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The Largest Topological Subcategory of Countably-based Equiological Spaces

Matías Menni¹ Alex Simpson²

*Laboratory for Foundations of Computer Science
Division of Informatics, University of Edinburgh
Edinburgh, Scotland*

Abstract

There are two main approaches to obtaining “topological” cartesian-closed categories. Under one approach, one restricts to a full subcategory of topological spaces that happens to be cartesian closed — for example, the category of *sequential* spaces. Under the other, one generalises the notion of space — for example, to Scott’s notion of *equiological* space. In this paper we show that the two approaches are equivalent for a large class of objects. We first observe that the category of countably-based equiological spaces has, in a precisely defined sense, a largest full subcategory that can be simultaneously viewed as a full subcategory of topological spaces. This category consists of certain “ ω -projecting” topological quotients of countably-based topological spaces, and contains, in particular, all countably-based spaces. We show that this category is cartesian closed with its structure inherited, on the one hand, from the category of sequential spaces, and, on the other, from the category of equiological spaces.

1 Introduction

It is important in computer science to reconcile topological and type-theoretic structure. On the one hand, as has often been stressed, see e.g. Smyth [28], topological structure accounts for an abstract notion of observable property, and continuity provides a mathematical alternative to computability, emphasising the finitary aspect of computation whilst avoiding the technicalities

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of recursion theory. On the other hand, type constructors, such as function space, arise fundamentally in both the syntax and semantics of programming languages. The challenge for reconciliation is provided by the well-known mathematical anomaly: the category, **Top**, of topological spaces is *not* cartesian closed.

Of course very many reconciliations of this situation have been proposed. One possibility is to cut down the category of topological spaces to a full subcategory that is cartesian closed. Some well-known examples are: Steenrod’s category of compactly-generated Hausdorff spaces [19]; the category, **Seq**, of *sequential* spaces (which contains many computationally important non Hausdorff spaces) [13]; or the even larger category of quotients of exponentiable spaces considered in [6]. However, the received wisdom about such categories is that their function spaces are topologically hard to understand. It is much quoted that the exponential $\mathbf{N}^{\mathbf{N}}$ can never be first-countable [13], whereas an ideal approach from a computational viewpoint would allow effectivity issues to be addressed, and the stricter requirement of second-countability is often claimed to be necessary for such (see e.g. Smyth [28]).

A second alternative is to expand the category **Top** by adding new objects and hence new potential exponentials. Again there are many ways of doing this. A very elegant construction is to take the *regular completion* of **Top** (as a left-exact category) or the related *exact completion* [24,4,23]. The regular completion has a straightforward description as a category of equivalence relations on topological spaces, whose importance (in the case of T_0 spaces) was first recognised by Dana Scott [1]. Following Scott, we call such structures, consisting of spaces together with equivalence relations, *equilogical spaces* (although we do not make the restriction to T_0 spaces), and we call the associated category **Equ**. Not only is **Equ** cartesian closed, but recent investigations have shown that other approaches to expanding **Top** to a cartesian-closed category (such as the *filter space* approach of Hyland [13]) can be naturally embedded within **Equ** [12,11,23]. A further important feature of **Equ** is that its full subcategory, $\omega\mathbf{Equ}$, of countably-based equilogical spaces is also a cartesian-closed category (with its structure inherited from **Equ**). This fact allows equilogical spaces to support an analysis of effectivity at higher types. It is also the basis of an interesting connection with realizability semantics. The category $\omega\mathbf{Equ}$ is equivalent to the category of assemblies over Scott’s combinatory algebra $\mathcal{P}\omega$ [27].

In this paper we demonstrate an interesting connection between the subcategory and supercategory approaches to achieving cartesian closure. We firstly show that the categories **Top** and $\omega\mathbf{Equ}$ share, in a precisely defined sense, a largest common full subcategory. This category, **PQ**, has an explicit description as the full subcategory of **Top** consisting of certain “ ω -projecting” quotient spaces of countably-based topological spaces. By its very definition, **PQ** contains all countably-based spaces. The remarkable fact is that **PQ** is also bicartesian closed (with finite limits). As a category of topological

spaces, \mathbf{PQ} inherits its bicartesian-closed structure from \mathbf{Seq} (which contains all quotients of countably-based spaces). Similarly, as a category of equilogical spaces, \mathbf{PQ} inherits its bicartesian-closed structure from \mathbf{Equ} . Thus one may conclude that, at least for (iterated) exponentials over countably-based spaces, the subcategory approach, as exemplified by \mathbf{Seq} , and the supercategory approach, as exemplified by \mathbf{Equ} , give equivalent ways of modelling continuity at higher types. This is surprising because it means that the inherently intensional construction of exponentials in \mathbf{Equ} can (in the case of \mathbf{PQ} objects) be replaced by a manifestly extensional construction in \mathbf{Seq} .

2 Topological subcategories of equilogical spaces

Equilogical spaces were introduced by Scott [1], as a very simple way of expanding the category of T_0 spaces. Rather than taking spaces themselves as objects of the category, Scott took equivalence relations on spaces instead. Of course, the idea generalizes immediately from T_0 spaces to arbitrary spaces. In the present paper we use the term *equilogical space* to mean this natural generalization. (Although historically inaccurate, this use of terminology seems consistent with the original conception.)

Definition 2.1

- (i) An *equilogical space* is a pair (X, \sim) where X is a topological space and \sim is an arbitrary equivalence relation on the underlying set of X .
- (ii) An *equivariant map*³ $\phi : (X, \sim_X) \rightarrow (Y, \sim_Y)$ is a function ϕ from the quotient set X/\sim_X to the quotient set Y/\sim_Y that is *realized* by some continuous $f : X \rightarrow Y$ that preserves the equivalence relations (i.e. the diagram below commutes).

$$\begin{array}{ccc}
 X & \overset{f}{\dashrightarrow} & Y \\
 \downarrow & & \downarrow \\
 X/\sim_X & \xrightarrow{\phi} & Y/\sim_Y
 \end{array}$$

We write \mathbf{Equ} for the category of equilogical spaces and equivariant maps.

Scott's interesting insight was that (in the T_0 case) the category of equilogical spaces is cartesian closed [1]. The original proof made use of his old result that the injective objects in the category of T_0 spaces, the *continuous lattices*, themselves form a cartesian-closed category [26]. Subsequently, Carboni and Rosolini realised that the construction is an example of a *regular completion*

³ In [1] the equivariant maps are defined as equivalence classes of equivalence-relation-preserving continuous functions, rather than as functions between quotient sets.

of a left-exact category, and that cartesian closure (and even *local* cartesian closure) are obtained for very general reasons [24,4,23]. The original T_0 version of equilogical spaces is just the regular completion of \mathbf{Top}_0 (the category of T_0 spaces). Similarly, the category \mathbf{Equ} defined above is the regular completion of \mathbf{Top} . This description enables a nice proof of the cartesian closure of \mathbf{Equ} itself. In Section 8 we sketch a direct proof using constructions from [1,23]. (Yet another proof is presented in [22].)

The evident functor $I : \mathbf{Top} \rightarrow \mathbf{Equ}$, mapping a topological space X to the equilogical space $(X, =)$, exhibits \mathbf{Top} as a full subcategory of \mathbf{Equ} . We call the objects (isomorphic to those) in its image the *topological* objects of \mathbf{Equ} . As \mathbf{Top} is not cartesian closed, it is clear that \mathbf{Equ} also contains many non-topological objects, and some such objects can be obtained by exponentiation from topological objects. (One example is the object $\mathbf{N}^{\mathbf{N}^{\mathbf{N}}}$ [13].)

The inclusion functor I has a left-adjoint $Q : \mathbf{Equ} \rightarrow \mathbf{Top}$ which maps an equilogical space (X, \sim) to the topological quotient X/\sim . Thus \mathbf{Top} is a full reflective subcategory of \mathbf{Equ} . This is well-explained by \mathbf{Equ} being the regular completion of \mathbf{Top} . However, the topological quotient functor Q has one additional important property: it is faithful! This fact motivates the following definition of when a full subcategory of \mathbf{Equ} can be viewed as a “topological” category (i.e. as a category of topological spaces and all continuous functions between them).

