Time varying and dynamic models for default risk in consumer loads

Time Varying and Dynamic Models for Default Risk in Consumer Loans

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Abstract

We review the incorporation of time varying variables into models of the risk of consumer default. Lenders typically have data which is of a panel format. This allows the inclusion of time varying covariates into models of account level default by including them in survival models, panel models or 'correction factor' models. The choice depends on the aim of the model and the assumptions that can be plausibly made. At the level of the portfolio, Merton-type models have incorporated macroeconomic and latent variables in mixed (factor) models and Kalman Filter models whilst reduced form approaches include Markov chains and stochastic intensity models. The latter models have mainly been applied to corporate defaults and considerable scope remains for application to consumer loans.

Keywords:, logistic regression, factor models, panel data, Kalman Filter, reduced form models, survival modelling, time varying covariates.

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1 Introduction

The aim of this paper is to critically review alternative dynamic approaches to consumer credit risk modelling. By 'dynamic' we mean models that relate aspects of credit risk to determining factors that vary over time. We use the term 'consumer credit' generically; we mean both unsecured credit, such as that extended on credit cards, personal loans, payment after use of utilities, and secured loans such as mortgages. Virtually all *application* risk models used by lenders have, until very recently, been static in the sense that they have related the probability that an applicant defaults in the first 12 or 18 months of holding a fixed term loan or a credit card to an applicant's characteristics which were known at the time of application only. Behavioural models include predictors that vary over time, notably recent repayment and account activity, but they rarely, if ever, include indicators of the macroeconomy. However there is considerable evidence that the state of a country's macroeconomy affects, on average, the chance an applicant will default in the future and the ranking in terms of risk of individuals who apply for a loan (Crook & Banasik 2005, Whitely et al 2004). It may also affect the value at risk of a portfolio of loans. The importance of such concerns is evidenced by the recent banking crises throughout the world. Consumer default modelling shows parallels with other statistical application areas such as medicine or educational attainment at the level of an individual but differs at the aggregate level. For example at the level of the individual, survival of credit worthiness parallels cancer survival in medicine. At the aggregate level (for example loan portfolio versus cancer prevalence), unlike medical applications default models have been used in regulatory requirements and this has led to different statistical models.

In this paper we firstly discuss models that predict the risk of default of an individual account and secondly, models of the risk associated with portfolios of loans. We conclude that consumer risk models can be made more accurate in their predictions of the probability a borrower will default and more informative about the level of value

at risk when states of the economy and dynamic behaviour are included, appropriately, in models of consumer risk.

2. Notational Conventions

Throughout the paper we adopt the following notational conventions. The letter *t* denotes *calendar* time and $t = a_i + \tau_i$ where a_i is origin time (date of account opening) and τ_i is *duration* time for borrower *i*, *i = 1…N*. Time has different possible granularities, for example, typically, monthly or annually. A term with a subscript *i* varies between cases but not over time; a term subscripted *t* varies over calendar time but not cases; and a term subscripted *it* may vary both over time and between cases. The term $\mathbf{x}_{it}$ denotes a vector of characteristics of an applicant *i* in time period *t* that are observed to vary over time, e.g. balance outstanding on a credit card. The term $\mathbf{w}_i$ denotes a vector of characteristics that are specific to a borrower but which are observed at only one point in time, and are not *observed* to change over time, e.g. variables from an application form (even if *de facto* their values do change over time). The term $\mathbf{z}_t$ denotes a vector of variables that vary over time, but which are not specific to an individual borrower e.g. macroeconomic variables. The term β_{0i} denotes an individual borrower specific constant. The term $\boldsymbol{\beta}_k$ is a vector of parameters to be estimated with the convention $\boldsymbol{\beta}_1$ relates to vector $\mathbf{w}_i$, $\boldsymbol{\beta}_2$ relates to vector $\mathbf{x}_{it}$, and $\boldsymbol{\beta}_3$ relates to vector $\mathbf{z}_t$. The terms $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$ denote matrices of parameters to be estimated. With one exception, a single variable is represented by a letter in upper, non-bold, case and its realisation in lower, non-bold, case. The exception is capital letter K which will refer to a constant. Later in the paper we write the general form $P(.|Z_t = z_t)$ as $p(z_t)$ for convenience. Whilst we express relationships mathematically, the finance literature sometimes uses imprecise terms. One example is to use capitalised symbols because of the mnemonic. We have followed this usage, for example PD, in section 3, albeit sparingly.

3 Credit Risk Models at the Level of the Individual Account

3.1 Generic Model

We begin by describing a very general and simplified statistical model of the probability of default of a borrower, *i*, during a discrete time period *t*. Let d_{it}^* be a latent continuous variable that represents 'utility from default' of borrower *i*, in period *t*. Define the default event $d_{it} = 1$ if $d_{it}^* > 0$ and non-default $d_{it} = 0$ if $d_{it}^* \leq 0$. Suppose

$$d_{it}^* = \beta_{0i} + \mathbf{w}_i^T \boldsymbol{\beta}_1 + \mathbf{x}_{it}^T \boldsymbol{\beta}_2 + \mathbf{z}_t^T \boldsymbol{\beta}_3 + \mathbf{x}_{it}^T \boldsymbol{\gamma}_1 \mathbf{w}_i + \mathbf{x}_{it}^T \boldsymbol{\gamma}_2 \mathbf{z}_t + \mathbf{w}_i^T \boldsymbol{\gamma}_3 \mathbf{z}_t + \eta_{it}, \quad (1)$$

and define

$$PD_{it} = P(d_{it} = 1 | \mathbf{x}_{it}, \mathbf{w}_i, \mathbf{z}_t), \quad (2)$$

so

$$P(d_{it} = 1 | \mathbf{x}_{it}, \mathbf{w}_i, \mathbf{z}_t) = F(\beta_{0i} + \mathbf{w}_i^T \boldsymbol{\beta}_1 + \mathbf{x}_{it}^T \boldsymbol{\beta}_2 + \mathbf{z}_t^T \boldsymbol{\beta}_3 + \mathbf{x}_{it}^T \boldsymbol{\gamma}_1 \mathbf{w}_i + \mathbf{x}_{it}^T \boldsymbol{\gamma}_2 \mathbf{z}_t + \mathbf{w}_i^T \boldsymbol{\gamma}_3 \mathbf{z}_t), \quad (3)$$

where *F* is a cumulative distribution function of η. The $\mathbf{x}_{it}^T \boldsymbol{\gamma}_1 \mathbf{w}_i$, $\mathbf{x}_{it}^T \boldsymbol{\gamma}_2 \mathbf{z}_t$ and $\mathbf{w}_i^T \boldsymbol{\gamma}_3 \mathbf{z}_t$ terms are linear combinations of interaction terms multiplied by their coefficients . The time subscripted variables may involve lags of differing lengths.

Suppose we know this model. By imposing restrictions, or making assumptions about various aspects of this model, we can show how various predictive models that currently are, or might be, used by lenders are encompassed within it (for a review of current methods see Crook et al: 2007 and Thomas et al: 2002). For example, by restricting the elements in each of the $\boldsymbol{\gamma}$ matrices and each of the $\boldsymbol{\beta}$ vectors, except for those in $\boldsymbol{\beta}_2$, to be all zeros and restricting β_{0i} to be a constant for all *i*, we gain a typical *application* risk model

$$P(d_{it} = 1 | \mathbf{w}_i) = F\left(\beta_{0i} + \mathbf{w}_i^T \boldsymbol{\beta}_1\right), \quad (4)$$

where in this case, *t* is a period extending, typically, from 0 to 12 months into the future, the $\mathbf{w}_i$ vector is of application characteristics, or credit bureau variables measured only at the time of application for credit, and d_{it} is as in equation (3). The function *F* is typically logistic (see Hosmer and Lemeshow 2002). Practitioners often apply a linear transformation to the term on the left hand side of equation (4) to gain a

'score', known as a 'credit score'. Alternatively they may gain a score by multiplying a predicted logit value by a constant.

In general there are three types of reasons why a borrower defaults. One reason is strategic, where the value of the debt outstanding exceeds the value of the asset which the debt was incurred to buy, plus transaction costs. The second is that an unexpected negative net income shock occurs, for example loss of job, divorce, health expenses, increases in interest payment etc. The third is simply mismanaging one's expenditure (see Chakravatti and Rhee (1999)). Equation (3) decomposes the variance of PD_{it} into that due to (a) variables the values of which may vary between cases but not over time, and variables whose values may vary between cases and over time but in practice are observed only at one point in time – the variables in $\mathbf{w}_i$; (b) variables that vary over time, are observed to do so and whose values are specific to the case – the variables in $\mathbf{x}_{it}$; and (c) time varying variables that are *not* specific to the case – the variables in $\mathbf{z}_t$. Examples of category (a) variables are time at address and net income. It is possible that both affect the chance that a borrower will default and both vary over time. But in practice neither is observed, except at the time of application. Examples of category (b) variables include, in the case of credit cards, balance outstanding in the previous period. Examples of category (c) variables include measures of the state of the economy. Thus if interest rates rise and this affects all borrowers, and we do not have a variable in $\mathbf{x}_{it}$ that describes the interest rate paid at time *t* by case *i*, then the increase in the interest rate is likely to affect all borrowers including *i* and so should be included. Similarly a rise in the unemployment rate is likely to be correlated with the chance that any one case becomes unemployed and less able to repay his/her debts. In addition, when macroeconomic variables change this may affect some applicants differentially and so interactions between $\mathbf{w}_i$ and $\mathbf{z}_t$ and between $\mathbf{x}_{it}$ and $\mathbf{z}_t$ are appropriate.

