (for example,  $\lambda x:\sigma. \text{rec } y:\tau. y$ ).

It is worth remarking that the techniques of this section can similarly be used to show that variant operational semantics for `Lily`, in which evaluation takes place under  $\Lambda$ - and/or  $\lambda$ -abstractions, also give rise to the same contextual equivalence.



$$\begin{array}{ccc}
\frac{M \rightarrow M'}{!M \dashrightarrow !M'} & \frac{M \dashrightarrow M'}{MN \dashrightarrow M'N} & \frac{N \dashrightarrow N'}{MN \dashrightarrow MN'} \\
\\
\frac{M \dashrightarrow M'}{\lambda x.M \dashrightarrow \lambda x.M'} & \frac{M \dashrightarrow M'}{\lambda !x.M \dashrightarrow \lambda !x.M'} & \frac{M \dashrightarrow M'}{!M \dashrightarrow !M'}
\end{array}$$

**Fig. 4.** Internal Reduction

## 5 Internal and Combined Reduction

In this section, we undertake a deeper study of reduction in our untyped linear  $\lambda$ -calculus. While surface reduction is computationally motivated, the disallowance of reduction inside thunks means that the conversion relation induced by surface reduction is not a congruence. To obtain a conversion relation that is a congruence, it is necessary to consider reduction inside thunks.

We implement reduction inside thunks using *internal reduction*,  $M \dashrightarrow M'$ , defined in Figure 4. *Combined reduction*  $M \Rightarrow M'$  is defined by:  $M \Rightarrow M'$  if  $M \rightarrow M'$  or  $M \dashrightarrow M'$ . Note that it is possible that both  $M \rightarrow M'$  and  $M \dashrightarrow M'$  (for example,  $\Omega(!\Omega) \rightarrow \Omega(!\Omega)$  and  $\Omega(!\Omega) \dashrightarrow \Omega(!\Omega)$ , where  $\Omega =_{\text{def}} \mu x.x$ , using the notation of Section 4). Accordingly, when we consider mixed reduction sequences containing both surface and internal reductions, we shall assume that each step comes with a distinguished status (as surface or internal).

The main technical effort of this section will go into the proof of Propositions 5.1 and 5.2 below.

### Proposition 5.1 (Confluence).

1. If  $M \dashrightarrow^* M_1$  and  $M \dashrightarrow^* M_2$  then there exists  $N$  such that  $M_1 \dashrightarrow^* N$  and  $M_2 \dashrightarrow^* N$ .
2. If  $M \Rightarrow^* M_1$  and  $M \Rightarrow^* M_2$  then there exists  $N$  such that  $M_1 \Rightarrow^* N$  and  $M_2 \Rightarrow^* N$ .

By the proposition, the conversion relation defined by  $M =_{\beta} M'$  if there exists  $N$  such that  $M \Rightarrow^* N$  and  $M' \Rightarrow^* N$  is an equivalence relation. It is, moreover, a congruence. Thus surface and internal reduction together provide an oriented decomposition of the natural  $\beta$ -conversion between terms of the untyped linear calculus. The next result exhibits natural structure within this decomposition.

**Proposition 5.2 (Internal Postponement).** *If  $M \Rightarrow^* N$ , by a reduction sequence containing  $k$  surface reductions, then there exists  $L$  such that  $M \rightarrow^* L \dashrightarrow^* N$ , where the surface reduction sequence  $M \rightarrow^* L$  contains at least  $k$  reductions.*

The proofs of the two propositions above make use of the (standard) technology of parallel reduction relations. Before giving these, we apply Proposition 5.2 to derive further properties of and interactions between surface, internal and combined reduction. The main result of the section is Theorem 5.5 below.

$$\begin{array}{c}
\frac{}{x \not\rightarrow x} \quad \frac{M \not\rightarrow M' \quad N \not\rightarrow N'}{(\lambda x.M)(N) \not\rightarrow M'[N'/x]} \quad \frac{M \not\rightarrow M' \quad N \not\rightarrow N'}{(\lambda!x.M)(!N) \not\rightarrow M'[N'/x]} \\
\\
\frac{M \not\rightarrow M' \quad N \not\rightarrow N'}{MN \not\rightarrow M'N'} \quad \frac{M \not\rightarrow M'}{\lambda x.M \not\rightarrow \lambda x.M'} \quad \frac{M \not\rightarrow M'}{\lambda!x.M \not\rightarrow \lambda!x.M'} \quad \frac{M \not\rightarrow M'}{!M \not\rightarrow !M'}
\end{array}$$

