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ABSTRACT

We discuss possible notions of conformal Lie algebras, paying particular attention to graded conformal Lie algebras with d -dimensional space isotropy: namely, those with a $\mathfrak{co}(d)$ subalgebra acting in a prescribed way on the additional generators. We classify those Lie algebras up to isomorphism for all $d \geq 2$ following the same methodology used recently to classify kinematical Lie algebras, as deformations of the “most abelian” graded conformal algebra. We find 17 isomorphism classes of Lie algebras for $d \neq 3$ and 23 classes for $d = 3$. Lie algebra contractions define a partial order in the set of isomorphism classes, and this is illustrated via the corresponding Hasse diagram. The only metric graded conformal Lie algebras are the simple Lie algebras, isomorphic to either $\mathfrak{so}(d+1, 2)$ or $\mathfrak{so}(d+2, 1)$. We also work out the central extensions of the graded conformal algebras and study their invariant inner products. We find that central extensions of a Lie algebra in $d = 3$ and two Lie algebras in $d = 2$ are metric. We then discuss several other notions of conformal Lie algebras (generalised conformal, Schrödinger, and Lifshitz Lie algebras), and we present some partial results on their classification. We end with some open problems suggested by our results.

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I. INTRODUCTION

In a recent series of papers,^{1–3} we classified kinematical Lie algebras in arbitrary dimensions via deformation theory. This extends to arbitrary dimension the classic results for $3 + 1$ dimensions by Bacry and Nuyts,⁴ following up from the earlier work of Bacry and Lévy-Leblond.⁵ A natural extension of the classification problem of kinematical Lie algebras is the classification of conformal Lie algebras, but one first has to decide what a conformal Lie algebra is.

Just like not all kinematical Lie algebras act by isometries on a pseudo-Riemannian spacetime, we do not expect all conformal Lie algebras (regardless of the definition) to act by conformal transformations on a conformal manifold. But by the same token, just like kinematical Lie algebras generalise the Lie algebras of isometries of the maximally symmetric Riemannian and Lorentzian spaces, whichever definition of conformal Lie algebra one adopts, one would expect that the Lie algebra of conformal Killing vector fields on one of these spaces qualifies as a conformal Lie algebra although as we shall see there are notions closely related to conformal Lie algebras (e.g., Lifshitz and Schrödinger algebras) which deviate from this requirement.

The Lie algebra of conformal Killing vector fields of $(d + 1)$ -dimensional Euclidean space and Minkowski spacetime (or indeed of any other simply connected conformally flat Riemannian or Lorentzian manifold of the same dimension) is isomorphic to $\mathfrak{so}(d+2, 1)$ in the Riemannian case or $\mathfrak{so}(d+1, 2)$ in the Lorentzian case. We let $J_{\mu\nu}$ denote the generators, where the index μ decomposes into $(a, +, -, 0)$, where $a = 1, \dots, d$. The inner product $\eta_{\mu\nu}$ has components

$$\eta_{ab} = \delta_{ab}, \quad \eta_{+-} = 1, \quad \text{and} \quad \eta_{00} = \varepsilon, \quad (1)$$

with $\varepsilon = \pm 1$ in the Riemannian and Lorentzian cases, respectively. Then the generators $J_{\mu\nu}$, which satisfy the Lie brackets

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\nu\rho}, \quad (2)$$

break up into J_{ab} , $D := J_{+-}$, $V_0 := J_{a0}$, $V_{\pm a} := J_{a\pm}$, and $S_{\pm} := J_{0\pm}$, with Lie brackets