Returning to equation (4), the assumption that the variables in $\mathbf{w}_i$ can appropriately represent the effects of the $\mathbf{x}_{it}$, $\mathbf{z}_t$ variables and interaction terms is highly unlikely to be valid, yet in practice is often made.

Now consider equation (3) but suppose we restrict the elements in the $\boldsymbol{\beta}$ vectors and $\boldsymbol{\gamma}$ matrices that relate to $\mathbf{z}_t$ to be zero, so that the right hand expression includes only the covariates in $\mathbf{w}_i$ and $\mathbf{x}_{it}$, and restrict β_{0i} to be a constant for all i. If the variables in $\mathbf{w}_i$ and $\mathbf{x}_{it}$ include covariates that represent aspects of the applicant's borrowing behaviour in periods prior to that for which we wish a prediction of PD_{it}, then we have a typical *behavioural model*. The covariates, *as used in practice*, consist of two types of variables: exogenous, like income and interest rates on borrower *i*'s loans, and also endogenous behaviour. Note also that this type of model, which is used extensively by banks for borrowers of some standing (Thomas et al: 2002), omits the effects of changes in the economy, which may not be directly represented by variables in $\mathbf{x}_{it}$.

It is important to notice that the model represented by equation (3) can be applied to an unbalanced panel dataset albeit with many variables, the variables in $\mathbf{w}_i$, remaining constant across all time periods and with the variables in $\mathbf{z}_t$ remaining constant across all of the cases. This is, of course, exactly the format of data typically held by lenders. The panel is unbalanced in that over calendar time some borrowers will be charged off and their subsequent performance will be missing and some will enter the data set at different calendar times. There are alternative ways of estimating the model represented by equation (3) and using it for prediction. One possibility is to use survival analysis (see Kalbfeisch and Prentice 2002 and Cox and Oakes 1984).

3.2 Survival Models

On the rare occasions in which survival analysis is used in practice for consumer credit risk modelling, time is regarded as continuous, whilst *de facto* it consists of discrete intervals. We begin by discussing the continuous time case. In a continuous time survival model we are interested in the probability at an instant in time of leaving one state, such as 'being up to date with payments', and moving into another state such as '90 days overdue'. Let T_i denote the amount of time until a borrower defaults. The probability of default during the next instant, conditional on not having defaulted before, is given by the hazard function (where τ is duration time)

$$\lambda_i(\tau) = \underset{\Delta t \to 0}{Lim} \frac{P(T_i \in [\tau, \tau + \Delta\tau) \mid T_i \geq \tau)}{\Delta\tau}. \tag{5}$$

The probability of surviving (i.e. not being in the default state) can be written in terms of the hazard function

$$S_i(\tau) = P(T_i \geq \tau) = \exp\left(-\int_0^{\tau} \lambda_i(q) dq \right). \tag{6}$$

Several papers have used Cox proportional hazard (PH) models to model the hazard function and associated survival probabilities. The PH model can be written as

$$\lambda_i(\tau, \mathbf{w}_i, \boldsymbol{\beta}_1) = \lambda_0(\tau) \exp(\mathbf{w}_i^T \boldsymbol{\beta}_1), \tag{7}$$

where the baseline hazard, $\lambda_0(\tau)$ is a function only of duration time and is the same for all borrowers. This is dynamic only in the sense that the predicted hazard value, and corresponding predicted survival probability, vary with duration time, with the entire baseline hazard function being shifted according to the static $\mathbf{w}_i$ variables, which in the credit risk modelling context are determined at the time of application (see Banasik et al: 1999, Stepanova & Thomas: 2001, Andreeva et al: 2005, 2007, Ma et al: 2009). However Cox PH models also allow the inclusion of time varying covariates. Then such a model can be written as

$$\begin{aligned} \lambda_i(a_i, \tau, \mathbf{x}_i, \mathbf{w}_i, \mathbf{z}, \boldsymbol{\beta}) = \lambda_0(\tau) \exp\{\beta_{0i} + + \mathbf{w}_i^T \boldsymbol{\beta}_1 + \mathbf{x}_i(\tau)^T \boldsymbol{\beta}_5 + \mathbf{z}(a_i + \tau_i)^T \boldsymbol{\beta}_6 + \\ \mathbf{x}_i(\tau)^T \boldsymbol{\gamma}_4 \mathbf{w}_i + \mathbf{x}_i(\tau)^T \boldsymbol{\gamma}_5 \mathbf{z}(a_i + \tau_i) + \mathbf{w}_i^T \boldsymbol{\gamma}_6 \mathbf{z}(a_i + \tau_i)\}, \end{aligned} \tag{8}$$

where covariates $\mathbf{x}_i(\tau)$ take on values specific to the case and vary over duration time, and $\mathbf{z}(a_i + \tau_i)$ are covariates that vary over calendar time, such as macroeconomic variables. The terms $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_5$ and $\boldsymbol{\beta}_6$ are vectors, and $\boldsymbol{\gamma}_4, \boldsymbol{\gamma}_5,$ and $\boldsymbol{\gamma}_6$ are matrices, of parameters to be estimated. In this specification predicted changes in the economic environment after the opening of an account affect the predicted hazard and survival probability in each future time period. Interaction terms, involving $\mathbf{z}_t$, for

example $\mathbf{w}_i^T \boldsymbol{\gamma}_1 \mathbf{z}(a_i + \tau_i)$, allow changes in the macroeconomic variables to alter the ranking of the hazards and of the survival probabilities.

This model, albeit without the $\mathbf{x}_i(\tau)$ variables, has been used by Bellotti and Crook (2008). Using a range of costs of type I and of type II errors they compared the predictive performance of three types of model when used to predict whether an applicant defaulted within 12 months of opening a credit card account. The three types of model were a survival model with seven macroeconomic variables, a survival model and a logistic regression, each without such variables. They found that the survival model with the macroeconomic variables outperformed the other two models. The most influential variables were interest rates, real earnings and consumer confidence respectively.

Survival models have a number of advantages over static logistic regression (LR) models. First they allow the prediction of the probability of default over any time horizon not just that for which the dependent variable for the LR was defined. In addition they predict the probability of default conditional on not having defaulted before, static LR does not do this. Third, because the survival probability for each period can be predicted it can be used to predict profitability (see Ma et al: 2009).

3.3 Panel Models

Equation (3) is a binary choice panel model where the time periods are *discrete*. As mentioned earlier, financial institutions typically have data which is in panel format. One might then use the time variation in the panel to incorporate time varying covariates.

In panel models, the dependent variable can have very different definitions compared to survival models or models of the occurrence of a once only event (See Diggle et al, 2002 and Baltagi, 2008 for explanations of panel models). In a panel we may attempt to predict whether a borrower will miss a single payment in a time period, conditional on $\mathbf{w}_i$, $\mathbf{x}_{it}$ and $\mathbf{z}_t$ covariates (though not necessarily conditional on the event never having happened before). Missing a single payment does not necessarily imply being

three payments behind, a common definition of default, though it would if in the previous period the borrower was already two payments behind. Conventional predictions from random effects models assume the out of training sample random effects term (β_{0i} in equation (1)) is zero.

Another possibility is that the dependent variable indicates whether the borrower reached 3 payments overdue in a month, and we model the occurrence of a missed third payment (though not necessarily in successive months) conditional on never having missed a third payment before.

Given that panel data are measured at discrete time intervals, with an appropriately set up data matrix we can estimate a discrete survival model. To see this, (and omitting the **z** variables for simplicity), the discrete hazard function is

$$h_i^d = P(T_i \in [\tau - 1, \tau) \mid T_i \geq \tau - 1) = 1 - \frac{S(\tau, \mathbf{w}_i, \mathbf{x}_{i\tau})}{S(\tau - 1, \mathbf{w}_i, \mathbf{x}_{i\tau-1})}, \quad (9)$$

where h_i^d denotes the discrete hazard for case *i*, and *S(•)* denotes the probability of survival. One specification of this relationship is due to Cox (1972):

$$\text{logit}(h_i^d(\tau, \mathbf{w}_i, \mathbf{x}_{i\tau})) = \text{logit}(h_0^d(\tau)) + \mathbf{w}_i^T \boldsymbol{\beta}_1 + \mathbf{x}_{i\tau}^T \boldsymbol{\beta}_2, \quad (10)$$

where $h_0^d(\tau)$ is a discrete baseline hazard function.