**Fig. 5.** Parallel Combined Reduction

$$\begin{array}{c}
\frac{}{x \not\rightarrow x} \quad \frac{M \not\rightarrow M' \quad N \not\rightarrow N'}{MN \not\rightarrow M'N'} \\
\\
\frac{M \not\rightarrow M'}{\lambda x.M \not\rightarrow \lambda x.M'} \quad \frac{M \not\rightarrow M'}{\lambda!x.M \not\rightarrow \lambda!x.M'} \quad \frac{M \not\rightarrow M'}{!M \not\rightarrow !M'}
\end{array}$$

**Fig. 6.** Parallel Internal Reduction

**Lemma 5.3.** *If  $M \rightarrow N$  and  $M \dashrightarrow M'$  then there exists  $N'$  such that  $M' \rightarrow N'$ .*

*Proof.* By induction on the derivation of  $M \rightarrow N$ . We consider one case.

Suppose  $M \equiv (\lambda!x.M_1)(!M_2) \rightarrow M_1[M_2/x] \equiv N$ . Either  $M' \equiv (\lambda!x.M'_1)(!M_2)$  where  $M_1 \dashrightarrow M'_1$ , or  $M' \equiv (\lambda!x.M_1)(!M'_2)$  where  $M_2 \Rightarrow M'_2$ . In the first case,  $M' \rightarrow M'_1[M_2/x]$ . In the second,  $M' \rightarrow M_1[M'_2/x]$ .  $\square$

**Corollary 5.4.** *If  $V$  is in surface normal form then:*

1.  $V \dashrightarrow N$  implies  $N$  is in surface normal form.
2.  $M \dashrightarrow V$  implies  $M$  is in surface normal form.

*Proof.* Statement 1 follows from Proposition 5.2, and statement 2 from Lemma 5.3.  $\square$

**Theorem 5.5.** *If  $M \Rightarrow^* V$ , where  $V$  is in surface normal form, then each infinite  $\Rightarrow$  reduction sequence from  $M$  contains only finitely many  $\rightarrow$  reductions.*

*Proof.* By Proposition 5.2, there exists  $U$  such that  $M \rightarrow^* U \dashrightarrow^* V$ . By Corollary 5.4.2,  $U$  is in surface normal form. Let  $k$  be the number of reductions in the sequence  $M \rightarrow^* U$ . We show that every  $\Rightarrow$  reduction sequence from  $M$  contains at most  $k$  surface reductions. Consider any reduction sequence  $M \Rightarrow^* N$  with  $l$  surface reductions. By Proposition 5.2, there exists  $L$  such that  $M \rightarrow^* L$  with at least  $l$  reductions. But, by Corollary 3.4, any  $\rightarrow$  reduction sequence from  $M$  has at most  $k$  reductions. Thus indeed  $l \leq k$ .  $\square$

We now turn to the proofs of Propositions 5.1 and 5.2, which use the parallel versions of combined and internal reduction defined in Figs. 5 and 6 respectively.

**Lemma 5.6.**

1.  $M \not\equiv M$  and  $M \dashv\rightarrow M$ .
2. If  $M \Rightarrow M'$  then  $M \not\equiv M'$ . Conversely, if  $M \not\equiv M'$  then  $M \Rightarrow^* M'$ .
3. If  $M \dashv\rightarrow M'$  then  $M \dashv\rightarrow M'$ . Conversely, if  $M \dashv\rightarrow M'$  then  $M \dashv\rightarrow^* M'$ .
4. If  $M \dashv\rightarrow M'$  then  $M \not\equiv M'$ .

**Lemma 5.7.**

1. If  $M \not\equiv M'$  and  $N \not\equiv N'$  then  $M[N/x] \not\equiv M'[N'/x]$ .
2. If  $M \dashv\rightarrow M'$  and  $N \dashv\rightarrow N'$  then  $M[N/x] \dashv\rightarrow M'[N'/x]$ .

