Heat-flow monotonicity related to the Hausdorff-Young inequality

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Heat-flow monotonicity related to the Hausdorff–Young inequality

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Dedicated to the memory of Laura Wisewell, 1975–2007

Abstract

It is known that if q is an even integer, then the $L^q(\mathbb{R}^d)$ norm of the Fourier transform of a superposition of translates of a fixed gaussian is monotone increasing as their centres 'simultaneously slide' to the origin. We provide explicit examples to show that this monotonicity property fails dramatically if $q > 2$ is not an even integer. These results are equivalent, upon rescaling, to similar statements involving solutions to heat equations. Such considerations are natural given the celebrated theorem of Beckner concerning the gaussian extremisability of the Hausdorff–Young inequality.

1. *Introduction*

For $d \in \mathbb{N}$ we let H_t denote the heat kernel on $\mathbb{R}^d$ given by

$$H_t(x) = t^{-d/2} e^{-\pi|x|^2/t}$$

and we define the Fourier transform $\hat{\mu}$ of a finite Borel measure μ on $\mathbb{R}^d$ by

$$\hat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}\, d\mu(x).$$

In what follows, for $p \in [1,\infty]$, we denote by p' the dual exponent satisfying $(1/p) + (1/p') = 1$. For μ, a positive finite Borel measure on $\mathbb{R}^d$, and $2 \leqslant q \leqslant p' \leqslant \infty$, let $Q_{p,q} : (0,\infty) \to \mathbb{R}$ be given by

$$Q_{p,q}(t) = t^{d(1/q-1/p')/2} \left\| \widehat{u(t,\cdot)^{1/p}} \right\|_q,$$

where $u(t,\cdot) = H_t * \mu$. If $q = 2k$ is an even integer, then by Plancherel's theorem one may write $Q_{p,q}$ in terms of a k-fold convolution given by

$$Q_{p,q}(t) = t^{d(1/q-1/p')/2} \| u(t,\cdot)^{1/p} * \cdots * u(t,\cdot)^{1/p} \|_2^{2/q}. \tag{1.1}$$

Expressions of this type are by now well known to be nondecreasing for $t > 0$ and this follows from the heat-flow approach to generalised Young's inequalities developed in [**6**, **8**] (see also [**4**] for an alternative approach). For the convenience of the reader, in the Appendix we have included a sketch of how this monotonicity follows from [**6**]. We note that, for $p = 1$, this is a particularly straightforward exercise using the fact that $\widehat{H_t}(\xi) = e^{-\pi t|\xi|^2}$. The purpose of this article is to show that this heat-flow monotonicity fails dramatically if q is not an even integer.

Theorem 1.1. *Let $d \in \mathbb{N}$, let $2 \leqslant q \leqslant p' \leqslant \infty$, and suppose q is not an even integer. Then there exists a positive finite Borel measure μ on $\mathbb{R}^d$ such that if $u(t,\cdot) = H_t * \mu$, then*

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$$Q_{p,q}(t) := t^{d(1/q-1/p')/2}\left\|\widehat{u(t,\cdot)^{1/p}}\right\|_q$$

is strictly decreasing for sufficiently small $t > 0$*.*

By making an appropriate rescaling, one may rephrase the above results in terms of 'sliding gaussians' in the following way. Let μ be a positive finite Borel measure on $\mathbb{R}^d$, and define $f : (0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ by

$$f(t,x) = \int_{\mathbb{R}^d} e^{-\pi|x-tv|^2}\, d\mu(v).$$

We interpret f as a superposition of translates of a fixed gaussian that simultaneously slide to the origin as t tends to zero. For $2 \leqslant q \leqslant p' \leqslant \infty$ define the quantity $\tilde{Q}_{p,q}(t)$ by

$$\tilde{Q}_{p,q}(t) = \|\widehat{f(t,\cdot)^{1/p}}\|_q.$$

The nondecreasingness of $Q_{p,q}$ for q an even integer tells us that $\tilde{Q}_{p,q}(t)$ is nonincreasing, and Theorem 1.1 tells us that, whenever q is not an even integer, there exist such measures μ for which $\tilde{Q}_{p,q}(t)$ is *strictly increasing* for sufficiently large t. It is interesting to note that the quantity $\|f(t,\cdot)\|_{q'/p}^{1/p}$, related to $\tilde{Q}_{p,q}(t)$ via the Hausdorff–Young inequality

$$\tilde{Q}_{p,q}(t) \leqslant \|f(t,\cdot)^{1/p}\|_{q'} = \|f(t,\cdot)\|_{q'/p}^{1/p},$$

is nonincreasing *for all* $2 \leqslant q \leqslant p' \leqslant \infty$, whether q is an even integer or not; see [**6**].

