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Mathematical Optimization Approaches for Facility Layout Problems: The State-of-the-Art and Future Research Directions

Miguel F. Anjos[a,1], Manuel V.C. Vieira[b,2]

[a] *GERAD & Department of Mathematics and Industrial Engineering, Polytechnique Montreal, Canada*
[b] *Departamento de Matemática, Faculdade de Ciências e Tecnologia & CMA, Universidade Nova de Lisboa, Portugal*

Abstract

Facility layout problems are an important class of operations research problems that has been studied for several decades. Most variants of facility layout are NP-hard, therefore global optimal solutions are difficult or impossible to compute in reasonable time. Mathematical optimization approaches that guarantee global optimality of solutions or tight bounds on the global optimal value have nevertheless been successfully applied to several variants of facility layout. This review covers three classes of layout problems, namely row layout, unequal-areas layout, and multifloor layout. We summarize the main contributions to the area made using mathematical optimization, mostly mixed integer linear optimization and conic optimization. For each class of problems, we also briefly discuss directions that remain open for future research.

Keywords: Facilities planning and design, unequal-areas facility layout, row layout, mixed integer linear optimization, semidefinite optimization.

Email addresses: anjos@stanfordalumni.org (Miguel F. Anjos), mvcv@fct.unl.pt (Manuel V.C. Vieira)

URL: http://www.miguelanjos.com (Miguel F. Anjos)

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1. Introduction

Facility layout problems (FLPs) are a general class of operations research problems concerned with finding the optimal arrangement of a given number of nonoverlapping indivisible departments within a given facility. The objective is to minimize the total expected cost of inter-departmental flows inside the facility, where the cost incurred for each pair of departments is equal to the rectilinear distance between the centroids of the departments multiplied by their pairwise cost. This cost, generally non-negative, accounts in the aggregate for adjacency preferences as well as costs that may arise from transportation, the construction of a material-handling system, or connection wiring. The facility and the departments are rectangular, and the area of each department is specified, but if the department's dimensions can vary, then determining them is also part of the FLP.

FLPs have a variety of applications. Much of the work was motivated by the physical organization of manufacturing systems, see e.g. [71]. The FLP is particularly relevant in flexible manufacturing systems that produce an array of different parts. The layout of the production components has a significant impact on the costs and the productivity of these systems, see e.g. [39]. Other applications of FLPs include balancing hydraulic turbine runners [60], algorithm initialization in numerical analysis [26], VLSI fixed-outline floorplanning [66], and optimal data memory layout generation for digital signal processors [95].

FLPs have been extensively studied in the literature since the 1960s. Numerous variations on the basic problem described above have been considered, and different models have been proposed for each variation. Examples of such variations are: specially structured instances of the problem (e.g. layouts on rows or on loops); dynamic FLPs with time-dependencies; FLPs under uncertainty in the data; and multi-objective FLPs. We refer the reader to the books [59, 41] and survey papers [71, 91] for more information about the FLP and its variations. A growing collection of FLP benchmark instances is available online [14].

The FLP is NP-hard in general, so solving it to global optimality in reasonable time is generally difficult. Indeed the restricted version where the dimensions of the departments are all equal and fixed, and the optimization is taken over a fixed set of possible locations for the departments, is known as the quadratic assignment problem, a combinatorial optimization problem well known for its computational difficulty, see e.g. [64].

The constraints of the basic FLP can be grouped into two sets:

- *Department shape requirements* include the required area, and restrictions on the dimensions (height and width) such as bounds on the ratios height/width and width/height, called aspect ratios. These requirements generally lead to convex constraints but still pose some challenges. In particular, requiring small aspect ratios, while desirable in real-world applications, generally makes the problem harder. On the other hand, while the area constraint traditionally required a careful linearization approach, it can be modeled exactly using conic optimization, see e.g. [18].

- *Department location requirements* include the nonoverlap of departments, fitting every department within the facility, assigning certain departments to, or forbidding them from, particular locations within the facility. The main challenge here are the nonoverlap constraints that are inherently nonconvex and combinatorial.

This review is focused on FLPs with the following properties:

1. the departments have different areas
2. the facility can be one-, two-, or three-dimensional.

The different dimensions lead to the three broad classes of FLPs covered in this review, namely row FLPs (Section 2), unequal-areas FLPs (Section 3), and multifloor FLPs (Section 4).

One-dimensional facilities lead to row FLPs, and we categorize them in terms of the number of rows: single-row, double-row, or multi-row. Single-row and double-row problems commonly occur in practical applications, as we discuss in Sections 2.1 and 2.2 respectively. Multi-row problems are a natural extension of the problem to three or more rows, and are considered in Section 2.3.

Unequal-areas FLPs have two-dimensional facilities with a single floor, and we assume that the facility is rectangular and that all the departments fit inside the facility. Unlike in the case of row layouts, not only the position but also the dimensions of each department are optimized. After discussing models and approaches for the basic two-dimensional problem in Sections 3.1 to 3.4, we consider in Section 3.5 the special case of flexible bay layouts, a type of layout that resembles row FLPs but with the fundamental difference that the width of the bays can vary, depending on the total area of the departments in each bay.

Three-dimensional facilities give rise to multifloor FLPs in which departments are to be placed over two or more floors. This is the focus of Section 4. The survey in Section 4.1 shows that most of the literature proposes models for specific applications rather than for the general problem. For this reason we propose in Section 4.2 a formulation for a generic form of the problem that we hope will motivate further research into multifloor FLP.

Regarding the choice of methodologies, we limit the scope of this review to mathematical optimization-based approaches. These include exact methods, but as the problems increase in difficulty very rapidly, we also include heuristic methods that use mathematical optimization approximations and/or relaxations. While there is a rich literature on heuristic algorithms for FLPs (see e.g. [71], [91], [57]), our focus here is on mathematical optimization approaches, primarily mixed integer linear optimization (MILO), often refereed to as mixed integer programming or MIP, semidefinite optimization (SDO), also called semidefinite programming or SDP, and nonlinear optimization. Because of their importance to the success of these approaches, we also include brief discussions of symmetry breaking (Section 5) and valid inequalities (Section 6) as these are essential ingredients for solving the resulting relaxations efficiently.

We conclude with a summary of directions for future research in Section 7.

2. Row FLPs

Row FLPs share the following common problem statement: Given a set of rectangular departments each of a given length, a number of rows, and a pairwise non-negative weight for each pair of departments, determine (i) an assignment of departments to rows, and (ii) the positions of the departments in each row, so that the total of the weighted center-to-center distances is minimized. Row FLPs arise in practical contexts where the departments are to be placed in rows with a predetermined separation between the rows due to factors such as the material-handling system or the flows of people. Moreover, within each row, a minimum clearance between departments is needed to satisfy safety and operational requirements. We assume that this clearance is included in the lengths of the departments. We also assume that the rows and the departments all have the same height, that any department can be assigned to any row, and that the distances between adjacent rows are equal. Under these assumptions, solving an instance of the row FLP means resolving three questions:

1. Assign each department to exactly one row;
2. Express mathematically the center-to-center distance between departments (that may or may not be in the same row);
3. Handle possible empty space between departments.

Section 2.1 is concerned with the simplest version of row FLP, namely the single-row FLP. Section 2.2 covers the double-row FLP, and Section 2.3 extends the coverage to the general multirow FLP.

2.1. The Single-Row FLP

An instance of the Single-Row FLP (SRFLP) consists of n one-dimensional departments with given positive lengths $\ell_1, \ldots, \ell_n$ and pairwise costs c_{ij}. The problem is to find a permutation of the departments that minimizes the weighted sum of the pairwise distances. Figure 1 provides an illustration of the SRFLP in the context of placing the departments along the path of an automated guided vehicle (AGV) transporting material between the departments; in this context the objective is to minimize the distance travelled by the AGV.

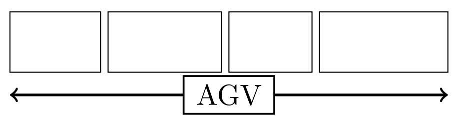

Figure 1: SRFLP along the path of an AGV

The SRFLP is the most studied of the row FLPs. Sometimes called the one-dimensional space allocation problem, it has interesting connections to well-known combinatorial optimization problems such as maximum-cut, quadratic linear ordering, and linear arrangement (see [18]).

Because there is only one row, there is no need to assign departments to rows. Moreover, $c_{ij} \geq 0$ ensures that there is no empty space between departments at optimality. Hence the remaining question is to express the center-to-center distance between departments.

A key observation, first made by Simmons [90], is that the SRFLP can be expressed as

$$\min_{\pi \in \Pi_n} \sum_{i<j} c_{ij} \left[\frac{1}{2} (\ell_i + \ell_j) + D_\pi(i,j) \right],$$

where Π_n denotes the set of all permutations of $\{1, 2, \ldots, n\}$, and $D_\pi(i,j)$ is the center-to-center distance between departments i and j under permutation π.

A first observation here that if π' denotes the permutation symmetric to π, defined by $\pi'_i = \pi_{n+1-i}, i = 1, \ldots, n$, then $D_\pi(i,j) = D_{\pi'}(i,j)$. In other words, the order of the departments in a particular layout can be reversed without changing the value of the objective function. Hence, it is possible to simplify the problem by considering only the permutations that have a particular facility, say facility 1, in the left half of the arrangement. Alternatively, we can require that a specific facility be to the left of another; this is known as the position $p-k$ method, see Section 5. This type of symmetry-breaking strategy can help reduce the computational cost of a mathematical optimization algorithm for SRFLP and for other types of layout problems, see Section 5. One aspect unique to the SDO-based approach is that it implicitly accounts for these symmetries, and thus does not require the use of additional explicit symmetry-breaking constraints, see Section 2.1.2.

A second observation is that *it is not necessary to know the position of each department*; it suffices to know for each pair of departments which departments are *between* them. Hence the key here is the concept of *betweenness.*

There is a large amount of literature on the SRFLP. For more detailed expositions on the state-of-the-art for the SRFLP, including extensions, metaheuristics, and exact approaches, we refer the reader to [57] and to the recent review paper [54] in this journal.

To give the reader a sense of the mathematical optimization approaches to the SRFLP, we present here two different ways to model betweenness. One is based on MILO and the other based on SDO.

2.1.1. MILO Model

The approach sketched here was originally proposed in [5]. Other MILO models for SRFLP include, in chronological order, [65], [43], [3], and [4].