One way of estimating the parameters in the first term on the right hand side is to represent it by a series of dummies, one for each time interval (Jenkins 1995), but functions of the duration time index itself are also legitimate (Singer and Willett: 1993). The value of the default indicator is set to zero for all intervals in which default is not observed. The value of the indicator is equal to one in the single period in which default is observed and the case is removed from the dataset thereafter.

The papers referred to in the previous section have all estimated survival models assuming time is continuous. But lenders hold data that is measured over discrete time

intervals, typically months. Despite this Stepanova and Thomas (2002) found very similar results when they compared models that assumed continuous time with those based on discrete time. Note also that as the discrete time intervals tend to zero the discrete time model tends to the continuous time model (Kalfeisch and Prentice 2002).

No parameterisations of these two models have been published using consumer loan data. A variant, followed by Saurina and Trucharte (2007), is to use yearly time periods and to model the probability of missing the third monthly payment in a year. Saurina and Trucharte used as predictors whether the borrower had defaulted in the past, whether the borrower is liquidity constrained and the GDP growth rate, all of which vary with time. They used a sample of 2.94 million mortgages in Spain, but pooled the data across cases and time. A further limitation of this work is that to predict risk, lenders typically require monthly rather than annual predictions. Nevertheless they gain an area under the receiver operating characteristic curve (AUROC) of 0.78. (This curve plots the proportion of defaulters predicted to default against the proportion of non-defaulters who are predicted to default, for every possible cut-off score. That is $F(s_i \mid i \in \text{default})$ against $F(s_i \mid i \in \text{non-default})$. (See Crook et al: 2007).

In another example Vallés (2006) used a random effects panel estimator to model the probability of default (90 days overdue) in a year using corporate data. A random effects model has the form of equation (3) but where β_{0i} is a random variable with an assumed common distribution. GDP growth and the inflation rate were significant and negatively related to default probability whereas the unemployment rate was positively related. She found that there was too much variation in the estimated model parameters between years to build a Through-The-Cycle (TTC) model (where PD_{it} does not vary over the business cycle – see section 4.5). However she did not include interaction terms between borrower characteristics and macroeconomic variables and it is not clear how well her model would predict out of sample.

We estimated a model of the first definition: missing a single payment in a month. The data, from a financial institution, were a random sample of holders of a credit

card that were issued with the card sometime between the late 1990s and early 2000s. The duration in the panel per borrower varied from under 10 to around 100 months. The cases joined and departed the panel at various times and so it is unbalanced. Since we wished to make predictions for borrower *i* we assumed a random effects model. The results for a model which has, as covariates, only information known at the time of application ($\mathbf{w}_i$ variables), linear and quadratic terms for duration time, and macroeconomic variables that could, in principle, be predicted at the time of application. The variables were chosen for inclusion based on a priori reasoning and previous estimates of credit scoring models. The results showed that the macroeconomic variables all have the expected sign and are significant. When interest rates or unemployment are high, so is the probability a borrower will miss a payment. We found that when house prices are high the probability of missing a payment is low. This may reflect the state of the economy more than the value of wealth householders have, since houses are not normally liquidated to pay a credit card bill. Duration time (and squared) were both highly significant. The proportion of variance which was explained by the random effect was large (49%) and highly significant indicating that pooling the data across time and cases would have resulted in inefficient estimates.

We subsequently also added behavioural, $\mathbf{x}_{it}$, variables, for example (balance/credit limit). These were all highly significant and had plausible signs. Again high interest rates and unemployment index increase the chance a payment is missed. The estimated value of ρ indicated that pooling the data would have resulted in inefficiently estimated parameters. Further details of these results are available from the authors on request.

3.4 Correction Factor Models

The common characteristic of these methods is that they involve taking a score that has been predicted from an estimated model and subsequently applying a "correction" which is specific to the state of the economy at the time the predicted PD is required. Two approaches have been suggested. Zandi (1998) suggested estimating a two stage model:

$$CS_{it} = g\{F(\beta_0 + \mathbf{w}_i^T \boldsymbol{\beta}_1 + \mathbf{x}_{it}^T \boldsymbol{\beta}_2)\}$$

and

$$PD_{it} = \beta_7 + \beta_8 CS_{it} + \beta_9 X_{it} \ , \qquad (11)$$

where CS_{it} is the predicted 'credit score' for borrower i, beginning of period t and X_{it} is a leading regional macroeconomic indicator for borrower i at time t with $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \beta_8$ and β_9 to be estimated. The g function is typically a linear transformation of a PD_{it} into a 'credit score'. F would typically be logistic. Here an *additive* correction factor, $\beta_9 X_{it}$, for the economy is a term which is separate from the predicted credit score. The first equation is parameterised before the second. A weakness of this functional form is that it does not allow a re-ranking of probabilities when there are changes in the macroeconomic indicators; instead only the intercept for all cases changes. In addition it is highly likely that the model is misspecified since time varying macroeconomic variables should be included directly in the function that represents CS.

In a second approach de Andrade (2007) built upon de Andrade and Thomas (2007). Here the estimated probability of default for case *i* in segment *s*, in time *t*, PD_{ist}, conditional on the state of the economy, is modelled as

$$PD_{ist} = F\left[\mathbf{x}_{ist}^T \boldsymbol{\beta}_s + \ln\left(\frac{\bar{d}_{st}}{1-\bar{d}_{st}}\right)\left(\frac{1-\overline{PD}_{st}}{\overline{PD}_{st}}\right)\right], \qquad (12)$$

where $\bar{d}_{st}$ denotes the estimated default rate in a risk segment *s* e.g. mortgage loans, period *t*; $\overline{PD}_{st}$ equals the mean predicted PD for a portfolio in segment *s*, period *t*; $\mathbf{x}_{ist}$ is a vector of covariates in the scoring model; F is the cumulative logistic distribution function. Both $\bar{d}_{st}$ and $\overline{PD}_{st}$ are parts of the $\mathbf{z}_t$ vector defined in section 2.

The $\boldsymbol{\beta}_s$ vector, if estimated conventionally, will be affected by the economy in several ways (Kelly et al 1999, de Andrade 2007). For example the $\boldsymbol{\beta}_s$ values may vary with

the cycle as, conceivably, previous repayments become more influential in a macroeconomic downturn than in a period of growth or stability. Second, the distribution of the observed covariate values may change over the cycle, affecting the $\overline{PD}_{st}$. Third, the average level of the predicted PD_{ist} values may change, analogous to a change in the prior probabilities of default. Equation (12) effectively alters the intercept of the logistic regression according to whether, for a specific segment, the observed default rate relative to the predicted default rate (predicted using macroeconomic variables) is high or low.

De Andrade (2007) estimates an autoregressive distributed lag (ADL) function of correction factors for each segment using up to 14 macroeconomic variables to gain PD_{ist} by applying the factors to the scores gained from unknown models (the $\mathbf{x}_{ist}^{T}\boldsymbol{\beta}$ terms).The predicted values of PD_{it}, using equation (4), were applied to Brazilian small and medium sized enterprises and tested on the training sample of 12 segments (in this case industries) to reveal an increase in the AUROC when macroeconomic variables were included. This particular approach has some scope for improvement. For example the ADL was not a co-integrating relationship, simultaneity between the variables was ignored and the equation does not allow the economic variables to alter the ranking of the borrowers, interaction being only at the level of the segment.

4 Portfolio Models

4.1 Loss Distributions

Lending institutions hold capital in case of losses resulting from unexpected default behaviour. According to the Basel II Accord (BIS 2006) (see below) for any segment s of similarly risky borrowers, the expected loss in period *t*, EL_{st} , may be calculated as the product of the average predicted probability that a borrower in segment *s* defaults in period *t*, denoted PD_{st}; the expected proportion of the debt outstanding by a typical borrower in segment s at time t that is never recovered by a lender, denoted LGD_{st} ; and the average expected exposure or debt outstanding by a borrower in

segment *s* at the time of default, denoted EAD_{st} .A necessary condition for this to be correct is that the random variables explaining all three terms are uncorrelated.

Realised losses may exceed these *predicted* amounts. For each borrower there is a distribution of possible values of *EAD*, *LGD* and *PD*. Taking every possible value of each, finding their product and summing over all borrowers gives a distribution of losses (see Bluhm et al 2003). Lenders are interested in this distribution for each segment of a portfolio. This distribution is often called the 'loss distribution' and its mean 'expected loss', The 'unexpected losses' from a segment of a portfolio may be represented by the difference between expected loss and the value of losses such that the probability of gaining a smaller loss than this value is α (other definitions are possible). This particular value of losses, the α-percentile of the loss distribution, is known as value at risk (VaR_α). Figure 1 illustrates. In this section we are concerned with the distribution of losses (or default fractions if *EAD* and *LGD* are fixed for each borrower) and in particular with VaR_α.