**Lemma 5.8.**

1. If  $M \not\equiv M_1$  and  $M \not\equiv M_2$  then there exists  $N$  such that  $M_1 \not\equiv N$  and  $M_2 \not\equiv N$ .
2. If  $M \dashv\rightarrow M_1$  and  $M \dashv\rightarrow M_2$  then there exists  $N$  such that  $M_1 \dashv\rightarrow N$  and  $M_2 \dashv\rightarrow N$ .

*Proof.* The proof, which is by induction on the structure of  $M$ , is a routine analysis of all possible cases, cf. [9].  $\square$

Proposition 5.1 is a straightforward consequence the last lemma.

The remaining lemmas are directed towards the proof of Proposition 5.2.

**Sub-lemma 5.9.** *If  $M \dashv\rightarrow M'$ ,  $N \not\equiv N'$  and  $N \rightarrow^* N'' \dashv\rightarrow N'$  then there exists  $L$  such that  $M[N/x] \rightarrow^* L \dashv\rightarrow M'[N'/x]$ .*

*Proof.* By a straightforward induction on the derivation of  $M \dashv\rightarrow M'$ .  $\square$

**Lemma 5.10.** *If  $M \not\equiv M'$  then there exists  $L$  such that  $M \rightarrow^* L \dashv\rightarrow M'$ .*

*Proof.* By induction on the derivation of  $M \not\equiv M'$ . The most interesting case is when  $M \equiv (\lambda!x.M_1)(!M_2) \not\equiv M'_1[M'_2/x] \equiv M'$ , where  $M_1 \not\equiv M'_1$  and  $M_2 \not\equiv M'_2$ . Then, by induction hypothesis, there exist  $L_1, L_2$  such that  $M_1 \rightarrow^* L_1 \dashv\rightarrow M'_1$  and  $M_2 \rightarrow^* L_2 \dashv\rightarrow M'_2$ . By Sub-lemma 5.9, there exists  $L$  such that  $L_1[M_2/x] \rightarrow^* L \dashv\rightarrow M'_1[M'_2/x]$ . Thus  $M \equiv (\lambda!x.M_1)(!M_2) \rightarrow M_1[M_2/x] \rightarrow^* L_1[M_2/x] \rightarrow^* L \dashv\rightarrow M'_1[M'_2/x] \equiv M'$ , as required.  $\square$

**Lemma 5.11.** *If  $M \dashv\rightarrow L \rightarrow N$  then there exists  $L'$  such that  $M \rightarrow L' \not\equiv N$ .*

*Proof.* By induction on the derivation of  $L \rightarrow N$ . We consider two cases.

If  $L \equiv (\lambda!x.L_1)(!L_2) \rightarrow L_1[L_2/x] \equiv N$ , then  $M \equiv (\lambda!x.M_1)(!M_2)$  where  $M_1 \dashv\rightarrow L_1$  and  $M_2 \not\equiv L_2$ . Thus  $M \rightarrow M_1[M_2/x]$  and, by Lemmas 5.6 and 5.7, we have that  $M_1[M_2/x] \not\equiv L_1[L_2/x] \equiv N$ . Hence the result holds with  $L' =_{\text{def}} M_1[M_2/x]$ .

If  $L \equiv L_1L_2 \rightarrow N_1L_2 \equiv N$ , where  $L_1 \rightarrow N_1$ , then  $M \equiv M_1M_2$  where  $M_1 \dashv\rightarrow L_1$  and  $M_2 \dashv\rightarrow L_2$ . By induction hypothesis, there exists  $L'_1$  such that  $M_1 \rightarrow L'_1 \not\equiv N_1$ . Thus  $M \rightarrow L'_1M_2 \not\equiv N_1L_2$ , hence the result holds with  $L' =_{\text{def}} L'_1M_2$ .  $\square$