For three distinct departments i, j, k, define the betweenness variables ζ_{ijk} as:

$$\zeta_{ijk} = \begin{cases} 1, & \text{if department } k \text{ lies between departments } i \text{ and } j, \\ 0, & \text{otherwise.} \end{cases}$$

Using these variables, the objective function of the SRFLP is expressed as:

$$\sum_{i<j} c_{ij} \left[\frac{1}{2} (\ell_i + \ell_j) + \sum_{k \neq i,j} \ell_k \zeta_{ijk} \right]$$

and this is optimized subject to the following constraints:

$$\zeta_{ijk} + \zeta_{ikj} + \zeta_{jki} = 1, \quad \text{for all } \{i, j, k\} \subseteq \{1, \dots, n\}, \tag{1}$$

$$\zeta_{ijd} + \zeta_{jkd} - \zeta_{ikd} \geq 0, \quad \text{for all } \{i, j, k, d\} \subseteq \{1, \dots, n\}, \tag{2}$$

$$\zeta_{ijd} + \zeta_{jkd} + \zeta_{ikd} \leq 2, \quad \text{for all } \{i, j, k, d\} \subseteq \{1, \dots, n\}, \tag{3}$$

$$\zeta_{ijk} \in \{0, 1\}, \quad \text{for all } \{i, j, k\} \subseteq \{1, \dots, n\}. \tag{4}$$

A polyhedral study concerning this formulation can be found in [83]. When 4 is relaxed to $0 \leq \zeta_{ijk} \leq 1$, the resulting linear optimization (LO) relaxation is weak. Thus an additional class of valid inequalities that improve the relaxation is proposed in [5].

Proposition 1. *[5] Let $\beta \leq n$ be a positive even integer and let $S \subseteq \{1,\dots,n\}$ such that $|S| = \beta$. For each $r \in S$, and for any partition (S_1, S_2) of $S \backslash \{r\}$ such that $|S_1| = \frac{1}{2}\beta$, the inequality*

$$\sum_{t<q, t \in S_1, q \in S_1} \zeta_{tqr} + \sum_{t<q, t \in S_2, q \in S_2} \zeta_{tqr} - \sum_{t \in S_1, q \in S_2} \zeta_{\min\{t,q\}, \max\{t,q\}, r} \leq 0 \quad (5)$$

is valid for the above formulation of the SRFLP.

It is straightforward to check that for $\beta = 4$, (5) is of the form (2). it is shown in [5] that the size of the LO relaxation can be reduced by projecting the feasible set into a lower-dimensional space.

2.1.2. SDO Model

To present an SDO-based relaxation, we begin by introducing $\{\pm 1\}$ binary variables as in customary in SDO (see [18]). For each pair of departments ij with $1 \leq i < j \leq n$, define

$$R_{ij} := \begin{cases} 1, & \text{if } i \text{ is to the right of } j, \\ -1, & \text{otherwise.} \end{cases}$$

In this definition, the order of the subscripts matters, and $R_{ij} = -R_{ji}$.

For an assignment of ± 1 values to the R_{ij} variables to represent a permutation, it is necessary to enforce the transitivity condition:

if i is to the right of j and j is to the right of k, then i is to the right of k.

Equivalently, if $R_{ij} = R_{jk}$ then $R_{ik} = R_{ij}$. This condition can be formulated using quadratic constraints:

$$R_{ij}R_{jk} - R_{ij}R_{ik} - R_{ik}R_{jk} = -1 \text{ for all triples } 1 \leq i < j < k \leq n. \quad (6)$$

Using the R_{ij} variables, it is straightforward to express betweenness after observing that $R_{ki}R_{kj} = -1$ if and only if facility k is between i and j. Hence the objective function can be expressed as

$$\sum_{i<j} c_{ij} \left[\frac{1}{2}(\ell_i + \ell_j) + \sum_{k \neq i,j} \ell_k \left(\frac{1 - R_{ki}R_{kj}}{2} \right) \right],$$

and the consequent formulation of SRFLP is:

$$
\begin{array}{ll}
\min & K - \sum\limits_{i<j} \frac{c_{ij}}{2} \left[\sum\limits_{k<i} \ell_k R_{ki} R_{kj} - \sum\limits_{i<k<j} \ell_k R_{ik} R_{kj} + \sum\limits_{k>j} \ell_k R_{ik} R_{jk} \right] \\
\text{s.t.} & \\
& R_{ij} R_{jk} - R_{ij} R_{ik} - R_{ik} R_{jk} = -1 \text{ for all triples } i<j<k \\
& R_{ij}^2 = 1 \text{ for all } i<j
\end{array}
\tag{7}
$$

where $K := \left(\sum\limits_{i<j} \frac{c_{ij}}{2} \right) \left(\sum\limits_{k=1}^{n} \ell_k \right)$.

Applying standard techniques from SDO, this formulation leads to the following SDO relaxation [17]:

$$
\begin{array}{ll}
\min & K - \sum\limits_{i<j} \frac{c_{ij}}{2} \left[\sum\limits_{k<i} \ell_k X_{ki,kj} - \sum\limits_{i<k<j} \ell_k X_{ik,kj} + \sum\limits_{k>j} \ell_k X_{ik,jk} \right] \\
\text{s.t.} & \\
& X_{ij,jk} - X_{ij,ik} - X_{ik,jk} = -1 \text{ for all triples } i<j<k \\
& X_{ii} = 1, \text{ for } i = 1, \ldots, n \\
& X \succeq 0, \ X \in \mathcal{S}^{\binom{n}{2}}
\end{array}
\tag{8}
$$

where $X \succeq 0$ denotes that X is symmetric positive semidefinite, and $\mathcal{S}^{\binom{n}{2}}$ is the set of symmetric matrices of dimension $\binom{n}{2}$. The interpretation of the entries of X is that $X_{p_i,p_j} = R_{p_i} R_{p_j}$ for any two pairs p_i, p_j.

Note that if every R_{ij} variable is replaced by its negative, then there is no change whatsoever to the formulation. In this way, the formulation (7) and the corresponding SDO relaxation (8) implicitly account for the symmetry of the SRFLP.

Subsequent improvements to the relaxation (8) were given in [48]. We refer the reader to that paper and to [54] for more details.

2.2. The Double-Row FLP

The Double-Row FLP (DRFLP) is an extension of the SRFLP in which departments can be placed on both sides of a central corridor. The distance between the two rows is assumed to be negligible, and thus the center-to-center distance between two departments is measured parallel to the corridor. Figure 2 illustrates the DRFLP with the corridor as the operating space for an AGV. Another application for the DRFLP is the arrangement of rooms in buildings, see e.g. [2].

To the best of our knowledge, the first reference to double-row layouts is in [42] where a nonlinear optimization model is proposed and used to

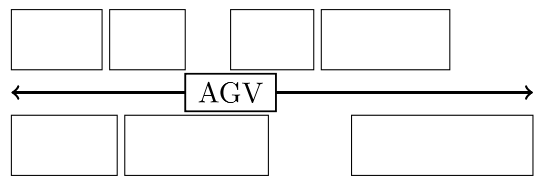

Figure 2: DRFLP with a corridor for an AGV

find locally optimal solutions. Most of the subsequent mathematical optimization approaches in the literature use either MILO (with the first model introduced in [30] and a recent new model in [10]) or SDO [47].

Unlike for the SRFLP, there is in the DRFLP a need to address all three questions for row FLPs. The assignment of departments to rows is somewhat simplified by the fact that there are only two rows: it suffices to determine which departments are placed in the first row, because the remaining departments must be in the second row. On the other hand, betweenness no longer suffices to determine center-to-center distances, and the optimal layout may involve some empty space between departments.

2.2.1. MILO models

In this section we describe two approaches that extend in different ways the MILO models proposed for the SRFLP. Both extensions involve a combination of discrete and continuous variables, where the former represent the assignment of departments to rows and the relative position of two departments, and the latter give the positions of the department centers with respect to a fixed origin. Without loss of generality the corridor is placed along the x-axis, and the origin is at the left end of the corridor.

A Model with $O(n^2)$ Binary Variables

Consider the binary vector $y = (y_{ij})_{1 \le i,j \le n}$ such that

$$y_{ij} = \begin{cases} 1, & \text{if department } i \text{ is to the left of department } j \\ & \quad \text{and both } i \text{ and } j \text{ are in the same row;} \\ 0, & \text{otherwise.} \end{cases}$$

The following inequalities are valid for all y-incidence vectors representing a partition of the n departments into two ordered subsets:

$$y_{ik} + y_{ki} + y_{jk} + y_{kj} - y_{ij} - y_{ji} \le 1, \ 1 \le i,j,k \le n, \ i < j, \ k \ne i,j \qquad (9)$$

$$y_{ik} + y_{ji} + y_{kj} - y_{ki} - y_{ij} - y_{jk} \le 1, \ 1 \le i,j,k \le n, \ i,k < j, \ k \ne i \qquad (10)$$

$$y_{ij} + y_{ik} + y_{jk} + y_{ji} + y_{ki} + y_{kj} \ge 1, \ \{i,j,k\} \subset \{1,\dots,n\}. \qquad (11)$$

Constraints (9) are transitivity constraints with respect to row assignments. They ensure that if i and k are in the same row ($y_{ik} + y_{ki} = 1$) and k and j are in the same row ($y_{jk} + y_{kj} = 1$), then $1 + 1 - (y_{ij} + y_{ji}) \le 1$, implying $y_{ij} + y_{ji} \ge 1$, i.e., i and j are in the same row.

Constraints (10) are three-cycle constraints. They forbid a solution where k is placed to the right of i, i is to the right of j, and j is to the right of k (thus forming an impossible cycle).

Constraints (11) require that at least two of i, j, k must be in the same row. It also ensures that no more than two rows are used.

We now state the MILO model of [10]:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} d_{ij} && (12)\\
\text{s.t.} \quad & d_{ij} \ge x_i - x_j,\ d_{ij} \ge x_j - x_i,\ 1 \le i < j \le n && (13)\\
& x_i + \left(\frac{\ell_i + \ell_j}{2} \right) \le x_j + L(1 - y_{ij}),\ 1 \le i, j \le n,\ i \ne j && (14)\\
& d_{ij} - \left(\frac{\ell_i + \ell_j}{2} \right) y_{ij} - \left(\frac{\ell_i + \ell_j}{2} \right) y_{ji} \ge 0,\ 1 \le i < j \le n && (15)\\
& y \in Q_n && (16)\\
& y_{ij} \in \{0, 1\},\ 1 \le i, j \le n,\ i \ne j && (17)\\
& \frac{\ell_i}{2} \le x_i \le L - \frac{\ell_i}{2},\ 1 \le i \le n && (18)
\end{aligned}
$$

where we use the continuous variables

- x_i representing the position of the center of i ($1 \le i \le n$) along the corridor,
- d_{ij} representing the distance between (the centers of) i and j ($1 \le i < j \le n$) measured parallel to the corridor.

Also $L = \sum_{i=1}^{n} \ell_i$, and

$$Q_n = \{y \in R^{n(n-1)} : (9), (10), (11), 0 \le y_{ij} \le 1,\ 1 \le i, j \le n,\ i \ne j\}.$$

The integral points of the polytope Q_n are precisely the y-incidence vectors of interest [10, 31].

Constraints (13) give the rectilinear distance between each pair of departments. Constraints (16) and (17) characterize the y-incidence vectors, and constraints (18) are bounds on the x variables. Constraints (14) ensure that departments assigned to the same row do not overlap.

Constraints (15) ensure that if department i is placed in the same row as department j, then the distance between their centers is at least $(\ell_i + \ell_j)/2$. Note that constraints (15) are redundant in the presence of constraints (13) and (14), but they may be helpful for a branching algorithm.