The quantities $Q_{p,q}$ have a more direct relation with the Hausdorff–Young inequality when $q = p'$. Suppose that $d\mu(x) = |f(x)|^p\, dx$ for some sufficiently well-behaved function f on $\mathbb{R}^d$ (such as bounded with compact support). In this case, if $Q_{p,q}$ is nondecreasing, then it is straightforward to verify that

$$\|\widehat{|f|}\|_{p'} = \lim_{t\to 0} Q_{p,q}(t) \leqslant \lim_{t\to\infty} Q_{p,q}(t) = \left\|\widehat{H_1^{1/p}}\right\|_{p'}\|f\|_p,$$

where H_1 is the heat kernel at time $t = 1$. Now, if p' is an even integer, then

$$\|\hat{f}\|_{p'} \leqslant \|\widehat{|f|}\|_{p'},$$

and so one recovers the sharp form of the Hausdorff–Young inequality on $\mathbb{R}^d$

$$\|\hat{f}\,\|_{p'} \leqslant \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{d/2} \|f\|_p \tag{1.2}$$

for p' an even integer, by Babenko [**1**]. We note that since (1.2) is not in general valid for nonnegative f when $p' < 2$, it follows that $Q_{p,q}(t)$ cannot possibly be nondecreasing for $t > 0$ when $q = p' < 2$.

Theorem 1.1 is of course a significant obstacle to finding a proof based on heat flow of the sharp Hausdorff–Young inequality due to Beckner [**2**, **3**]; that is, for *all* $p' \in [2,\infty)$. It should also be remarked that, whenever $p' \in [2,\infty)$ is not an even integer, there exists $f \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|\hat{f}\|_{p'} > \|\widehat{|f|}\|_{p'}. \tag{1.3}$$

Thus, in general, it is not without loss of generality that one considers nonnegative initial data for the heat flow. Inequality (1.3) may be seen as a consequence of an observation due to Hardy and Littlewood [**9**] concerning a majorant problem in the context of classical Fourier series. In fact, the counterexamples in our proof of Theorem 1.1 are somewhat in the spirit of the Hardy–Littlewood majorant counterexample in [**9**].

The idea of looking for monotone quantities underlying inequalities in analysis is of course not new. One way of constructing such quantities, which has been successful in recent years, is via heat-flow methods of the type we consider here. As we have already mentioned, this heat-flow perspective applies to a wide variety of so-called generalised Young's inequalities (or Brascamp–Lieb inequalities), which include the classical Young's convolution, multilinear Hölder and Loomis–Whitney inequalities; in particular, see [6, 8]. Among other notable (and closely related) examples from harmonic analysis are certain multilinear analogues of Kakeya maximal inequalities [7] and adjoint restriction inequalities for the Fourier transform; see the forthcoming [5].

2. *Proof of Theorem* 1.1

It suffices to handle $d = 1$, since if μ is a one-dimensional counterexample, then its d-fold tensor product is a d-dimensional counterexample. The case $p = 1$ will turn out to be pivotal and so we deal with that first of all.

Observe that if μ is a finite sum of Dirac delta measures, each supported at an integer, then $\hat{\mu}$ is a trigonometric polynomial, and thus a bounded periodic function on $\mathbb{R}$ with period 1. If c_n denotes the nth Fourier coefficient of $|\hat{\mu}|^q$, then

$$\begin{aligned}
Q_{1,q}(t)^q &= t^{1/2} \int_{\mathbb{R}} \widehat{H_t}(\xi)^q |\hat{\mu}(\xi)|^q \, d\xi \\
&= \sum_{n \in \mathbb{Z}} c_n t^{1/2} \int_{\mathbb{R}} \widehat{H_t}(\xi)^q e^{2\pi i n \xi} \, d\xi \\
&= \sum_{n \in \mathbb{Z}} c_n t^{1/2} \int_{\mathbb{R}} e^{-q\pi t \xi^2} e^{2\pi i n \xi} \, d\xi \\
&= q^{-1/2} \sum_{n \in \mathbb{Z}} c_n e^{-\pi n^2/qt}.
\end{aligned}$$