Definition 2.2 We say that a full subcategory \mathbf{C} of \mathbf{Equ} is *topological* if the (faithful) composite functor $\mathbf{C} \hookrightarrow \mathbf{Equ} \xrightarrow{Q} \mathbf{Top}$ is full.

In other words \mathbf{C} is topological if $Q : \mathbf{Equ} \longrightarrow \mathbf{Top}$ cuts down to an equivalence between \mathbf{C} and a full subcategory of \mathbf{Top} .

It is easily seen that the full subcategory of topological objects of \mathbf{Equ} gives one topological subcategory of \mathbf{Equ} . Moreover, this category can be shown to be a *maximal* (but not *the maximum* — see below!) topological subcategory of \mathbf{Equ} — any strictly larger full subcategory of \mathbf{Equ} is not topological.

Such remarks are hardly surprising. However, what is interesting about the notion of topological subcategory is that there exist other topological subcategories of \mathbf{Equ} that contain non-topological equilogical spaces amongst their objects (and are hence incomparable with the maximal topological subcategory identified above). We shall see that one such subcategory arises in a very natural way.

Let us consider what happens when equilogical spaces are restricted to equivalence relations over countably-based spaces. (We say that a topological space is *countably based* if there exists some countable base for its topology [28]. Such spaces are also known as second-countable spaces.) We write $\omega\mathbf{Top}$ for the category of countably-based topological spaces and $\omega\mathbf{Equ}$ for the category of equilogical spaces (X, \sim) where X is countably based. As mentioned in the introduction, $\omega\mathbf{Equ}$ is cartesian closed with its cartesian-closed structure inherited from \mathbf{Equ} (see Section 8). From a computer science viewpoint, the

restriction to countably-based spaces is natural, allowing $\omega\mathbf{Equ}$ to be used to formalise issues of effectivity at higher types.

Clearly the functor $I : \mathbf{Top} \rightarrow \mathbf{Equ}$ cuts down to a functor $I : \omega\mathbf{Top} \rightarrow \omega\mathbf{Equ}$, identifying (up to isomorphism) the topological objects in $\omega\mathbf{Equ}$. We also have the topological quotient functor $Q : \omega\mathbf{Equ} \rightarrow \mathbf{Top}$. (Note that the image of Q does not land in $\omega\mathbf{Top}$ as topological quotients of countably-based spaces are not in general countably based.)

As with \mathbf{Equ} , the topological objects of $\omega\mathbf{Equ}$ form a topological subcategory of $\omega\mathbf{Equ}$. The difference this time is that the topological objects do *not* form a maximal topological subcategory. Instead, there is a unique maximal topological subcategory of $\omega\mathbf{Equ}$, including all topological objects, but also containing certain non-topological equilogical spaces.

Definition 2.3 We say that a full subcategory \mathbf{C} of $\omega\mathbf{Equ}$ *contains* $\omega\mathbf{Top}$ if the functor $I : \omega\mathbf{Top} \longrightarrow \omega\mathbf{Equ}$ factors through the inclusion $\mathbf{C} \hookrightarrow \omega\mathbf{Equ}$.

Theorem 2.4 *There exists a unique largest topological full subcategory, \mathbf{C} , of $\omega\mathbf{Equ}$ containing $\omega\mathbf{Top}$. (I.e. for any other topological full subcategory, \mathbf{C}' , of $\omega\mathbf{Equ}$ also containing $\omega\mathbf{Top}$, the inclusion $\mathbf{C}' \hookrightarrow \omega\mathbf{Equ}$ factors through the inclusion $\mathbf{C} \hookrightarrow \omega\mathbf{Equ}$.)*

In order to prove the theorem, we define the largest topological subcategory explicitly.

Definition 2.5 We say that an object A (in any category) is *projective with respect to* a map $r : B \rightarrow R$ if for every $f : A \rightarrow R$ there exists some $\bar{f} : A \rightarrow B$ such that $r.\bar{f} = f$.

Definition 2.6 We say that a morphism $r : B \rightarrow R$ in \mathbf{Top} is ω -*projecting* if every countably-based space is projective with respect to it.

Definition 2.7 We write \mathbf{EPQ} for the full subcategory of $\omega\mathbf{Equ}$ consisting of those objects (A, \sim) for which the induced quotient $A \rightarrow (A/\sim)$ in \mathbf{Top} is ω -projecting.

The acronym EPQ stands for Equilogical ω -Projecting Quotient.

Proof of Theorem 2.4. We show that \mathbf{EPQ} is the category characterised by the theorem. First, $\omega\mathbf{Top}$ is trivially contained in \mathbf{EPQ} . For the fullness of $Q : \mathbf{EPQ} \rightarrow \mathbf{Top}$ suppose we have $f : (A/\sim_A) \rightarrow (B/\sim_B)$ in \mathbf{Top} where (B, \sim_B) is in \mathbf{EPQ} . Then, as $q_B : B \rightarrow (B/\sim_B)$ is ω -projecting, there exists $g : A \rightarrow B$ such that $q_B.g = f.q_A$. Then g realizes the equivariant map $f : (A, \sim_A) \rightarrow (B, \sim_B)$. Thus \mathbf{EPQ} is a topological subcategory containing $\omega\mathbf{Top}$.

It remains to show that \mathbf{EPQ} is the largest such subcategory. Suppose that an object (B, \sim) of $\omega\mathbf{Equ}$ lies in some other such category \mathbf{C}' . To show the quotient $q : B \rightarrow (B/\sim)$ is ω -projecting, suppose A is countably based and take any $f : A \rightarrow (B/\sim)$ in \mathbf{Top} . As \mathbf{C}' contains $\omega\mathbf{Top}$, the object

$(A, =)$ is in \mathbf{C}' . As \mathbf{C}' is a topological subcategory, the continuous function $f : A \rightarrow (B/\sim)$ gives an equivariant map $\phi_f : (A, =) \rightarrow (B, \sim)$ in $\omega\mathbf{Equ}$. Then any realizer $\bar{f} : A \rightarrow B$ for ϕ_f will satisfy $q_B \cdot \bar{f} = f$. Thus indeed $q_B : B \rightarrow (B/\sim)$ is ω -projecting. \square

It is not immediately obvious that \mathbf{EPQ} is not just the category of all topological objects of $\omega\mathbf{Equ}$. That this is not the case is given by the following surprising theorem, whose proof will eventually be given in Section 8. (A consequence of the theorem is that the non-topological equilogical space $\mathbf{N}^{\mathbf{N}^{\mathbf{N}}}$ is an object of \mathbf{EPQ} .)

Theorem 2.8 *The category \mathbf{EPQ} is bicartesian closed with finite limits. Moreover the inclusion functor $\mathbf{EPQ} \hookrightarrow \omega\mathbf{Equ}$ preserves this structure.*

By its definition as a topological subcategory of \mathbf{Equ} , we have that \mathbf{EPQ} is equivalent to a full subcategory of \mathbf{Top} which, because of the equivalence, must itself be cartesian closed. Thus, even though exponentiation in \mathbf{EPQ} goes outside the world of the topological objects of \mathbf{Equ} , it can nonetheless be viewed as a purely topological phenomenon. Accordingly, it is of interest to give an explicit description of the equivalent topological category.

Definition 2.9 We write \mathbf{PQ} for the full subcategory of \mathbf{Top} consisting of those spaces Q for which there exists a countably-based space A together with an ω -projecting topological quotient $q : A \twoheadrightarrow Q$.

The acronym PQ stands for ω -Projecting Quotient spaces. It is immediate from the definitions that the functor $Q : \mathbf{Equ} \rightarrow \mathbf{Top}$ cuts down to the claimed equivalence of categories $Q : \mathbf{EPQ} \rightarrow \mathbf{PQ}$. In Sections 3–7 we shall prove the bicartesian closure of \mathbf{PQ} directly, finally using this in Section 8 to prove Theorem 2.8.