In attempts to reduce the chance of systematic bank failures, the G10 countries have adopted various capital requirements regulations, the latest being the Basel II Accord (BIS 2006). This allows banks to estimate the minimum amount of capital ('regulatory capital') they are required to hold, subject to regulatory approval. Potential contributory factors to the current banking crisis include the possibility that the Accord did not require lenders to hold sufficient capital in the event of their assets falling in value. We note some theoretical weaknesses of the Merton model which may underlie the Basel II formula, below.

The amount of regulatory capital that a bank must hold to cover for defaults on loans differs according to the types of loans held. For retail loans, the subject of this paper, the regulatory capital to cover for credit risk is:

$$RC_{st} = EAD_{st} * LGD_{st} * \{\Phi[g_1(\rho).\Phi^{-1}(PD_{st}) + g_2(\rho).\Phi^{-1}(0.999)] - PD_{st}\}, \quad (13)$$

where ρ is said to be the correlation between asset values over borrowers and Φ^{-1}. denotes the inverse of the cumulative distribution function for a standard normal

random variable. The value $\Phi[\bullet]$ represents the $VaR_{99.9}$ (as a multiplier of EAD, LGD). The specified values of ρ are 0.15, 0.04 and a function of PD for mortgages, revolving credit and other retail exposures, respectively. The final term of equation (13) is included because banks are expected to price loans to include expected losses. A bank then aims to build statistical models to predict PD_{st}, LGD_{st}, and EAD_{st} if it uses the Basel II advanced internal ratings based (IRB) approach.

We can classify empirical models that are concerned with the distributions of probabilities of default and/or of default rates into Merton-type models, econometric models, actuarial models, markov chain models and stochastic intensity models. Actuarial models yield closed form distributions of losses and are exemplified by Credit Risk+ (Credit Suisse:1997). We know of no published applications of actuarial models to consumer loans and for space reasons we omit them here. Merton-type models are generally called 'structural' models and the remaining models are known as 'reduced form' models.

4.2 Merton-type Models

4.2.1 The Vasicek Formula

Following Merton's model for a bond (Merton 1974), Schonbucher (2000) assumes a borrower, *i*, defaults at the end of a given time horizon, T^*, if at that time the value of his assets, V_{iT^*}, a random variable, falls below a threshold K_i. That is $V_{iT^*} < K_i$. Suppose the end of the horizon occurs in a calendar time period *t*. Default then occurs when $V_{it} < K_i$. Suppose the current value of assets is indexed at zero. Then a value of V_{it} implies a return over periods 0 to *t*, and many recent papers are expressed in terms of the return $((V_{it} - V_{it-1})/V_{it-1})$ rather than V_{it}. We explain them in terms of V_{it} rather than return, for simplicity. We initially assume K_i to be the same for all borrowers.

A potentially major determinant of the default rate for a portfolio of loans is the correlation between the default probabilities of the individual borrowers. Default probabilities may be correlated because of indirect links between them, for example several borrowers may be employed by the same employer or by employers in the

same industry. They may also be subject to the same interest rate changes or legislative or bank policy changes. Many of these shocks can be represented by observed changes in the state of the macroeconomy or by different regional identifiers. We call a group of borrowers that are subject to variations in the same risk drivers a segment. From a pragmatic perspective, a typical retail portfolio simply has too many borrowers for a lender to specify and evaluate the complete set of joint probabilities of default. Instead a simulation model may be used.

It is assumed that the value of the borrower's assets is determined by a common factor, Z_t , and a borrower specific noise component, ε_{it} as follows:

$$V_{it} = \sqrt{\rho} Z_t + \sqrt{1-\rho}\, \varepsilon_{it}\,, \tag{14}$$

where Z_t and ε_{it} are independent of each other and standard fnormally distributed with mean of zero and unit variance, ε_i are i.i.d. and Z_t is serially uncorrelated. Following Hamerle et al (2003, Hamerle et al 2004, Hamerle and Rosch (2005) we make Z explicitly time dependent and we also treat time as discrete. Notice that given Z_t, V_{it} is independent between borrowers. Z_t could be an observable macroeconomic variable, $Z_t^{[o]}$, or an unobservable latent variable, $Z_t^{[u]}$, that affects all borrowers equally. The correlation between V_{it} and V_{jt} is ρ (known as 'asset correlation') and between V_{it} and Z_t is $\sqrt{\rho}$. Of course the assumption that the ε_{it} values are iid is unlikely to be realistic. Note that if the right hand side of equation (1) equals the right hand side of equation (14) then $d_{it}^* = V_{it}$.

Assuming that all borrowers in a risk segment have an equal probability of default and the same threshold, $K_i = K \quad \forall i$, Schonbucher proves that the probability that borrower *i* defaults, conditional on the realisation z_t of Z_t in period *t*, is

$$p_i(z_t) = P(V_{it} < K \mid Z_t = z_t) = \Phi\left(\frac{K - \sqrt{\rho}\, z_t}{\sqrt{1-\rho}}\right), \tag{15}$$

where the denominator is due to a scale change in ε_{it}.

If, as is usually the case in a portfolio of retail borrowers, the total number of borrowers, *N*, is very large indeed it may be more useful to work with the fraction of borrowers that default. We now redefine *B*, a random variable, to be this fraction. Conditional on the realization of Z_t, when *N* tends to infinity the law of large numbers implies that the proportion of borrowers who default equals the probability that any individual will default, *PD*, so *PD = B = n/N.* It can then be shown (Schonbucher op cit) by integrating over the density function of Z_t that the cumulative distribution of *B*, *F(b),* is given by

$$F(b) = P(B \le b) = \Phi\left(\frac{1}{\sqrt{\rho}} \left(\sqrt{1-\rho}\, \Phi^{-1}(b) - K \right) \right), \tag{16}$$

from which, by differentiation, the density function can be derived.

Lenders, policy makers and researchers are typically interested in the probability that the fraction of loans that default is less than a particular number, b_α, and, as shown bySmithson (2003), by inverting equation (16) one can derive an expression for b_α:

$$b_\alpha = \Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(\alpha) + \Phi^{-1}(PD)}{\sqrt{1-\rho}} \right), \tag{17}$$

where K in equation (15) is $\Phi^{-1}(PD)$. Equation (17) is the Vasicek formula (Vasicek: 1987) in the Basel II Accord (BIS 2006), the $\Phi[\bullet]$ function in equation (13), and when multiplied by the proportion of loans that is not recovered gives, after the deduction of expected losses, the capital requirement for unexpected losses per dollar of exposure at default. In the Accord, for retail exposures, α=0.999. This type of model is similar to the commercial product CreditMetrics™ which is used extensively by commercial banks. Both assume that a borrower will default when the value of his assets falls below a threshold where the value of assets is related to a common risk factor and an idiosyncratic, i.e. ε_{it}, term (Finger 1999, Bucay and Rosen 2001, Gupton 1997).

The model can be extended to include multiple latent factors, non-homogeneous borrowers and multiple observable factors to gain variants of equations (15) and (17) (see Schonbucher: 2000) .For example Rösch (2003) makes the default threshold a function of observable systematic factors – macroeconomic variables, $\mathbf{z}_t^{[o]}$. Thus he writes the default condition as $V_{it} < K_0 + \mathbf{K}^T \mathbf{z}_t^{[o]}$. where $\mathbf{K}$ is a (M x 1) vector of parameters, and $\mathbf{z}_t^{[o]}$ is a (M x 1) vector of observable systematic factors. Then, conditional on the realisation of $z_t^{[u]}$ for $Z_t^{[u]}$, the probability of default is

$$p_t(z_t^{[u]}) = \Phi\left(\frac{K_0 + \mathbf{K}^T \mathbf{z}_t^{[o]} - \sqrt{\rho} z_t^{[u]}}{\sqrt{1-\rho}}\right). \tag{18}$$

The $Z_t^{[u]}$ represents omitted correlations between borrower defaults and omitted observables.

4.2.2 Estimation

Following Hamerle and Rösch (2006) we assume borrowers may not be homogeneous. We write the probability of default, conditional on the realisation of $z_t^{[u]}$ for $Z_t^{[u]}$, equation (15), including lagged observable macroeconomic factors and allowing the threshold to depend on lagged observable individual risk factors, $\mathbf{x}_{it-1}$, as well as time invariant individual factors, $\mathbf{w}_i$, thus

$$p_{it}(z_t^{[u]}) = \Phi\left(\frac{K_0 + \mathbf{K}_1^T \mathbf{w}_i + \mathbf{K}_2^T \mathbf{x}_{it-1} + \mathbf{K}_3^T \mathbf{z}_{t-1}^{[o]} - \sqrt{\rho}\, z_t^{[u]}}{\sqrt{1-\rho}}\right). \tag{19}$$

Notice that this is of the same form as equation (3) with an additional term: $Z_t^{[u]}$ and scaling factor $\sqrt{1-\rho}$.