Since $q > 2$, it follows that $|\hat{\mu}|^q$ is continuously differentiable everywhere and thus the Fourier coefficients of $|\hat{\mu}|^q$ are absolutely summable. This is sufficient to justify the above interchange of summation and integration. Furthermore, note that

$$\sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} n^2 c_n t^{-2} e^{-\pi n^2/qt}$$

is uniformly convergent because, trivially, each summand is bounded in modulus by an absolute constant (that is, independent of $t > 0$ and $n \neq 0$) multiple of $1/n^2$. Again, by standard results, it follows that we may differentiate the above expression for $Q_{1,q}(t)^q$ term by term to obtain

$$\begin{aligned}
\frac{d}{dt}[Q_{1,q}(t)^q] &= \frac{\pi}{q^{3/2}} t^{-2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} n^2 c_n e^{-\pi n^2/qt} \\
&= \frac{\pi}{q^{3/2}} t^{-2} \sum_{n=1}^{\infty} n^2 (c_n + c_{-n}) e^{-\pi n^2/qt} \\
&= \frac{\pi}{q^{3/2}} t^{-2} e^{-\pi/qt} \left(c_1 + c_{-1} + \sum_{n=2}^{\infty} n^2 (c_n + c_{-n}) e^{-\pi(n^2-1)/qt} \right).
\end{aligned}$$

Since $(c_n)_{n \in \mathbb{Z}}$ is, in particular, a bounded sequence, it follows that

$$\sum_{n=2}^{\infty} n^2 (c_n + c_{-n}) e^{-\pi(n^2-1)/qt} \longrightarrow 0$$

as t tends to zero. Thus, to prove Theorem 1.1 when $p = 1$, it suffices to find a μ formed out of a finite sum of Dirac delta measures, each supported at an integer and such that

$$c_1 + c_{-1} < 0. \tag{2.1}$$

To this end, we let $m, n \in \mathbb{Z}$ be coprime, let $r \in (0, 1/2)$ and let

$$\mu := \delta_0 + r\delta_m + r\delta_n, \tag{2.2}$$

so that

$$\hat{\mu}(\xi) = 1 + re^{-2\pi i m\xi} + re^{-2\pi i n\xi}.$$

Since $|\hat{\mu}(\xi)|^2 = \hat{\mu}(\xi)\overline{\hat{\mu}(\xi)}$, we have that

$$|\hat{\mu}(\xi)|^q = \sum_{k=0}^{\infty} a_k r^k (e^{-2\pi i m\xi} + e^{-2\pi i n\xi})^k \sum_{k'=0}^{\infty} a_{k'} r^{k'} (e^{2\pi i m\xi} + e^{2\pi i n\xi})^{k'},$$

where a_k is the kth binomial coefficient in the expansion of $(1 + x)^{q/2}$; that is,

$$a_k = \frac{(q/2)((q/2) - 1) \cdot \ldots \cdot ((q/2) - k + 1)}{k!}.$$

Observe that if $k < q/2 + 1$, then $a_k > 0$, and thereafter a_k is strictly alternating in sign. Now, we have

$$\begin{aligned} c_1 + c_{-1} &= \int_0^1 |\hat{\mu}(\xi)|^q (e^{-2\pi i\xi} + e^{2\pi i\xi})\, d\xi \\ &= \sum_{k,k'=0}^{\infty} a_k a_{k'} r^{k+k'} \int_0^1 (e^{-2\pi i m\xi} + e^{-2\pi i n\xi})^k (e^{2\pi i m\xi} + e^{2\pi i n\xi})^{k'} (e^{-2\pi i\xi} + e^{2\pi i\xi})\, d\xi \end{aligned}$$

