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**Citation for published version:**

Sierra, S, Špenko, Š, Vancliff, M, Veerapen, P & Wiesner, E 2019, 'Associating Geometry to the Lie Superalgebra  $sl(1|1)$  and to the Color Lie Algebra  $sl_2^c(k)$ ', *Proceedings of the american mathematical society*, vol. 147, 10, pp. 4135-4146. <https://doi.org/10.1090/proc/14647>

**Digital Object Identifier (DOI):**

[10.1090/proc/14647](https://doi.org/10.1090/proc/14647)

**Link:**

[Link to publication record in Edinburgh Research Explorer](#)

**Document Version:**

Peer reviewed version

**Published In:**

Proceedings of the american mathematical society

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**ASSOCIATING GEOMETRY TO THE LIE SUPERALGEBRA  $\mathfrak{sl}(1|1)$   
AND TO THE COLOR LIE ALGEBRA  $\mathfrak{sl}_2^c(\mathbb{k})$**

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ABSTRACT. In the 1990s, in work of Le Bruyn and Smith and in work of Le Bruyn and Van den Bergh, it was proved that point modules and line modules over the homogenization of the universal enveloping algebra of a finite-dimensional Lie algebra describe useful data associated to the Lie algebra ([5, 6]). In particular, in the case of the Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$ , there is a correspondence between Verma modules and certain line modules that associates a pair  $(\mathfrak{h}, \phi)$ , where  $\mathfrak{h}$  is a two-dimensional Lie subalgebra of  $\mathfrak{sl}_2(\mathbb{C})$  and  $\phi \in \mathfrak{h}^*$  satisfies  $\phi([\mathfrak{h}, \mathfrak{h}]) = 0$ , to a particular type of line module. In this article, we prove analogous results for the Lie superalgebra  $\mathfrak{sl}(1|1)$  and for a color Lie algebra associated to the Lie algebra  $\mathfrak{sl}_2$ .

INTRODUCTION

The ideas of algebraic geometry that were introduced in [1] often provide useful tools in studying noncommutative algebras, particularly Artin-Schelter regular algebras. In this setting, one can associate geometric objects (points, lines, etc.) of an appropriate projective space to certain classes of graded modules (namely, point modules, line modules, etc.) over the regular algebra.

Le Bruyn and Smith in [5] and, later, Le Bruyn and Van den Bergh in [6] proved that this framework can be applied to Lie algebras. In [5], Le Bruyn and Smith studied the homogenization  $\mathcal{H}(\mathfrak{sl}_2(\mathbb{C}))$  of the universal enveloping algebra of the Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$ , and they showed that most of the line modules over  $\mathcal{H}(\mathfrak{sl}_2(\mathbb{C}))$  can be realized as homogenizations of Verma modules over  $\mathfrak{sl}_2(\mathbb{C})$ . Taking this further, in [6], Le Bruyn and Van den Bergh showed that these results extend to finite-dimensional Lie algebras  $\mathfrak{g}$  of dimension  $n$ . In particular, they showed a correspondence between certain  $d$ -dimensional linear subschemes of  $\mathbb{P}^n$  and modules induced from  $d$ -codimensional subalgebras of  $\mathfrak{g}$ . In the special case of  $\mathfrak{g} = \mathfrak{sl}_2$ , where  $n = 3$  and

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2010 *Mathematics Subject Classification.* 14A22, 17B70, 17B75.

*Key words and phrases.* Lie algebra, Lie superalgebra, color Lie algebra, line module.

<sup>1</sup>S.J. Sierra & Š. Špenko were supported in part by EPSRC grant EP/M008460/1.

<sup>2</sup>Some of this work was completed while Š. Špenko was a post-doctoral scholar at the University of Edinburgh.

<sup>3</sup>M. Vancliff was supported in part by NSF grant DMS-1302050.

$d = 1$ , this work associates a pair  $(\mathfrak{h}, \phi)$ , where  $\mathfrak{h}$  is a two-dimensional Lie subalgebra of  $\mathfrak{g}$  and  $\phi \in \mathfrak{h}^*$  is such that  $\phi([\mathfrak{h}, \mathfrak{h}]) = 0$ , to a particular type of line module over  $\mathcal{H}(\mathfrak{sl}_2)$ .

The results in [5, 6] for Lie algebras suggest that similar techniques for analogous algebras – such as Lie superalgebras and color Lie algebras – might be productive. In this article, we investigate this idea for the Lie superalgebra  $\mathfrak{sl}(1|1)$  and for a color Lie algebra  $\mathfrak{sl}_2^c$  associated to  $\mathfrak{sl}_2$ . In particular, we prove in Theorem 2.3, Proposition 2.5, Theorem 2.7 and Theorem 3.3 that, for  $\mathfrak{g} \in \{\mathfrak{sl}(1|1), \mathfrak{sl}_2^c\}$ , there is a one-to-one correspondence between the collection of certain line modules over an appropriate graded algebra associated to  $\mathfrak{g}$  and pairs  $(\mathfrak{h}, \phi)$ , where  $\mathfrak{h}$  is a two-dimensional subalgebra of  $\mathfrak{g}$  and  $\phi \in \mathfrak{h}^*$  satisfies certain conditions. Moreover, most line modules can be realized as homogenizations of modules induced from one-dimensional  $\mathfrak{h}$ -modules. We conclude the paper by discussing the challenge in applying similar ideas to the Lie superalgebra  $\mathfrak{sl}(2|1)$ .

**Acknowledgments.** This project was initiated at the first WINART (Women in Noncommutative Algebra and Representation Theory) workshop held at the Banff International Research Station (BIRS) in the Spring of 2016. The environment at BIRS was particularly stimulating during the workshop and, for that, the authors warmly thank BIRS and the organizers and the participants of the workshop.

## 1. THE LIE ALGEBRA $\mathfrak{sl}_2(\mathbb{C})$

In this section, we establish notation to be used throughout the paper, and summarize the results of [5, 6] to be generalized to the Lie superalgebra  $\mathfrak{sl}(1|1)$  and to the color Lie algebra  $\mathfrak{sl}_2^c$  discussed in the Introduction.

Throughout the article,  $\mathbb{k}$  denotes an algebraically closed field, and, unless otherwise indicated,  $\text{char}(\mathbb{k}) \neq 2$ . We use  $M(n, \mathbb{k})$  to denote the vector space of  $n \times n$  matrices with entries in  $\mathbb{k}$ . For a graded  $\mathbb{k}$ -algebra  $B$ , the span of the homogeneous elements in  $B$  of degree  $i$  will be denoted  $B_i$ , and the dual of any vector space  $V$  will be denoted  $V^*$ . For homogeneous polynomials  $f_1, \dots, f_m$ , we use  $\mathcal{V}(f_1, \dots, f_m)$  to denote their zero locus in projective space.