A Model with $O(n^3)$ Binary Variables

For this model, we define two sets of binary variables:

$$y_{ik} = \begin{cases} 1, & \text{if department } i \text{ is assigned to row } k \\ 0, & \text{otherwise.} \end{cases}$$

$$z_{kij} = \begin{cases} 1, & \text{if department } j \text{ is placed to the right of department } i \text{ in row } k \\ 0, & \text{otherwise.} \end{cases}$$

As in the previous model, we use continuous variables to determine the location of the departments. Specifically we let x_{ik} denote the absolute location of department i in row k, and set it to zero if i is not assigned to row k.

These definitions support the model proposed in [30]. This model explicitly accounts for clearances between departments. As corrected in [97], the model is:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} \left(v_{ij}^{+} + v_{ij}^{-} \right) \\
\text{s.t.} \quad & \sum_{k \in K} x_{ik} - \sum_{k \in K} x_{jk} + v_{ij}^{+} - v_{ij}^{-} = 0, \ i \in I_1, j \in I_2 && (19) \\
& x_{ik} \leq M y_{ik}, \ i = 1, \ldots, n, \ k \in K && (20) \\
& \sum_{k \in K} y_{ik} = 1, \ i = 1, \ldots, n && (21) \\
& \frac{\ell_i y_{ik} + \ell_j y_{ik}}{2} + a_{ik} z_{kji} \leq x_{ik} - x_{jk} + M(1 - z_{kji}) && (22) \\
& \qquad\qquad\qquad\qquad\qquad\qquad i \in I_1, j \in I_2, k \in K \\
& \frac{\ell_i y_{ik} + \ell_j y_{ik}}{2} + a_{ik} z_{kij} \leq x_{jk} - x_{ik} + M(1 - z_{kij}), && (23) \\
& \qquad\qquad\qquad\qquad\qquad\qquad i \in I_1, j \in I_2, k \in K \\
& z_{kij} + z_{kji} \leq \frac{1}{2}(y_{ik} + y_{jk}), \ i \in I_1, j \in I_2, k \in K && (24) \\
& z_{kij} + z_{kji} + 1 \geq y_{ik} + y_{jk}, \ i \in I_1, j \in I_2, k \in K && (25) \\
& \left.
\begin{aligned}
& x_{ik} \geq 0, \ i \in I, k \in K \\
& v_{ij}^{+}, v_{ij}^{-} \geq 0, i \in I_1, j \in I_2 \\
& y_{ik} \in \{0, 1\}, i \in I, k \in K \\
& z_{kij} \in \{0, 1\}, i \in I, j \in I \setminus \{i\}, k \in K
\end{aligned}
\right. && (26)
\end{aligned}
$$

where a_{ij} is the required clearance between departments i and j, $I_1 = \{1, \ldots, n-1\}$, $I_2 = \{i+1, \ldots, n\}$, $K = \{1, 2\}$ is the set of rows, and the constant $M = \sum_{i \in I} (\ell_i + \max_{j \in I} a_{ij})$ is analogous to L in the previous model but also includes the clearances.

Constraints (19) compute the distances between departments. Constraints (20) set $x_{ik} = 0$ when department i is not assigned to row k. Constraints (21) ensure that a department is assigned to just one row. Constraints (22) and (23) prevent departments from overlapping if they are located in the same row.

Constraints (24) and (25) ensure consistency between the variables y and z as follows: If $y_{ik} = 1$ and $y_{jk} = 1$ then (24) and (25) together ensure that exactly one of z_{kij} and z_{kji} is equal to one. Otherwise, i.e., if at least one of y_{ik} and y_{jk} is equal to zero, then (24) sets both z_{kij} and z_{kji} to zero. Constraints (25) force either z_{kij} or z_{kji} to be 1 if i and j are both in row k.

Note that the $O(n^2)$ model has significantly fewer variables than the $O(n^3)$ model, and that the meaning of the continuous variables x_{ik} differs between the two models. Finally, it is important to observe that while the $O(n^2)$ model is specific to the DRFLP, the $O(n^3)$ model can be applied directly to the MRFLP by increasing the cardinality of K.

2.2.2. SDO Model

An SDO-based approach for the MRFLP was developed in [47] and also applied to the DRFLP. This approach is presented in Section 2.3.1.

2.3. The Multirow FLP

The MRFLP is a natural extension of row layout to three or more rows. An instance of the MRFLP has a given number of rows to which the departments can be assigned, the departments all have the same height (equal to the row height), the distances between adjacent rows are equal, and departments can in general be assigned to any row.

The MRFLP has received very limited attention in the operations research literature to date. In terms of practical applications, it captures the basic structure of contexts where the departments are to be arranged in well-defined rows because the separation between the rows is predetermined. It is thus a problem that is discrete in one dimension and continuous in the other. Heuristic algorithms were proposed in [42], and a nonlinear optimization formulation was given in [36] and solved using a genetic algorithm (GA).

In terms of approaches using MILO and SDO, as noted in Section 2.2.1, the $O(n^3)$ MILO formulation of [97] for the DRFLP can be easily extended to the MRFLP (this was not specifically done in that paper). More recently, an SDO-based approach was introduced in [47], and it is this approach that we present here. To the best of our knowledge, this is the only global optimization approach for the general row FLP with more than two rows.

2.3.1. SDO Model

The SDO model presented in [47] for the MRFLP is based on the SDO formulation for the SRFLP presented in Section 2.1.2. The idea is to first assume that the assignment of departments to rows is fixed and that no spaces are allowed between departments in the same row. This restricted version of the MRFLP is called the k-Parallel Row Ordering Problem (k-PROP), see Section 2.3.2 and the references therein for more details.

Consider the k-PROP with n departments and m rows, and let the assignment of departments to rows be specified by $r: \{1, \dots, n\} \to \{1, \dots, m\}$.

Define the binary variables R_{ij} as in Section 2.1.2, and let d_{ij} represent the center-to-center distance between i and j measured parallel to the rows. If i and j are assigned to the same row, i.e., if $r(i) = r(j)$, then

$$\begin{aligned} d_{ij} = \frac{1}{2}(\ell_i + \ell_j) + \sum_{\substack{k\in N,\ k<i \\ r(k)=r(i)}} \ell_k \frac{1 - R_{ki}R_{kj}}{2} + \sum_{\substack{k\in N,\ i<k<j \\ r(k)=r(i)}} \ell_k \frac{1 + R_{ik}R_{kj}}{2} \\ + \sum_{\substack{k\in N,\ k>j \\ r(k)=r(i)}} \ell_k \frac{1 - R_{ik}R_{jk}}{2}, \end{aligned} \tag{27}$$

while if $r(i) \neq r(j)$

$$\begin{aligned} d_{ij} = R_{ij} \Bigg[\Bigg(\frac{\ell_j}{2} + \sum_{\substack{k\in N,\ k<j \\ r(k)=r(j)}} \ell_k \frac{1 + R_{kj}}{2} + \sum_{\substack{k\in N,\ k>j \\ r(k)=r(j)}} \ell_k \frac{1 - R_{jk}}{2} \Bigg) \\ - \Bigg(\frac{\ell_i}{2} + \sum_{\substack{k\in N,\ k<i \\ r(k)=r(i)}} \ell_k \frac{1 + R_{ki}}{2} + \sum_{\substack{k\in N,\ k>i \\ r(k)=r(i)}} \ell_k \frac{1 - R_{ik}}{2} \Bigg) \Bigg]. \end{aligned} \tag{28}$$

The above relations, plus the triangle inequalities relating the distances between every triplet of departments i, j, k:

$$z_{ij} + z_{ik} \geq z_{jk},\ \ z_{ij} + z_{ik} \geq z_{jk},\ \ z_{ik} + z_{jk} \geq z_{ij},\ \ 1 \leq i < j < k \leq n, \tag{29}$$

are used in [47] to extend the SDO formulation for the SRFLP to an SDO formulation for the k-PROP. For the sake of brevity here, we refer the reader to [47] for the technical details.

Once an SDO formulation of the k-PROP is obtained, the possibility of spaces is handled using the following results:

Theorem 1 ([47])**.** *If all the department lengths ℓ_i are integer, then there is always an optimal solution to the MRFLP on the half-integer grid.*

Corollary 1 ([47])**.** *If all the department lengths ℓ_i are integer, then for each instance of the MRFLP, we obtain an equivalent instance of the k-PROP by adding spacing departments of length 0.5 such that the length of each row becomes equal to $M := \sum_{i=1}^{n} \ell_i$.*

The strategy is thus to add spacing departments of length 0.5 and with all involved connectivities equal to zero, and then apply the SDO approach for k-PROP. Because the number of spacing departments needed will normally be too large for practical computation, several results are proved in Hungerländer and Anjos [47] to reduce the number of spacing departments needed.

Finally, to remove the restriction that the assignment of departments to rows is fixed, Hungerländer and Anjos [47] obtain global optimal solutions (respectively bounds) by using this approach for all possible assignments (respectively for a subset of them).

2.3.2. Special Cases of the MRFLP

The difficulty in solving the general MRFLP has motivated the study of a number of special cases with simplifying assumptions and/or specific structure that allow for more effective modeling and solution approaches.

The Equidistant MRFLP

A first such special case is the equidistant version of the MRFLP, denoted MREFLP, in which all departments have the same length. This structure makes it possible to prove many interesting results. The single-row case is known in the literature as the linear arrangement problem, see e.g. [63], [11], [12], [80], [6], and is well known to be NP-hard even if all the pairwise costs are binary [35].

The double-row case was considered in [7] where a MILO formulation based on the quadratic assignment problem is given.

For the general MREFLP, it is shown in [16] that the problem has an optimal solution on the integer grid (although the lengths of the spaces are in general continuous quantities). This implies that only spaces of unit length need to be used when modeling the MREFLP, and hence that the problem can be formulated as a purely discrete optimization problem, as is the case for the SRFLP in Section 2.1. Moreover, exact results were proved in [16] for the minimum number of spaces that must be added so as to preserve at least one optimal solution. These results lead to both MILO and SDO models for the MREFLP.

The Space-Free MRFLP

Another important special case of the MRFLP is the Space-Free MRFLP (SF-MRFLP) in which no spaces are allowed within the rows, all rows have a common left origin, and the leftmost department in each row is flush with the left end of the row. When there is only one row, the SF-MRFLP is equivalent to the SRFLP. Where there are two rows, the SF-MRFLP is also called the Space-Free DRFLP or the Corridor Allocation Problem, for which a MILO formulation was proposed in [8], and an SDO approach in [46].

A special case of the SF-MRFLP that has attracted attention is the k-PROP introduced in Section 2.3.1. Because the assignment of departments to rows is given, and no spaces are allowed within the rows, the k-PROP reduces to finding the optimal permutation of the departments within each row. An SDO approach for k-PROP was mentioned in Section 2.3.1, and another was given in [45]. When the number of rows equals two, this problem is simply called PROP, and a MILO formulation for it was given in [9].

2.4. Computational Performance of the Models

Row FLPs remain highly challenging problems. We summarize here the state-of-the-art in terms of the computational performance of the approaches preserved above.

For both the SRFLP and the single-row MREFLP, the largest instances solved to optimality had 42 departments, see [48] and [44] respectively.