(of course, since μ is a real measure, it follows that $|\hat{\mu}|$ is even and therefore $c_1 = c_{-1}$; nevertheless, it is slightly more convenient to consider $c_1 + c_{-1}$ in order to preserve a certain symmetry later in the proof). To justify the above interchange of summation and integration, it suffices to show that $\sum_{k\geqslant 0} |a_k|(2r)^k$ is finite. This follows immediately because $(a_k)_{k\geqslant 0}$ is a bounded sequence and $r \in (0, 1/2)$. Therefore, we have

$$c_1 + c_{-1} = \sum_{k,k'=0}^{\infty} a_k a_{k'} r^{k+k'} \sum_{(\mathbf{j},\mathbf{j}')\in\Lambda_{k,k'}} \binom{k}{j_1}\binom{k'}{j_1'},$$

where

$$\begin{aligned} \Lambda_{k,k'} := \{(\mathbf{j}, \mathbf{j}') = ((j_1, j_2), (j_1', j_2')) \in (\mathbb{N}_0^2)^2 : j_1 + j_2 = k,\ \ j_1' + j_2' = k' \text{ and} \\ m(j_1 - j_1') + n(j_2 - j_2') = \pm 1\} \end{aligned}$$

and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We claim that, by choosing m and n appropriately (depending on q), we can ensure that $\Lambda_{k,k'}$ is empty *whenever* $a_k a_{k'} > 0$. It will only remain to check that $\Lambda_{k,k'}$ is nonempty for some k and k' for which $a_k a_{k'} < 0$. The proof of the claim proceeds as follows. First, a simple argument shows that if $n - m$ is even, then the sets $\Lambda_{k,k'}$ are empty whenever k and k' have the same parity. A second argument shows that $\Lambda_{k,k'}$ is empty whenever one of k and k' is less than $q/2 + 1$ upon an appropriate choice of m and n. This leaves a contribution from summands with k and k' greater than $q/2 + 1$ and, as long as one summand is nonzero, it is clear that $c_1 + c_{-1} < 0$, as required.

We now turn to the details. Since m and n are coprime, it implies that there exist integers α_0 and β_0 such that

$$\alpha_0 m + \beta_0 n = 1; \tag{2.3}$$

moreover, if $\alpha m + \beta n = \pm 1$ for integers α and β, then

$$(\alpha, \beta) = \pm(\alpha_0, \beta_0) + N(n, -m) \quad \text{for some } N \in \mathbb{Z}.$$

Therefore, if $(\mathbf{j}, \mathbf{j}') \in \Lambda_{k,k'}$, then

$$j_1 - j_1' = \pm\alpha_0 + Nn \tag{2.4}$$

and

$$j_2 - j_2' = \pm\beta_0 - Nm \tag{2.5}$$

for some $N \in \mathbb{Z}$. Also we have

$$j_1 + j_2 = k, \tag{2.6}$$

and

$$j_1' + j_2' = k'. \tag{2.7}$$

Lemma 2.1. *Suppose $n - m$ is even and that k and k' have the same parity. Then $\Lambda_{k,k'}$ is empty.*

Proof. Let $(\mathbf{j}, \mathbf{j}') \in \Lambda_{k,k'}$. By summing equations (2.4)–(2.7), it follows that

$$2(j_1 + j_2) = \pm(\alpha_0 + \beta_0) + (n - m)N + k + k'. \tag{2.8}$$

Thus $n - m$ even implies that $\alpha_0 + \beta_0$ is even. On the other hand, $n - m$ even and

$$(\alpha_0 + \beta_0)m + (n - m)\beta_0 = 1$$

imply that $\alpha_0 + \beta_0$ is odd. Hence $\Lambda_{k,k'} = \emptyset$. □

For fixed integers α_0 and β_0 satisfying (2.3), define

$$\alpha_* := \min\{|\alpha_0 + Nn| : N \in \mathbb{Z}\}$$

and

$$\beta_* := \min\{|\beta_0 - Nm| : N \in \mathbb{Z}\}.$$

Lemma 2.2. *Suppose that m and n are positive integers. Then the set $\Lambda_{k,k'}$ is empty whenever*

$$k' \geqslant 0 \quad \text{and} \quad 0 \leqslant k \leqslant \min\{\alpha_*, \beta_*\} - 1 \tag{2.9}$$

or

$$k \geqslant 0 \quad \text{and} \quad 0 \leqslant k' \leqslant \min\{\alpha_*, \beta_*\} - 1.$$