We conclude the present section with some remarks and questions. The definitions above of the objects of \mathbf{EPQ} and \mathbf{PQ} are not particularly satisfying. It would be very nice to have descriptions in terms of properties intrinsic to the quotient $A \rightarrow (A/\sim)$, and properties internal to the quotient space itself. By definition, all objects of \mathbf{PQ} are topological quotients of countably-based spaces. We do not, at present, know any example of a quotient of a countably-based space that cannot be shown to be in \mathbf{PQ} .

The fact that \mathbf{PQ} is a cartesian-closed category consisting entirely of quotients of countably-based spaces is important as it means that Weihrauch’s approach to “Type 2” computability via representations [17,29] is applicable. This is interesting because it has always been unclear how to apply Weihrauch’s approach to higher-type computation. Furthermore, Scott’s approach to computability in countably-based algebraic lattices [27] can be applied to $\omega\mathbf{Equ}$ and hence to \mathbf{EPQ} . Thus the equivalence between \mathbf{PQ} and \mathbf{EPQ} opens up the possibility of comparing Weihrauch’s and Scott’s approaches. We believe that such an investigation would provide a very interesting exercise.

Finally, we further explain the choice of $\omega\mathbf{Top}$ as the basis for the identification of \mathbf{EPQ} as the category characterised by Theorem 2.4. For any full subcategory \mathbf{T} of \mathbf{Top} , we can form the evident full subcategory $\mathbf{Equ}_{\mathbf{T}}$ of equilogical spaces over \mathbf{T} . For any such category \mathbf{T} , the proof of Theorem 2.4 generalises to determine a largest topological subcategory $\mathbf{LT}_{\mathbf{T}}$ of $\mathbf{Equ}_{\mathbf{T}}$ containing \mathbf{T} itself. As already mentioned, in the case that \mathbf{T} is \mathbf{Top} , then $\mathbf{LT}_{\mathbf{Top}}$ is equivalent to \mathbf{Top} itself, hence $\mathbf{LT}_{\mathbf{Top}}$ is not cartesian closed. Why is it then that in the case that \mathbf{T} is $\omega\mathbf{Top}$, we do obtain a cartesian-closed category for $\mathbf{LT}_{\mathbf{T}}$? We do not know a good general answer to this question, but the choice of $\omega\mathbf{Top}$ seems very constrained. For example, one can show that if \mathbf{T} is the full subcategory κ -based topological spaces for any cardinal $\kappa \geq 2^\omega$ then $\mathbf{LT}_{\mathbf{T}}$ is *not* cartesian closed. However, there may be other ways of obtaining categories \mathbf{T} such that $\mathbf{LT}_{\mathbf{T}}$ is cartesian-closed. Two possibilities for \mathbf{T} that could be worth investigating are: the category of first-countable spaces (cf. [7]); and the category of exponentiable spaces (cf. [6]).

3 Sequential spaces and limit spaces

In this section we introduce the category of \mathbf{Seq} sequential spaces which is a full subcategory of \mathbf{Top} [7]. We also introduce the category \mathbf{Lim} of limit spaces in the sense of Kuratowski [18]. Although this category is not a subcategory of \mathbf{Top} , it does embed the category of sequential spaces. It is easy to prove that \mathbf{Lim} is cartesian closed because products and exponentials have straightforward definitions. We use this to prove the known result that \mathbf{Seq} is also cartesian closed and that it inherits this structure from that in \mathbf{Lim} [6,13]. These properties of \mathbf{Seq} and \mathbf{Lim} will be used in Sections 4–7 to prove the cartesian closure of \mathbf{PQ} .

3.1 Sequential spaces

The sequential spaces are those topological spaces whose topologies are determined by sequence convergence. Explicitly, say that a sequence (x_i) of elements of a set X is *eventually in* a subset $O \subseteq X$ if there exists l such that, for all $i \geq l$, $x_i \in O$. Recall that, in an arbitrary topological space X , a sequence (x_i) is said to converge to a point x if, for every neighbourhood of x , the sequence is eventually in the neighbourhood.

There is another way of viewing convergent sequences. Let \mathbb{N}^+ denote the one point compactification of the natural numbers. This has $\mathbb{N} \cup \{\infty\}$ as underlying set and its topology is given by the following base $\{\{n\} | n \in \mathbb{N}\} \cup \{\{n, n+1, \dots, \infty\} | n \in \mathbb{N}\}$. That is, a sequence converges to some $n \in \mathbb{N}$ if and only if the sequence is eventually equal to n . On the other hand, a sequence converges to ∞ if and only if, for all n , the sequence is eventually greater than n . It is easily checked that, for any topological space X , the convergent sequences in X are in one-to-one correspondence with the continuous functions

from \mathbb{N}^+ to X .

Definition 3.1 Let X be a topological space.

- (i) A subset O of X is *sequentially open* if every sequence converging to a point in O is eventually in O .
- (ii) A subset O of X is *sequentially closed* if no sequence in O converges to a point not in O .
- (iii) X is *sequential* if every sequentially open subset is open or, equivalently, if every sequentially closed subset is closed.

Let **Seq** denote the category of sequential spaces and continuous functions. For sequential spaces, the notion of continuity has a natural reformulation. It is easily checked that a function $f : X \rightarrow Y$ between sequential spaces is continuous if and only if it preserves convergent sequences.

It is easily checked that every countably-based space is sequential (as, indeed, is any first-countable space). Thus $\omega\mathbf{Top}$ is a full subcategory of **Seq**. Moreover, the embedding $\omega\mathbf{Top} \hookrightarrow \mathbf{Seq}$ preserves (finite) products, (finite) coproducts and subspaces (equalizers).

The set of sequentially open subsets of any topological space is a sequential topology. This fact induces a functor $\mathbf{Top} \rightarrow \mathbf{Seq}$ which is right adjoint to the embedding in the opposite direction. That is, **Seq** is a full coreflective subcategory of **Top**. This shows that **Seq** is complete and cocomplete and explains why, in **Top**, coproducts and quotients of sequential spaces are again sequential spaces [7]. It follows that every quotient of a countably based space is sequential. Thus, in particular, **PQ** is a full subcategory of **Seq**.

On the other hand, in contrast to the countably-based case, subspaces and (even finite) products (in **Top**) of sequential spaces, need not be sequential in general. This implies that products in **Seq** do not always coincide with topological products. Similarly, regular subobjects in **Seq** do not in general have the subspace topology.

3.2 Limit spaces

In order to gain better understanding of the structure of **Seq**, we introduce the related notion of (Kuratowski) limit space [18].

Definition 3.2

- (i) A *limit space* consists of a set X together with a distinguished family of functions $(\mathbb{N} \cup \{\infty\}) \rightarrow X$, called *convergent sequences* in X . We say that (x_i) *converges to* x_∞ in X if the induced function $(\mathbb{N} \cup \{\infty\}) \rightarrow X$ is one of the convergent sequences in X . The convergent sequences must satisfy the following axioms:
 - (a) the constant sequence (x) converges to x ;
 - (b) if (x_i) converges to x , then so does every subsequence of (x_i) ;

- (c) if (x_i) is a sequence such that every subsequence of (x_i) contains a subsequence converging to x , then (x_i) converges to x .
- (ii) A function between limit spaces is said to be *continuous* if it preserves convergent sequences.

We usually write $(x_i) \rightarrow x$ as a shorthand for (x_i) converges to x .

It is easy to see that **Seq** is a full subcategory of **Lim**. The embedding assigns to each sequential space, the limit space with same underlying set and as convergent sequences those that converge topologically.

Viewed as a limit space, the one point compactification of the natural numbers, \mathbb{N}^+ , acts as a generic convergent sequence in **Lim**: convergent sequences, in any limit space X , are in one-to-one correspondence with the continuous functions from \mathbb{N}^+ to X . This fact will be useful later in the proofs of Propositions 3.3 and 5.2.