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac} + \delta_{ad} J_{bc} & [D, S_{\pm}] &= \pm S_{\pm}, \\
[J_{ab}, V_{\pm c}] &= \delta_{bc} V_{\pm a} - \delta_{ac} V_{\pm b} & [J_{ab}, S_{\pm}] &= 0, \\
[J_{ab}, V_{0c}] &= \delta_{bc} V_{0a} - \delta_{ac} V_{0b} & [J_{ab}, D] &= 0, \\
[D, V_{\pm a}] &= \pm V_{\pm a} & [D, V_{0a}] &= 0.
\end{aligned} \tag{3}$$

and, in addition,

$$\begin{aligned}
[S_{\pm}, V_{0a}] &= V_{\pm a} & [V_{0a}, V_{0b}] &= -\epsilon J_{ab}, \\
[S_{\pm}, V_{\mp a}] &= V_{0a} & [V_{0a}, V_{\pm b}] &= \epsilon \delta_{ab} S_{\pm}, \\
[S_{+}, S_{-}] &= D & [V_{+a}, V_{-b}] &= \epsilon (J_{ab} + \delta_{ab} D).
\end{aligned} \tag{4}$$

What properties of the above Lie algebras shall we take as characterising the notion of a “conformal Lie algebra?”

The obvious geometrical answer is that the above Lie algebras are isomorphic to Lie algebras of conformal vector fields on pseudo-Riemannian manifolds. The Lie algebra of conformal Killing vectors in a pseudo-Riemannian manifold has a precise algebraic structure shared with the Lie algebra of isometries and, more generally, the Killing superalgebra of a supergravity background.⁶ Conformal Killing vectors share, with Killing vectors and Killing spinors in some supergravity theories, the property that they are parallel sections of a vector bundle with connection. The connection in the conformal case is the one that gives rise to the notion of conformal Killing transport,⁷ an extension to conformal Killing vectors of earlier results of Kostant⁸ for Killing vectors. All these (super)algebras have in common that they are filtered deformations of graded subalgebras of a graded Lie (super)algebra: the (super)algebra associated with the “flat” model for the geometry. For conformal Killing transport, the flat model is that of any (simply connected) conformally flat manifold which, in the present context, would be one of the simple conformal Lie algebras above. Of course, if we insist on the dimension being $\frac{1}{2}(d+3)(d+2)$, then the algebra is that of the flat model, but we could drop this requirement and insist only that it should contain an $\mathfrak{so}(d)$ subalgebra and perhaps that it acted transitively. This is an interesting problem which we will not address in this paper.

Although this property of acting like conformal transformations on a pseudo-Riemannian manifold is the property dictated by geometric orthodoxy, it is a sense too narrow. Just like not all kinematical Lie algebras are isomorphic to a Lie algebra of isometries in a pseudo-Riemannian manifold, we do not wish to restrict attention to Lie algebras of conformal Killing vectors. Kinematical Lie algebras with d -dimensional space isotropy, for $d \geq 2$, are symmetries of non-metrical structures such as Galilean (or Newton–Cartan), Carrollian, and Aristotelian.⁹ By analogy with the case of Lorentzian and Riemannian spacetimes, one could define a *conformal vector field* to be one which generates conformal rescalings of the relevant structures. Since Newton–Cartan and Carrollian structures are degenerate, these conformal vector fields typically span an infinite-dimensional Lie algebra, which in at least one Carrollian case has been shown to coincide with the BMS algebra.¹⁰

Another property of the above simple conformal Lie algebras, and one that forms the kinematical basis for the AdS/CFT correspondence, is that they are isomorphic to the Lie algebra of isometries of a Lorentzian manifold in one higher dimension. This suggests re-interpreting kinematical Lie algebras with $(d+1)$ -dimensional space isotropy as *holographic conformal algebras* with d -dimensional space isotropy.

Moving away from the geometric conformal algebras, we may instead concentrate on algebraic properties. At the surface lies the observation that the simple conformal algebras above have a Lie subalgebra isomorphic to $\mathfrak{so}(d)$, relative to which the additional generators transform as 3 copies of the vector d -dimensional representation and 3 copies of the scalar one-dimensional representation. We shall call such Lie algebras *generalised conformal algebras* (see Definition 2) although it is not clear that they have the right to be called conformal. We consider them because they form a large class of Lie algebras which encompass many of the conformal Lie algebras we consider in this paper.