Proof. By symmetry, it suffices to check that $\Lambda_{k,k'}$ is empty when (2.9) holds. Suppose that $(\mathbf{j}, \mathbf{j}') \in \Lambda_{k,k'}$ and set $(\alpha, \beta) := (j_1 - j_1', j_2 - j_2')$ so that $\alpha m + \beta n = \pm 1$. By (2.4)–(2.7), it follows that

$$k \geqslant j_1 \geqslant j_1 - j_1' = \pm\alpha_0 + Nn$$

and

$$k \geqslant j_2 \geqslant j_2 - j_2' = \pm\beta_0 - Nm$$

for some $N \in \mathbb{Z}$. Since $m(j_1 - j_1') + n(j_2 - j_2') = \pm 1$, it follows that $\pm\alpha_0 + Nn$ and $\pm\beta_0 - Nm$ must have opposing signs. Therefore, we have

$$k \geqslant \min_{N \in \mathbb{Z}} \min\{|\alpha_0 + Nn|, |\beta_0 - Nm|\} = \min\{\alpha_*, \beta_*\}.$$

Hence $\Lambda_{k,k'}$ is empty when (2.9) holds. □

To complete the proof of Theorem 1.1 when $p=1$, let $k(q)$ denote the smallest integer greater than $q/2+1$ and consider the following particularly simple choice of m and n:

$$m = 2k(q)+1 \quad \text{and} \quad n = m+2.$$

Now we have

$$\left(\frac{m+1}{2}\right)m + \left(-\frac{m-1}{2}\right)n = 1 \tag{2.10}$$

and an easy calculation shows that

$$\alpha_* = k(q)+1 \quad \text{and} \quad \beta_* = k(q).$$

Therefore, by Lemmas 2.1 and 2.2, we have

$$c_1 + c_{-1} = \sum_{\substack{k,k' \geqslant k(q) \\ k,k' \text{opposing parity}}} a_k a_{k'} r^{k+k'} \sum_{(\mathbf{j},\mathbf{j}') \in \Lambda_{k,k'}} \binom{k}{j_1}\binom{k'}{j_1'}.$$

Moreover, (2.10) trivially implies that $\Lambda_{k(q)+1,k(q)}$ is nonempty and hence $c_1 + c_{-1} < 0$, as required.

We remark that there are many choices of the integers m and n that would have worked when $p=1$. Our argument for $p>1$ below capitalises on the fact that there exist m and n separated by a distance $\mathrm{O}(m)$ and such that $\Lambda_{k,k'}$ continues to be empty whenever $a_k a_{k'} > 0$. Moreover, we shall require that m can be chosen as large as we please. To see that such a choice of m and n is possible, suppose that

$$m = 3k_0 + 1 \quad \text{and} \quad n = 2m+3, \tag{2.11}$$

where k_0 is an even integer greater than $q/2+1$. A straightforward computation shows that

$$\left(\frac{2m+1}{3}\right)m + \left(-\frac{m-1}{3}\right)n = 1 \tag{2.12}$$

and consequently

$$\alpha_* = 2k_0 + 1 \quad \text{and} \quad \beta_* = k_0.$$

Furthermore, Λ_{2k_0+1,k_0} is nonempty by (2.12). Since k_0 is even, it again follows from Lemmas 2.1 and 2.2 that $c_1 + c_{-1} < 0$. We emphasise that k_0 can be as large as we please in this argument.

Now suppose $p>1$ and let m and n be given by (2.11). The idea behind the remainder of the proof is the following. For sufficiently large m and small $t>0$, it is clear that $H_t * \mu$ is a finite sum of 'well-separated' gaussians causing $(H_t * \mu)^{1/p}$ to be 'very close' to $H_t^{1/p} * \tilde{\mu}$, where

$$\tilde{\mu} := \delta_0 + r^{1/p}\delta_m + r^{1/p}\delta_n.$$

Given this, $Q_{p,q}(t)^q$ should be 'very close' to

$$p^{dq/2} t^{d/2} \|\widehat{H_{pt} * \tilde{\mu}}\|_q^q.$$

Furthermore, if $r \in (0, 1/2^p)$, this last quantity, as we have seen, is strictly decreasing for sufficiently small t with derivative bounded above in modulus by a constant multiple of $t^{-2} e^{-\pi/pqt}$. Thus, to conclude our proof of Theorem 1.1 when $p>1$, it suffices to check that the error

$$\begin{aligned} E_{p,q}(t) &:= Q_{p,q}(t)^q - p^{dq/2} t^{d/2} \|\widehat{H_{pt} * \tilde{\mu}}\|_q^q \\ &= t^{(1-q/p')/2} \int_{\mathbb{R}} (|((H_t * \mu)^{1/p})\widehat{\ }|^q - |(H_t^{1/p} * \tilde{\mu})\widehat{\ }|^q) \end{aligned}$$

has a derivative bound of the form

$$|E'_{p,q}(t)| \leqslant Ct^{-\gamma}e^{-cm^2/t}, \tag{2.13}$$

for sufficiently large m and small t. Here $C = C_{p,q,m}$ denotes a constant that may depend on p, q and m, and $\gamma = \gamma_{p,q}$ and $c = c_{p,q}$ constants that may depend on p and q.