The probability of default, unconditional on $z_t^{[u]}$, is

$$P_{it} = \int_{-\infty}^{\infty} p_{it}(z_t^{[u]}) \phi(z_t^{[u]}) dz_t^{[u]}, \tag{20}$$

where $p_{it}(z_t^{[u]})$ is as equation (19), and where we integrate over all possible realisations of $Z_t^{[u]}$.

If we observe a default pattern over individuals in time period *t* of $\{d_{1t},....d_{it}....d_{N_t t}\}$, each d_{it} taking on a value of 1 if borrower i defaults, and zero otherwise, which must be conditional on the realisation $z_t^{[u]}$ of $Z_t^{[u]}$, we can write

$$P(d_{1t},.....d_{N_t t} \mid Z_t^{[u]} = z_t^{[u]}) = \prod_{i=1}^{N_t} [\, p_{it}(z_t^{[u]})^{d_{it}}\,][1 - p_{it}(z_t^{[u]})]^{1-d_{it}}\,. \qquad (21)$$

By integrating over the realisations $z_t^{[u]}$, taking logs and summing over *t* we gain the log-likelihood function.

Equation (20) is a random effects probit model (see Liang and Zeger:1986 and Verbeke and Molenberghs: 2000 for a description of random effects probit models). But unlike conventional panel random effects models where the random effect concerns the case, here it concerns the time period. In general terms this is a Generalised Linear Mixed Model, GLMM, (McNeil and Wendin 2007) and the likelihood can be optimised using ML techniques from which estimates of $\rho, K_0, \mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$ and the likelihood can be derived.

Given estimates of $\rho, K_0, \mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$ in period *t*+1, forecasts of individual default probabilities conditional on values of $\mathbf{w}_i, \mathbf{x}_{it}$ and $\mathbf{z}_t^{[o]}$ can be made using equation (19) (where $\mathbf{z}_{t-1}^{[o]}$ has been included instead of $\mathbf{z}_t^{[o]}$). The forecasts are functions of $Z_{t+1}^{[u]}$ since $Z_{t+1}^{[u]}$ is unknown in period t. Unconditional expected default probabilities can be found by integrating over realizations of $Z_{t+1}^{[u]}$ as in equation (20).

The predicted loss distribution can then be estimated for period *t*+1. To do this we require values of EAD_{it+1} and LGD_{it+1} as well as the predicted relative frequency of

defaults. Hamerle and Rösch (2006) assume these fixed at 1 and 100% respectively for convenience.

In period t+1 the number of defaults, n_{t+1}, is $\sum_{i=1}^{N_t} d_{it+1}$ so the relative frequency of defaults, or default rate, is $B_{t+1} = n_{t+1}/N_{t+1}$ and its distribution is found by integrating equation (21) over all realizations of $Z_t^{[u]}$ (with t+1 replacing t)

$$\int \prod_{i=1}^{N_{t+1}} [\hat{p}_{it+1}(z_{t+1}^{[u]})^{d_{it+1}}][1-\hat{p}_{it+1}(z_{t+1}^{[u]})]^{1-d_{it+1}} \phi(z_{t+1}^{[u]}) dz_{t+1}^{[u]}. \qquad (22)$$

Notice this distribution depends on both the observed macroeconomic variables and the unobserved factor, $Z^{[u]}$, because $\hat{p}_{it+1}(z_{t+1}^{[u]})$ is given by equation (19) but with t+1 replacing t. This distribution can be simulated by Monte Carlo simulation.

A weakness of Merton type models applied to consumer loans is that the assumption that a consumer will default on, say, a credit card loan, when his/her assets fall below a threshold is questionable. It may be more applicable to a mortgage loan. However one might restore the plausibility of the barrier condition by interpreting it as occurring when a borrower's 'credit worthiness' falls below a certain level.

An alternative Merton-type model was proposed by de Andrade and Thomas (2007) who assumed the 'creditworthiness' of a borrower followed a jump diffusion process of Zhou (1997) and where default occurred if a borrower's credit worthiness fell below a threshold. They simulated probabilities of default corrected for states of the economy, where the latter were assumed to follow a first order Markov chain between four states. However the condition for default is questionable and there are difficulties in the empirical application of this model. The identification of jumps is difficult and the assumption the economy is in one of only four states might be rather inaccurate.

4.3 Econometric Models

This type of model is a regression model with the right hand side of similar form to equation (3) but the dependent variable is the default *rate* in a market segment,

$\beta_{0i} = \beta_0$ and all of the coefficients in the $\boldsymbol{\beta}$ vectors and $\boldsymbol{\gamma}$ matrices are constrained to be zero except for those relating to the $\mathbf{z}_t$ vector. The $\mathbf{z}_t$ vector contains macroeconomic variables, possibly lagged. The link function could be logistic. However the analysis is typically carried out at the level of a segment of borrowers, which we assume here. Credit Portfolio View ™ is an example of this type of model (Wilson 1997a and b).

An autoregressive distributed lag function for each macroeconomic variable is parameterised to give

$$Z_{st} = \sum_{l}^{L} Z_{st-l}\gamma_l + \eta_{st}, \tag{23}$$

where Z_{st-l} is the value, for segment *s*, normally distributed, with lag *l* and η_{st} is a random value assumed *N(0,∑).*

The default rate in a market segment in period *t, B_{st},* is related to the vector of macroeconomic variables using a logit link function. Thus

$$\begin{aligned} B_{st} &= \frac{1}{1+\exp(Y_{st})} \\ Y_{st} &= \mathbf{z}_{st}^T\boldsymbol{\beta}_{\mathbf{s}} + \varepsilon_{st}, \end{aligned} \tag{24}$$

where Y_{st} is a 'credit worthiness index'

B_{st} is a default rate in segment *s* period *t* (see previous section)

$\mathbf{z}_{st}$ is a [M x 1] vector of M macroeconomic variables for segment s, in period *t*;

$\boldsymbol{\beta}_s$ is a [M x 1] vector of parameters to be estimated;

ε_{st} is a random variable assumed N(0,∑).

The $\boldsymbol{\beta}_s$ vector is estimated for each segment separately using time series data on default rates and Z_{st} values. See Hamilton (1994) for a review of time series analysis.

Variants of this model are outlined by Bucay and Rosen (2001). One variant is that Y_{st} is additionally made a function of variables that are specific to the segment.

Alternatively the entire analysis could be performed for an individual borrower whereby each *s* subscript would be replaced by an *i* subscript and B_{it} becomes the individual borrower's probability of default. A third variant is that Z_{st} in equation (24) could be replaced by a vector of principal components extracted from the macroeconomic variables. The distribution of losses can be gained by Monte Carlo simulation (Koyluoglu and Hickman: 1998) of the η_{st} and ε_{st} terms whilst preserving their covariance.

A further variation is to relate a segment's default rate to *both* observable *and* unobservable latent factors, but without assuming the Merton model. The Kalman Filter (see Harvey: 1990) may be used to estimate values of the latent factor recursively and the default rate model subsequently parameterised (Jiménez and Menciá: 2007). To explain briefly, consider an *observation* equation, where $\mathbf{B_t}$ is a vector of default rates and

$$\mathbf{B}_t = \mathbf{z}_t^T \boldsymbol{\beta}_3 + (\mathbf{z}_t^{[u]})^T \boldsymbol{\gamma}_t + \boldsymbol{\upsilon}_t , \tag{25}$$

where

$$\mathbf{z}_t^{[u]} = (\mathbf{z}_{t-1}^{[u]})^T \boldsymbol{\theta} + \boldsymbol{\omega}_t \tag{26}$$

is called a *state equation* because it represents how states of the system in *t*-1 transition into states in *t*, and $\boldsymbol{\theta}$ is a transition matrix. The terms $\boldsymbol{\upsilon}_t$ and $\boldsymbol{\omega}_t$ are vector white noise. We observe $\mathbf{B}_t$ but not $\mathbf{z}_t^{[u]}$. Under suitable assumptions, the parameters of the model can be estimated by ML (see Hamilton 1994).

Notice that in econometric models the correlation between default probabilities of borrowers is not modelled as a separate term, but is implicit in the model because borrowers in a segment are subject to the same macroeconomic variables.