More closely related to conformality is the fact that the simple conformal algebras above have a Lie subalgebra isomorphic to $\mathfrak{co}(d) = \mathfrak{so}(d) \oplus \mathbb{R}D$, where the adjoint action of the *dilatation* D defines a $\mathbb{Z}$ -grading, where J_{ab} , V_0 , and D have degree 0, and $V_{\pm}$ and $S_{\pm}$ have degree ± 1 . In this paper, we will take this gradation by “conformal weight” to be the defining property of a conformal algebra. We will not just demand that the Lie algebra be graded by D , but we are also specifying the conformal weights.

We may generalise this class of Lie algebras along at least two directions. First, we can still demand on the $\mathfrak{co}(d)$ subalgebra with D a grading element but allowing for arbitrary conformal weights. This leads to the notion a *generalised Lifshitz algebra* (see Definition 3), discussed briefly in Sec. VII, where as a first step in their classification, we classify possible $\mathbb{Z}$ -gradings on kinematical Lie algebras.

The second generalisation is to demand that the dilatation D should be part of an $\mathfrak{sl}(2, \mathbb{R})$ subalgebra. In other words, the $\mathfrak{co}(d)$ -subalgebra extends to an $\mathfrak{so}(d) \oplus \mathfrak{sl}(2, \mathbb{R})$ subalgebra in such a way that the additional generators transform according to a representation $\mathbb{V} \otimes E$ of $\mathfrak{so}(d) \oplus \mathfrak{sl}(2, \mathbb{R})$, where $\mathbb{V}$ is the d -dimensional vector representation of $\mathfrak{so}(d)$ and E is a representation of $\mathfrak{sl}(2, \mathbb{R})$ of dimension 2 or 3. In the case where E is the fundamental representation (so $\dim E = 2$), we will show that for $d \geq 3$ the Lie algebra is unique up to isomorphism and admits a unique central extension which is isomorphic to the Schrödinger Lie algebra. In general, we shall call the central extensions of such Lie algebras *generalised Schrödinger algebras* (see Definition 4) and we will discuss their classification in Sec. VIII.

We shall now define the main characters in this paper: the graded conformal Lie algebras.

Let $d \geq 2$. Recall that $\mathfrak{so}(d)$ is the Lie algebra of skew-symmetric linear transformations of d -dimensional Euclidean space and $\mathfrak{co}(d) = \mathfrak{so}(d) \oplus \mathbb{R}D$ is the extension of $\mathfrak{so}(d)$ by dilatations.

Definition 1. By a **graded conformal algebra** (with d -dimensional space isotropy), we mean a real Lie algebra $\mathfrak{g}$ satisfying the following properties:

- (1) $\mathfrak{g}$ has a subalgebra $\mathfrak{h} \cong \mathfrak{co}(d)$ and
- (2) under the adjoint action of $\mathfrak{h}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{V}_+ \oplus \mathbb{V}_- \oplus \mathbb{V}_0 \oplus \mathbb{S}_+ \oplus \mathbb{S}_-$, where $\mathbb{V}_\pm, \mathbb{V}_0$ are vectors under $\mathfrak{so}(d)$ and have weights ± 1 and 0, respectively, under D , whereas $\mathbb{S}_\pm$ are scalars under $\mathfrak{so}(d)$ and have weights ± 1 under D .

In other words, a graded conformal algebra is the real span of $J_{ab} = -J_{ba}$, $V_{\pm a}$, $V_0 a$, $S_\pm$, with $a, b \in \{1, \dots, d\}$, satisfying Eq. (3) and any other Lie brackets not mentioned here, subject only to the axioms of a Lie algebra.

Notice that the simple conformal Lie algebras whose brackets are given by (3) and (4) are not just $\mathbb{Z}$ -graded but actually $(\mathbb{Z}_2 \times \mathbb{Z})$ -graded, with J_{ab} , D , $S_\pm$ even under $\mathbb{Z}_2$ and $V_{\pm a}$, $V_0 a$ odd under $\mathbb{Z}_2$. This parity is maintained in all graded conformal algebras in $d \neq 3$ but not necessarily in $d = 3$ due to the existence of the parity-violating vector product in $\mathbb{R}^3$.