Let **Lim** denote the category of limit spaces and continuous maps. In [15], it is shown that it arises as the full and reflective subcategory of $\dashv\dashv$ -separated sheaves of a Grothendieck topos (Johnstone calls limit spaces *subsequential spaces*). This fact implies that **Lim** is a quasitopos. Although we shall mainly use properties of the categorical structure of **Lim** true in any quasitopos it is instructive to give explicit description of finite limits, finite colimits and exponentials.

There is an evident forgetful functor **Lim** \rightarrow **Set**. It has a “chaotic” right adjoint ∇ which assigns to each set, the limit space with this underlying set and where every sequence converges to every point. It also has a “discrete” left adjoint which assigns to each set, the limit space with this underlying set but where a sequence converges to a point if and only if the sequence is eventually the constant sequence of that point.

We are going to use ∇ later, but for our present purpose we just mention that the existence of these adjoints imply that the forgetful functor preserves limits and colimits. This gives us the underlying sets of many constructions among limit spaces.

Let X and Y be limit spaces. A sequence $((x_i, y_i))$ of pairs converges to (x, y) in $X \times Y$ iff $(x_i) \rightarrow x$ in X and $(y_i) \rightarrow y$ in Y .

A sequence (z_i) converges in $X + Y$ to an $x \in X$ if there exists a k such that for each $j \geq k$, $z_j \in X$ (i.e.: (z_i) is eventually in X) and $(z_j)_{j \geq k}$ converges to x in X . Similarly for $y \in Y$.

The underlying set of Y^X is the set of continuous functions from X to Y and $(f_i) \rightarrow f$ if for each $(x_i) \rightarrow x$ in X , $(f_i x_i) \rightarrow f x$ in Y .

Monos are exactly those morphisms with injective underlying functions and epis are exactly those morphisms with surjective underlying functions.

A mono $m : A \rightarrow X$ is regular if and only if $(ma_i) \rightarrow ma$ in X implies $(a_i) \rightarrow a$ in A .

An epi $q : X \rightarrow Q$ is regular if and only if for each $(z_i) \rightarrow z$ in Q it holds that for every subsequence $(z_{\alpha i})$ there exists a subsequence $(z_{\beta \alpha i})$ and a

sequence $(x_i) \rightarrow x$ in X such that, for each i , $qx_i = z_{\beta\alpha i}$ and $qx = z$.

3.3 **Seq** as a reflective subcategory of **Lim**

We say that a limit space is *topological* if it lies in the image of the embedding of **Seq** in **Lim**. Such limit spaces are easily characterised explicitly. We say that a subset U of the underlying set of a limit space X is *sequentially open* if every sequence in X converging to a point in U is eventually in U . We say that a sequence (x_i) *topologically converges* to a point x in X if, for every sequentially open subset U containing x , the sequence (x_i) is eventually in U . Clearly, $(x_i) \rightarrow x$ implies (x_i) topologically converges to x . The limit space X is topological if and only if the converse holds, i.e. X is topological if and only if convergence agrees with topological convergence.

Underlying the above characterisation is a reflection functor from **Lim** to **Seq**. The family of sequentially open subsets of a limit space forms a topology and the resulting topological space is sequential. This operation determines a functor $F : \mathbf{Lim} \rightarrow \mathbf{Seq}$ that is left adjoint to the embedding in the opposite direction [15,13]. An immediate consequence of this is that the embedding preserves products and equalizers. Also, using the explicit description of coproducts in **Lim**, it is easy to see that the embedding also preserves coproducts. We shall use these facts later.

In the proof of Corollary 10.2 of [13], the following property of the reflection is stated as obvious. We thought it worth giving a direct proof (it also follows from results in [6]).

Proposition 3.3 *The left adjoint $F : \mathbf{Lim} \rightarrow \mathbf{Seq}$ preserves finite products.*

Proof. It is clear that $F(X \times_{\mathbf{Lim}} Y)$ and $FX \times_{\mathbf{Seq}} FY$ have the same underlying set and that the identity function $F(X \times_{\mathbf{Lim}} Y) \rightarrow FX \times_{\mathbf{Seq}} FY$ is continuous. So we need only prove that every open in $F(X \times_{\mathbf{Lim}} Y)$ is open in $FX \times_{\mathbf{Seq}} FY$, i.e. that every sequentially open subset of $X \times_{\mathbf{Lim}} Y$ is a sequentially open subset of $FX \times_{\mathbf{Lim}} FY$ (as the inclusion from **Seq** to **Lim** preserves products).

By the symmetry of product, it suffices to prove that $W \subseteq X \times_{\mathbf{Lim}} Y$ sequentially open implies W sequentially open in $X \times_{\mathbf{Lim}} FY$. Suppose then that $((a_i, b_i)) \rightarrow (a, b)$ in $X \times_{\mathbf{Lim}} FY$ where $(a, b) \in W$. As $\{x \in X \mid (x, b) \in W\}$ is sequentially open in X , there exists an m such that, for all $i \geq m$, $(a_i, b) \in W$. Write a_∞ for a and define:

$$V = \{y \in Y \mid \text{for all } j \text{ with } m \leq j \leq \infty, (a_j, y) \in W\}.$$

We now prove that $V \subseteq Y$ is sequentially open. Suppose for contradiction that, in Y , $(y_i) \rightarrow y \in V$ but (y_i) is not eventually in V . Then, there exists a subsequence $y_{g_i} \rightarrow y$ in Y with each y_{g_i} not in V . So for each i there exists f_i with $m \leq f_i \leq \infty$, such that (a_{f_i}, y_{g_i}) is not in W . The sequence (f_i) is an arbitrary sequence of elements of \mathbb{N}^+ . By the compactness of \mathbb{N}^+ , (f_i) has a converging subsequence in \mathbb{N}^+ , $(f_{h_i}) \rightarrow j$ for some j with $m \leq j \leq \infty$.

But then we have that $(a_{fhi}) \rightarrow a_j$ in X and that $(y_{ghi}) \rightarrow y$ in Y . So $((a_{fhi}, y_{ghi})) \rightarrow (a_j, y)$ in $X \times_{\mathbf{Lim}} Y$.

But $(a_j, y) \in W$, as $y \in V$. Yet for no i is (a_{fhi}, y_{ghi}) in W . This contradicts that W is sequentially open in $X \times_{\mathbf{Lim}} Y$. So V is sequentially open.

Then (b_i) is eventually in V . Hence indeed $((a_i, b_i))$ is eventually in W , proving that W is sequentially open. \square

By an elementary categorical argument [8], it follows that **Seq** is an exponential ideal of **Lim** (i.e. if X is a sequential space and Y is a limit space then the object X^Y of **Lim** is topological). This means, in particular, that **Seq** is a cartesian-closed category, and that the embedding $\mathbf{Seq} \hookrightarrow \mathbf{Lim}$ preserves the cartesian-closed structure.

4 Pre-embeddings and pre-extensional spaces

In this section we introduce the notion of a pre-embedding and we use it to give an abstract characterisation of sequential spaces as a subcategory of **Lim**. Pre-embeddings will also be important later on for obtaining injectivity results.

A continuous $f : X \rightarrow Y$ between topological spaces is a (topological) *pre-embedding* if for every open U in X there exists an open V in Y such that $f^{-1}V = U$. Let us notice now that if $f : X \rightarrow Y$ is a pre-embedding and Y is countably based then X is countably based. Also consider the following fact whose easy proof we omit.

Proposition 4.1 *Let $f : X \rightarrow Y$ be a pre-embedding. If (fx_i) converges to fx in Y then (x_i) converges to x in X .*

This proposition suggests how to formulate the notion of pre-embedding between limit spaces.

We say that a map $f : X \rightarrow Y$ in **Lim** is a **Lim-pre-embedding** if $(fx_i) \rightarrow fx$ in Y implies $(x_i) \rightarrow x$ in X . Note that a map in **Lim** is a regular mono if and only if it is both mono and a **lim-pre-embedding**. In fact, **Lim-pre-embeddings** in general share many of the properties of regular monos.