Differentiating and grouping terms, we obtain

$$E'_{p,q}(t) = \frac{1}{2}\left(1 - \frac{q}{p'}\right) t^{-(1+q/p')/2}(\mathrm{I} + \mathrm{II}) + t^{(1-q/p')/2}(\mathrm{III} + \mathrm{IV}),$$

where

$$\begin{aligned}
\mathrm{I} &:= \int |((H_t * \mu)^{1/p})\hat{}|^{q-2}((H_t * \mu)^{1/p})\hat{}(((H_t * \mu)^{1/p})\hat{} - (H_t^{1/p} * \tilde{\mu})\hat{}),\\
\mathrm{II} &:= \int (H_t^{1/p} * \tilde{\mu})\hat{}(|((H_t * \mu)^{1/p})\hat{}|^{q-2}((H_t * \mu)^{1/p})\hat{} - |(H_t^{1/p} * \tilde{\mu})\hat{}|^{q-2}(H_t^{1/p} * \tilde{\mu})\hat{}),\\
\mathrm{III} &:= \int |((H_t * \mu)^{1/p})\hat{}|^{q-2}((H_t * \mu)^{1/p})\hat{}((\partial_t (H_t * \mu)^{1/p})\hat{} - (\partial_t H_t^{1/p} * \tilde{\mu})\hat{}),\\
\mathrm{IV} &:= \int (\partial_t H_t^{1/p} * \tilde{\mu})\hat{}(|((H_t * \mu)^{1/p})\hat{}|^{q-2}((H_t * \mu)^{1/p})\hat{} - |(H_t^{1/p} * \tilde{\mu})\hat{}|^{q-2}(H_t^{1/p} * \tilde{\mu})\hat{}).
\end{aligned}$$

Bounds of the form (2.13) are easily obtained for I, II, III and IV by elementary estimates, such as the Cauchy–Schwarz and Hausdorff–Young inequalities, along with pointwise estimates on the heat kernel H_t. We illustrate this for the term I, leaving the remaining terms to the reader. Applying the Cauchy–Schwarz inequality, the Hausdorff–Young inequality and Plancherel's theorem, we have

$$\begin{aligned}
\mathrm{I} &\lesssim \left(\int |((H_t * \mu)^{1/p})\hat{}|^{2(q-1)}\right)^{1/2}\left(\int |((H_t * \mu)^{1/p})\hat{} - (H_t^{1/p} * \tilde{\mu})\hat{}|^2\right)^{1/2}\\
&\leqslant \left(\int (H_t * \mu)^{2(q-1)/(2q-3)p}\right)^{(2q-3)/2}\left(\int |(H_t * \mu)^{1/p} - H_t^{1/p} * \tilde{\mu}|^2\right)^{1/2}.
\end{aligned}$$

Here we have used the fact that $q \geqslant 2$. Now, the first integral factor above is bounded by a power of t (which is permissible). For the second integral factor we split the integration over $\mathbb{R}$ into $\bigcup_0^6 I_j$, where

$$\begin{aligned}
&I_0 = (-\infty, -\epsilon m], \quad I_1 = [-\epsilon m, \epsilon m], \quad I_2 = [\epsilon m, (1-\epsilon)m], \quad I_3 = [(1-\epsilon)m, (1+\epsilon)m],\\
&I_4 = [(1+\epsilon)m, (1-\epsilon)n], \quad I_5 = [(1-\epsilon)n, (1+\epsilon)n] \quad \text{and} \quad I_6 = [(1+\epsilon)n, \infty),
\end{aligned}$$

for some sufficiently small positive absolute constant ϵ. We claim that a bound of the form (2.13) holds for each term given by

$$\int_{I_j} |(H_t * \mu)^{1/p} - H_t^{1/p} * \tilde{\mu}|^2.$$

For $j = 0, 2, 4, 6$ this is a simple consequence of the triangle inequality combined with elementary estimates on the heat kernel H_t. For $j = 1$ this follows from the facts that, for $x \in I_1$, we have

$$H_t * \mu(x) = H_t(x) + \mathrm{O}(t^{-1/2}e^{-cm^2/t})$$

and

$$H_t^{1/p} * \tilde{\mu}(x) = H_t(x)^{1/p} + \mathrm{O}(t^{-1/2p}e^{-cm^2/t}),$$

and the mean value theorem applied to the function $x \mapsto x^{1/p}$. The cases $j = 3$ and $j = 5$ are similar.