All graded conformal Lie algebras share the brackets (3) and are distinguished by the additional brackets between the generators $V_{\pm a}$, $V_0 a$, $S_\pm$. If these additional brackets are zero, we have the **static graded conformal algebra**, which we will denote $\mathfrak{g}$ from now on. Any other graded conformal algebra is, by definition, a deformation of $\mathfrak{g}$, and this is the approach we will take toward the classification. As in the case of kinematical Lie algebras, the problem depends on the value of d , the case $d \geq 4$ being generic, whereas $d \leq 3$ needs special care. In this paper, we will deal with $d \geq 2$ although the discussion breaks up into several cases: $d \geq 5$, $d = 4$, $d = 3$, and $d = 2$. We refer to Ref. 1 for the methodology and to Ref. 3 for the rationale of working with complex Lie algebras in $d = 2$.

This paper is organised as follows: In Sec. II, we discuss the deformations of the static graded conformal algebra for $d \geq 4$, which is the generic range. The bulk of the discussion is about $d \geq 5$, but in Sec. II E, we simply observe that the generic case also includes $d = 4$ since there is no substantial change in the calculations. All that happens in $d = 4$ is that $\mathfrak{so}(4)$ is not simple, but this does not change the results. After introducing the deformation complex in Sec. II A, we calculate the second cohomology and thus determine the infinitesimal deformations in Sec. II B. In Sec. II C, we work out the obstructions to integrating the infinitesimal deformations, and after solving the resulting system of quadratic equations, in Sec. II D, we exploit the action of automorphisms of $\mathfrak{g}$ in order to classify the isomorphism classes of deformations. The results are summarised in Sec. II F and particularly in Table VI. In Sec. II G, we show that the only metric Lie algebras in this classification are the simple conformal algebras: $\mathfrak{so}(d+1, 2)$ and $\mathfrak{so}(d+2, 1)$. In Sec. II H, we show how all graded conformal algebras for $d \geq 4$ can be obtained as contractions of the simple conformal algebras and we exhibit explicit contractions for each case.

In Sec. III, we discuss the case of $d = 3$, which is substantially different due to the existence of an $\mathfrak{so}(3)$ -invariant vector product (or, equivalently, the Levi-Civita symbol ϵ_{abc}) which can now appear in brackets. The panorama of graded conformal algebras in $d = 3$ is somewhat richer than in $d \geq 4$, resulting in new graded conformal algebras which have no analog in $d \geq 4$. In Sec. VI A, we introduce the differential complex, which is larger than the one for $d \geq 4$. Its second cohomology is computed in Sec. III B, and in this way, determine the space of infinitesimal deformations. The obstructions to integrability are determined in Sec. III C. There we see that the solution of the integrability conditions leads to several branches of solutions: one of them being the $d = 3$ analog of the $d \geq 4$ deformations. The other branches are unique to $d = 3$, and their study concludes in Sec. III D with the determination of the isomorphism classes of Lie algebras for $d = 3$. The graded conformal algebras unique to $d = 3$ are listed in Table X. In Sec. III E, we discuss the contractions and we summarise the results in Fig. 1, which illustrates the Hasse diagram of the partial order defined by contraction on the set of isomorphism classes of graded conformal algebras.

In Sec. IV, we discuss the case of $d = 2$, which although superficially different from $d > 2$ actually results in formally the same classification as $d \geq 4$. The complex for $d = 2$ is again different from the generic d due to the existence of the $\mathfrak{so}(2)$ -invariant symplectic structure on the vector representation. This manifests itself in a larger endomorphism ring (namely, $\mathbb{C}$ as opposed to $\mathbb{R}$), and it is therefore convenient to work with a complexified Lie algebra, while ensuring that at every moment the deformations are real. In Sec. IV A, we introduce the complex Lie algebra whose deformation complex is described in Sec. IV B. Infinitesimal deformations are determined in Sec. IV C, and the obstructions are determined in Sec. IV D. Finally, the isomorphism classes of Lie algebras are determined in Sec. IV E, with results which are no different from those of $d \geq 4$. This is in sharp contrast to the case of kinematical Lie algebras,³ where the panorama in $d = 2$ was substantially richer than for generic d .