Proposition 4.2 *Let $f : X \rightarrow Y$ be a **Lim-pre-embedding**.*

- (i) *If $g : Y \rightarrow Z$ is also a **Lim-pre-embedding**, the composition $g.f$ is too.*
- (ii) *For an arbitrary $h : Z \rightarrow Y$, the pullback h^*f of f along h is a **Lim-pre-embedding**.*
- (iii) *If $f' : X' \rightarrow Y'$ is a **Lim-pre-embedding**, the product $f \times f'$ also is.*
- (iv) *For any object Z , $f^Z : X^Z \rightarrow Y^Z$ is a **Lim-pre-embedding**.*

Proof. The first two are easy calculations, and the third follows from them. The last is also easy but we present it as an example. Let $(f^Z h_i) \rightarrow f^Z h$ in Y^Z . We want to prove that $(h_i) \rightarrow h$ in X^Z . To do this, let $(z_i) \rightarrow z$ in Z .

Then $((f^Z h_i)z_i) \rightarrow (f^Z h)z$. That is, $(f(h_i z_i)) \rightarrow f(hz)$. As f is a **Lim**-pre-embedding, $(h_i z_i) \rightarrow hz$. So in fact, $(h_i) \rightarrow h$. Hence, as required, f^Z is also a **Lim**-pre-embedding. \square

It is worth noting that **Lim**-pre-embeddings have a nice categorical characterisation from which the above properties follow. Recall the “chaotic” inclusion $\nabla : \mathbf{Set} \rightarrow \mathbf{Lim}$ and for any limit space X , let ∇X be the corresponding chaotic limit space; also let $X \rightarrow \nabla X$ be the unit of the adjunction and $\nabla f : \nabla X \rightarrow \nabla Y$ be the reflection of f . A map $f : X \rightarrow Y$ is a **Lim**-pre-embedding if and only if the following square is a pullback.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \nabla X & \xrightarrow{\nabla f} & \nabla Y \end{array}$$

As we said, in **Top**, subspaces of sequential spaces need not be sequential. The following may then come as a surprise.

Proposition 4.3 *Let X be a sequential space and let $f : A \rightarrow X$ be a **Lim**-pre-embedding. Then:*

- (i) *A is topological.*
- (ii) *If X is countably based then f is a topological pre-embedding.*

Proof. To prove (i) we are going to show that if (a_i) is eventually in every sequentially open neighbourhood of a then $(a_i) \rightarrow a$ in A . In order to do this let U be an open neighbourhood of fa . As $f^{-1}U$ is sequentially open, (a_i) is eventually in $f^{-1}U$. Then, (fa_i) is eventually in U . As X is topological, this means that $(fa_i) \rightarrow fa$ in X . As f is a pre-embedding, $(a_i) \rightarrow a$ in A .

To prove (ii) we are going to use the following property of countably based spaces: the closure of any subset is obtained by adding the limits of all convergent sequences in the subset. Moreover, we are going to use the characterization of sequential spaces in terms of closed sets.

By the previous item we know that A is topological. We now show that $U \subseteq A$ is sequentially closed implies that there exists $V \subseteq X$ sequentially closed such that $f^{-1}V = U$.

Suppose U is sequentially closed. Now take the closure \overline{fU} of fU , the image of U under f . We are going to prove that $U = f^{-1}\overline{fU}$. Trivially $U \subseteq f^{-1}\overline{fU}$. For the other inclusion, let $fa \in \overline{fU}$. As X is countably based, there exists a sequence (fa_i) in fU such that $(fa_i) \rightarrow fa$. As f is a pre-embedding, $(a_i) \rightarrow a$. As U is closed, $a \in U$. So $U = f^{-1}\overline{fU}$. \square

Actually, property (ii) holds for every space that satisfies the condition mentioned in the proof. Such spaces are known as *Fréchet* spaces [7].

By Propositions 4.1 and 4.3 it follows that it is irrelevant to distinguish between topological and **Lim**-pre-embeddings into countably-based spaces.

Corollary 4.4 *In Lim:*

- (i) *Regular subobjects of topological objects are topological (they may not have the subspace topology).*
- (ii) *However, regular subobjects of countably-based spaces are in one-to-one correspondence with topological subspaces.*

Although not essential for the main results of the paper, we conclude this section with an application of pre-embeddings to obtain an abstract characterisation of the topological objects in **Lim**.

Let Σ be Sierpinski space (i.e. the two element space $\{\perp, \top\}$ with the singleton $\{\top\}$ as the only non-trivial open). It is an easy fact in topology that the continuous functions from any topological space X to Σ are in one-to-one correspondence with the open subsets of X . Similarly, Σ is also a limit space and the maps from any limit space X to Σ are in one-to-one correspondence with the sequentially open subsets of X .

By the last observation, Σ^X in **Lim** is an object of sequentially open subsets of a limit space X . Moreover, as **Seq** is an exponential ideal of **Lim**, the object Σ^X is topological. (Warning — its topology is *not* in general the Scott topology!) For any limit space X let $\Omega : X \rightarrow \Sigma^{\Sigma^X}$ denote the transpose of the evaluation map. If X is topological then it is easily checked that Ω is mono if and only if X is a T_0 space.

The definition below originated in synthetic domain theory [14].

Definition 4.5 A limit space X is *extensional* if $\Omega : X \rightarrow \Sigma^{\Sigma^X}$ is a regular mono. It is *pre-extensional* if the map is a **Lim**-pre-embedding.

As Σ^{Σ^X} is topological, by Proposition 4.3, it follows that so is any pre-extensional object. Moreover, if X is extensional then $\Omega : X \rightarrow \Sigma^{\Sigma^X}$ is also mono and so X is T_0 .

Recall that $F : \mathbf{Lim} \rightarrow \mathbf{Seq}$ is the reflection functor.

Proposition 4.6 *If $(\Omega x_i) \rightarrow \Omega x$ in Σ^{Σ^X} then $(x_i) \rightarrow x$ in FX .*

Proof. Let O be sequentially open in X and $x \in O$. It is clear that $(O) \rightarrow O$ in Σ^X . Then, as $(\Omega x_i) \rightarrow \Omega x$, $((\Omega x_i)O) \rightarrow (\Omega x)O$. That is, $((\Omega x_i)O)$ must be eventually \top . In other words, (x_i) must be eventually in O . So $(x_i) \rightarrow x$ in FX . \square

So, if X is a sequential space then $\Omega : X \rightarrow \Sigma^{\Sigma^X}$ is a **Lim**-pre-embedding.

Corollary 4.7 *In Lim:*

- (i) *The full subcategory of pre-extensional objects is equivalent to Seq.*
- (ii) *The full subcategory of extensional objects is equivalent to the category of T_0 sequential spaces.*

5 Projectivity

Recall the notion of ω -projecting map used to define **PQ** in Section 2. As $\omega\mathbf{Top}$ is a full subcategory of **Seq** and hence also of **Lim**, it is clear that we can also define the ω -projecting maps in any of these categories. We shall be mainly interested in the ω -projecting maps in **Lim**, and their relationship to ω -projecting quotients in **Top**.

We first prove some closure properties of ω -projecting maps.

Proposition 5.1 *Let $f : X \rightarrow Y$ be an ω -projecting map in **Lim**.*

- (i) *If $g : Y \rightarrow Z$ is also ω -projecting, the composition $g.f$ is too.*
- (ii) *For an arbitrary $h : Y' \rightarrow Y$, the pullback h^*f of f along h is ω -projecting.*
- (iii) *If $f : X' \rightarrow Y'$ is ω -projecting, the product $f \times f'$ also is.*
- (iv) *If B is a countably based space, $f^B : X^B \rightarrow Y^B$ is ω -projecting.*

Proof. Statement (i) is straightforward.