For the DRFLP, the $O(n^2)$ model was used in [10] to obtain solutions of instances with up to 12 departments within one hour. The $O(n^3)$ model was also tested in [10] but was unable to solve instances with more than 10 departments within three hours. The corrected model of [97] was used in [76] for asymmetric flows. The constraints are (20)–(26), and the objective function is

$$\sum_{i \in I_1} \sum_{j \in I_2} (c_{ij} + c_{ji}) \left(v_{ij}^+ + v_{ij}^- \right).$$

The conclusion of the computational tests is that with a time limit of 10 minutes, most of the heuristic algorithms perform better than CPLEX on instances with more than 20 departments.

Finally for the MRFLP, tight global bounds were computed in [47] for instances with up to 12 departments. The authors adapted an approach originally proposed in [34] for the max-cut problem and several ordering problems. The SDO-based approach was applied to instances with up to 5 rows and up to 8 departments. The results show that the SDO approach is most effective for 4 or 5 rows. There may be an intuitive explanation for this: as an extreme example, note that it is easier to partition 5 departments into 5 rows than into 2 rows. This is in part because the model does not take into account the distance between rows, so assigning department 1 to row 1 is exactly the same as assigning it to row 4. Accounting for the distances between rows may change the nature of the results, but has not yet been done to the best of our knowledge.

3. Unequal-Areas FLP

The Unequal-Areas FLP (UA-FLP) is concerned with finding the optimal arrangement of a given number of nonoverlapping indivisible departments with varying areas so as to minimize the total expected cost of flows inside the facility. Unlike in the row FLPs, the dimensions of each department are optimized (subject to the area requirement).

The UA-FLP, sometimes called the single-floor FLP, has received much attention in the literature. It was first stated in [23], and one of the first MILO formulations was proposed in [74] using binary variables to prevent overlap.

We begin with an exact formulation of the UA-FLP in Section 3.1. This allows us to establish notation, and more importantly to explicitly show where the main difficulties are for solving UA-FLP. Exact MILO models are covered in Section 2.2.1. This includes sequence-pair formulations in Section 3.2.1, one of which solved instances with up to 11 departments to global optimality, the largest such results to date [72].

Most of the approaches reviewed here are two-stage frameworks, where the first stage determines the relative location of the departments, and the second stage obtains a final layout via a mathematical optimization model. Two-stage approaches are mathematical optimization-based heuristics that are not guaranteed to find the global optimal layout but they seem to be the most promising for handling large-scale instances of UA-FLP. The main differences between the approaches are in the first-stage algorithms. We present in Section 3.3 approaches that are entirely based on nonlinear optimization models, one of which was recently shown to be able to compute layouts for instances with up to 100 departments in less than 15 min of computation time [22]. Other two-stage approaches are summarized in Section 3.4.

A MILO formulation for the important special case of flexible bay UA-FLP is discussed in Section 3.5

A number of heuristics for the UA-FLP make use of a slicing-tree structure. This is a binary tree that represents the floor plan after applying a recursive partitioning process. Each node of the tree contains either a department or a cut operator, thus each slicing tree corresponds to a particular layout. This strategy was first used in [79] in the context of VLSI design and later extended to the UA-FLP in [93]. It was also used in [85, 33, 84, 55, 29].

3.1. An Exact Formulation of the UA-FLP

We begin by presenting an exact formulation that uses only continuous variables. The reasons for doing so are two-fold: we establish some notation that will be common for the remainder of this section, and we explicit point out where the difficulties lie in solving UA-FLP, thus motivating the solution approaches subsequently presented.

We assume that we are given the height and width of the facility as h_F and w_F respectively, and that for each department i we have lower and upper bounds $w_i^{\min}$ and $w_i^{\max}$ on its width, and $h_i^{\min}$ and $h_i^{\max}$ on its height. We also assume that β_i, an upper bound on the aspect ratio of department i, is given for each department i. It is necessary that $\beta_i \geq 1$, and the closer β_i is to unity, the closer the shape of department i will be to a square.

With this notation, the UA-FLP can be formulated as follows (see [94]):

$$\min_{x_i,y_i,h_i,w_i} \quad \sum_{1\leq i<j\leq n} c_{ij}(|x_i - x_j| + |y_i - y_j|) \tag{30}$$

$$\text{s.t.} \quad w_i^{\min} \leq w_i \leq w_i^{\max}, \text{ for } i = 1, \dots, n \tag{31}$$

$$h_i^{\min} \leq h_i \leq h_i^{\max}, \text{ for } i = 1, \dots, n \tag{32}$$

$$w_i h_i = A_i, \quad \text{for } i = 1, \dots, n \tag{33}$$

$$\max\left\{\frac{w_i}{h_i}, \frac{h_i}{w_i}\right\} \leq \beta_i, \text{ for } i = 1, \dots, n \tag{34}$$

$$x_i + \frac{1}{2}w_i \leq \frac{1}{2}w_F \quad \text{and} \quad \frac{1}{2}w_i - x_i \leq \frac{1}{2}w_F, \text{ for } i = 1, \dots, n \tag{35}$$

$$y_i + \frac{1}{2}h_i \leq \frac{1}{2}h_F \quad \text{and} \quad \frac{1}{2}h_i - y_i \leq \frac{1}{2}h_F, \text{ for } i = 1, \dots, n \tag{36}$$

$$|x_i - x_j| \geq \frac{1}{2}(w_i + w_j) \quad \text{or} \quad |y_i - y_j| \geq \frac{1}{2}(h_i + h_j), \quad \text{for all } 1 \leq i < j \leq n. \tag{37}$$

The first four sets of constraints enforce the shape requirements. Constraints (31) and (32) enforce the bounds on the width and height of each department. Constraints (33) enforce the area requirement for each department. Note that these constraints can be relaxed to $w_i h_i \geq A_i$. This relaxed form has the advantage of being convex, and in fact it can be formulated as a conic constraint (see Section 3.2). Because the optimization will push this relaxed form towards equality, in general $w_i h_i$ will equal A_i at optimality. Moreover Theorem 3.1 in [92] states that if $\sum_{i=1}^n A_i = h_F w_F$ then the constraints 33 must hold at every feasible solution. Constraints (34) enforce the maximum aspect ratio; it is straightforward to write them as two linear inequality constraints.

The last two sets of constraints enforce the location requirements. Constraints (35)–(36) ensure that the departments are inside the facility. Finally, constraints (37) prevent overlapping; these constraints are disjunctive and nonconvex, and are the hardest ones to handle. If the relative position of each pair of departments is known, then the constraints (37) can be written as linear inequalities, and the formulation becomes a (convex) conic optimization problem that is straightforward to solve. This observation motivates the two-stage philosophy in several of the approaches in the literature; we present the most prominent in Sections 3.3 and 3.4.

Note that this formulation locates the center of the facility at the origin, while some of the models below locate the origin at the bottom left-hand corner of the facility. This difference is otherwise of no consequence.

3.2. MILO Models

We begin with the MILO model introduced by [73] and enhanced in [88]. Define the binary variables

$$z_{ij}^h = \begin{cases} 1 & \text{if } i \text{ must precede } j \text{ horizontally,} \\ 0 & \text{otherwise,} \end{cases}$$

$$z_{ij}^v = \begin{cases} 1 & \text{if } i \text{ must precede } j \text{ vertically,} \\ 0 & \text{otherwise.} \end{cases}$$

The MILO formulation is as follows:

$$\min \quad \sum_{1 \le i < j \le n} c_{ij}(u_{ij} + v_{ij}) \tag{38}$$

$$\text{s.t.} \quad \begin{array}{l} u_{ij} \ge x_i - x_j \text{ and } u_{ij} \ge x_j - x_i,\ 1 \le i < j \le n \\ v_{ij} \ge y_i - y_j \text{ and } v_{ij} \ge y_j - y_i,\ 1 \le i < j \le n \end{array} \tag{39}$$

$$\begin{array}{l} \frac{1}{2}w_i \le x_i \le w_F - \frac{1}{2}w_i,\ i = 1, \ldots, n \\ \frac{1}{2}h_i \le y_i \le h_F - \frac{1}{2}h_i,\ i = 1, \ldots, n \end{array} \tag{40}$$

$$\begin{array}{l} w_i^{\min} \le w_i \le w_i^{\max},\ i = 1, \ldots, n \\ h_i^{\min} \le h_i \le h_i^{\max},\ i = 1, \ldots, n \end{array} \tag{41}$$

$$\begin{array}{l} a_i w_i + 4\left(w_i^{\min} + \frac{\lambda}{\Delta - 1}(w_i^{\max} - w_i^{\min})\right)^2 h_i \ge \\ \qquad 2a_i\left(w_i^{\min} + \frac{\lambda}{\Delta - 1}(w_i^{\max} - w_i^{\min})\right),\ \lambda = 0, 1, \ldots, \Delta - 1 \end{array} \tag{42}$$

$$z_{ij}^h + z_{ji}^h + z_{ij}^v + z_{ji}^v = 1,\ 1 \le i < j \le n \tag{43}$$

$$\begin{array}{l} x_i + \frac{1}{2}w_i \le x_j - \frac{1}{2}w_j + w_F(1 - z_{ij}^h),\ i \ne j \\ y_i + \frac{1}{2}h_i \le y_j - \frac{1}{2}h_j + h_F(1 - z_{ij}^v),\ i \ne j \end{array} \tag{44}$$

$$z_{ij}^h,\ z_{ij}^v \in \{0, 1\},\ i, j \in N. \tag{45}$$

Constraints (39) provide a linearization of the objective function (30) above. Constraints (40) ensure that each department is within the facility; they differ from (35)–(36) because this formulation places the origin at the bottom left-hand corner of the facility. Constraints (41) are lower and upper bounds for the widths and heights of the departments.

Constraints (42) are the polyhedral outer approximation on Δ points of (33). This approximation was introduced in [88] and also used in [72] and [62] (see Section 3.2.1 below). This approximation is effective in practice but less efficient that using the aforementioned convex conic relaxation that is supported by most current MILO solvers. This is because $w_i h_i \ge A_i$ is equivalent to a second-order cone constraint:

$$\begin{bmatrix} h_i & \sqrt{A_i} \\ \sqrt{A_i} & w_i \end{bmatrix} \succeq 0 \Leftrightarrow w_i + h_i \ge \left\| \begin{pmatrix} w_i - h_i \\ 2\sqrt{A_i} \end{pmatrix} \right\|_2 . \tag{46}$$

Finally, constraints (43)–(45) use the relative-location variables z_{ij}^h and

z_{ij}^v to prevent overlapping: depending on which of the variables $z_{ij}^h, z_{ij}^v, z_{ji}^h, z_{ji}^v$ is set to 1, i is to the left of, to the right of, below, or above j.

An alternative MILO representation of the relative positions of departments is given in the next section.

3.2.1. Sequence-Pair Formulations

Sequence-pair approaches determine the relative positions of the departments using the so-called *sequence-pair representation*, and combine this representation with a MILO model similar to the one above to obtain the optimal layout. The sequence-pair representation was first used for VLSI design in [75], and for the FLP in [72] and [62].

A sequence-pair is a pair of sequences of departments, denoted Γ_+ and Γ_-, that together encode the relative location of the departments. The following theorem indicates how to translate a sequence-pair into a layout.