In [5], Le Bruyn and Smith show how noncommutative algebraic geometry, in the spirit of Artin, Tate, and Van den Bergh (cf. [1]), can be applied to the Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$ . In particular, since the homogenization  $A$  of the universal enveloping algebra of  $\mathfrak{sl}_2(\mathbb{C})$  by a central element  $t$  is an Artin-Schelter regular algebra, the authors prove that there is a one-to-one correspondence between the collection of line modules over  $A$  on which  $t$  acts without torsion and the collection

of pairs  $(\mathfrak{b}, \lambda)$ , where  $\mathfrak{b}$  is a Borel subalgebra of  $\mathfrak{sl}_2(\mathbb{C})$  and  $\lambda \in \mathbb{C}$ . This result is stated in Theorem 1.2 below. To this end, consider the Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$  with basis  $\{e, f, h\}$  and Lie bracket

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Since  $\mathfrak{sl}_2(\mathbb{C})$  embeds in  $M(2, \mathbb{C})$  via the map

$$e \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we may define the determinant function,  $\det$ , on elements of  $\mathfrak{sl}_2(\mathbb{C})$ . Identifying  $\mathbb{P}^3$  with  $\mathbb{P}((\mathbb{k}e \oplus \mathbb{k}f \oplus \mathbb{k}h \oplus \mathbb{k}t)^*)$ , we define the pencil of quadrics  $Q(\delta) = \mathcal{V}(\det + \delta^2 t^2)$ , for all  $\delta \in \mathbb{P}^1$  (where  $Q(\infty) = \mathcal{V}(t^2)$ ), and lines  $l_{(\mathfrak{b}, \lambda)}$  in  $\mathbb{P}^3$  by

$$l_{(\mathfrak{b}, \lambda)} = \mathcal{V}(E, H - \lambda t),$$

where  $\lambda \in \mathbb{C}$  and  $\mathfrak{b}$  is a Borel subalgebra of  $\mathfrak{sl}_2(\mathbb{C})$  with standard basis  $\{E, H\}$ . As observed in [5],  $l_{(\mathfrak{b}, \lambda)}$  is independent of the choice of standard basis.

In order to define a line module over  $A$ , we first note that, by viewing  $e, f, h, t \in A$  as having degree one, the algebra  $A$  is a graded  $\mathbb{C}$ -algebra.

**Definition 1.1.** [1] A *line module* over  $A$  is a graded cyclic  $A$ -module with Hilbert series  $H(x) = 1/(1-x)^2$ .

**Theorem 1.2.** [5, Theorems 1 and 2] *As above, let  $A$  denote the homogenization of the universal enveloping algebra of  $\mathfrak{sl}_2(\mathbb{C})$  by a central element  $t$ .*

- (a) *The lines that lie on the quadrics  $Q(\delta) = \mathcal{V}(\det + \delta^2 t^2)$  ( $\delta \in \mathbb{P}^1$ ) are*
  - (i) *the lines on the plane  $\mathcal{V}(t)$ , and*
  - (ii) *the lines  $l_{(\mathfrak{b}, \lambda)}$  where  $\mathfrak{b}$  is a Borel subalgebra and  $\lambda \in \mathbb{C}$ .*
- (b) *The lines in  $\mathbb{P}^3$  that determine the line modules over  $A$  are precisely the lines on the quadrics  $Q(\delta)$ , for all  $\delta \in \mathbb{P}^1$ .*
- (c) *The line modules over  $A$  are of two types:*
  - (i) *those corresponding to the lines on the plane  $\mathcal{V}(t)$ , that is,  $A/(At + Aa)$  for all  $a \in A_1 \setminus \mathbb{C}t$ , and*
  - (ii) *those corresponding to the lines  $l_{(\mathfrak{b}, \lambda)}$  where  $\mathfrak{b}$  is a Borel subalgebra and  $\lambda \in \mathbb{C}$ , that is,  $A/(AE + A(H - \lambda t))$ , where  $\{E, H\}$  is a standard basis of  $\mathfrak{b}$ .*

**Remark 1.3.** For comparison with the techniques in [6] and with our methods in Sections 2 and 3, we note that  $\lambda \in \mathbb{C}$  defines a linear functional  $f_\lambda \in \mathfrak{b}^*$  by  $f_\lambda(E) = 0$  and  $f_\lambda(H) = \lambda$ , where  $\mathfrak{b}$ ,  $E$  and  $H$  are as above. In particular,  $f_\lambda([\mathfrak{b}, \mathfrak{b}]) = 0$  and  $\lambda$  defines a 1-dimensional  $\mathfrak{b}$ -module via  $f_\lambda$ . Moreover, given a Borel subalgebra  $\mathfrak{b}$  of  $\mathfrak{sl}_2(\mathbb{C})$ , any  $f \in \mathfrak{b}^*$  such that  $f([\mathfrak{b}, \mathfrak{b}]) = 0$  satisfies  $f = f_\lambda$ , for some  $\lambda \in \mathbb{C}$ .

2. THE LIE SUPERALGEBRA  $\mathfrak{sl}(1|1)$ 

In this section, we consider the Lie superalgebra  $\mathfrak{sl}(1|1)$ . We generalize to  $\mathfrak{sl}(1|1)$  the relationship between line modules over a certain graded algebra and subalgebras of Lie algebras given in [5, 6]. In particular, for two-dimensional subalgebras  $\mathfrak{h}$  of  $\mathfrak{sl}(1|1)$  and linear functionals  $\phi \in \mathfrak{h}^*$ , we show that a correspondence, analogous to that for  $\mathfrak{sl}_2(\mathbb{C})$ , between pairs  $(\mathfrak{h}, \phi)$  and line modules exists. However, only for certain pairs  $(\mathfrak{h}, \phi)$  and certain line modules can the correspondence for  $\mathfrak{sl}(1|1)$  be realized via a homogenization and dehomogenization process similar to that used in [5, 6]. Our main results are Theorem 2.3, Proposition 2.5 and Theorem 2.7.

In this section, we write  $\mathfrak{g}$  for  $\mathfrak{sl}(1|1)$ . Recall that  $\mathfrak{g}$  is the set of all  $2 \times 2$  matrices with supertrace equal to zero; that is

$$\mathfrak{g} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2, \mathbb{k}) : a - d = 0 \right\}.$$

There is a  $\mathbb{Z}_2$ -grading on  $\mathfrak{g}$  that is given by  $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$  where

$$\mathfrak{g}_{\bar{0}} = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} : a, d \in \mathbb{k}, a = d \right\} \quad \text{and} \quad \mathfrak{g}_{\bar{1}} = \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} : b, c \in \mathbb{k} \right\}.$$

The Lie (super)bracket is given by  $[X, Y] = XY - (-1)^{|X||Y|}YX$  for homogeneous elements  $X, Y \in \mathfrak{g}$ , where  $|X|$  and  $|Y|$  indicate the  $\mathbb{Z}_2$ -degree of  $X$  and  $Y$  respectively. It follows that  $\mathfrak{g}$  has basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

with relations

$$[e, f] = h, \quad 0 = [h, e] = [h, f] = [e, e] = [f, f] = [h, h].$$

We call any subspace of  $\mathfrak{g}$  that is closed under the Lie (super)bracket a *subalgebra* of  $\mathfrak{g}$ .