*Proof (of Proposition 5.2).* We have a reduction sequence  $M \Rightarrow^* N$ , possibly consisting of both  $\rightarrow$  and  $\dashrightarrow$  rewrites. This can equally well be viewed as a sequence of  $\rightarrow$  and  $\dashrightarrow$  rewrites. We begin by associating a complexity measure to any such reduction sequence of  $\rightarrow$  and  $\dashrightarrow$  rewrites. To do this, first assign to each  $\dashrightarrow$  rewrite in the sequence the number of  $\rightarrow$  rewrites that occur to the right of it. We thus obtain a sequence of numbers, one for each  $\dashrightarrow$  rewrite, which we write in ascending order (equivalently, we write in sequence starting with the rightmost  $\dashrightarrow$  rewrite and working leftwards). For example, the rewrite sequence

$$M \equiv M_0 \dashrightarrow M_1 \dashrightarrow M_2 \rightarrow M_3 \rightarrow M_4 \dashrightarrow M_5 \rightarrow M_6 \dashrightarrow M_7 \equiv N$$

gets assigned the sequence 0, 1, 3, 3. This sequence is our complexity measure.

Now take the sequence of  $\rightarrow$  and  $\dashrightarrow$  rewrites reducing  $M$  to  $N$ . If this sequence does not contain a subsequence  $M_i \dashrightarrow M_{i+1} \rightarrow M_{i+2}$ , then we have  $M \rightarrow^* M' \dashrightarrow^* N$ , and hence  $M \rightarrow^* M' \dashrightarrow^* N$  as required.

Otherwise, select a two-step subsequence  $M_i \dashrightarrow M_{i+1} \rightarrow M_{i+2}$ . Using Lemma 5.11 followed by 5.10, replace this with a sequence  $M_i \rightarrow M' \rightarrow^* M'' \dashrightarrow M_{i+2}$ . One thus obtains a new reduction sequence from  $M$  to  $N$  containing the same number of  $\dashrightarrow$  rewrites and at least as many  $\rightarrow$  rewrites (possibly more). However, because the identified  $\dashrightarrow$  rewrite is shifted to the right, the complexity measure of the new sequence is below that of the original in the lexicographic ordering. Thus by repeatedly selecting two-step subsequences, we repeatedly reduce the complexity measure until we obtain a reduction sequence  $M \rightarrow^* M' \dashrightarrow^* N$  containing at least as many surface rewrites as the original sequence. Therefore  $M \rightarrow^* M' \dashrightarrow^* N$ , as required.  $\square$

## 6 Call-by-need Operational Semantics for Lily

In the Lily expressions  $\text{let } !x = s \text{ in } t$  and  $\text{rec } x : \sigma. t$ , the variable  $x$  may occur zero, one or several times in  $t$ . Because of this, the natural implementation mechanism is call-by-need, whereby the evaluation of the terms substituted for such variables is shared. (In contrast, in an application  $(\lambda x : \sigma. t)(s)$ , the variable  $x$  occurs exactly once in  $t$ , and there is no call for sharing.) An operational semantics implementing such a call-by-need evaluation strategy is presented in [3], and the authors have proved that the call-by-need semantics does not affect the notion of contextual equivalence (private communication). In this section, we outline a straightforward proof of this result.

Again, rather than using the big-step operational semantics of [3], which is based on [5, 12], it is convenient for our purposes to use a small-step version, following [13]. We use  $S, \dots$  to range over *variable/frame stacks*, which are sequences of items of two forms: (i)  $\langle F \rangle$ , where  $F$  is an “evaluation frame”,

$$F ::= (-)(t) \mid \text{let } !x = (-) \text{ in } t \mid (-)(\sigma) ;$$

(ii)  $\langle x \rangle$ , for a variable  $x$ . We use  $H$  to range over *heaps*, which are finite sequences of assignments of the form  $[x \mapsto t]$ , with all variables  $x$  distinct.