Appendix. *The even integer case*

We appeal to the following general theorem from [**6**, Proposition 8.9].

THEOREM A.1. *Let $m, n \in \mathbb{N}$, $n_1, \ldots, n_m \in \mathbb{N}$ and $p_1, \ldots, p_m > 0$. Suppose that, for each $1 \leqslant j \leqslant m$, there are linear surjections $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ and $A_j : \mathbb{R}^{n_j} \to \mathbb{R}^{n_j}$ such that the mapping $M = \sum_{j=1}^m (1/p_j) B_j^* A_j B_j$ is invertible and*

$$B_j M^{-1} B_j^* \leqslant A_j^{-1}$$

for all $1 \leqslant j \leqslant m$. Also, for each $1 \leqslant j \leqslant m$ let u_j be a solution to the heat equation

$$\partial_t u_j = \frac{1}{4\pi} \operatorname{div}(A_j^{-1} \nabla u_j). \tag{A.1}$$

Then the quantity

$$t^{(\sum_{j=1}^m n_j/p_j - n)/2} \int_{\mathbb{R}^n} \prod_{j=1}^m u_j(t, B_j x)^{1/p_j} \, dx$$

is nondecreasing for $t > 0$.

Multiplying out the L^2 norm in (1.1), we see that

$$Q_{p,2k}(t)^{2k} = t^{(\sum_{j=1}^m n_j/p_j - n)/2} \int_{\mathbb{R}^n} \prod_{j=1}^m u_j(t, B_j x)^{1/p_j} \, dx,$$

where $n = (2k-1)d$, $m = 2k$, $n_j = d$, $p_j = p$ and

$$B_j : \mathbb{R}^{(2k-1)d} \longrightarrow \mathbb{R}^d,$$
$$(x_1, \ldots, x_{2k-1}) \longmapsto \begin{cases} x_j & \text{for } j = 1, \ldots, 2k-1, \\ \displaystyle\sum_{j=1}^k x_j - \sum_{j'=k+1}^{2k-1} x_{j'} & \text{for } j = 2k. \end{cases}$$

Furthermore, for each $j = 1, \ldots, 2k$, it is clear that $u_j = u$ satisfies the heat equation (A.1), where A_j is the identity mapping. Since $p_j = p$ for all j, by homogeneity it suffices to verify the remaining hypotheses of Theorem A.1 when $p = (2k)'$.

It is straightforward to verify that, with respect to the canonical bases, the mapping $M = (1/(2k)') \sum_{j=1}^{2k} B_j^* B_j$ is given in block form by

$$M = \frac{1}{(2k)'} \begin{pmatrix} \mathbf{1}(k,k) + I_{kd} & -\mathbf{1}(k, k-1) \\ -\mathbf{1}(k-1, k) & \mathbf{1}(k-1, k-1) + I_{(k-1)d} \end{pmatrix},$$

where $\mathbf{1}(r, s)$ denotes the $rd \times sd$ matrix given by

$$\mathbf{1}(r,s) = \begin{pmatrix} I_d & I_d & \cdots & I_d \\ \vdots & \vdots & & \vdots \\ I_d & I_d & \cdots & I_d \end{pmatrix}$$

and I_l is the $l \times l$ identity matrix. A direct computation shows that M is invertible with

$$M^{-1} = \frac{1}{2k-1} \begin{pmatrix} -\mathbf{1}(k,k) + 2kI_{kd} & \mathbf{1}(k, k-1) \\ \mathbf{1}(k-1, k) & -\mathbf{1}(k-1, k-1) + 2kI_{(k-1)d} \end{pmatrix}$$

and $B_j M^{-1} B_j^* = I_d$ for each $j = 1, \ldots, k$. It now follows from Theorem A.1 that $Q_{p,2k}(t)$ is nondecreasing for each $t > 0$.

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