The universal enveloping algebra,  $U(\mathfrak{g})$ , of  $\mathfrak{g}$  is defined to be

$$U(\mathfrak{g}) = \mathbb{k}\langle e, f, h \rangle / \langle ef + fe - h, he - eh, hf - fh, e^2, f^2 \rangle,$$

and the homogenization of  $U(\mathfrak{g})$  by a central element  $t$ , analogous to that used in [5], is a quadratic algebra  $H$ , where

$$H = \frac{\mathbb{k}\langle e, f, h, t \rangle}{\langle ef + fe - ht, he - eh, hf - fh, e^2, f^2, et - te, ft - tf, ht - th \rangle},$$

such that  $H/H(t-1) \cong U(\mathfrak{g})$ . In particular,  $e$  and  $f$  are zero divisors in  $U(\mathfrak{g})$ , so  $H$  is not a domain, and hence not regular ([2, 7]). In spite of this,  $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$  (cf. Remark 2.6). In order to address the presence of zero divisors in  $H$ , we use a quadratic algebra  $\hat{H}$  that maps onto  $H$ , namely

$$\hat{H} = \frac{\mathbb{k}\langle e, f, h, t \rangle}{\langle ef + fe - ht, he - eh, hf - fh, et - te, ft - tf, ht - th \rangle},$$

and  $H = \hat{H}/\langle e^2, f^2 \rangle$ . The algebra  $\hat{H}$  is defined in [3] and is proved therein to be an AS-regular algebra. We carry over the  $\mathbb{Z}_2$ -grading from  $\mathfrak{g}$  to  $\hat{H}$  by taking  $|e| = |f| = \bar{1}$  and  $|h| = |t| = \bar{0}$ . Furthermore,  $\hat{H}$  has also the standard  $\mathbb{Z}$ -grading, with respect to which each of the generators  $e, f, h, t$  has degree one.

In [3], the lines in  $\mathbb{P}^3$  that correspond to the right line modules over  $\hat{H}$  are determined. That work entails identifying  $\mathbb{P}^3$  with  $\mathbb{P}((\mathbb{k}e \oplus \mathbb{k}f \oplus \mathbb{k}h \oplus \mathbb{k}t)^*)$ . We note that the symmetry in the relations of  $\hat{H}$  implies that the lines corresponding to the right line modules also correspond to the left line modules.

**Proposition 2.1.** [3] *The lines in  $\mathbb{P}^3$  that correspond to the (left) line modules over  $\hat{H}$  are*

- (a) *all lines in  $\mathbb{P}^3$  that meet the line  $\mathcal{V}(h, t)$ , and*
- (b) *all lines on the quadric  $\mathcal{V}(ht - 2ef)$ .*

*If  $\mathcal{V}(u, v)$  is such a line, where  $u, v \in \hat{H}_1$  are linearly independent, then the corresponding line module is isomorphic to the  $\hat{H}$ -module  $\hat{H}/(\hat{H}u + \hat{H}v)$ .*

**Lemma 2.2.**

- (a) *The 2-dimensional subalgebras of  $\mathfrak{g}$  are the subspaces  $\mathbb{k}h \oplus \mathbb{k}(\alpha e + \beta f)$  for all  $(\alpha, \beta) \in \mathbb{P}^1$ .*
- (b) *All 2-dimensional subalgebras of  $\mathfrak{g}$  are  $\mathbb{Z}_2$ -graded.*
- (c) *If  $\mathfrak{h} = \mathbb{k}h \oplus \mathbb{k}(\alpha e + \beta f)$ , where  $(\alpha, \beta) \in \mathbb{P}^1$ , and if  $\phi \in \mathfrak{h}^*$ , then  $\phi$  determines a 1-dimensional  $\mathfrak{h}$ -module  $\mathbb{k}_\phi$  (via  $a \cdot y = \phi(a)y$  for all  $a \in \mathfrak{h}$ ,  $y \in \mathbb{k}_\phi$ ) if and only if  $(\phi(\alpha e + \beta f))^2 = \alpha\beta\phi(h)$ .*

**Proof.**

(a) For  $(\alpha, \beta) \in \mathbb{P}^1$ , it is straightforward to check that  $\mathbb{k}h \oplus \mathbb{k}(\alpha e + \beta f)$  is a subalgebra of  $\mathfrak{g}$  of dimension two.

In order to show that all 2-dimensional subalgebras are of this form, let  $\mathfrak{h}$  be such a subalgebra, and suppose  $\alpha e + \beta f + \gamma h \in \mathfrak{h}$  for some  $\alpha, \beta, \gamma \in \mathbb{k}$ . Since  $\mathfrak{h}$  is a subalgebra, we have that

$$2\alpha\beta h = [\alpha e + \beta f + \gamma h, \alpha e + \beta f + \gamma h] \in \mathfrak{h}.$$

Thus, if  $\alpha\beta \neq 0$ , then  $h \in \mathfrak{h}$ , and so  $\alpha e + \beta f \in \mathfrak{h}$ , proving that  $\mathfrak{h}$  has the desired form. On the other hand, if  $\alpha\beta = 0$  for all elements  $\alpha e + \beta f + \gamma h$  of  $\mathfrak{h}$ , then it follows that either  $\mathfrak{h} = \mathbb{k}h \oplus \mathbb{k}e$  or  $\mathfrak{h} = \mathbb{k}h \oplus \mathbb{k}f$ .

(b) This follows from (a), since  $\alpha e + \beta f$  is  $\mathbb{Z}_2$ -homogeneous for all  $(\alpha, \beta) \in \mathbb{P}^1$ .

(c) Let  $\phi \in \mathfrak{h}^*$  and write  $\phi(h) = \lambda \in \mathbb{k}$  and  $\phi(\alpha e + \beta f) = \gamma \in \mathbb{k}$ . The map  $\phi$  determines a representation of  $\mathfrak{h}$  on  $\mathbb{k}$  if and only if

$$[x, y] \cdot 1 = \phi(x)\phi(y) - (-1)^{|x||y|}\phi(y)\phi(x)$$

for all homogeneous  $x, y \in \mathfrak{h}$ . Checking this criterion, we obtain

$$\begin{aligned} [h, h] \cdot 1 &= 0 \cdot 1 = 0 = \lambda^2 - (-1)^0 \lambda^2, \\ [h, \alpha e + \beta f] \cdot 1 &= 0 \cdot 1 = 0 = \lambda \gamma - (-1)^0 \gamma \lambda, \end{aligned}$$

whereas

$$[\alpha e + \beta f, \alpha e + \beta f] \cdot 1 = 2\alpha\beta h \cdot 1 = 2\alpha\beta\lambda,$$

and

$$(\phi(\alpha e + \beta f))^2 - (-1)^1 (\phi(\alpha e + \beta f))^2 = 2\gamma^2,$$

which completes the proof.  $\blacksquare$

For a subalgebra  $\mathfrak{h} = \mathbb{k}h \oplus \mathbb{k}(\alpha e + \beta f)$ , where  $(\alpha, \beta) \in \mathbb{P}^1$ , and  $\phi \in \mathfrak{h}^*$ , we consider the  $\hat{H}$ -module

$$L_{(\mathfrak{h}, \phi)} = \frac{\hat{H}}{\hat{H}(h - \phi(h)t) + \hat{H}(\alpha e + \beta f - \phi(\alpha e + \beta f)t)}.$$