The call-by-need evaluation relation is defined in Fig. 7. It implements a single-step relation of the form  $(S, t, H) \rightarrow_{\text{nd}} (S', t', H')$ . Roughly, this is interpreted as saying

1.  $(S, s(t), H) \rightarrow_{\text{nd}} (S \langle (-)(t) \rangle, s, H)$
2.  $(S, \text{let } !x = s \text{ in } t, H) \rightarrow_{\text{nd}} (S \langle \text{let } !x = (-) \text{ in } t \rangle, s, H)$
3.  $(S, t(\sigma), H) \rightarrow_{\text{nd}} (S \langle (-)(\sigma) \rangle, t, H)$
- 4.\*  $(S, \text{rec } x : \sigma. t, H) \rightarrow_{\text{nd}} (S \langle x \rangle, t, [x \mapsto t]H)$
- 5.\*  $(S \langle (-)(t) \rangle, \lambda x : \sigma. s, H) \rightarrow_{\text{nd}} (S, s[t/x], H)$
- 6.\*  $(S \langle \text{let } !x = (-) \text{ in } t \rangle, !s, H) \rightarrow_{\text{nd}} (S, t, [x \mapsto s]H)$
- 7.\*  $(S \langle (-)(\sigma) \rangle, \Lambda \alpha. t, H) \rightarrow_{\text{nd}} (S, t[\sigma/\alpha], H)$
- 8.\*  $(S, x, H) \rightarrow_{\text{nd}} (S \langle x \rangle, H(x), H)$
9.  $(S \langle x \rangle, v, H) \rightarrow_{\text{nd}} (S, v, H[v/x])$

\* *active reductions*, see Appendix A.

**Fig. 7.** Call-by-need Evaluation for `Lily`

that the `Lily` term built up from  $t$  using the nested evaluation frames in  $S$  evaluates in a single step to the term built from  $t'$  using the frames in  $S'$ . In Fig. 7, when we write  $[x \mapsto t]H$ , we assume that  $x$  is not in the domain of  $H$ . We treat heaps  $H$  as functions, writing  $H(x)$  for the value assigned to  $x$ , and writing  $H[v/x]$  for the heap obtained from  $H$  by replacing the existing term assigned to  $x$  (which is assumed to be in the domain of  $H$ ) with  $v$ .

The call-by-need evaluation of a `Lily` program  $t : \sigma$  starts off with the configuration  $(\varepsilon, t, \varepsilon)$  (where  $\varepsilon$  is the empty sequence) and then proceeds deterministically according to the rules in Fig. 7. Either an infinite sequence of  $\rightarrow_{\text{nd}}$  reductions results, or the evaluation terminates in a configuration of the form  $(\varepsilon, v, H)$  for some (possibly open) value  $v$ . If the latter case holds, we write  $t \downarrow_{\text{nd}}$ . The main result of this section states that, for programs of arbitrary type, the call-by-need semantics terminates if and only if the call-by-name semantics does.

**Theorem 6.1.** *If  $t : \sigma$  then  $t \downarrow_{\text{nd}}$  if and only if  $t \downarrow_{\text{nm}}$ .*

The sharing of recursion implemented in Fig. 7, introduces cycles into the heap, and this makes it hard to give a direct operational proof of the equivalence of call-by-name and call-by-need, see [12] for discussion. This difficulty has, in fact, been overcome by the authors of [3], but their proof is highly involved (private communication).

We give a significantly simpler proof that call-by-name and call-by-need coincide. First, we define an almost trivial translation of `Lily` into itself, which serves the purpose of “padding out” the call-by-name semantics (sic) for the purpose of facilitating its comparison with the call-by-need semantics. The remaining step is to prove that the “almost trivial” translation really is trivial. For this last step, we again translate into the untyped linear  $\lambda$ -calculus of Sec. 3, this time applying Theorem 5.5.

The almost trivial translation from `Lily` to itself, is the identity everywhere, except for the translation of `thunks`, which are padded with a dummy recursion, acting as delay.

$$(!s)^\dagger =_{\text{def}} !(\text{rec } z : \tau. s^\dagger) \quad z \notin \text{fv}(s) .$$

Here, we are translating well-typed terms  $\Gamma; \Delta \vdash t : \sigma$ , to well-typed terms  $\Gamma; \Delta \vdash t^\dagger : \sigma$ , and the type  $\tau$  introduced above is determined by this requirement.