Theorem 2 ([72])**.** *Given a sequence-pair* (Γ_+, Γ_-) *and two departments* i *and* j *in* (Γ_+, Γ_-)*,* i *and* j *satisfy the following horizontal/vertical relationship in the FLP:*

- *if* j *succeeds* i *in both* Γ_+ *and* Γ_-*, then* j *is to the right of* i*;*
- *if* j *precedes* i *in both* Γ_+ *and* Γ_-*, then* j *is to the left of* i*;*
- *if* j *precedes* i *in* Γ_+ *and succeeds* i *in* Γ_-*, then* j *is above* i*;*
- *if* j *succeeds* i *in* Γ_+ *and precedes* i *in* Γ_-*, then* j *is below* i*.*

The sequence-pair structure can be incorporated in a MILO model as follows. Given a sequence-pair (Γ_+, Γ_-) and departments i and j, define the binary variables:

$$z_{ij}^+ = \begin{cases} 1 & \text{if } i \text{ precedes } j \text{ in } \Gamma_+, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_{ij}^- = \begin{cases} 1 & \text{if } i \text{ precedes } j \text{ in } \Gamma_-, \\ 0 & \text{otherwise.} \end{cases}$$

These definitions lead to Theorem 3:

Theorem 3 ([72])**.** *For any two departments* i *and* j*, the following hold:*

- *if* $z_{ij}^+ = 1$ *and* $z_{ij}^- = 1$*, then* i *precedes* j *horizontally;*
- *if* $z_{ij}^+ = 0$ *and* $z_{ij}^- = 0$*, then* j *precedes* i *horizontally;*
- *if* $z_{ij}^+ = 0$ *and* $z_{ij}^- = 1$*, then* i *precedes* j *vertically;*
- *if* $z_{ij}^+ = 1$ *and* $z_{ij}^- = 0$*, then* j *precedes* i *vertically.*

A different MILO model (see [72]) for the FLP can now be formulated:

$$\begin{aligned}
\min \quad & \sum_{1\le i<j\le n} c_{ij}(u_{ij}+v_{ij}) \\
\text{s.t.} \quad & (39)\text{–}(42) \\
& \left.\begin{aligned} & z^+_{ij}+z^+_{ji}=1,\ 1\le i<j\le n \\ & z^-_{ij}+z^-_{ji}=1,\ 1\le i<j\le n \end{aligned}\right. && (47) \\
& \left.\begin{aligned} & z^+_{ik}+z^+_{kj}-z^+_{ij}\le 1,\ 1\le i<j\le n \\ & z^-_{ik}+z^-_{kj}-z^-_{ij}\le 1,\ 1\le i<j\le n \end{aligned}\right. && (48) \\
& \left.\begin{aligned} & x_i+\frac{1}{2}w_i\le x_j-\frac{1}{2}w_j+w_F(2-z^+_{ij}-z^-_{ij}),\ i,j\in N,\ i\neq j \\ & y_i+\frac{1}{2}h_i\le y_j-\frac{1}{2}h_j+h_F(1+z^+_{ij}-z^-_{ij}),\ i,j\in N,\ i\neq j \end{aligned}\right. && (49) \\
& z^+_{ij},\ z^-_{ij}\in\{0,1\},\ i,j\in N. && (50)
\end{aligned}$$

Constraints (47) ensure that every department appears exactly once in each sequence, and constraints (48) are transitivity constraints for the two sequences. Together these constraints ensure that the binary variables represent valid sequences. Constraints (49) express nonoverlapping in terms of the sequence-pair variables.

Using the MILO model above with additional valid inequalities, including $p-k$ symmetry-breaking constraints (see Section 5), instances of UA-FLP with up to 11 departments were solved to global optimality in [72]. The computational time reached almost 17 hours for the 11-department instance.

Castillo and Westerlund [27] proposed a MILO model that satisfies the area requirements within a given accuracy ε using cutting planes. We omit the details for this because the area constraints can be handled more effectively using conic optimization, as mentioned above. We point out however that Castillo and Westerlund [27] used the following alternative formulation of nonoverlap that is essentially the sequence-pair representation:

$$\frac{1}{2}(w_i+w_j)-(x_i-x_j)\le w_F(X_{ij}+Y_{ij}),\ 1\le i<j\le n \tag{51}$$

$$\frac{1}{2}(w_i+w_j)-(x_j-x_i)\le w_F(1+X_{ij}-Y_{ij}),\ 1\le i<j\le n \tag{52}$$

$$\frac{1}{2}(h_i+h_j)-(y_i-y_j)\le h_F(1-X_{ij}+Y_{ij}),\ 1\le i<j\le n \tag{53}$$

$$\frac{1}{2}(h_i+h_j)-(y_j-y_i)\le h_F(2-X_{ij}-Y_{ij}),\ 1\le i<j\le n \tag{54}$$

$$X_{ij},Y_{ij}\in\{0,1\},\ 1\le i\le n, \tag{55}$$

where the variables X_{ij} and Y_{ij} are a binary codification of the relative position of departments. A GA was implemented in [62] where each sequence-pair is a chromosome; the sequence-pair gives the relative position of the departments (first stage), and then a LO model is solved to find the best layout (second stage). They achieved the best results up to then for instances with up to 35 departments; the computational time was 26 hours for the 35-department instance.

A branch-and-bound algorithm that uses the sequence-pair representation was presented in [96]. A minimum-cost network flow problem is solved to obtain a feasible layout from the sequence-pair representation of the relative position layout. However, this is only valid for a restricted version of the UA-FLP in which the department widths and heights are fixed, and the facility has no limitations. Such an UA-FLP with just the nonoverlapping constraints can be transformed into a network flow problem. The advantage is that network flow algorithms can be several orders of magnitude faster than general LO algorithms.

It was observed in [72] and [62] that the MILO model with constraints (43) and (45) has $2^{n(n-1)}$ possible combinations of the binary variables, while the sequence-pair-based MILO formulation has $(n!)^2$ sequences generating the same set of relative-position combinations. The authors claim that this difference is key to the effectiveness of the sequence-pair approach. The difference is indeed significant: Using Stirling's approximation, we have that $(n!)^2 = \theta(e^{2(n\ln(n)-n)})$ while $2^{n(n-1)} = e^{0.693(n^2-n)}$. However, a comparison of the two formulations shows that another important difference is the presence of the transitivity constraints (48) in the sequence-pair model. It is not entirely clear to what extent each of these differences contributes to the efficiency of the sequence-pair approach. It would be interesting to carry out a computational study to clarify this question.

3.3. Two-Stage Approaches Using Nonlinear Optimization

A two-stage approach based on the attractor-repeller (AR) technique for VLSI floorplanning was introduced in [19]. The first stage uses the AR technique to establish the relative positions of the departments, and the second stage finds a feasible layout satisfying the relative positions specified by the solution to the first stage. The objective of this approach is not to achieve global optimality but rather to efficiently compute competitive solutions to large-scale instances of UA-FLP.

The AR model approximates each department by a circle with radius r_i proportional to the square root of A_i. The model places the circles inside the facility while allowing some overlapping. The amount of overlapping is controlled via a so-called *target distance*: given $\alpha > 0$, the target distance t_{ij} for circles i and j is set as $t_{ij} = \alpha(r_i + r_j)^2$. The AR model is:

$$\begin{aligned}
\min_{x,y} \quad & \sum_{1\le i<j\le n} c_{ij} D_{ij} + f\left(\frac{D_{ij}}{t_{ij}}\right) \\
\text{s.t.} \quad & x_i + r_i \le \frac{1}{2} w_F,\ i = 1,\dots,n \\
& x_i - r_i \ge -\frac{1}{2} w_F,\ i = 1,\dots,n \\
& y_i + r_i \le \frac{1}{2} h_F,\ i = 1,\dots,n \\
& y_i - r_i \ge \frac{1}{2} h_F,\ i = 1,\dots,n,
\end{aligned}$$

where (x_i, y_i) is the center of circle i, $D_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2$ and $f(z) = \frac{1}{z} - 1$ is a penalty function. The constraints keep the circles inside the facility whose center is at the origin. The objective function is a trade-off between the attractor term $c_{ij}D_{ij}$ and the repeller term $f\left(\frac{D_{ij}}{t_{ij}}\right)$.

While the constraints are linear, the objective function is nonlinear and nonconvex. It was convexified in [19] by replacing the term $c_{ij}D_{ij} + f\left(\frac{D_{ij}}{t_{ij}}\right)$ with the following piecewise function:

$$f_{ij}(x_i, x_j, y_i, y_j) = \begin{cases} c_{ij} z + \frac{t_{ij}}{z} - 1, & z \ge \sqrt{\frac{t_{ij}}{c_{ij}}} \\ 2\sqrt{c_{ij} t_{ij}} - 1, & 0 \le z < \sqrt{\frac{t_{ij}}{c_{ij}}} \end{cases} \tag{56}$$

where $z = (x_i - x_j)^2 + (y_i - y_j)^2$, and it is assumed that $c_{ij} > 0$. Note that the second branch of f_{ij} is constant, and that by construction, f_{ij} attains its minimum whenever the positions of i and j satisfy $D_{ij} \le \sqrt{t_{ij}/c_{ij}}$. This includes the case where $D_{ij} = 0$, i.e., the two circles completely overlap. Of course, such a placement is undesirable. The ideal arrangement of the circles has $D_{ij} \approx \sqrt{t_{ij}/c_{ij}}$, i.e., close to the boundary of the flat portion of f_{ij}. At these points, the minimum of f_{ij} is attained at the same time as the overlap is minimized. This motivates the introduction in [20] of a *generalized target distance*:

$$T_{ij} = \sqrt{\frac{t_{ij}}{c_{ij} + \epsilon}},\ 1 \le i, j \le n, \tag{57}$$

where $\epsilon > 0$ is chosen sufficiently small so that $T_{ij} \approx \sqrt{t_{ij}/c_{ij}}$. This modification also removes the need for the assumption that $c_{ij} > 0$.

In practice, attaining a solution with $D_{ij} \approx T_{ij}$ is not easy. The approach in [20] sacrifices convexity and proposes a modified AR model with the objective:

$$\min \sum_{1 \le i < j \le n} f_{ij}(x_i, x_j, y_i, y_j) - K \log \left\{ \frac{D_{ij}}{T_{ij}} \right\}, \tag{58}$$

where

$$f_{ij}(x_i, x_j, y_i, y_j) = \begin{cases} c_{ij} z + \frac{t_{ij}}{z} - 1, & z \ge T_{ij} \\ 2\sqrt{c_{ij} t_{ij}} - 1, & 0 \le z < T_{ij}, \end{cases}$$

and the logarithmic term steers the optimization away from solutions with $D_{ij} \approx 0$. Indeed the minima of this (nonconvex) function satisfy $D_{ij} \approx T_{ij}$. This can be viewed as a compromise in the sense that convexity is lost, but computational efficiency is gained because a suitable choice of starting point and nonlinear optimization solver makes it possible to compute a solution close to these known minima.