**Theorem 2.3.** *The line modules over  $\hat{H}$  that correspond to the lines in  $\mathbb{P}^3 \setminus \mathcal{V}(t)$  that meet  $\mathcal{V}(h, t)$  are precisely the  $\hat{H}$ -modules  $L_{(\mathfrak{h}, \phi)}$  for all 2-dimensional subalgebras  $\mathfrak{h}$  of  $\mathfrak{g}$  and linear functionals  $\phi \in \mathfrak{h}^*$ .*

**Proof.** If  $\ell$  is a line in  $\mathbb{P}^3 \setminus \mathcal{V}(t)$  that meets  $\mathcal{V}(h, t)$ , then  $\ell = \mathcal{V}(h - \lambda t, \alpha e + \beta f - \gamma t)$  for some  $\lambda, \gamma \in \mathbb{k}$  and  $(\alpha, \beta) \in \mathbb{P}^1$ . By Proposition 2.1, such a line corresponds to a line module  $L$  over  $\hat{H}$  and  $L \cong \hat{H}/(\hat{H}(h - \lambda t) + \hat{H}(\alpha e + \beta f - \gamma t))$ . To the module  $L$ , we associate the pair  $(\mathfrak{h}, \phi)$  where  $\mathfrak{h} = \mathbb{k}h \oplus \mathbb{k}(\alpha e + \beta f)$  is a 2-dimensional subalgebra of  $\mathfrak{g}$ , by Lemma 2.2(a), and  $\phi \in \mathfrak{h}^*$  satisfies  $\phi(h) = \lambda$  and  $\phi(\alpha e + \beta f) = \gamma$ . Conversely, given  $(\mathfrak{h}, \phi)$  as in the statement, and noting Lemma 2.2(a), the reverse process yields the line  $\ell' = \mathcal{V}(h - \phi(h)t, \alpha e + \beta f - \phi(\alpha e + \beta f)t)$ , where  $(\alpha, \beta) \in \mathbb{P}^1$  and  $\mathfrak{h} = \mathbb{k}h \oplus \mathbb{k}(\alpha e + \beta f)$ , and all such lines meet  $\mathcal{V}(h, t)$  but do not lie on  $\mathcal{V}(t)$ . By Proposition 2.1, the line module corresponding to  $\ell'$  is the module  $L_{(\mathfrak{h}, \phi)}$ .  $\blacksquare$

**Definition 2.4.** Let  $\mathfrak{h} = \mathbb{k}h \oplus \mathbb{k}(\alpha e + \beta f)$  for some  $(\alpha, \beta) \in \mathbb{P}^1$ , and let  $\phi \in \mathfrak{h}^*$ . We call  $\phi$   $\mathbb{Z}_2$ -graded if  $\phi(\alpha e + \beta f) = 0$ .

**Proposition 2.5.** *For any 2-dimensional subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and  $\mathbb{Z}_2$ -graded linear functional  $\phi \in \mathfrak{h}^*$ , the  $\hat{H}$ -module  $L_{(\mathfrak{h}, \phi)}$  is a  $\mathbb{Z}_2$ -graded line module over  $\hat{H}$ . Conversely, for any  $\mathbb{Z}_2$ -graded line module  $L$  over  $\hat{H}$  on which  $t$  acts without torsion,  $L \cong L_{(\mathfrak{h}, \phi)}$  for some 2-dimensional subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  and  $\mathbb{Z}_2$ -graded linear functional  $\phi \in \mathfrak{h}^*$ .*

**Proof.** Let  $(\mathfrak{h}, \phi)$  be as in Definition 2.4. By Theorem 2.3,  $L_{(\mathfrak{h}, \phi)}$  is a line module over  $\hat{H}$ . Since  $\phi$  is  $\mathbb{Z}_2$ -graded,  $\alpha e + \beta f - \phi(\alpha e + \beta f)t = \alpha e + \beta f$ , which is  $\mathbb{Z}_2$ -homogeneous. As  $h - \phi(h)t$  is also  $\mathbb{Z}_2$ -homogeneous, it follows that  $L_{(\mathfrak{h}, \phi)}$  is  $\mathbb{Z}_2$ -graded.

For the converse, suppose  $L$  is a  $\mathbb{Z}_2$ -graded line module over  $\hat{H}$  on which  $t$  acts without torsion. By Proposition 2.1,  $L \cong \hat{H}/\hat{H}\mathfrak{h}'$  where  $\mathfrak{h}'$  is a 2-dimensional subspace of  $\hat{H}_1$ . As  $L$  is  $\mathbb{Z}_2$ -graded, it follows that either  $\mathfrak{h}' = \mathbb{k}e \oplus \mathbb{k}f$  or  $\mathfrak{h}' = \mathbb{k}h \oplus \mathbb{k}t$  or  $\mathfrak{h}' = \mathbb{k}(\delta_1 h - \delta_2 t) \oplus \mathbb{k}(\alpha e + \beta f)$ , where  $(\delta_1, \delta_2), (\alpha, \beta) \in \mathbb{P}^1$ . In the first case,  $L$  corresponds to the line  $\mathcal{V}(e, f)$ , which does not meet  $\mathcal{V}(h, t)$  and does not lie on  $\mathcal{V}(2ef - ht)$ ; hence, by Proposition 2.1,  $\mathfrak{h}' \neq \mathbb{k}e \oplus \mathbb{k}f$ . In the second case,  $t$  acts by torsion on  $L$ , which contradicts the hypothesis. It follows that

$$L \cong \frac{\hat{H}}{\hat{H}(\delta_1 h - \delta_2 t) + \hat{H}(\alpha e + \beta f)},$$

where  $(\delta_1, \delta_2), (\alpha, \beta) \in \mathbb{P}^1$ . However, the case that  $\delta_1 = 0$  yields a module on which  $t$  acts with torsion, and so we may assume  $\delta_1 = 1$ . Noting Lemma 2.2 and defining  $\mathfrak{h} = \mathbb{k}h \oplus \mathbb{k}(\alpha e + \beta f)$  and  $\phi \in \mathfrak{h}^*$  by  $\phi(h) = \delta_2$  and  $\phi(\alpha e + \beta f) = 0$ , completes the proof.  $\blacksquare$

As mentioned earlier, our goal is to extend the results of [5, 6] to  $\mathfrak{g}$ , where  $\mathfrak{g} = \mathfrak{sl}(1|1)$ . Extending the methods of [6] to our setting entails using the module  $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{k}_\phi$  where  $\mathfrak{h}$  is a 2-dimensional subalgebra of  $\mathfrak{g}$ ,  $U(\mathfrak{g})$  and  $U(\mathfrak{h})$  are the universal enveloping algebras of  $\mathfrak{g}$  and  $\mathfrak{h}$  respectively, and  $\mathbb{k}_\phi$  is a 1-dimensional  $\mathfrak{h}$ -module (cf. Lemma 2.2(c)). Applying the methods of [6, Section 2] in our setting produces a graded  $H$ -module from the  $U(\mathfrak{g})$ -module  $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{k}_\phi$  using the central generator  $t$ . The idea is as follows. One observes that  $U(\mathfrak{g})$  is a filtered algebra by taking  $U(\mathfrak{g})_0 = \mathbb{k}$ ,  $U(\mathfrak{g})_1 = \mathbb{k} \oplus \mathfrak{g}$ ,  $U(\mathfrak{g})_2 = U(\mathfrak{g})_1 + \mathfrak{g}^2$ ,  $\dots$ , for the filtration, and hence that  $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{k}_\phi$  is a filtered  $U(\mathfrak{g})$ -module, where the filtration is given by  $M_i = U(\mathfrak{g})_i \otimes \mathbb{k}_\phi$  for all  $i$ . The graded  $H$ -module,  $\mathcal{H}(M)$ , obtained from this data is given by  $\mathcal{H}(M) = \bigoplus_{i=0}^{\infty} M_i t^i$ . In this case,  $t \in H$  acts on the graded module via  $t \cdot ((a \otimes b)t^i) = (a \otimes b)t^{i+1}$  and  $x \in \mathbb{k}e \oplus \mathbb{k}f \oplus \mathbb{k}h \subset H$  acts on the module via  $x \cdot ((a \otimes b)t^i) = (xa \otimes b)t^{i+1}$  for all  $i$ .