**Lemma 6.2.** *If  $t : \sigma$  then  $t \downarrow_{\text{nd}}$  if and only if  $t^\dagger \downarrow_{\text{nm}}$ .*

To prove Lemma 6.2, one shows that the call-by-name evaluation of  $t^\dagger$  simulates the call-by-need evaluation of  $t$ . Crucially, the padding of thunks ensures that rule 8 of Fig. 7 always corresponds to a  $\rightarrow_{\text{nm}}$  reduction for the term generated from  $t^\dagger$  by inserting it in the context determined by  $F$  and substituting, for each variable  $x$  with associated heap assignment  $[x \mapsto s]$ , a term  $\text{rec } x : \sigma. s_0$ , where  $s_0$  is the term originally assigned to  $x$  when it was first added to the heap. More details are given in Appendix A.

Theorem 6.1 now follows from the lemma below, which is an easy application of Theorem 5.5.

**Lemma 6.3.** *If  $t : \sigma$  then  $t \downarrow_{\text{nm}}$  if and only if  $t^\dagger \downarrow_{\text{nm}}$ .*

*Proof.* We give another translation from `Lily` into our untyped linear  $\lambda$ -calculus.

$$\begin{array}{ll}
x^\ddagger =_{\text{def}} x & (\text{let } !x = s \text{ in } t)^\ddagger =_{\text{def}} (\lambda !x. t^\ddagger)(s^\ddagger) \\
(\lambda x : \sigma. t)^\ddagger =_{\text{def}} !(\lambda !x. t^\ddagger) & (\Lambda \alpha. t)^\ddagger =_{\text{def}} !t^\ddagger \\
(s(t))^\ddagger =_{\text{def}} (\lambda !w. w(!t^\ddagger))(s^\ddagger) & (t(\sigma))^\ddagger =_{\text{def}} (\lambda !z. z)(t^\ddagger) \\
(!t)^\ddagger =_{\text{def}} !t^\ddagger & (\text{rec } x : \sigma. t)^\ddagger =_{\text{def}} \mu x. t^\ddagger
\end{array}$$

It is easily established that, for  $t : \sigma$  we have that  $t \downarrow_{\text{nm}}$  if and only if  $t^\dagger$  is surface normalizing. However, we have  $(t^\dagger)^\ddagger \dashrightarrow^* t^\ddagger$ . Therefore, by Theorem 5.5,  $t^\ddagger$  is surface normalizing if and only if  $(t^\dagger)^\ddagger$  is. Thus indeed  $t \downarrow_{\text{nm}}$  if and only if  $t^\dagger \downarrow_{\text{nm}}$ .  $\square$

More generally, a similar application of Theorem 5.5 shows that call-by-name termination is preserved by the congruence relation on `Lily` terms generated by the call-by-name reductions. In other words, the natural ‘‘conversion relation’’ on `Lily` terms is correct with respect to contextual equivalence. Of course, the use of rewriting methods for establishing such simple results goes back to [9].

## 7 Linear Combinatory Algebras

The aim of this short final section is to demonstrate that our untyped linear  $\lambda$ -calculus is the  $\lambda$ -calculus counterpart of Abramsky’s *linear combinatory algebras*, see [1]. This gives some evidence that our calculus arises reasonably naturally, independently of its applications to operational semantics.

**Definition 7.1.** A *!-applicative structure* is an algebra  $(A, \cdot, !)$  where  $\cdot$  is a binary operation on the set  $A$  and  $!$  is a unary operation.

As is standard, we usually omit the ‘‘application’’ operation  $\cdot$ , using a simple juxtaposition  $xy$  for  $x \cdot y$ . Application associates to the left (i.e.  $xyz = (xy)z$ ).

**Definition 7.2 ([1]).** A *linear combinatory algebra* is a  $!$ -applicative structure  $(A, \cdot, !)$  in which there exist elements  $I, B, C, K, W, D, \delta, F \in A$  satisfying:

$$\begin{array}{ll} Ix = x & Wx(!y) = x(!y)(!y) \\ Bxyz = x(yz) & D(!x) = x \\ Cxyz = xzy & \delta(!x) = !!x \\ Kx(!y) = x & F(!x)(!y) = !(xy) . \end{array}$$

The main result of this section asserts that linear combinatory algebras are characterized by a form of combinatory completeness in which the forms of implicit  $\lambda$ -abstraction available correspond to the two forms  $\lambda x.M$  and  $\lambda !x.M$  of our untyped linear  $\lambda$ -calculus. Moreover, the equalities associated with the implicit abstractions agree with the two redex forms in Fig. 3.