In the second stage, the nonoverlapping constraints (37) are formulated as *complementarity constraints*. For each pair i, j, we introduce new variables X_{ij} and Y_{ij} satisfying

$$X_{ij} \ge \frac{1}{2}(w_i + w_j) - |x_i - x_j|, \;\; X_{ij} \ge 0,$$

$$Y_{ij} \ge \frac{1}{2}(h_i + h_j) - |y_i - y_j|, \;\; Y_{ij} \ge 0,$$

$$X_{ij} Y_{ij} = 0$$

This last constraint enforces nonoverlap by requiring that at least one of X_{ij} and Y_{ij} equal zero. Using the coordinates of the centers of the circles in the optimal solution of the modified AR model to initialize the nonlinear optimization solver, this approach improved on the then-best-known solutions for large instances, in particular for the Armour-Buffa 20-department instance.

There are some challenges with this approach so far. First a nonconvex model with the repeller function $\frac{1}{z} - 1$ was proposed in [19]; this model was then modified in [20] to achieve convexity, but then the addition of a new penalty term resulted again in a loss of convexity, though in a more controlled manner. Moreover, the optimization problem with complementarity constraints is difficult to solve for large-scale instances.

This motivated the significant improvements to this approach carried out in [50]. For the first stage, $f_{ij}(x_i, x_j, y_i, y_j)$ is replaced by a more complicated expression that also integrates information about the aspect ratio constraints. We refer the reader to [50] for details on the first stage, and instead present below the recent further improvements in [22].

The more significant contributions in [50] are their improved second stage, and the linking of the two stages. They introduce the following convex second-stage model that can be solved efficiently:

$$\min_{(x_i,y_i),w_i,h_i} \quad \sum_{1\le i<j\le n} (u_{ij}+v_{ij}) \tag{59}$$

$$\text{s.t.} \quad u_{ij} \ge x_i - x_j, \text{ for } 1 \le i < j \le n \tag{60}$$

$$u_{ij} \ge x_j - x_i, \text{ for } 1 \le i < j \le n \tag{61}$$

$$v_{ij} \ge y_i - y_j, \text{ for } 1 \le i < j \le n \tag{62}$$

$$v_{ij} \ge y_j - y_i, \text{ for } 1 \le i < j \le n \tag{63}$$

$$w_i^{\min} \le w_i \le w_i^{\max}, \text{ for } 1 \le i < j \le n \tag{64}$$

$$h_i^{\min} \le h_i \le h_i^{\max}, \text{ for } 1 \le i < j \le n \tag{65}$$

$$w_i h_i \ge A_i, \text{ for } 1 \le i < j \le n \tag{66}$$

$$\beta_i w_i - h_i \ge 0, \text{ for } 1 \le i < j \le n \tag{67}$$

$$\beta_i h_i - w_i \ge 0, \text{ for } 1 \le i < j \le n, \tag{68}$$

plus appropriately chosen linear inequality constraints to ensure nonoverlap.

These nonoverlap constraints are obtained as follows. Consider the coordinates of the centers of the circles in the optimal solution to the first stage as a set of points on the plane, and compute their Delaunay triangulation. One of the properties of this triangulation is that it maximizes the minimum angle over all the angles of the triangles; in practice this means that thin triangles are less likely. The edges of the Delaunay triangulation are taken to represent the relative positions of the departments, and these positions are then enforced by the appropriate linear constraints. For example, if the centers of i and j are connected in the triangulation and j is to the right of i, then the constraint $x_j - x_i \ge \frac{1}{2}(w_i + w_j)$ is added to the model. The result is a second stage model that is a conic optimization problem and can be solved efficiently.

The overall approach in [50] provided further improved layouts for the classical Armour-Buffa instance, and computed high-quality layouts for several 30-department instances in 5 minutes or less of computation time.

Most recently, [22] further developed the AR concept. As the second stage of [50] is highly effective, the novelty in [22] is the formulation of the first stage. Specifically they propose a more precise formulation that models the departments as rectangles instead of approximating them by circles. The aspect ratio constraints can therefore be exactly enforced at the first stage, instead of being approximated as in [50]. They still forego convexity and use the simple objective function

$$c_{ij}D_{ij}^2 + K\frac{\theta_{ij}^2}{D_{ij}^2} - 1,$$

where $\theta_{ij}^2 = \frac{1}{4}\left((w_i + w_j)^2 + (h_i + h_j)^2\right)$. Note that $D_{ij}/\theta_{ij} \approx 1$ indicates that some of the borders of the rectangles are close, regardless of whether the rectangles are overlapping (by a small amount) or not.

The resulting first-stage model is:

$$\begin{aligned}
\min_{x_i,y_i,h_i,w_i} \quad & \sum_{1\le i<j\le n} \left(c_{ij}D_{ij}^2 + K\frac{\theta_{ij}^2}{D_{ij}^2} - 1 \right) \\
\text{s.t.} \quad & x_i + \frac{1}{2}w_i \le \frac{1}{2}w_F \text{ and } \frac{1}{2}w_i - x_i \le \frac{1}{2}w_F, \quad \text{for } i = 1, \ldots, n, \\
& y_i + \frac{1}{2}h_i \le \frac{1}{2}h_F \text{ and } \frac{1}{2}h_i - y_i \le \frac{1}{2}h_F, \quad \text{for } i = 1, \ldots, n, \\
& w_i h_i \ge A_i, \quad \text{for } i = 1, \ldots, n, \\
& \beta w_i - h_i \ge 0, \quad \text{for } i = 1, \ldots, n, \\
& \beta h_i - w_i \ge 0, \quad \text{for } i = 1, \ldots, n, \\
& w_i^{\min} \le w_i \le w_i^{\max}, \quad \text{for } i = 1, \ldots, n, \\
& h_i^{\min} \le h_i \le h_i^{\max}, \quad \text{for } i = 1, \ldots, n.
\end{aligned}$$

where $K = \alpha \sum_{1\le i<j\le n} c_{ij}$, and $0 < \alpha \le 1$. By solving for different choices of α (and hence of K), the authors of [22] improved on the best solutions by earlier techniques. Furthermore, they computed layouts for instances with up to 100 departments in less than 15 min of computation time. This is the only approach entirely based on mathematical optimization models that has been able to reach such large-scale instances of UA-FLP.

3.4. Other Two-Stage Approaches

A heuristic based on a graph-pair representation and a simulated annealing technique was proposed in [25]. One of the graphs represents the horizontal separation of the departments, and the other represents the vertical separation. Their results are generally good, but the authors make changes in the areas of the facilities or departments. These modifications are reasonable from a practical point of view, but they make it difficult to compare with other techniques.

A LO-based GA approach, which differs from [62] in the chromosome coding, was introduced in [58]. The idea is that the GA searches for the relative locations of the departments, and the LO model determines their exact locations and shapes. In particular, a new *location/shape representation* is proposed to encode the relative locations. Specifically, the relative location of department i is represented as (x_i, y_i, α_i), where $\alpha_i = h_i/w_i$. For each (x_i, y_i, α_i), define two straight lines, one passing through (x_i, y_i) and the upper right corner $(x_i + w_i/2, y_i + h_i/2)$, and the other passing through (x_i, y_i) and the upper left corner $(x_i - w_i/2, y_i + h_i/2)$. These lines split the facility into four regions with reference to department i, so every other department is above or below or left or right of i. Like the sequence-pair representation, the location/shape representation always generates a consistent assignment of the binary decision variables. Note that while the MILO-model does not contain the transitivity constraints for the integer variables, this encoding (based on continuous variables) encapsulates transitivity. The results in [58] show that this approach outperforms previous techniques: the cost function is reduced and the computational time is lower.

A similarly structured approach was proposed in [37] using a random-key GA in the first stage and a LO model in the second stage. The authors report results on several instances of UA-FLP from the literature, and find slightly better solutions in a considerably shorter computational time, in comparison with the other GA approaches. They also applied their approach to larger instances with up to 125 departments, but without restrictions on the dimensions of the facility. The computational time is reduced because they do not solve all the LO problems originating from the relative-position solutions: they only solve the problems that provably yield a feasible solution with a cost not exceeding 40% of that of the previous best solution.

3.5. Flexible Bay Structure

A flexible bay structure is a continuous layout where the departments are located in parallel bays with flexible widths. This special case of the UA-FLP arises in manufacturing facilities [69]. The bay structure is similar to the row structure in row FLPs, but a fundamental difference is that the width of each bay depends on the total area of the departments in that bay, whereas in row FLPs, the heights of the rows and of the departments are equal and fixed. The bays have straight aisles on both sides, and departments are not allowed to span multiple bays. This structure restricts the set of feasible solutions, but it has advantages in practice: the bay boundaries form the basis of an aisle structure that facilitates the transfer of the layout solution to an actual facility design.

A MILO formulation for this problem was proposed in [56]. The continuous variables x_i, y_i represent the location of department i, and h_{ik} represent the height of department i in bay k. The binary variables are defined as follows:

$$z_{ik} = \begin{cases} 1, & \text{if department } i \text{ is assigned to bay } k \\ 0, & \text{otherwise;} \end{cases}$$

$$r_{ij} = \begin{cases} 1, & \text{if department } i \text{ is above department } j \text{ in the same bay} \\ 0, & \text{otherwise;} \end{cases}$$

$$\delta_k = \begin{cases} 1, & \text{if bay } k \text{ is occupied} \\ 0, & \text{otherwise.} \end{cases}$$

The MILO model is:

$$
\begin{aligned}
\min \quad & \sum_{1 \le i < j \le n} c_{ij}(u_{ij} + v_{ij}) \\
\text{s.t.} \quad & u_{ij} \ge x_i - x_j \text{ and } u_{ij} \ge x_j - x_i,\ 1 \le i < j \le n \\
& v_{ij} \ge y_i - y_j \text{ and } v_{ij} \ge y_j - y_i,\ 1 \le i < j \le n \\
& \sum_{k \in K} z_{ik} = 1,\ i \in N && (69) \\
& w_k = \frac{1}{h_F} \sum_{i \in N} z_{ik} A_i,\ k \in K && (70) \\
& w_i^{\min} z_{ik} \le w_k \le w_i^{\max} + w_F(1 - z_{ik}),\ k \in K, i \in N && (71) \\
& \left.
\begin{aligned}
& x_i \ge \sum_{j \le k} w_j - 0.5 w_k - (w_F - w_i^{\min})(1 - z_{ik}),\ k \in K, i \in N \\
& x_i \le \sum_{j \le k} w_j - 0.5 w_k + (w_F - w_i^{\min})(1 - z_{ik}),\ k \in K, i \in N
\end{aligned}
\right. && (72) \\
& \left.
\begin{aligned}
& \frac{h_{ik}}{A_i} - \frac{h_{jk}}{a_j} - \max\left\{ \frac{\ell_i^{\max}}{A_i}, \frac{\ell_j^{\min}}{a_j} \right\} (2 - z_{ik} - z_{jk}) \le 0,\ i < j \\
& \frac{h_{ik}}{A_i} - \frac{h_{jk}}{a_j} + \max\left\{ \frac{\ell_i^{\max}}{A_i}, \frac{\ell_j^{\min}}{a_j} \right\} (2 - z_{ik} - z_{jk}) \le 0,\ i < j
\end{aligned}
\right. && (73) \\
& \sum_{i \in N} h_{ik} = h_F \delta_k,\ k \in K && (74) \\
& h_i^{\min} z_{ik} \le h_{ik} \le h_i^{\max} z_{ik},\ i \in N, k \in K && (75) \\
& \sum_{i \in N} h_{ik} = h_i,\ i \in N && (76) \\
& y_i - 0.5 h_i \ge y_j + 0.5 h_j - w_H(1 - r_{ij}),\ i \ne j && (77) \\
& r_{ij} + r_{ji} = 1,\ 1 \le i < j \le n && (78) \\
& r_{ij} + r_{ji} \ge z_{ik} + z_{jk} - 1,\ 1 \le i < j \le n, k \in K && (79) \\
& 0.5 h_i \le y_i \le w_H - 0.5 h_i,\ i \in N. && (80)
\end{aligned}
$$

where K is the set of bays, and N is the set of departments.