However, in  $U(\mathfrak{g})$ , we have that  $e^2 = 0 = f^2$ , and so  $U(\mathfrak{g})$  does not have a PBW basis that contains  $e^i f^j h^k$  for all  $i, j, k \geq 0$ . Thus, there is no reason to expect the graded  $H$ -module obtained via this method to be a line module (indeed, in general, it will not have the desired Hilbert series). Instead, we replace  $U(\mathfrak{g})$ ,  $U(\mathfrak{h})$  and  $H$  in the above construction with algebras  $\widehat{U(\mathfrak{g})}$ ,  $\widehat{U(\mathfrak{h})}$ , and  $\hat{H}$ , respectively, where  $\hat{H}$  is given above and

$$\widehat{U(\mathfrak{g})} = \mathbb{k}\langle e, f, h \rangle / \langle ef + fe - h, he - eh, hf - fh \rangle,$$

and  $\widehat{U(\mathfrak{h})}$  is the subalgebra of  $\widehat{U(\mathfrak{g})}$  generated by  $\mathfrak{h}$ . By construction,  $\widehat{U(\mathfrak{g})} \xrightarrow{\chi} U(\mathfrak{g})$  (where  $\chi$  is the canonical map) and  $\chi|_{\widehat{U(\mathfrak{h})}} : \widehat{U(\mathfrak{h})} \rightarrow U(\mathfrak{h})$ . Moreover, we replace the  $U(\mathfrak{h})$ -module  $\mathbb{k}_\phi$  by a  $\widehat{U(\mathfrak{h})}$ -module  $\widehat{\mathbb{k}_\phi}$  where  $\widehat{\mathbb{k}_\phi} = \mathbb{k}_\phi$  as a vector space, and the action is given by  $y \cdot a = \chi(y)a$  for all  $y \in \widehat{U(\mathfrak{h})}$ ,  $a \in \widehat{\mathbb{k}_\phi}$ . The algebra  $\widehat{U(\mathfrak{g})}$  has a filtration analogous to that for  $U(\mathfrak{g})$ ; namely,  $\widehat{U(\mathfrak{g})}_0 = \mathbb{k}$ ,  $\widehat{U(\mathfrak{g})}_i = \widehat{U(\mathfrak{g})}_{i-1} + \mathfrak{g}^i$ , for all  $i \geq 1$ . With these replacements, we obtain the



filtered  $\widehat{U(\mathfrak{g})}$ -module  $N = \widehat{U(\mathfrak{g})} \otimes_{\widehat{U(\mathfrak{h})}} \widehat{\mathbb{k}_\phi}$  and the graded  $\hat{H}$ -module  $\mathcal{H}(N) = \bigoplus_{i=0}^{\infty} N_i t^i$ , where  $N_i = \widehat{U(\mathfrak{g})}_i \otimes_{\widehat{U(\mathfrak{h})}} \widehat{\mathbb{k}_\phi}$ .

**Remark 2.6.** Before continuing we remark that, although  $U(\mathfrak{g})$  and  $\widehat{U(\mathfrak{g})}$  are not graded algebras, the above discussion implicitly assumes that  $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$  and  $\mathfrak{g} \hookrightarrow \widehat{U(\mathfrak{g})}$ . These facts both follow from the observation that  $\widehat{U(\mathfrak{g})}$  is the Ore extension  $\widehat{U(\mathfrak{g})} = \mathbb{k}[h, e][[f; \sigma, \delta]]$ , where  $\mathbb{k}[h, e]$  is the polynomial ring on two generators,  $\sigma(e) = -e$ ,  $\sigma(h) = h$ ,  $\delta(e) = h$ ,  $\delta(h) = 0$ , and so  $\widehat{U(\mathfrak{g})}$  has PBW basis  $\{e^i f^j h^k : i, j, k \geq 0\}$ , and that  $U(\mathfrak{g}) = \widehat{U(\mathfrak{g})}/\langle e^2, f^2 \rangle$ , a factor of  $\widehat{U(\mathfrak{g})}$  by monomial relations. These embeddings allow  $\mathcal{H}(M)$  and  $\mathcal{H}(N)$  to be well defined.

In attempting to adapt the methods of [6] from the setting of  $\mathfrak{sl}_2(\mathbb{C})$  (where  $f_\lambda([\mathfrak{b}, \mathfrak{b}]) = 0$  as in Remark 1.3) to our setting, we find that assuming the analogous condition regarding  $\phi \in \mathfrak{h}^*$  (that is,  $\phi$  is  $\mathbb{Z}_2$ -graded) typically does not yield a 1-dimensional  $\mathfrak{h}$ -module (cf. Lemma 2.2(c)). Hence, “forgetting” the  $\mathbb{Z}_2$ -grading might be beneficial, and this is the approach used in the next result.

**Theorem 2.7.**

- (a) Let  $\mathfrak{h} = \mathbb{k}h \oplus \mathbb{k}(\alpha e + \beta f)$  be a subalgebra of  $\mathfrak{g}$  where  $(\alpha, \beta) \in \mathbb{P}^1$ . If  $\phi \in \mathfrak{h}^*$ , where  $(\phi(\alpha e + \beta f))^2 = \alpha\beta\phi(h)$ , then  $\mathcal{H}(\widehat{U(\mathfrak{g})} \otimes_{\widehat{U(\mathfrak{h})}} \widehat{\mathbb{k}_\phi})$  is isomorphic to the line module  $L_{(\mathfrak{h}, \phi)}$ .
- (b) Let  $L$  be the line module  $L = \hat{H}/(\hat{H}(h - \lambda t) + \hat{H}(\alpha e + \beta f - \gamma t))$ , where  $\lambda, \gamma \in \mathbb{k}$  and  $(\alpha, \beta) \in \mathbb{P}^1$ . If  $\gamma^2 = \alpha\beta\lambda$ , then  $L \cong \mathcal{H}(\widehat{U(\mathfrak{g})} \otimes_{\widehat{U(\mathfrak{h})}} \widehat{\mathbb{k}_\phi})$ , where  $\mathfrak{h} = \mathbb{k}h \oplus \mathbb{k}(\alpha e + \beta f)$ ,  $\phi(h) = \lambda$  and  $\phi(\alpha e + \beta f) = \gamma$ .