4.4 Empirical Results

4.4.1 Merton type models

Whilst almost all of the applications of these methodologies have related to corporate loans (Hamerle et al: 2004, Hamerle and Rösch: 2006, Rösch: 2003, Dullman and

Trapp: 2004) there are several examples of their application to consumer loans. They estimate asset correlations and VaR values. Concerning the former Rösch and Scheule (2004) applied the Merton type model to the charge-off rate (the proportion of loans that are written off by lenders) for 100,000 borrowers from US commercial banks. Using data from 1991 to 2001 the asset correlation (ρ in equation (14)) was 0.012, 0.0098 and 0.0073 for credit card loans, real estate loans and other consumer loans respectively when macroeconomic variables were omitted. Clearly all are well below the correlations assumed in the Basel II Accord. When macroeconomic variables were included (so giving PiT correlations) the correlations were even lower.

Parameterisations of Merton-type models for consumer loans generally suggest the VaR values of predicted loss (or default rate) distributions are lower than are implied by the Basel II formula. This was found by Rosch and Scheule (op cit) and by De Andrade and Thomas (2007) who applied their jump diffusion process model to a sample of Brazilian consumer credit loans. On the other hand, Perli and Nayda (2004) . considered six market segments from two credit cards issued by Capital One. They calculated economic losses making each term in their calculation a function of macroeconomic variables. They found the predicted VaR was much lower (higher) for the higher (lower) risk segments than was the required capital under Basel II.

4.4.2 Econometric Models

Different studies have addressed different issues. Bucay and Rosen (2001) estimated econometric models for a sample of credit cards issued between 1995 and 1999. The portfolio was divided into 11 risk segments based on application score. They found, using segment specific variables as well as macroeconomic variables that the proportion of the variance in the credit worthiness index that was explained by the latter varied between 38% and 73% depending on the segment. Macroeconomic variables generally explained a greater proportion of this variance in lower risk segments. Values of the macroeconomic variables were simulated and the predicted loss distribution constructed for the portfolio. They found that the estimated VaR(99.9%) was 12.5% higher when the model included only segment specific variables rather than both these and macroeconomic variables. They also estimated a

Merton type model and found the VaR(99.9%) was the same as predicted by the CPV model, but the expected loss was lower.

An example of the application of the Kalman Filter (KF) in a credit risk model is Jiménez and Menciá (2007). Jiménez and Menciá assume a vector autoregressive (VAR) model for the growth in the number of loans in month *t* and for the increase in default frequency in month *t*. In both cases lagged endogenous variables were included as was an unobserved factor. Values for the factors were estimated using the KF and the parameters of the VARs estimated using these values. Quarterly data, 1984 to 2006, relating to all loans over €6,000 in Spain for each of ten commercial sectors plus consumer loans and mortgages were used. For each sector the growth in default rates was significantly negatively related to lagged GDP and significantly positively related to the latent factor; but real interest rates, even with three lagged terms, were not related. They then simulated the loss distribution and found that when the latent factors were included the VaR (99.9%) after three years was 5% and 2% lower, respectively, for consumer loans and mortgages than when latent factors were omitted.

Rodriguez and Trucharte (2007) follow Carey's non-parametric simulation method (Carey: 1998, 2001) to generate loss distributions for Spanish mortgages First they pool the simulated loans across all years (1990 – 2004) and compare the loss rate as a percentage of exposure at the 99^{th}, 99.5^{th} and 99.9^{th} percentiles to find that the simulated rates were higher, except at the 99.9^{th} percentile, than the rates implied by, and so covered by, the Basel II formula. When looking over an economic cycle the distribution of losses implied by Basel had a fatter tail than the simulated distribution above the 99.5^{th} percentile. Second, they take a reference portfolio, 2004, and stress values of the predictors to gain a new distribution of PD_{it}s and so of losses. They found that the loss rates, at all the percentiles, for the worst year in the data period are considerably larger than those implied by the Basel II IRB approach using average PD estimates over the cycle.

4.5 Point in Time versus Through The Cycle Ratings Systems

As Brough (2007) notes when calculating economic capital under an IRB advanced approach, 'a firm must estimate PDs by obligor grade or pool from long-run averages of one year default rates (BIRU4.6.24)' and the long run average must be calculated from default rates in a representative sample of years from throughout an economic cycle. However there are alternative ways of calculating the long run average PDs. A lender is required to classify borrowers into risk grades according to the predicted probability of default $\hat{P}D_{it}$. Two methods are possible. A TTC rating system is one where the $\hat{P}D_{it}$ used to allocate a borrower to a grade does not depend on the state of the macroeconomy because this state has been hypothetically fixed at a stressed level representing a severe recession. We denote this $PD_{it}^{[s],TTC}$. A PiT rating system is one where the $\hat{P}D_{it}$ used to grade a borrower *does* depend on the likely future state of the macroeconomy. We denote this $PD_{it}^{[us],PiT}$. To explain further (following Heitfield: 2004 and 2005) we write equation (1) (omitting interaction terms for simplicity) as

$$d_{it}^{*} = \beta_0 + \mathbf{w}_i^T \boldsymbol{\beta}_1 + \mathbf{x}_{it}^T \boldsymbol{\beta}_2 + (\mathbf{z}_t^{[o]})^T \boldsymbol{\beta}_3 + \sqrt{\rho_d}\, Z_t^{[u]} + \sqrt{1-\rho_d}\, \varepsilon_{it}, \tag{27}$$

where ρ_d denotes the correlation between PD_{it} and PD_{jt}, the $\mathbf{z}_t$ vector has been partitioned into observed and unobserved components and the sensitivity of d^* to the unobserved time varying variable is determined by the value of ρ_d. We can represent a stressed state of the economy by setting $(\mathbf{z}^{[o]})_t^T \boldsymbol{\beta} + \sqrt{\rho_d}\, z_t^{[u]} = \varsigma$ where ς is a constant. Then the PD_{it} in a TTC system used to grade a borrower can be written as

$$PD_{it}^{[s],TTC} = F\left(\frac{\beta_0 + \mathbf{w}_i^T \boldsymbol{\beta}_1 + \mathbf{x}_{it}^T \boldsymbol{\beta}_2 + \varsigma}{\sqrt{1-\rho_d}} \right), \tag{28}$$

and in a PiT system it is

$$PD_{it}^{[us],PiT} = F\left(\beta_0 + \mathbf{w}_i^T \boldsymbol{\beta}_1 + \mathbf{x}_{it}^T \boldsymbol{\beta}_2 + (\mathbf{z}_t^{[o]})^T \boldsymbol{\beta}_3\right). \tag{29}$$

So a PiT risk grade is defined as

$$\Omega^{[us],PiT} = \{i \mid \beta_0 + \mathbf{w}_i^T \boldsymbol{\beta}_1 + \mathbf{x}_{it}^T \boldsymbol{\beta}_2 + (\mathbf{z}_t^{[o]})^T \boldsymbol{\beta}_3 = \varpi_{PiT}\}, \tag{30}$$

where ϖ denotes a constant and a TTC risk grade is defined as

$$\Omega^{[s],TTC} = \left\{ i \mid \frac{\beta_0 + \mathbf{w}_i^T \boldsymbol{\beta}_1 + \mathbf{x}_{it}^T \boldsymbol{\beta}_2 + \zeta}{\sqrt{1-\rho_d}} = \varpi_{TTC} \right\}. \tag{31}$$

In each case, all members of a grade have the same corresponding PD.

Suppose that for each type of system a risk grade is defined as a range of the corresponding above probabilities. Consider borrowers that are rated using a PiT system. If the economy went into recession, $PD_{it}^{[us],PiT}$ for each borrower would increase and borrowers would be allocated to a lower grade. But the mean observed PD in any one grade would be unchanged; the grade simply has a different set of borrowers. The risk of the portfolio has increased and by equation (13) the capital requirement has risen. Now consider borrowers that are rated using a TTC system. The economic downturn does not affect $PD_{it}^{[s],TTC}$ so no borrower would change grade, but the mean observed PD in each grade increases. In practice, according to Heitfield, for corporate ratings, agencies often use a TTC system where the grade is altered in the light of the likely future states of the economy.

Heitfield (2004) uses the above models to show the expected pooled PDs for each combination of rating methodology and stressed or unstressed scenarios. He shows that if a TTC system is used then the expected stressed pooled PDs will be stable over an economic cycle as will expected unstressed pooled PDs if a PiT system is used. However pooled PDs, which are estimated in a way that makes their expected values unstable, are difficult to estimate using observed past default rates.

The FSA (2006) suggest that in practice lenders often try to transform PDs estimated by PiT models for a portfolio into long run average PDs, instead of estimating long run average default rates for individual grades, as intended in the Basel II Accord. One possible reason is a lack of long term historical data on default rates by grade.

According to the FSA (2006) the most common approach to this transformation is to use a variable scalar method.