Constraints (69) ensure that each department is assigned to a single bay. Constraints (70) calculate the width of each bay as the total area of the departments assigned to that bay divided by the facility height. Note that under the assumption that $\sum_{i \in N} A_i \le w_F h_F$, we have $\sum_{k \in K} w_k \le w_F$. Constraints (71) impose bounds on the bay widths, based on the width bounds of the departments assigned to each bay. Constraints (72) determine the horizontal locations of the department centroids. In this model, the x-coordinate is located at the middle of the bays. Therefore, if department i is assigned to bay k, x_i is calculated as $x_i = \sum_{j=1}^{k} w_j - 0.5w_k$. This agrees with constraints (72) with $z_{ik} = 1$. If i and j are in the same bay k, then constraints (73) ensure that the widths of the two departments are the same and equal to the bay width. Constraints (74) set the total heights of the departments in a bay equal to h_F if the bay is used, and zero if the bay is empty. Constraints (75) are bounds on the department heights. They also enforce $h_{ik} = 0$ when department i is not located in bay k. Constraints (76) define the heights of the departments. Constraints (78)–(79) ensure that department i is either above or below department j. Constraints (77) prevent departments in the same bay from overlapping. Constraints (80) ensure that the departments are inside the facility.

By adding symmetry-breaking constraints (see Section 5) and valid inequalities (see Section 6), instances with up to 14 departments were solved to optimality in [56]. The 14-department instance needed around 120 hours of computational time.

4. Multifloor FLP

The multifloor FLP (MF-FLP) involves finding the optimal arrangement of departments in a facility with multiple floors. Practical applications include production facilities, hotels, office buildings, and hospitals. This problem has added complexity in comparison to the UA-FLP because we must also consider the interactions between departments on different floors. Furthermore, elevators and/or stairwells are required to transfer people and/or material between the floors, and these need to be placed at coherent locations in every floor that they reach.

Globally optimal algorithms for MF-FLP work in general only for small instances [38]. The problem was first investigated in [51] and later in [70], but most of the subsequent models in the literature are designed for specific types of MF-FLP, as the literature survey in Section 4.1 shows. Indeed there is no commonly agreed definition of the MF-FLP because different authors make their own assumptions about the structure of the problem. This lack of a common definition makes it hard to compare the approaches. We therefore propose in Section 4.2 a general formulation for the MF-FLP that we hope will gain acceptance as a standard formulation, and will lead to increased research activity on this problem.

Some approaches first distribute the departments over the floors, minimizing the vertical interaction costs. This is essentially the first stage of a two-stage approach, where the second stage then optimizes the layout of each floor independently; see [70] and [24]. Specifically the following MILO formulation is used in [69] to assign departments to floors:

$$\min \quad \sum_{1\leq i<j\leq n} c_{ij} d^v_{ij} \tag{81}$$

$$\text{s.t.} \quad \sum_{k=1}^{p} z_{ik} = 1, \ 1 \leq i \leq n \tag{82}$$

$$d^v_{ij} \geq \delta \sum_{k=1}^{p} k(z_{ik} - z_{jk}), \ 1 \leq i < j \leq n \tag{83}$$

$$d^v_{ij} \geq \delta \sum_{k=1}^{p} k(z_{jk} - z_{ik}), \ 1 \leq i < j \leq n \tag{84}$$

$$\sum_{i=1}^{n} A_i z_{ik} \leq w_F \, h_F, \ 1 \leq k \leq p \tag{85}$$

$$z_{ik} \in \{0, 1\}, \ 1 \leq i \leq n, \ 1 \leq k \leq p, \tag{86}$$

where p is the number of floors. The variable z_{ik} equals 1 if department i is assigned to floor k, and equals 0 otherwise. Constraints (82) assign each department to exactly one floor. Constraints (83)-(84) compute the vertical distance d^v_{ij} between each pair i, j of departments, where δ is the floor height. Note that

$$\left| \sum_{k=1}^{p} k(z_{ik} - z_{jk}) \right|$$

is precisely equal to the number of floors separating i and j. Constraints (85) ensure that the departments assigned to each floor fit into that floor.

Each floor then becomes an instance of UA-FLP with some additional constraints to ensure coherence in the location of the elevators. Computing the vertical costs still remains a challenge and was addressed in [24].

Another possible simplification is to restrict all the departments to have the same shape and to require that they be assigned to specific locations in the building. This reduces the problem to a quadratic assignment problem. Such a formulation was used in [38], and was solved using the RLT linearization technique [1, 86] within a branch-and-bound algorithm.

A mathematical formulation of MF-FLP for process plant layout was presented in [82]. Its objective function considers the construction and land costs to decide the number of floors and the floor area. Another model for a processing plant was proposed in [32]; it incorporates many structural and operational issues, but becomes unwieldy.

A GA is also used in [61] to find a layout with inner walls and passages. The connections between the departments, passages, and elevators are represented as an adjacency graph, and the distances are calculated using Dijkstra's algorithm. This representation allows the measurement of the distances of paths that use corridors and elevators. The bi-objective model minimizes the total cost of transporting the materials and maximizes the adjacency achieved. It is applied to a multideck ship layout with inner walls.

Another bi-objective model is proposed in [40] for a MF-FLP formulation that minimizes not only the material handling costs (as usual) but also the facility construction costs. This model is similar to the one we present in Section 4.2, but a major difference is that the length and width of the facility, the number of elevators, and the number of floors are decision variables.

For completeness, we also mention the robust model in [49] in which some of the usual parameters are considered to be uncertain, and the model in [81] that takes into account safety distances in the event of an explosion.

4.2. A MF-FLP Formulation

We assume that the following parameters are given: the number of departments and their areas, the number of floors, the dimensions and height of the floors, the interconnection costs, and the number and size of the elevators. We consider the elevators to be a general system (incorporating elevators, stairs, pipes, etc.) for vertical movement. We want to determine the locations of the elevators and the locations and dimensions of the departments. The horizontal distance is the rectilinear distance (which is a reliable measure, as in the single-floor case), and the vertical distance will be measured using the elevators. This makes the formulation complex. The number of floors and elevators is assumed to be fixed; if necessary, we could run the model for several different options. The floor dimensions are fixed, but they could easily be treated as decision variables.

Let δ denote the ceiling height, p the number of floors, and e the number of elevators. Let also $M = w_F + h_F + \delta p$. Define the following variables:

$z_{ik} = 1$ if department i is assigned to floor k, 0 otherwise;

$Z_{ij} = 1$ if departments i and j are allocated to the same floor, 0 otherwise;

X_{ij}, Y_{ij}: nonoverlapping binary variables;

(x_i, y_i): coordinates of the centroid of department i;

d^v_{ij}: vertical distance between i and j;

d^h_{ij}: horizontal distance between i and j located on the same floor;

d^e_{ij}: horizontal distance between i and j located on different floors, where the path includes an elevator.

Note that the indices $n+1, \ldots, n+e$ correspond to the elevators.

The formulation is as follows:

$$
\begin{aligned}
\min \quad & \sum_{1\le i<j\le n} c_{ij}(d^e_{ij} + d^v_{ij}) \\
\text{s.t.} \quad & \sum_{k=1}^{p} z_{ik} = 1,\ 1 \le i \le n && (87) \\
& d^v_{ij} = \delta \left| \sum_{k=1}^{p} k(z_{ik} - z_{jk}) \right|,\ 1 \le i < j \le n \\
& d^h_{ij} = |x_i - x_j| + |y_i - y_j|,\ 1 \le i < j \le n \\
& d^e_{ij} \ge d^h_{ij},\ 1 \le i < j \le n && (88) \\
& d^e_{ij} \ge |x_i - x_\ell| + |y_i - y_\ell| + |x_j - x_\ell| + |y_j - y_\ell| - MZ_{ij}, \\
& \ 1 \le i < j \le n,\ n+1 \le \ell \le n+e \\
& Z_{ij} \ge z_{ik} + z_{jk} - 1,\ 1 \le i < j \le n, k = 1, \ldots, p \\
& Z_{ij} \le 1 - z_{ik} + z_{jk},\ 1 \le i < j \le n, k = 1, \ldots, p && (89) \\
& Z_{ij} \le 1 + z_{ik} - z_{jk},\ 1 \le i < j \le n, k = 1, \ldots, p \\
& x_i + \frac{1}{2}w_i \le \frac{1}{2}w_F,\ x_i - \frac{1}{2}w_i \ge -\frac{1}{2}w_F,\ 1 \le i \le n+e \\
& y_i + \frac{1}{2}h_i \le \frac{1}{2}h_F,\ y_i - \frac{1}{2}h_i \ge -\frac{1}{2}h_F,\ 1 \le i \le n+e && (90)
\end{aligned}
$$

$$
\begin{aligned}
& w_i h_i = A_i, \ 1 \leq i \leq n \\
& w_i - \beta h_i \leq 0, \ h_i - \beta w_i \leq 0, \ 1 \leq i \leq n
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
& x_i - x_j \geq \frac{1}{2}(w_i + w_j) - w_F(1 - Z_{ij} + X_{ij} + Y_{ij}), \ 1 \leq i < j \leq n + e \\
& x_j - x_i \geq \frac{1}{2}(w_i + w_j) - w_F(2 - Z_{ij} - X_{ij} + Y_{ij}), \ 1 \leq i < j \leq n + e \\
& y_i - y_j \geq \frac{1}{2}(h_i + h_j) - h_F(2 - Z_{ij} + X_{ij} - Y_{ij}), \ 1 \leq i < j \leq n + e \\
& y_j - y_i \geq \frac{1}{2}(h_i + h_j) - h_F(3 - Z_{ij} - X_{ij} - Y_{ij}), \ 1 \leq i < j \leq n + e
\end{aligned} \tag{92}
$$

$$
\begin{aligned}
& z_{ik} = 1, \ n + 1 \leq i \leq n + e, \ 1 \leq k \leq p \\
& Z_{ij} = 1, \ n + 1 \leq i < j \leq n + e
\end{aligned} \tag{93}
$$

$$
X_{ij}, Y_{ij}, Z_{ij}, z_{ik} \in \{0, 1\}, \ 1 \leq i < j \leq n + e, \ 1 \leq k \leq p \tag{94}
$$

$$
h_i, w_i, \geq 0, \ 1 \leq i \leq n. \tag{95}
$$

Constraints (87) allocate each department to exactly one floor. Constraints (88) compute the distances between each pair of departments; if two departments are on different floors, the distance depends on the elevator position. Constraints (89) set $Z_{ij} = 1$ if i and j are on the same floor, and 0 otherwise. Constraints (92) prevent the overlapping of departments and elevators on the same floor. Constraints (89) and (92) have been taken from [82]. Constraints (93) ensure that each elevator covers all the floors and every pair of elevators shares the same floor.