**Proof.**

(a) Let  $\psi$  denote the surjective map  $\psi : \widehat{U(\mathfrak{g})} \rightarrow \widehat{U(\mathfrak{g})} \otimes_{\widehat{U(\mathfrak{h})}} \widehat{\mathbb{k}_\phi}$ , where  $\psi(x) = x \otimes 1$  for all  $x \in \widehat{U(\mathfrak{g})}$ . Since  $\widehat{U(\mathfrak{g})}$  has a PBW basis (cf. Remark 2.6) and since  $\widehat{U(\mathfrak{g})} \otimes_{\widehat{U(\mathfrak{h})}} \widehat{\mathbb{k}_\phi}$  is a filtered  $\widehat{U(\mathfrak{g})}$ -module, we obtain  $\dim_{\mathbb{k}}(\psi(\widehat{U(\mathfrak{g})}_i)) = i + 1$ , for all  $i \geq 0$ , analogous to the situation in the proof of [6, Theorem 2.2(2)]. It follows that each nonzero homogeneous component of degree  $i$  of  $\mathcal{H}(\widehat{U(\mathfrak{g})} \otimes_{\widehat{U(\mathfrak{h})}} \widehat{\mathbb{k}_\phi})$  has dimension  $i + 1$ , so that the Hilbert series of  $\mathcal{H}(\widehat{U(\mathfrak{g})} \otimes_{\widehat{U(\mathfrak{h})}} \widehat{\mathbb{k}_\phi})$  equals that of the polynomial ring on two variables. Moreover,  $\mathcal{H}(\widehat{U(\mathfrak{g})} \otimes_{\widehat{U(\mathfrak{h})}} \widehat{\mathbb{k}_\phi}) = \hat{H}(1 \otimes 1)$ , and so is cyclic. Hence,  $\mathcal{H}(\widehat{U(\mathfrak{g})} \otimes_{\widehat{U(\mathfrak{h})}} \widehat{\mathbb{k}_\phi})$  is a line module. The left annihilator in  $\hat{H}$  of  $1 \otimes 1$  contains  $h - \phi(h)t$  and  $\alpha e + \beta f - \phi(\alpha e + \beta f)t$ . Hence,  $\mathcal{H}(\widehat{U(\mathfrak{g})} \otimes_{\widehat{U(\mathfrak{h})}} \widehat{\mathbb{k}_\phi})$  is a homomorphic image of the  $\hat{H}$ -module  $L_{(\mathfrak{h}, \phi)}$ . By Theorem 2.3,  $L_{(\mathfrak{h}, \phi)}$  is a line module, so the two modules have the same Hilbert series, and thus are isomorphic.

(b) By Proposition 2.1,  $L$  is the line module corresponding to the line  $\mathcal{V}(h - \lambda t, \alpha e + \beta f - \gamma t)$ . If we take  $\mathfrak{h} = \mathbb{k}h \oplus \mathbb{k}(\alpha e + \beta f)$  and define  $\phi \in \mathfrak{h}^*$  by  $\phi(h) = \lambda$ ,  $\phi(\alpha e + \beta f) = \gamma$ , then, by Lemma 2.2(a),  $\mathfrak{h}$  is a subalgebra of  $\mathfrak{g}$ , and, by Lemma 2.2(c), the condition  $\gamma^2 = \alpha\beta\lambda$  guarantees

the existence of the 1-dimensional  $\mathfrak{h}$ -module  $\mathbb{k}_\phi$  and its counterpart  $\widehat{\mathbb{k}}_\phi$ , so the result follows from (a).  $\blacksquare$

In contrast with Theorem 1.2, where each line module on which  $t$  acts without torsion corresponds to a line  $l_{(b,\lambda)}$ , and hence to a 1-dimensional  $\mathfrak{b}$ -module (cf. Remark 1.3), the last result suggests that, in the setting of  $\mathfrak{sl}(1|1)$ , some of the line modules  $L_{(b,\phi)}$  appear not to correspond to any 1-dimensional  $\mathfrak{h}$ -module. This is likely a consequence of  $\hat{H}$  “forgetting” that  $e^2 = 0 = f^2$  in  $U(\mathfrak{g})$ ; that is,  $\mathfrak{h}$  (and hence,  $U(\mathfrak{h})$ ) has fewer 1-dimensional modules than  $\widehat{U(\mathfrak{h})}$  has.

### 3. THE COLOR LIE ALGEBRA $\mathfrak{sl}_2^c(\mathbb{k})$

Owing to the results used from [3], we assume that  $\text{char}(\mathbb{k}) = 0$  in this section. Our goal is to extend the results of [5, 6] to the setting of the color Lie algebra  $\mathfrak{sl}_2^c(\mathbb{k})$  mentioned in the Introduction. We prove results for  $\mathfrak{sl}_2^c(\mathbb{k})$  that are analogous to those proved in the previous section for  $\mathfrak{sl}(1|1)$ , with the main result of this section being Theorem 3.3.

In this section, we consider the color Lie algebra  $\mathfrak{g} = \mathfrak{sl}_2^c(\mathbb{k})$  that is derived from the Lie algebra  $\mathfrak{sl}_2(\mathbb{k})$  via a process that is described in [4]. Recall that  $\mathfrak{g}$  has basis  $\{a_1, a_2, a_3\}$  and color-Lie bracket defined by

$$\langle a_1, a_2 \rangle = a_3 = \langle a_2, a_1 \rangle, \quad \langle a_2, a_3 \rangle = a_1 = \langle a_3, a_2 \rangle, \quad \langle a_3, a_1 \rangle = a_2 = \langle a_1, a_3 \rangle, \quad \langle a_i, a_i \rangle = 0,$$

for all  $i$ . Moreover,  $\mathfrak{g}$  is  $G$ -graded, where  $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ ,  $\mathfrak{g} = \bigoplus_{\alpha \in G} \mathfrak{g}_\alpha$  and

$$\mathfrak{g}_{(0,0)} = \{0\}, \quad \mathfrak{g}_{(1,0)} = \mathbb{k}a_1, \quad \mathfrak{g}_{(0,1)} = \mathbb{k}a_2, \quad \mathfrak{g}_{(1,1)} = \mathbb{k}a_3.$$

As in [4],  $\mathfrak{g}$  can be viewed as a cocycle twist of  $\mathfrak{sl}_2(\mathbb{k})$ . Continuing our terminology from the previous section, we refer to any subspace of  $\mathfrak{g}$  that is closed under the color-Lie bracket as a *subalgebra* of  $\mathfrak{g}$ .

The universal enveloping algebra,  $U(\mathfrak{g})$ , of  $\mathfrak{g}$  is defined to be

$$U(\mathfrak{g}) = \frac{\mathbb{k}\langle a_1, a_2, a_3 \rangle}{\langle a_1a_2 + a_2a_1 - a_3, a_2a_3 + a_3a_2 - a_1, a_3a_1 + a_1a_3 - a_2 \rangle}.$$

We denote the homogenization of  $U(\mathfrak{g})$  by a single central element  $a_4$  by the algebra

$$H = \mathbb{k}\langle a_1, a_2, a_3, a_4 \rangle / \langle a_1a_2 + a_2a_1 - a_3a_4, a_2a_3 + a_3a_2 - a_1a_4, \\ a_3a_1 + a_1a_3 - a_2a_4, a_1a_4 - a_4a_1, a_2a_4 - a_4a_2, a_3a_4 - a_4a_3 \rangle.$$

If we define  $|a_4| = (0, 0) \in G$ , then  $H$  is a  $G$ -graded algebra. However, Theorem 3.3 below suggests that the  $G$ -grading contributes nothing towards accomplishing our objective.