To prove (ii), let $f' : X' \rightarrow Y'$ be the pullback of f along any $h : Y' \rightarrow Y$. For any $g : A \rightarrow Y'$ where A is countably based, let $g' : A \rightarrow X'$ be such that $f.g' = h.g$ (as given by f being ω -projecting). Let $\bar{g} : A \rightarrow X'$ be given by the universal property of the pullback. Then $f'.\bar{g} = g$ as required.

Statement (iii) is a consequence of (i) and (ii).

To prove (iv), consider any map $g : A \rightarrow Y^B$ where A is countably based. Take the exponential transpose $h : A \times B \rightarrow Y$ and extend to $\bar{h} : A \times B \rightarrow X$ such that $f.\bar{h} = h$ (as f is ω -projecting). Defining $\bar{g} : A \rightarrow X^B$ as the exponential transpose of \bar{h} we have that $f^B.\bar{g} = g$ as required. \square

Now observe that, by our explicit description of regular epis in **Lim** (given in Section 3.2), if \mathbb{N}^+ is projective with respect to a map h then h is a regular epi. As \mathbb{N}^+ is countably based, we obtain the following.

Proposition 5.2 *If f is ω -projecting then it is a regular epi.*

By the two previous propositions, if $f : X \rightarrow Y$ is ω -projecting in **Lim** then, for every countably-based space A , the map $f^A : X^A \rightarrow Y^A$ is a regular epi. (Note that the converse holds trivially.) Thus we have that $f : X \rightarrow Y$ is ω -projecting if and only if, for every countably-based space A , the following property holds in the internal logic of **Lim**.⁴

$$\mathbf{Lim} \models (\forall f \in Y^A)(\exists \bar{f} \in X^A)(f = q.\bar{f})$$

Thus we have shown that the original *external* notion of being ω -projecting is equivalent to its natural *internal* analogue.

⁴ As **Lim** is a quasitopos it has a full first-order intuitionistic internal logic. However, the property in question can be interpreted more generally in any cartesian-closed regular category.

In section 7 we are going to prove the cartesian closure of **PQ**, by working inside **Lim** and using the closure properties of ω -projecting maps. In order to do this we need to study what projecting quotients in **Top** look like from the perspective of **Lim**.

Proposition 5.3 *Let $r : B \rightarrow R$ be a continuous function between sequential spaces. The following are equivalent:*

- (i) $r : B \rightarrow R$ is ω -projecting in **Top**.
- (ii) $r : B \rightarrow R$ is an ω -projecting quotient in **Top**.
- (iii) $r : B \rightarrow R$ is ω -projecting in **Lim**.

Proof. As **Seq** is a full subcategory of both **Top** and **Lim**, it is clear that (i) and (iii) are equivalent and that (ii) implies both of them.

We now prove that (iii) implies (ii). By the previous proposition, r is a regular epi in **Lim**. But the functor **Lim** \rightarrow **Seq** \rightarrow **Top** has a right adjoint and so preserves regular epis. As B and R are sequential spaces, the functor maps r to the continuous function $r : B \rightarrow R$ in **Top**. Therefore r is a regular epi in **Top**, i.e. it is a topological quotient. \square

Beware, in **Top** (unlike in **Seq**), there exist ω -projecting maps (necessarily not between sequential spaces) that are not topological quotients.

6 Injectivity

In order to prove the cartesian closure of **PQ** we need to investigate injectivity, the dual notion to projectivity.

Definition 6.1 In any category, we say that an object X is *injective with respect to* a map $\underline{g} : Y \rightarrow Z$ if for every $f : Y \rightarrow X$ there exists $\bar{f} : Z \rightarrow X$ such that $f = \bar{f}.g$.

We shall be interested, in particular, in objects that are injective with respect to all pre-embeddings between countably-based spaces. (Recall from Section 4 that topological pre-embeddings and **Lim**-pre-embeddings agree between countably-based spaces.) In **Lim**, such injective objects are related to ω -projecting maps as follows.

Proposition 6.2 *In **Lim**, E is injective with respect to pre-embeddings between countably-based spaces if and only if, for every pre-embedding $a : A \rightarrow B$ between countably-based spaces, $E^a : E^B \rightarrow E^A$ is ω -projecting.*

Proof. For the “if” direction, suppose E^a is ω -projecting. Then, given any $f : A \rightarrow E$, we obtain $g : \mathbf{1} \rightarrow E^A$ (by exponential transpose) then $\bar{g} : \mathbf{1} \rightarrow E^B$ (because E^a is ω -projecting) then $\bar{f} : B \rightarrow E$ (again by exponential transpose). The equation $\bar{f}.a = f$ is easily verified.

For the “only if” direction, suppose E is injective with respect to pre-embeddings between countably-based spaces, and let $a : A \rightarrow B$ be a pre-

embedding between two countably-based spaces. Take any $f : C \rightarrow E^A$ where C is countably-based. We then obtain $g : A \times C \rightarrow E$ (by exponential transpose), whence $\bar{g} : B \times C \rightarrow A$ (because $a \times id_C : A \times C \rightarrow B \times C$ is a pre-embedding between countably-based spaces by Proposition 4.2), whence $\bar{f} : C \rightarrow E^B$ (again by exponential transpose). The equation $E^a.\bar{f} = f$ is easily verified. \square

In [26], Dana Scott introduced the *continuous lattices*, and characterised these as the injective objects with respect to subspace embeddings in the category of T_0 topological spaces. Martín Escardó pointed out to us that, in **Top** itself, the continuous lattices are, more generally, injective with respect to topological pre-embeddings. (Note that the topological pre-embeddings between T_0 spaces are exactly the subspace embeddings.)

For our purposes, we require only a convenient collection of injective objects in $\omega\mathbf{Top}$. Although we could work with countably-based continuous lattices, it suffices to restrict attention to the (even more manageable) algebraic lattices. We assume that the reader is familiar with the definition of these [5,10]. We shall only sketch the various constructions on algebraic lattices that we require.

Proposition 6.3 *Every algebraic lattice is injective with respect to every topological pre-embedding.*

Proof. Let $a : X \rightarrow Y$ be any topological pre-embedding. Suppose D is an algebraic lattice. Consider any $f : X \rightarrow D$. Then the extension $\bar{f} : Y \rightarrow D$ is defined by:

$$\bar{f}(y) = \bigsqcup \{ \bigsqcap f(a^{-1}U) \mid U \text{ is an open neighbourhood of } y \}$$

The proof that this is a continuous extension of f is identical to the standard proof of the injectivity of continuous lattices with respect to subspace embeddings between T_0 spaces [26]. \square

Proposition 6.4 *Every topological space can be topologically pre-embedded into an algebraic lattice. Moreover, every countably-based space can be pre-embedded in a countably-based algebraic lattice.*

Proof. For any topological space X , construct the algebraic lattice D as the set of all filters of opens ordered by inclusion. The function mapping x to its neighbourhood filter is a topological pre-embedding (with respect to the Scott topology on D).

For a countably based space, choose a countable base containing the empty set and the whole set. Construct D as the set of filters of basic opens ordered by inclusion. The pre-embedding is given by the function mapping x to its filter of basic open neighbourhoods. \square

7 Bicartesian closure

In this section we state and prove our main result, Theorem 7.2.

We write $\omega\mathbf{ALG}$ for the category of countably-based algebraic lattices. It is well known that $\omega\mathbf{ALG}$ is cartesian closed [5,10]. We assume that the reader is familiar with the construction of exponentials in this category. In particular, for compact elements $a \in D$ and $b \in E$ of some D, E in $\omega\mathbf{ALG}$, we let $(a \searrow b) : D \rightarrow E$ denote the related step function. Explicitly:

$$(a \searrow b)(d \in D) = \begin{cases} b, & \text{if } a \leq d \\ \perp, & \text{otherwise} \end{cases}$$

Lemma 7.1 *The embedding $S : \omega\mathbf{ALG} \rightarrow \mathbf{Lim}$ is a cartesian closed functor.*

Proof. The embedding S assigns to each countably based algebraic lattice, the corresponding space with the Scott topology. It is easy to see that it preserves products. Now, for D, E countably based algebraic lattices, it is also clear that $S(E^D)$ and SE^{SD} have the same underlying set so we need only prove that they have the same convergent sequences. So let $(f_i) \rightarrow f$ in $S(E^D)$ and let $(x_i) \rightarrow x$ in SD . We must show that $(f_i x_i) \rightarrow fx$ in SE . In order to do this, given any compact $e \leq fx$ we will prove that $(f_i x_i)$ is eventually above e .