5. Symmetry-Breaking Constraints

Many versions of the FLP have symmetric solutions. For example, it is clear that flipping a solution to UA-FLP by 180 degrees gives exactly the same solution. This matters because the presence of symmetry is often problematic when solving mixed integer optimization problems. We briefly summarize here the main symmetry-breaking strategies in the literature, primarily from the point of view of the UA-FLP because this is the problem for which they have most been used. However the strategies can be extended in a straightforward manner to many of the MILO models discussed in this review.

One way to break the symmetry in the UA-FLP [73] is to require some department k to be located in a specific quarter of the facility by adding the pair of constraints $x_k \leq 0.5 w_F, \ y_k \leq 0.5 h_F$ (where it is assumed that the origin is at the bottom left corner of the facility). This is called the position

k method in [88]. However, if department k has its centroid located at the facility centroid, then this method does not work. It is straightforward to extend this method to multirow and multifloor layouts.

An alternative strategy is the position $p-k$ method [88] that considers a given pair of departments p and k and requires the centroid of p to be below and to the left of the centroid of k by adding the following four constraints:

$$x_p \leq x_k, \quad y_p \leq y_k, \quad z^h_{kp} = z^v_{kp} = 0, \text{ and}$$

$$(x_k - x_p) + (y_k - y_p) \geq \min\{w^{\min}_k + w^{\min}_p, h^{\min}_k + h^{\min}_p\}.$$

The departments p and k can be chosen in different ways; a common criterion is to choose them to satisfy $c_{pk} = \max_{i,j \in N} c_{ij}$. It is claimed in [27] that simply choosing departments 1 and 2 works just as well, and there the constraints $x_1 - x_2 \geq 0$ and $y_2 - y_1 \geq 0$ are used. From [88], it is not clear whether the position k method or the position $p - k$ method is better. For the DRFLP, the $p - k$ method was used in [10] with p and k chosen such that $c_{pk} = \min_{i,j \in N} c_{ij}$.

Finally, several classes of hierarchical constraints that are applicable to general symmetric MILO problems were considered in [89] for the UA-FLP. Those authors study the effect of one such class of constraints:

$$4\sum_{i=1}^{n} i x_i \leq n(n+1)w_F, \quad 4\sum_{i=1}^{n} y_i \leq n(n+1)h_F.$$

They find that these constraints can break the symmetry effectively, but their dense structure renders the CPLEX enumeration procedure relatively ineffective.

6. Valid inequalities

As already mentioned, valid inequalities are essential for solving mathematical optimization models efficiently in practice, especially MILO problems. In this Section we gather a number of valid inequalities used in the literature to improve MILO models. As in Section 5, these results are mostly about the UA-FLP, but unlike the symmetry-breaking constraints, they are mostly specific to the problem at hand. A noteworthy exception are the transitivity constraints, often called triangle inequalities, introduced for the first in this review in the form (1)–(4), and mentioned subsequently throughout, see e.g. constraints (6), (9) and (48). Transitivity can be applied to nearly every variant of the FLP.

For row layout problems, some valid inequalities have been proposed for the SRFLP. Proposition 1 contains a description of valid inequalities, and Amaral and Letchford [13] presented several large classes of valid inequalities. For the DRFLP, the inequalities (15) in Section 2.2.1 are redundant but may be helpful for a branching algorithm; hence they can be viewed as valid inequalities. However, very little is known with respect to valid inequalities for DRFLP and MRFLP.

Meller et al. [73] were the first to investigate valid inequalities for the UA-FLP. The inequalities reduced the number of nodes in the branch and bound tree but increased the computational time. Sherali et al. [88] determined that the best results were obtained by incorporating only the B2 and V2 constraints of [73]. Using the notation of model (38)–(45), these inequalities are

$$
\begin{aligned}
&\text{(B2)} \quad u_{ij} \geq (w_i^{\min} + w_j^{\min})(z_{ij}^h + z_{ji}^h) \\
&\text{(B2)} \quad v_{ij} \geq (h_i^{\min} + h_j^{\min})(z_{ij}^v + z_{ji}^v) \\
&\text{(V2)} \quad u_{ij} \geq (w_i + w_j) - \min\{w_i^{\max} + w_j^{\max}, w_F\}(1 - z_{ij}^h - z_{ji}^h) \\
&\text{(V2)} \quad v_{ij} \geq (w_i + w_j) - \min\{h_i^{\max} + h_j^{\max}, w_F\}(1 - z_{ij}^v - z_{ji}^v).
\end{aligned}
$$

These constraints do not reduce the feasible set of the relaxed MILO model because they are redundant, and they do not enforce the separation of the departments. They are useful in branch and bound algorithms because they improve the lower bounds. In the linear relaxation, if

$$z_{ij}^h, z_{ji}^h = (w_F - w_i - w_j)/w_F \text{ and } z_{ij}^v, z_{ji}^v = (h_F - h_i - h_j)/h_F$$

then (44) leads to $x_i \approx x_j$, $y_i \approx y_j$, i.e., departments i and j overlap. Thus the root lower bound of (38)–(45) is typically zero.

Taking this into account, [88] model the constraint $u_{ij} = |x_i - x_j|$ in an unusual way. Define the variables

$$t_{ij}^h = \begin{cases} 1 & \text{if } x_i \leq x_j, \\ 0 & \text{if } x_i \geq x_j, \end{cases}$$

where the choice of 0 or 1 is inconsequential when $x_i = x_j$. An upper bound on u_{ij} is $U_{ij} = w_F - w_i^{\min} - w_j^{\min}$, and it is proved in [88] that $u_{ij} = |x_i - x_j|$ can be modeled by

$$
\begin{aligned}
&0 \leq u_{ij} + x_i - x_j \leq 2U_{ij}(1 - t_{ij}),\ i < j \\
&0 \leq u_{ij} - x_i + x_j) \leq 2U_{ij}t_{ij},\ i < j \\
&t_{ij} \in \{0, 1\},\ i < j.
\end{aligned}
$$

A similar set of inequalities exists for $v_{ij} = |y_i - y_j|$.

An alternative set of nonoverlapping constraints is proposed in [88]:

$$
\begin{aligned}
& x_i - x_j \geq w_i + w_j - M_{ij}(1 - z_{ij}^h) \\
& y_i - y_j \geq h_i + h_j - M_{ij}(1 - z_{ij}^v) \\
& -(w_F - w_i^{\min} - w_j^{\min}) \leq x_i - x_j \leq w_F - w_i^{\min} - w_j^{\min} \\
& -(h_F - h_i^{\min} - h_j^{\min}) \leq y_i - y_j \leq h_F - h_i^{\min} - h_j^{\min} \\
& w_i^{\min} + w_j^{\min} \leq w_i + w_j \leq w_i^{\max} + w_j^{\max} \\
& h_i^{\min} + h_j^{\min} \leq h_i + h_j \leq h_i^{\max} + h_j^{\max} \\
& z_{ij}^h + z_{ji}^h + z_{ij}^v + z_{ji}^v = 1 \\
& z_{ij}^{h(v)} \in \{0, 1\}.
\end{aligned}
$$

They construct a convex-hull representation of the above constraint set in a higher dimensional space. This convex hull can also be derived using the reformulation-linearization technique (RLT) of [87]. Because of the size of this representation, they use it for just one pair of departments, the positively interacting (nonfixed) pair with the largest total area. Using this, together with constraints (B2) and (V2), symmetry-breaking constraints (see Section 5), and a new branching priority rule, [88] solve instances with up to 9 departments to global optimality. [72] use inequalities (B2) and (V2) in the context of the sequence-pair representation formulation, plus symmetry-breaking constraints, and the same branching priority rule to solve instances with 11 departments within 24 hours.

Finally, symmetry-avoidance constraints and a tightening of the nonoverlapping constraints via

$$
\begin{aligned}
\frac{1}{2}(w_i + w_j) - u_{ij} \leq w_F X_{ij},\ 1 \leq i < j \leq n \\
\frac{1}{2}(h_i + h_j) - v_{ij} \leq h_F(1 - X_{ij}),\ 1 \leq i < j \leq n
\end{aligned}
$$

are used in [27]. Note that these are the same as (B2) and (V2).

7. Directions for Future Research

Facility layout continues to be the focus of much research, as evidenced for example in the bibliography of this review that includes more than 30 research articles published since 2010. We covered three classes of layout problems, namely row layout, unequal-areas layout, and multifloor layout, primarily from the perspective of mathematical optimization techniques, mostly MILO and SDO.

Within row layout problems, the single-row FLP is the one most studied in the literature, and major progress has been achieved in recent years since the first papers that modeled and solved this problem using SDO [17, 21]. The recent review in this journal by [54] provides a detailed exposition of this progress. The double-row and multirow cases have also attracted attention recently, but have generally been less studied. The multirow FLP in particular sets a challenge to the research community in terms of providing new ideas for models and algorithms, including new classes of valid inequalities, to compute global solutions.

Unequal-area layout is by far the most studied class of FLPs. Nevertheless only instances with up to 11 departments have been solved to global optimality using MILO formulations. Other mathematical optimization-based research has focused on two-stage heuristics that can provide good solutions for instances with up to 100 departments. Moreover, unlike for row FLPs, little work has been done with respect to applying semidefinite optimization to this class of problems beyond the observation that the department area constraint can be relaxed in a conic form (see equation 46) that is handled efficiently by conic optimization software. Indeed, to the best of our knowledge, the only approach entirely based on semidefinite optimization was given in [92] where it is applied to obtain global bounds for benchmark instances in the area of VLSI floorplanning. The important question of computing global bounds for instances of UA-FLP thus remains open.

Multifloor layout has received the least attention in the literature. While there has been an increased interest in it in different contexts, most of the models are motivated by application-specific assumptions that are not commonly used in the literature. This state of affairs makes it difficult to compare the performance of different approaches. Because there is no commonly agreed definition for the MF-FLP, we proposed a general formulation in Section 4.2 that we hope will gain acceptance in the community and will motivate further research into this most challenging version of facility layout. At the very least, we hope that this review prompts a discussion of the assumptions that should be made in defining a standard version of the problem. This could then lead not only to the development of novel models and solution techniques, including classes of valid inequalities, but also to more effective comparisons of them, which is essential to help the research community make further progress on this difficult but important problem.

Finally, symmetry remains a key issue for the computational solution of FLPs. General methods for handling symmetry in MILO, such as isomorphism pruning [67, 68] and orbital branching [77], have proven advantageous for general problems with general symmetry groups. Problem-specific techniques have also been proposed, e.g. orbitopal fixing [52, 53] is an efficient way to break symmetry in bin packing problems, and modified orbital branching was show to be effective for for problems with structured symmetry via the unit commitment problem [15, 78]. It remains to be seen how these results may have an impact for the solution of certain classes of facility layout.

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