We may recover  $U(\mathfrak{g})$  from  $H$  via  $U(\mathfrak{g}) \cong H/H(a_4 - 1)$ . Using Bergman’s Diamond Lemma, it is straightforward to see that  $U(\mathfrak{g})$  has PBW basis  $\{a_1^i a_2^j a_3^k : i, j, k \geq 0\}$ ; thus,  $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$ , and  $\mathfrak{h} \hookrightarrow U(\mathfrak{h}) \hookrightarrow U(\mathfrak{g})$  for all subalgebras  $\mathfrak{h}$  of  $\mathfrak{g}$ .

We identify  $\mathbb{P}^3$  with  $\mathbb{P}((\mathbb{k}a_1 \oplus \cdots \oplus \mathbb{k}a_4)^*)$ . Owing to the symmetry of the relations defining  $H$ , the right line modules over  $H$  and the left line modules over  $H$  are parametrized by the same lines in  $\mathbb{P}^3$ . By [3],  $H$  is AS-regular and the line modules over  $H$  are given by the following result.

**Proposition 3.1.** [3] *The (left) line modules over  $H$  are given by the following 13 types of lines in  $\mathbb{P}^3$ , and conversely:*

- 1(a) all lines in  $\mathcal{V}(a_1 + a_2)$  that pass through  $(1, -1, 0, 0)$ ,
- 1(b) all lines in  $\mathcal{V}(a_1 - a_2)$  that pass through  $(1, 1, 0, 0)$ ,
- 2(a) all lines in  $\mathcal{V}(a_1 + a_3)$  that pass through  $(1, 0, -1, 0)$ ,
- 2(b) all lines in  $\mathcal{V}(a_1 - a_3)$  that pass through  $(1, 0, 1, 0)$ ,
- 3(a) all lines in  $\mathcal{V}(a_2 + a_3)$  that pass through  $(0, 1, -1, 0)$ ,
- 3(b) all lines in  $\mathcal{V}(a_2 - a_3)$  that pass through  $(0, 1, 1, 0)$ ,
- 4(a) all lines in  $\mathcal{V}(a_4 - 2a_3)$  that pass through  $(1, -1, 0, 0)$ ,
- 4(b) all lines in  $\mathcal{V}(a_4 + 2a_3)$  that pass through  $(1, 1, 0, 0)$ ,
- 5(a) all lines in  $\mathcal{V}(a_4 - 2a_2)$  that pass through  $(1, 0, -1, 0)$ ,
- 5(b) all lines in  $\mathcal{V}(a_4 + 2a_2)$  that pass through  $(1, 0, 1, 0)$ ,
- 6(a) all lines in  $\mathcal{V}(a_4 - 2a_1)$  that pass through  $(0, 1, -1, 0)$ ,
- 6(b) all lines in  $\mathcal{V}(a_4 + 2a_1)$  that pass through  $(0, 1, 1, 0)$ ,
7. all lines in  $\mathcal{V}(a_4)$ .

As in Section 2, if  $\mathfrak{h}$  is a subalgebra of  $\mathfrak{g}$ , then we denote an  $\mathfrak{h}$ -module of dimension one by  $\mathbb{k}_\phi$ , where  $\phi \in \mathfrak{h}^*$  and  $a \cdot y = \phi(a)y$  for all  $a \in \mathfrak{h}$ ,  $y \in \mathbb{k}_\phi$ . The next result shows that not all elements of  $\mathfrak{h}^*$  give rise to an  $\mathfrak{h}$ -module.

**Lemma 3.2.**

- (a) *The 2-dimensional subalgebras of  $\mathfrak{g}$  are precisely the subspaces  $\mathbb{k}a_i \oplus \mathbb{k}(a_j + \mu a_k)$  for all distinct  $i, j, k$  and  $\mu = \pm 1 \in \mathbb{k}$ . In particular, no 2-dimensional subalgebra of  $\mathfrak{g}$  is  $G$ -graded.*
- (b) *Let  $\mu = \pm 1 \in \mathbb{k}$  and  $\{i, j, k\} = \{1, 2, 3\}$ . If  $\mathfrak{h}$  is the subalgebra  $\mathfrak{h} = \mathbb{k}a_i \oplus \mathbb{k}(a_j + \mu a_k)$ , then  $\mathfrak{h}$  has exactly two one-parameter families of 1-dimensional modules:*

$$\{\mathbb{k}_\phi : \phi \in \mathfrak{h}^*, \phi(a_j + \mu a_k) = 0\} \quad \text{and} \quad \{\mathbb{k}_\phi : \phi \in \mathfrak{h}^*, \phi(a_i) = \mu/2\}.$$

**Proof.**

(a) We first observe that any subspace of the given form is a subalgebra. Conversely, suppose  $\mathfrak{h}$  is a 2-dimensional subalgebra and write  $\mathfrak{h} = \mathbb{k}(\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3) \oplus \mathbb{k}(\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3)$ ,

where  $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \in \mathbb{P}^2$ . Since  $\dim(\mathfrak{h}) = 2$ , the matrix  $X$  of coefficients, where

$$X = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix},$$

has rank two. By symmetry and row operations on  $X$ , it follows that we may assume  $\mathfrak{h} = \mathbb{k}v_1 \oplus \mathbb{k}v_2$ , where  $v_1 = a_1 + \alpha a_3$  and  $v_2 = a_2 + \beta a_3$ , for some  $\alpha, \beta \in \mathbb{k}$ . Since  $\mathfrak{h}$  is a subalgebra,

$$2\alpha a_2 = \langle v_1, v_1 \rangle \in \mathfrak{h}, \quad 2\beta a_1 = \langle v_2, v_2 \rangle \in \mathfrak{h} \quad \text{and} \quad \alpha a_1 + \beta a_2 + a_3 = \langle v_1, v_2 \rangle \in \mathfrak{h}.$$

However,  $\dim(\mathfrak{h}) = 2$ , so it follows that the matrix of coefficients of the elements  $v_1, v_2, 2\alpha a_2, 2\beta a_1, \alpha a_1 + \beta a_2 + a_3$  has rank at most two; that is, all  $3 \times 3$  minors of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 2\beta & \alpha \\ 0 & 1 & 2\alpha & 0 & \beta \\ \alpha & \beta & 0 & 0 & 1 \end{bmatrix}$$

are zero. Hence,  $2\alpha\beta = 0 = \alpha^2 + \beta^2 - 1$ , which implies  $(\alpha, \beta) = (0, \pm 1)$  or  $(\pm 1, 0)$  and so  $\mathfrak{h}$  is of the desired form. For  $\mu = \pm 1$ ,  $a_j + \mu a_k$  is not  $G$ -homogeneous for all distinct  $j$  and  $k$ , so  $\mathfrak{h}$  is not  $G$ -graded.