So let (a_i) an ascending sequence of compact elements such that $\bigsqcup a_i = x$. Then, $f(\bigsqcup a_i) = \bigsqcup f a_i = fx$. So there exists an m such that $e \leq f a_m$. That is, $(a_m \searrow e) \leq f$. As a_m is compact, there exists L such that for all $j \geq L, x_j \geq a_m$. On the other hand, as $(a_m \searrow e)$ is compact, there exists L' such that for all $j \geq L', f_j \geq (a_m \searrow e)$. Now, let $M = \text{Max}\{L, L'\}$. For all $j \geq M, f_j \geq (a_m \searrow e)$ and so $e \leq f_j a_m$. Also, $a_m \leq x_j$ and then $f_j a_m \leq f_j x_j$. So $e \leq f_j x_j$. That is, $(f_i x_i)$ is eventually above e .

We now prove the converse so assume $(f_i) \rightarrow f$ in SE^{SD} . For any compact $c \leq f$ we will show that (f_i) is eventually above c . It is known that $c = \bigsqcup \{(a_1 \searrow b_1), \dots, (a_k \searrow b_k)\}$ for some k where the a 's and b 's are compact elements in D and E respectively. Now, for any $n \in \{1, \dots, k\}$ consider the sequence constantly a_n . By hypothesis, $(f_i a_n) \rightarrow f a_n$. As $(a_n \searrow b_n) \leq f$ if and only if $b_n \leq f a_n$ it follows that $(f_i a_n)$ is eventually above b_n . That is, there exists L_n such that for all $j \geq L_n, f_j a_n \geq b_n$. Then, for all $j \geq L_n, (a_n \searrow b_n) \leq f_j$. Now, let $M = \text{Max}\{L_1, \dots, L_k\}$. It is clear that for all $j \geq M, f_j \geq (a_1 \searrow b_1)$ and \dots and $f_j \geq (a_k \searrow b_k)$. Then, for all $j \geq M, f_j \geq c$. That is, (f_i) is eventually above c . \square

Theorem 7.2 *The category \mathbf{PQ} is bicartesian closed with finite limits. Moreover the inclusion $\mathbf{PQ} \hookrightarrow \mathbf{Seq}$ preserves this structure.*

Proof. As the inclusion from \mathbf{Seq} to \mathbf{Lim} preserves finite limits, exponentials and coproducts, it suffices to show that \mathbf{PQ} inherits all the specified structure from \mathbf{Lim} .

Let Q and R be in **PQ**. Then, there exist ω -projecting maps $q : A \rightarrow Q$ and $r : B \rightarrow R$ in **Lim** with A and B countably based,

To prove that $Q \times R$ is in **PQ**, just recall that $Q \times R$ is topological, that $A \times B$ is countably based and that ω -projecting maps are closed under products in **Lim**. Thus, by Proposition 5.3, $Q \times R$ is an ω -projecting quotient of $A \times B$ in **Top**.

For equalizers, we show that any regular subobject $m : Q' \rightarrow Q$ (in **Lim**) of Q is in **PQ**. Construct the pullback

$$\begin{array}{ccc}
 A' & \xrightarrow{\quad} & A \\
 \downarrow q' & \lrcorner & \downarrow q \\
 Q' & \xrightarrow{m} & Q
 \end{array}$$

Then A' is countably based and Q' is topological, both by Corollary 4.4, and q' is ω -projecting, by Proposition 5.1. Again, by Proposition 5.3, Q' is an ω -projecting quotient of A' in **Top**.

For coproducts, $Q + R$ is topological and $A + B$ is countably based so, by Proposition 5.1, we need only prove that $(q + r) : A + B \rightarrow Q + R$ is ω -projecting. So let C be countably based and take any $h : C \rightarrow Q + R$.

As coproducts are stable we obtain that C is isomorphic to $F + G$ and h is isomorphic to $f + g$ in the following diagram:

$$\begin{array}{ccccc}
 F & \xrightarrow{in_F} & C & \xleftarrow{in_G} & G \\
 \downarrow f & \lrcorner & \downarrow h & \lrcorner & \downarrow g \\
 Q & \xrightarrow{in_Q} & Q + R & \xleftarrow{in_R} & R
 \end{array}$$

As C is countably based and the injections are regular monos, F and G are countably based. Then, as q and r are projecting there exist $\bar{f} : F \rightarrow A$ and $\bar{g} : G \rightarrow B$ such that $q.\bar{f} = f$ and $r.\bar{g} = g$. So we have $\bar{h} = \bar{f} + \bar{g} : C \cong F + G \rightarrow A + B$ such that $(q + r).\bar{h} = h$.

Concerning exponentials. Let $q : A \rightarrow Q$ and $r : B \rightarrow R$ be as before. As **Seq** is an exponential ideal of **Lim**, R^Q is topological. So, by Proposition 5.3, it suffices to construct an ω -projecting map $e : [A, B] \rightarrow R^Q$ from a countably based space $[A, B]$.

Using Proposition 6.4, let A and B arise as domains of pre-embeddings $a : A \rightarrow D$ and $b : B \rightarrow E$ into ω -algebraic lattices D and E .

We define $[A, B]$ by taking pullbacks as follows.

$$\begin{array}{ccccc}
 [A, B] & \xrightarrow{\chi} & \langle A, B \rangle & \xrightarrow{\psi} & E^D \\
 \downarrow e & \lrcorner & \downarrow d & \lrcorner & \downarrow E^a \\
 & & B^A & \xrightarrow{b^A} & E^A \\
 & & \downarrow r^A & & \\
 R^Q & \xrightarrow{R^q} & R^A & &
 \end{array}$$

As b is a pre-embedding between countably based spaces, it is a **Lim**-pre-embedding by Proposition 4.1. By Proposition 4.2, b^A also is and so ψ is too. By Lemma 7.1, E^D is countably based, so $\langle A, B \rangle$ is countably based too.

As E is injective with respect to pre-embeddings between countably-based spaces, we have by Proposition 6.2 that E^a is ω -projecting. It then follows by proposition 5.1, that d , r^A , $r^A.d$ and finally e are ω -projecting. So in order to prove that R^Q is in **PQ**, we need only prove that $[A, B]$ is countably based. To see this, notice that as the functor $R^{(-)}$ carries colimits to limits, and q is a regular epi (by Proposition 5.2), then $R^q : R^Q \rightarrow R^A$ is a regular mono. Then χ also is and as $\langle A, B \rangle$ is countably based, $[A, B]$ is countably based too. \square

As **EPQ** is equivalent to **PQ**, we obtain the following.

Corollary 7.3 *EPQ is bicartesian closed with finite limits.*

8 Relating to equiological spaces

To complete the proof of Theorem 2.8, it remains to show that the embedding of **EPQ** in $\omega\mathbf{Equ}$ preserves all the identified structure. By the description of finite limits and coproducts in $\omega\mathbf{Equ}$, and the fact that countably-based spaces are closed under these operations, it follows that **EPQ** inherits this structure from $\omega\mathbf{Equ}$. It remains to prove that the embedding $\mathbf{EPQ} \rightarrow \omega\mathbf{Equ}$ preserves exponentials. For this, we need to explicitly introduce the cartesian-closed structure on $\omega\mathbf{Equ}$. This is most easily done by considering an equivalent category, introduced in [1].