A  *$!$ -applicative polynomial* over a set  $A$  is a syntactic expression built up using elements of  $A$  as constants, variables  $x, y, \dots$ , and operator symbols ‘ $\cdot$ ’, and ‘ $!$ ’. Any  $!$ -applicative structure  $(A, \cdot, !)$  induces an evident equality relation between polynomials.

We say that a variable  $x$  is *linear* in a  $!$ -applicative polynomial  $e$ , if it occurs exactly once, and not within the scope of a ‘ $!$ ’-operator symbol. We write  $\text{vars}(e)$  for the set of variables occurring in  $e$ , and  $\text{linvars}(e)$  for the set of variables that are linear in  $e$ .

**Theorem 7.3 (Linear combinatory completeness).** *For any  $!$ -applicative structure  $(A, \cdot, !)$ , the following are equivalent.*

1.  $(A, \cdot, !)$  is a linear combinatory algebra.
2. For any  $!$ -applicative polynomial  $e$  over  $A$ ,
  - (a) if  $x \in \text{linvars}(e)$  then there exists a polynomial  $\lambda^*x.e$  with  $\text{vars}(\lambda^*x.e) = \text{vars}(e) - \{x\}$  and  $\text{linvars}(\lambda^*x.e) = \text{linvars}(e) - \{x\}$  such that the equality  $(\lambda^*x.e)(x) = e$  holds;
  - (b) there exists a polynomial  $\lambda !^*x.e$  with  $\text{vars}(\lambda !^*x.e) = \text{vars}(e) - \{x\}$  and  $\text{linvars}(\lambda !^*x.e) = \text{linvars}(e) - \{x\}$  such that  $(\lambda !^*x.e)(!x) = e$ .

It follows easily from the theorem that the closed linear terms of our untyped  $\lambda$ -calculus, considered modulo  $=_\beta$  (see Sec. 5), themselves form a linear combinatory algebra.

## References

1. S. Abramsky, E. Haghverdi, and P. Scott. Geometry of interaction and linear combinatory algebras. *Math. Struct. in Comp. Sci.*, 12:625–665, 2002.
2. A. Barber. *Linear Type Theories, Semantics and Action Calculi*. PhD thesis, Department of Computer Science, University of Edinburgh, 1997.
3. G.M. Bierman, A.M. Pitts, and C.V. Russo. Operational properties of Lily, a polymorphic linear lambda calculus with recursion. *Elect. Notes in Theor. Comp. Sci.*, 41, 2000.
4. D.J. Howe. Proving congruence of bisimulation in functional programming languages. *Inf. and Comp.*, 124:103–112, 1996.
5. J. Launchbury. A natural semantics for lazy evaluation. In *Proc. 20th POPL*, pages 144–154, 1993.

6. I.A. Mason and C.L. Talcott. Equivalence in functional languages with effects. *J. Functional Programming*, 1:287–327, 1991.
7. P.W. O’Hearn and J.C. Reynolds. From Algol to polymorphic linear lambda-calculus. *Journal of the ACM*, 47:167–223, 2000.
8. A.M. Pitts. Parametric polymorphism and operational equivalence. *Math. Struct. in Comp. Sci.*, 10:321–359, 2000.
9. G.D. Plotkin. Call-by-name, call-by-value and the  $\lambda$ -calculus. *Theor. Comp. Sci.*, 1:125–159, 1975.
10. G.D. Plotkin. Type theory and recursion. Invited talk at *8th Symposium on Logic in Comp. Sci.*, 1993.
11. J.C. Reynolds. Types, abstraction and parametric polymorphism. In *Information Processing ’83*, pages 513–523. North Holland, 1983.
12. J. Seaman and S.P. Iyer. An operational semantics of sharing in lazy evaluation. *Science of Computer Programming*, 27:289–322, 1996.
13. P. Setsoft. Deriving a lazy abstract machine. *J. Functional Programming*, 7:231–248, 1999.