(b) By symmetry, it suffices to consider  $(i, j, k) = (3, 1, 2)$ . Let  $\phi \in \mathfrak{h}^*$ . There is only one relation that needs to be checked, namely

$$a_3 \cdot ((a_1 + \mu a_2) \cdot 1) + (a_1 + \mu a_2) \cdot (a_3 \cdot 1) = \mu(a_1 + \mu a_2) \cdot 1.$$

Writing  $\phi(a_3) = \alpha \in \mathbb{k}$  and  $\phi(a_1 + \mu a_2) = \beta \in \mathbb{k}$ , and evaluating each side of the relation implies that  $\alpha\beta + \beta\alpha = \mu\beta$ ; that is,  $\mathbb{k}$  is an  $\mathfrak{h}$ -module if and only if  $\beta(2\alpha - \mu) = 0$ . This latter equation has solution set  $\{(\alpha', 0), (\mu/2, \beta') : \alpha', \beta' \in \mathbb{k}\}$ , so the result follows.  $\blacksquare$

Given  $\mathfrak{h}$  and  $\mathbb{k}_\phi$  as in Lemma 3.2, we may consider the  $U(\mathfrak{g})$ -module  $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{k}_\phi$  and the  $H$ -module  $\mathcal{H}(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{k}_\phi)$ , where the latter is defined in a manner analogous to that described in [6, Section 2] and in Section 2 of this article. Moreover, since  $U(\mathfrak{g})$  has a PBW basis, arguments similar to those provided in Section 2 prove the next result.

**Theorem 3.3.** *For each  $k \in \{1, 2, 3\}$ , the  $a_k$ -torsion (left) line modules over  $H$ , on which  $a_4$  acts without torsion, are homogenizations of induced modules. More precisely, we have the following.*

- (a) *There exists a one-to-one correspondence between all pairs  $(\mathfrak{h}, \phi)$ , such that  $\mathfrak{h} = \mathbb{k}a_i \oplus \mathbb{k}(a_j + \mu a_k)$ ,  $\{i, j, k\} = \{1, 2, 3\}$ ,  $\mu = \pm 1 \in \mathbb{k}$ , and  $\phi \in \mathfrak{h}^*$  where  $\phi(a_j + \mu a_k) = 0$ , and line modules given by 1(a)-3(b) in Proposition 3.1 on which  $a_4$  acts without torsion; this correspondence is given by*

$$\mathcal{H}(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{k}_\phi) \cong \frac{H}{H(a_i - \phi(a_i)a_4) + H(a_j + \mu a_k)}.$$

(b) *There exists a one-to-one correspondence between all pairs  $(\mathfrak{h}, \phi)$ , such that  $\mathfrak{h} = \mathbb{k}a_i \oplus \mathbb{k}(a_j + \mu a_k)$ ,  $\{i, j, k\} = \{1, 2, 3\}$ ,  $\mu = \pm 1 \in \mathbb{k}$ , and  $\phi \in \mathfrak{h}^*$  where  $\phi(a_i) = \mu/2$ , and line modules given by 4(a)-6(b) in Proposition 3.1 on which  $a_4$  acts without torsion; this correspondence is given by*

$$\mathcal{H}(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathbb{k}_\phi) \cong \frac{H}{H(a_4 - 2\mu a_i) + H(a_j + \mu a_k - \phi(a_j + \mu a_k)a_4)}. \quad \blacksquare$$

This last result suggests that the  $G$ -grading on  $\mathfrak{g}$  and the  $G$ -grading on  $H$  play no role in dictating the correspondence between line modules over  $H$  and pairs  $(\mathfrak{h}, \phi)$  as discussed in the statement.

#### 4. THE LIE SUPERALGEBRA $\mathfrak{sl}(2|1)$

In this section, we consider the Lie superalgebra  $\mathfrak{sl}(2|1)$ , and return to the assumption that  $\text{char}(\mathbb{k}) \neq 2$ . We show that, in this setting, a simple analogue of the algebra  $\hat{H}$  (respectively,  $H$ ) that was used in Section 2 (respectively, Section 3) yields an algebra with zero divisors and so is not a regular algebra.

Recall that  $\mathfrak{sl}(2|1)$  consists of the matrices  $(\alpha_{ij}) \in M(3, \mathbb{k})$  with supertrace  $\alpha_{11} + \alpha_{22} - \alpha_{33} = 0$ , where the Lie (super)bracket is defined as in Section 2, and the  $\mathbb{Z}_2$ -grading on  $\mathfrak{sl}(2|1)$  may be described as follows. Using the elementary matrices,  $E_{11}, E_{12}, \dots, E_{33}$ , as a basis for  $M(3, \mathbb{k})$ , let  $x_1, \dots, x_4, y_1, \dots, y_4$  be given by  $x_1 = E_{11} + E_{33}$ ,  $x_2 = E_{22} + E_{33}$ ,  $x_3 = E_{12}$ ,  $x_4 = E_{21}$ ,  $y_1 = E_{13}$ ,  $y_2 = E_{31}$ ,  $y_3 = E_{23}$ ,  $y_4 = E_{32}$ . With this notation,  $\mathfrak{sl}(2|1) = (\mathfrak{sl}(2|1))_{\bar{0}} \oplus (\mathfrak{sl}(2|1))_{\bar{1}}$ , where  $(\mathfrak{sl}(2|1))_{\bar{0}} = \bigoplus_{i=1}^4 \mathbb{k}x_i$  and  $(\mathfrak{sl}(2|1))_{\bar{1}} = \bigoplus_{i=1}^4 \mathbb{k}y_i$ .

Homogenizing the universal enveloping algebra of  $\mathfrak{sl}(2|1)$  by using a central element  $t$  and then deleting the relations  $y_i^2 = 0$ , for all  $i = 1, \dots, 4$ , yields a quadratic algebra  $\hat{H}$  on the nine generators  $x_1, \dots, x_4, y_1, \dots, y_4, t$  with 36 (i.e.,  $\binom{9}{2}$ ) defining relations, among which are the relations  $x_3y_1 = y_1x_3$ ,  $y_1y_3 = -y_3y_1$  and  $x_3y_3 - y_3x_3 = y_1t$ . These three defining relations imply that

$$\begin{aligned} y_1^2t - x_3y_1y_3 &= y_1^2t - y_1x_3y_3 \\ &= y_1(y_1t - x_3y_3) \\ &= -y_1y_3x_3 \\ &= y_3y_1x_3 \\ &= y_3x_3y_1 \\ &= (x_3y_3 - y_1t)y_1 \\ &= x_3y_3y_1 - y_1^2t \\ &= -y_1^2t - x_3y_1y_3, \end{aligned}$$

from which it follows that  $2y_1^2t = 0$  in  $\hat{H}$ . Thus,  $\hat{H}$  is not a domain, and hence not regular ([2, 7]).

Consequently, if results analogous to those in Sections 2 and 3 hold for  $\mathfrak{sl}(2|1)$ , then either an alternative graded algebra will need to be used, or a careful analysis with the zero divisors will need to be performed in order to discuss the line modules.

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