Definition 8.1

- (i) An *assembly* (over an algebraic lattice) M is a triple $M = (|M|, \delta_M, D_M)$ such that $|M|$ is a set, D_M is an algebraic lattice and δ_M is a function from $|M|$ to the set of nonempty subsets of D_M .
- (ii) A morphism between assemblies $f : M \rightarrow N$ is a function $f : |M| \rightarrow |N|$ such that there exists a continuous $\bar{f} : D_M \rightarrow D_N$ *realizing* f in the sense

that $(\forall m \in |M|)(\forall d \in \delta_M m)(\overline{fd} \in \delta_N(fm))$.

Let **Ass** be the category of assemblies over algebraic lattices and morphisms between them. The proposition below appears in Remark 3.1 of [23].

Proposition 8.2 *Ass and Equ are equivalent.*

Proof. First define a functor $E' : \mathbf{Equ} \rightarrow \mathbf{Ass}$. For any space X , let $\eta_X : X \rightarrow \widehat{X}$ be its representation as a chosen pre-embedding into an algebraic lattice. To each (X, \sim_X) assign $(X/\sim_X, \delta_X, \widehat{X})$ where δ_X assigns to each $[x]$ in X/\sim_X the nonempty subset $\{\eta x' | x' \sim x\}$ of \widehat{X} .

The action on maps is the identity (using Proposition 6.3 to see that this produces a morphism between assemblies). It is easy to see that this functor is full and faithful.

The functor $E : \mathbf{Ass} \rightarrow \mathbf{Equ}$ is defined as follows. For each assembly $M = (|M|, \delta_M, D_M)$ let E_M be the topological space with underlying set $\{(m, d) \in |M| \times D_M | d \in \delta_M m\}$ and with the unique topology that makes the projection $E_M \rightarrow D_M$ into a pre-embedding. Let \sim_{E_M} be the equivalence relation defined by:

$$(m, d) \sim_{E_M} (m', d') \quad \text{iff} \quad m = m'$$

Define $E(M) = (E_M, \sim_{E_M})$.

To define the action on arrows notice that E_M/\sim_{E_M} is isomorphic to M . So the action of E on arrows is the identity up to the evident isomorphism.

It is straightforward to check that this functor is also full and faithful and that together with E' they give an equivalence between **Ass** and **Equ**. \square

The advantage of **Ass** over **Equ** is that its exponentials have an easy description. For assemblies M, N let $|N^M|$ be the set of morphisms from M to N . Then, the exponential is defined by

$$N^M = (|N^M|, \delta_{N^M}, D_N^{D_M})$$

where $D_N^{D_M}$ is the exponential of algebraic lattices and $\delta_{N^M}(f : M \rightarrow N) = \{g : D_M \rightarrow D_N | g \text{ realizes } f\}$.

Let $\omega\mathbf{Ass}$ denote the category of assemblies between countably-based algebraic lattices. It is not difficult to see that the equivalence of the Proposition 8.2 cuts down to one between $\omega\mathbf{Ass}$ and $\omega\mathbf{Equ}$ (so long as the choice of pre-embedding in the definition of E' is chosen so as to preserve the countable base!). Also, the description of exponentials in $\omega\mathbf{Ass}$ is identical to that in **Ass**.

We can now prove that the embedding of **EPQ** in $\omega\mathbf{Equ}$ preserves exponentials. To calculate the exponential in **EPQ** we use its equivalence with **PQ**.

Given objects (A, \sim_A) and (B, \sim_B) in **EPQ**, we write $q : A \rightarrow Q$ and $r : B \rightarrow R$ for the induced ω -projecting regular epis in **Lim**. In Section 7, we constructed the ω -projecting regular epi $e : [A, B] \rightarrow R^Q$ and a pre-embedding $c : [A, B] \rightarrow \widehat{B}^{\widehat{A}}$. Writing \sim for the induced equivalence relation

on the countably-based space $[A, B]$ we have that the quotient $([A, B]/\sim)$ is isomorphic to the exponential R^Q . As the equivalence $\mathbf{EPQ} \rightarrow \mathbf{PQ}$ reflects exponentials we obtain the following.

Proposition 8.3 *In \mathbf{EPQ} , $(B, \sim_B)^{(A, \sim_A)}$ is isomorphic to $([A, B], \sim)$.* \square

So we must prove the proposition below.

Proposition 8.4 *In $\omega\mathbf{Equ}$, $(B, \sim_B)^{(A, \sim_A)}$ is isomorphic to $([A, B], \sim)$.*

Proof. We use the equivalence between $\omega\mathbf{Ass}$ and $\omega\mathbf{Equ}$. Calculate the exponential $E'(B, \sim_B)^{E'(A, \sim_A)} = (R, \delta_B, \widehat{B})^{(Q, \delta_A, \widehat{A})} = (R^Q, \delta, \widehat{B}^{\widehat{A}})$ where, for $k \in R^Q$, $\delta(k) = \{c(f) \mid f \in [A, B] \text{ and } e(f) = k\}$.

But E_{R^Q} is iso to $[A, B]$ and the projection $E_{R^Q} \rightarrow \widehat{B}^{\widehat{A}}$ is a pre-embedding. Moreover, $E_{R^Q}/\sim_{E_{R^Q}} \cong R^Q$ so the image of the exponential assembly above is isomorphic to $([A, B], \sim)$. As the functor E is part of an equivalence, it preserves exponentials. So $([A, B], \sim)$ is indeed the exponential of (A, \sim_A) and (B, \sim_B) in $\omega\mathbf{Equ}$. \square

Corollary 8.5 *The embedding $\mathbf{EPQ} \rightarrow \omega\mathbf{Equ}$ is a bicartesian-closed functor preserving finite limits.* \square

9 Conclusions

Our results have immediate applications. For example, one readily sees that the space of discrete natural numbers occurs as the natural numbers object in \mathbf{PQ} and that the inclusions to \mathbf{Seq} , \mathbf{Lim} and \mathbf{Equ} all preserve the nno. Thus one obtains that the type hierarchies over \mathbf{N} in both \mathbf{Lim} and \mathbf{Equ} agree. It has long been known that the type hierarchy over \mathbf{N} in \mathbf{Lim} is given by the Kleene/Kreisel continuous functionals [25]. Thus we have an alternative proof of the recent result from [1] that the continuous functionals arise as the full type hierarchy in \mathbf{Equ} . More interesting is that a similar analysis applies to the full type hierarchy over any countably-based space. For example, the type hierarchy over the Euclidean reals in \mathbf{Lim} coincides with the hierarchy over the topological (projective) reals in \mathbf{Equ} . Similar results relating type hierarchies in categories of filter spaces with type hierarchies in \mathbf{Equ} have appeared recently in [23,11].

Our results and techniques bear comparison with recent work by Berger and Normann on totality in type hierarchies, in which they relate intensional “totality” structure on Scott domains with extensional structure modelled either topologically or in limit spaces [2,3,20,21]. Our work is similar in motivation, although, by considering only equilogical spaces, we are restricting the carriers of our intensional hierarchy to lattices rather than to the more general Scott domains. Nonetheless, it seems that the techniques used in our proof of Theorem 7.2 generalise to give a categorical approach to proving some of their results. Also, our analysis of largest common subcategories shared by

the extensional and intensional approaches provides a conceptual basis for understanding the “lifting theorems” of Normann and Waagbø [21].

Another interesting connection is that the proof of Theorem 7.2 essentially gives a categorical approach to the logical relations known as partial surjective homomorphisms (which originated in Friedman’s completeness proof for the simply-typed λ -calculus [9]). It seems that the notions of injectivity and projectivity form an abstract basis for understanding such special logical relations.

One possible application of \mathbf{PQ} is to tame the “troublesome” probabilistic powerdomain [16]. Using ideas from synthetic domain theory [14], one can find a natural left-exact cartesian-closed full subcategory of predomains within \mathbf{PQ} . This structure can be used to give an “internal” definition of a predomain of continuous valuations on any predomain, i.e. a candidate probabilistic powerdomain. It seems plausible that, because of the representation of the objects of \mathbf{PQ} as quotients of countably-based spaces, this powerdomain will address the problems raised in [16].

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