The general variable scalar method is to predict $\hat{PD}_{it}$ over time and for each period apply an appropriate multiplier to transform PD_t into the long run PD. An example is Ingolfson and Elvarson (2007) who use a Kalman Filter technique whereby they have an observation equation relating to observed 'serious' defaults (defaults that subsequently are written off), $\mathbf{y}_t$: $\mathbf{y}_t = (\mathbf{z}_t^{[u]})^T \mathbf{A} + \boldsymbol{\upsilon}_t^T \mathbf{R}$ and a state-space equation $\mathbf{z}_t^{[u]} = (\mathbf{z}_{t-1}^{[u]})^T \boldsymbol{\theta} + \boldsymbol{\omega}_{t-1}^T \mathbf{Q}$ where $\mathbf{z}_t^{[u]}$ represents the time series pattern of the unobserved default 'cycle' and $\mathbf{v}_t$ and $\boldsymbol{\omega}_{t-1}$ are vectors of random terms. The structure of the $\boldsymbol{\theta}$ and $\mathbf{Q}$ matrices were set up assuming that the default 'cycle' i.e. state equation, has a cyclical component following a sine wave, a long term trend and a random element. The parameters of the matrices are estimated using the usual KF algorithm. The result is an extrapolative model of the time series of defaults (as a percentage of the number of loans). The predicted $\hat{PD}_t$ s of the portfolio, where the predictions are based on a PiT logistic regression model, is then multiplied by a scalar for that time period which is derived from a past relationship between the predicted default cycle and that predicted by the PiT logistic regression models. The model was fitted for loans to an Icelandic bank during 1990-2000 with a high degree of fit. A weakness of this paper is that it does not make corrections for each market segment separately as preferred by the FSA (2007). A general challenge for scalar methods is that according to the FSA (2007) they should adjust for changes in the macroeconomy only. Changes in the observed default rates due to changes in the mix of borrowers, changes in the propensity to default or changes in the acceptance policy of the lender should not be averaged away. See Loffler (2003) and Oung (2005) for applications of the KF to TTC ratings for corporate loans. Gordy and Howells (2006) discuss how regulators might adjust a PiT rating system to derive the minimum acceptable capital required over time using a smoothed AR(1) function.

4.6 Markov Chain Models

Now suppose d_t is not restriced to (0,1) but can take on a range of nominal positive integer values, each indicating a state of repayment delinquency such as the number of scheduled payments that are overdue. Let there be $C = 1, 2, \ldots V$ possible states. (In the credit risk modelling context the states could alternatively be aspects of a borrower's behaviour such as account balance, but then we would need a symbol other than d to denote these). Consider a matrix, $\mathbf{P}_t$, of transition probabilities for a borrower, *i*, between delinquency state *u* at time *t*, and delinquency state *v* at time *t*+1. A possible application is to have two of the *V* possible states as absorbing states, these being the loan is paid off and the loan has missed so many payments it is in default (Cyert et al 1962). If

$$P(d_{t_g} = v_{t_g} \mid d_{t_{g-1}} = v_{t_{g-1}}, \ldots\ldots d_1 = v_1) = P(d_t = v_t \mid d_{t-1} = v_{t-1}) \quad \forall v_t \ldots\ldots v_1, \qquad (32)$$

then $d_1 \ldots\ldots d_{t_g}$ is a first order MC. See Putterman (1994) or Stock (2005) for discussions of markov chains.

The transition matrix may be pre-multiplied by a vector of the number of accounts in each state to gain the expected distribution of accounts across all states in a future period. If the matrix is stationary the probability that an account moves from state *u* to state *v* over *t* steps is given by the *u.v* cell in the $\mathbf{P}^t$ matrix.

Notice that panel data that contains a nominal measure of repayment behaviour (i.e. delinquency) can be represented as a transition matrix. Observed values of the delinquency state are recorded in successive time periods, $t = 1 \ldots t_g$, for each borrower. Notice also that the MC represented by equation (32) is analogous to a linear model with an endogenous variable lagged by one period.

Define $p_{iuv}(t-1,t)$ as equal to $P(d_t = v_t \mid d_{t-1} = v_{t-1})$ for case *i*. If we make the transition probabilities functions of covariates, then we may write

$$p_{iuv}(t-1,t) = F\left(\beta_{0uv} + \mathbf{w}_{iuv}^T \boldsymbol{\beta}_{1uv} + \mathbf{x}_{ituv}^T \boldsymbol{\beta}_{2uv} + \mathbf{z}_{tuv}^T \boldsymbol{\beta}_{3uv} + \mathbf{x}_{ituv}^T \boldsymbol{\gamma}_{1uv} \mathbf{w}_{iuv} + \mathbf{x}_{ituv}^T \boldsymbol{\gamma}_{2uv} \mathbf{z}_{tuv} + \mathbf{w}_{iuv}^T \boldsymbol{\gamma}_{3uv} \mathbf{z}_{tuv}\right), \qquad (33)$$

where the covariates of the right hand side are as defined in equation (3) except that they may be specific to the states *u* and *v*. If we consider the simple case of *V*=2 and *v* is the default state and *u* is any other state then equation (33) is the same as equation (3). If we consider *u*=0 and *v*=1 then to perfectly define the model, when *V*=2, we require two equations, one with the lhs being $p_{i01}(t-1,t)$ and the other with the lhs being $p_{i11}(t-1,t)$, with corresponding changes to the rhs. For further discussion and estimators see Gourieroux (2000). Note that the literature below has not estimated this type of model.

Cyert et al (1962) gives one of the earliest applications of Markov chains. Cyert and Thompson (1968) estimate a different matrix for each of eight risk categories of borrower. Frydman et al (1985) test the applicability of the Mover Stayer model (MS) of Blumen et al (1962). The MS model assumes some individuals stay in their initial state e.g. up to date with payments, ('stayers'), whilst others move between states according to a stationary Markov chain ('movers'). Tests generally find that the MS model gives predicted transition matrices that are significantly closer to observed matrices than stationary Markov chains. For example Till and Hand (2001) found this for a sample of credit card holders and Frydman et al found this for revolving credit accounts.

Statistical tests of whether transition matrices are stationary and first order are given in Anderson and Goodman (1957). Both Till and Hand, using credit card accounts, and Ho et al (2004), using a sample of current accounts with borrowing facilities, find that the transition matrices are not first order; the probability of an account transitioning from one state to another depends on which of at least one of the previous states the account was in. Ho et al find their Markov chain was not stationary whilst Till and Hand did not test for this. Ho et al collapsed their ten state transition matrix into three states and rejected the hypothesis that the chain was second order rather than third order. They went on to find that the most significant segmentation out of many considered is not just into movers and stayers but into those who stay, those that move up to three times ('twitchers'), those that move four times ('shakers') and those that move five times ('movers') in a 48 month period. Till and Hand find

that if the assumption of stationarity is rejected, as statistical tests in other work suggest it should be, the time taken to reach three months overdue from 0,1,or 2 payments overdue is 124, 108 and 69 months respectively; much larger than if stationarity is assumed.

More recent applications of Markov chains have been to embed them into a Markov decision process (MDP) model to choose optimal strategies for each state so as to maximise expected profits. A good example is Trench et al (2003). In this paper each state is defined by a combination of (a) values of management control variables e.g. credit line and (b) variables representing customer behaviour. To reduce the dimensions implied by the use of two control variables and six behavioural variables cells were aggregated. Variables were defined in ways to increase the chance the transition matrix was first order. The MDP model to choose the optimal action, *a*, from a set of A_u possible actions for state of account *u* was set up as

$$V_t(u) = \underset{a \in A_u}{Max}\{NCF(u_a) + \beta \sum_{v \in U} P(v \mid u_a) V_{t+1}(v)\}, \qquad (34)$$

where $V_t(u)$ is the maximised discounted NPV of net cash flows in state *u* at time *t*, $NCF(u_a)$ is the net cash flow in state u_a when action *a* is taken, β is a discount factor, $P(v \mid u_a)$ is the transition probability giving the probability of transiting to state *v* from state u_a. Tests showed the MDP model increased NPV compared with the model currently in use by a bank. See White (1969) for dynamic programming methods.

4.7 Stochastic Intensity Models

These models have almost exclusively been applied to corporate loans and in this context contributions have been made by Jarrow and Turnbull (1995), Lando (1998) and Duffie and Singleton (1999). See also Crowder (2001) for discussions of intensity models. The crucial point is that these models can also be applied to consumer loans as well.

A large number of models fit this category. One of the most influential is that of Jarrow, Lando and Turnbull (1997) (JLT). JLT describe a matrix of transition probabilities between rating state *u* in period *t* and rating state *v* in period *t+1*, $\mathbf{P}_{uv}(t,t+1)$ over state space *C = 1…V*. Corporate ratings describe the chance the company will default on loans. Applied to consumer loans we could regard the states as delinquency states. State *V* represents default or bankruptcy which is an absorbing state. Thus

$$\mathbf{P}_{uv}(t,t+1) = \begin{pmatrix} p_{11}(t,t+1) & \ldots & p_{1V}(t,t+1) \\ p_{21}(t,t+1) & & \\ & & \\ & & \\ 0 & \ldots & 1 \end{pmatrix} \tag{35}$$

is a *V* x *V* matrix. Now move from discrete to continuous time. Consider a Poisson process which has value N_t at time *t* where *N* takes on integer values. Then the probability of a change in *N* in some very small time interval *dt* is

$$P(N_{t+dt} - N_t = 1) = \lambda dt, \tag{36}$$

where λ is the Poisson intensity parameter.