## A Outline proof of Lemma 6.2

The main technical lemma we need is Lemma A.1 below. This concerns configurations  $(S, t, H)$  arrived at by a sequence  $(\varepsilon, s, \varepsilon) \rightarrow_{\text{nd}}^* (S, t, H)$  for some program  $s : \sigma$ . Given such a sequence, and any term  $t'$  with  $\text{fv}(t')$  contained in the domain of  $H$ , (the term  $t$  is one such), we define  $t'[H]$  as follows. If  $H = \varepsilon$  then  $t'[H] =_{\text{def}} t$ . If  $H = [x \mapsto u]H'$  then  $t'[H] =_{\text{def}} (t'[(\text{rec } x : \sigma'. u_0)/x])[H']$ , where  $u_0$  is the first value assigned to  $x$  in a heap occurring along the sequence  $(\varepsilon, s, \varepsilon) \rightarrow_{\text{nd}}^* (S, t, H)$ , and  $\sigma'$  is the appropriate type. Here  $t'[H]$  is an abuse of notation since the value does not solely depend on  $H$ . In fact, for any two heaps  $H_1, H_2$  occurring in the sequence  $(\varepsilon, s, \varepsilon) \rightarrow_{\text{nd}}^* (S, t, H)$  and containing  $\text{fv}(t')$ , it holds that  $t'[H_1] \equiv t'[H_2]$ . Also, for any term  $t'$  we define  $[S]t'$  as follows. If  $S = \varepsilon$  then  $[S]t' =_{\text{def}} t'$ . If  $S = S' \langle (-)(s') \rangle$  then  $[S]t' =_{\text{def}} [S'](t'(s))$ . If  $S = S' \langle \text{let } !x = - \text{ in } s' \rangle$  then  $[S]t' =_{\text{def}} [S'](\text{let } !x = t' \text{ in } s')$ . If  $S = S' \langle (-)(\sigma') \rangle$  then  $[S]t' =_{\text{def}} [S'](t'(\sigma'))$ . If  $S = S' \langle x \rangle$  then  $[S]t' =_{\text{def}} [S']t'$ . Finally, we call reductions number 4–8, in Fig. 7, *active*, and the others *passive*.

**Lemma A.1.** *Suppose  $s : \sigma$  and  $(\varepsilon, s, \varepsilon) \rightarrow_{\text{nd}}^* (S, t, H)$ .*

1. *If  $x$  is declared in  $H$  then  $(x[H])^\dagger \rightarrow_{\text{nm}}^+ ((H(x))[H])^\dagger$ .*
2. *If  $S = S_0 \langle x \rangle S_1$  then  $(x[H])^\dagger \rightarrow_{\text{nm}}^+ (([S_1]t)[H])^\dagger$ .*
3. *If  $(S, t, H) \rightarrow_{\text{nd}} (S', t', H')$ , where  $S = S_0 S_1$  and  $S' = S_0 S'_1$ , then it holds that  $(([S_1]t)[H])^\dagger \rightarrow_{\text{nm}}^* (([S'_1]t')[H'])^\dagger$ . Moreover, if the call-by-need reduction step is active then the call-by-name sequence contains at least one reduction.*

All three statements are proved simultaneously, by induction on the length of the reduction sequence  $(\varepsilon, s, \varepsilon) \rightarrow_{\text{nd}}^* (S, t, H)$ . For space reasons, we omit the details.

*Proof (of Lemma 6.2).* If  $t \downarrow_{\text{nd}}$  then it follows easily from Lemma A.1.3 that  $t^\dagger \downarrow_{\text{nm}}$ . If  $t \not\downarrow_{\text{nd}}$  then there exists an infinite  $\rightarrow_{\text{nd}}$  reduction sequence from  $(\varepsilon, t, \varepsilon)$ . Because the four passive reductions either strictly reduce the size of the term component in a configuration, or retain the same term and reduce the size of the stack, the infinite sequence cannot contain infinitely many consecutive passive reductions. Therefore, it must contain infinitely many active reductions. Thus, again by Lemma A.1.3,  $t^\dagger$  has an infinite  $\rightarrow_{\text{nm}}$  reduction sequence. So indeed  $t^\dagger \not\downarrow_{\text{nm}}$ .  $\square$