We now regard the change in *N* as a jump from one state *u* to default, *V*. The time to default can be modelled as the first time the Markov chain of *V* x *V* states reaches the default state. The evolution of the chain can be represented by its generator matrix of intensities:

$$\mathbf{\Lambda} = \begin{pmatrix} -\lambda_{11} & \lambda_{12} & \ldots & \lambda_{1V} \\ \lambda_{21} & -\lambda_{22} & & \\ & & & \\ & & & \\ 0 & \ldots & & 0 \end{pmatrix}, \tag{37}$$

where $\lambda_{uu} - \sum_{u \neq v} \lambda_{uv} = 0$ and each λ_{uv} gives the probability that the chain is in state *v* at time *t*, given it was in state *u* at time 0. Put another way, if the chain starts at time 0 in state *u*, it will stay in that state for length of time exp (λ_{uu}) and then jump to *v* in the next instant with probability $\lambda_{vv} / \lambda_{uu}$. Clearly we are most interested in the $\lambda_{uV.}$

If the intensities, λ_{uv} s in equation (37), can be made functions of (duration) time we have a non-(time) homogeneous Poisson process. Schonbucher (2000) shows that in the period between τ_1 and τ_2, the probability of one jump is

$$P(N_{\tau_2} - N_{\tau_1} = 1) = \left(\int_{\tau_1}^{\tau_2} \lambda(s) ds \right) \exp\left(- \int_{\tau_1}^{\tau_2} \lambda(s) ds \right). \tag{38}$$

If we now also make $\lambda(\tau)$ stochastic, Schonbucher shows we have a Cox process and so $P(N_{\tau_2} - N_{\tau_1} = 1)$ equals the expectation of the right hand side of equation (38).

The intensities, $\lambda_{uv}(\tau)$, can be modelled as functions of random hazard functions of covariates, as explained in section 1. Thus, $\lambda_{uv}(\tau)$ is the intensity of transitioning from state *u* to state *v* in the next instant of time, conditional on having remained in state *u* until that time. Thus we could write

$$\lambda_{uv,i}(\tau, \mathbf{x_i}(\tau)) = \underset{\Delta t \to 0}{Lim} \frac{P(\tau \leq T_i < \tau + \Delta\tau, C = v, \mathbf{x_i}(\tau) = \mathbf{x}_i^*(\tau) \mid T_i \geq \tau)}{\Delta\tau}, \tag{39}$$

where $\mathbf{x}_i^*(\tau)$ is a realisation of the state variables in vector $\mathbf{x_i}(\tau)$ and *v* is the state to which the borrower transitions from state *u*. This is equation (5) with these two conditions added. The process is described as 'doubly stochastic' since the λ_{uv} values are determined from a stochastic model (a survivor model) and are then part of a second stochastic model (a Poisson model).

Lando and Skodeberg (2002) modelled corporate loan transitions in this way. They related intensities to time varying covariates using, essentially Cox Proportional Hazards. They test to see if the firm was previously upgraded to the present class (an

$X_i(\tau)$ covariate) to find it was significant. Of course the time varying covariates could be non-case specific and so could be macroeconomic variables.

Kavvathas (2000) models stochastic intensities using competing risks. A competing risks approach is appropriate because for any initial state, if the number of time intervals is sufficient, the firm could transition to any of *1…V* states in the next period. They model downgrade, upgrade and constant grade corporate transition intensities separately. They find that high spot interest rates are associated with higher probabilities of downgrading. They also find that intensities to downgrade are positively related to advantageous credit states and negatively related to stock returns. See also Crowder (2001) for discussions of competing risks models.

5. Conclusion

Considerable progress has been made in modelling consumer credit default risk in the last decade. Whilst dynamic models in the form of Markov chain models were discussed in the literature in the 1960s and behavioural scoring models in the 1990s, there has been considerable development in the last decade in the application of techniques to predicting the changing risk of both individuals and of portfolios of loans. The data that lenders collect is of a panel structure, albeit with missing values in certain places. This offers lenders considerable opportunities to incorporate covariates that vary over time, both those specific to the borrower and those which may affect everyone, and combinations of both. At the level of the account the use of survival analysis allows lenders to predict the probability of default in the next month taking into account predicted or *ex-post* observed macroeconomic indicators. The panel data structure of lenders allows them in principle to use panel techniques to estimate, for example, the probability of a missed payment in a particular month where this may or may not be a one-off event. This can be done more efficiently using random effects models than by data pooling. Alternative techniques, such as scalar techniques, still require further methodological development since they currently do not incorporate the possibility that changes in the state of the economy may alter the risk ranking of applicants or borrowers.

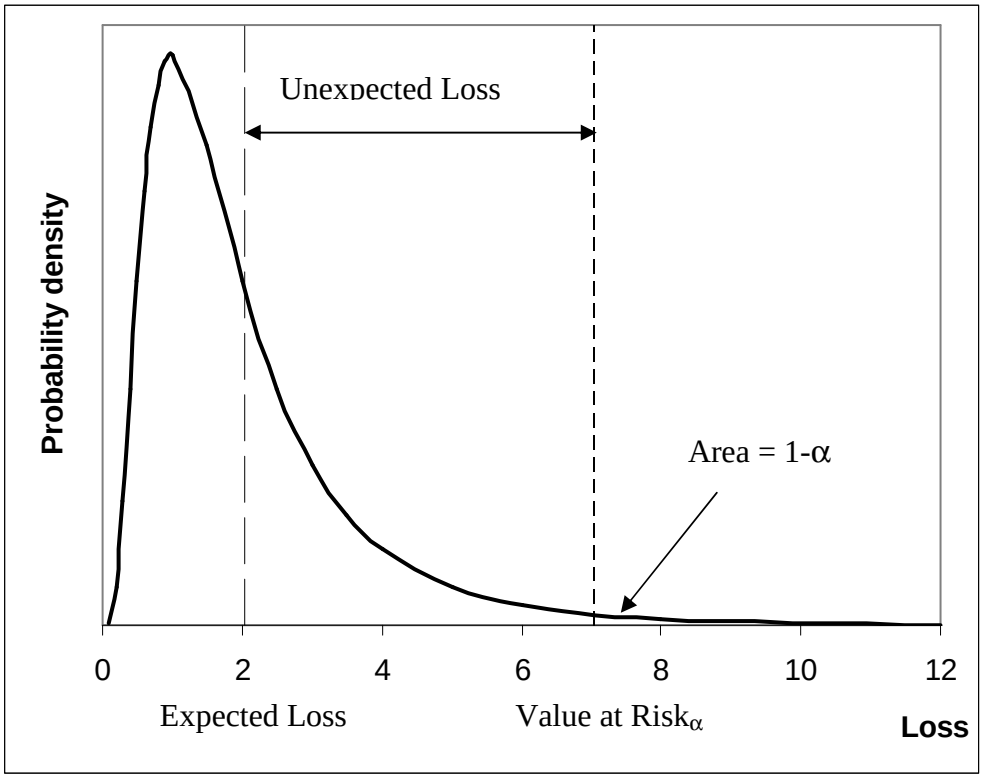
Models of the distributions of default probabilities which have been developed for application to corporate loans can also be applied to consumer loans. Corporate models which allow for inter-company default correlations in the form of unobserved factors, have considerable potential to be applied to consumer loans. They may be estimated using random (time) effects. The statistical significance of unobserved factors, that represent omitted risk covariates and asset correlations can be directly estimated. On the rare occasions when they have been applied to consumer loans we see that the asset correlations are very low and well below the correlations given in the Basel II formula. Macroeconomic variables can be incorporated into predictions of default rate distributions and when they are the estimated VaR values are likely to be more accurate. It is also possible to estimate values for latent unobservable factors in additional to the effects of macroeconomic variables using Kalman Filter techniques and on the few occasions on which it has been done it has been found that the implied VaR values are lower than when they are omitted. Again there is considerable scope for developing this work further. Pooling the data and omitting latent factors has also been tried as a way of incorporating macroeconomic variables and when this has been done it has been found that stressed loss rates are actually higher than under Basel II.

Finally, in the corporate literature stochastic intensity models have commonly replaced Merton type models and there is considerable potential for applying the former to consumer loans. Consumer default transition matrices appear not to be first or even second order. There is considerable opportunity to model the transition probabilities in terms of macroeconomic variables and to introduce macroeconomic variables into intensity models to examine the implications of different states of the economy for default distributions.

Figure 1

Loss Distribution

f



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