In Sec. V, we work out the central extensions of the graded conformal algebras. Again, the results depend on d . In Sec. V A, we classify the universal central extensions for the Lie algebras in Tables VI and X. The results are contained in Table XVII. In Sec. V B, we investigate whether any central extension admits an invariant inner product. We find that one Lie algebra (GCA_{16}), which exists only for $d = 3$, has a metric central extension. In Sec. V C, we classify the universal central extensions for the $d = 2$ avatars of the Lie algebras in Table VI and summarise the results in Table XVIII. The metricity of these central extensions is investigated in Sec. V D, where it is found that two $d = 2$ Lie algebras admit metric central extensions: GCA_1 and GCA_8 .

In Sec. VI, we discuss a generalisation of the notion of graded conformal algebras, where we dispense with the grading. It is questionable to consider them as conformal algebras at all, but they are introduced since they are a large class of algebras

FIG. 1. Contractions between graded conformal algebras: red dots are unique to $d = 3$.

with the right spectrum of generators, which encompass many of the other kinds of algebras discussed in this paper. We will not solve the classification problem here, but for $d \geq 4$ we will write down the most general deformation and the conditions for integrability, but will not solve them in general.

In Sec. VII, we discuss generalisations of the Lifshitz algebra (extended by boosts). To a first approximation, these Lie algebras are essentially a graded kinematical Lie algebra extended by the grading element. We classify the consistent gradings of kinematical Lie algebras where $\mathfrak{so}(d)$ is in degree zero, a result summarised in Table XXI.

In Sec. VIII, we discuss generalisations of the Schrödinger algebra, which can be understood as a central extension of a conformal algebra (which might be missing a vectorial generator). We define generalised Schrödinger algebras as those containing an $\mathfrak{so}(d) \oplus \mathfrak{sl}(2, \mathbb{R})$ subalgebra, relative to which the vectorial generators transform according to $V \otimes E$, where V is the vector representation of $\mathfrak{so}(d)$ and E is a representation of $\mathfrak{sl}(2, \mathbb{R})$ of dimension 2 or 3. We classify them for $d \geq 3$ when $\dim E = 2$ and for $d \geq 4$ when $\dim E = 3$.

Finally, in Sec. IX, we summarise the results and make some general comments.

II. DEFORMATIONS FOR $d \geq 4$

Let $d \geq 5$ to begin with. We will let $\mathbb{W}$ denote the real vector space spanned by $V_{\pm a}, V_0, S_{\pm}$ and let $\mathbb{W}^*$ denote its dual, with canonical dual basis $v_a^{\pm}, v_a^0, \sigma^{\pm}$. (Because of the existence of the $\mathfrak{so}(d)$ -invariant δ_{ab} we are free to identify vector subscripts and superscripts, explaining why we write v_a^0 and not, say, v^{0a} .) Recall that $\mathfrak{h} \cong \mathfrak{co}(d)$ is the Lie subalgebra spanned by J_{ab} and D .

A. The deformation complex

The differential complex for deformations of the static graded conformal algebra is denoted $(C^{\bullet}, \partial)$. The cochains are those of the static conformal algebra $\mathfrak{g}$ relative to $\mathfrak{h}$ and with values in the adjoint module $\mathfrak{g}$,

$$C^p := C^p(\mathfrak{g}, \mathfrak{h}; \mathfrak{g}) \cong \text{Hom}(\wedge^p \mathbb{W}, \mathfrak{g})^{\mathfrak{h}} \cong (\wedge^p \mathbb{W}^* \otimes \mathfrak{g})^{\mathfrak{h}}, \quad (5)$$

which is the space of $\mathfrak{h}$ -equivariant skewsymmetric p -multilinear maps $\mathbb{W} \times \cdots \times \mathbb{W} \rightarrow \mathfrak{g}$, and the differential $\partial: C^p \rightarrow C^{p+1}$ is defined on generators by

$$\begin{aligned}\partial J_{ab} &= v_a^+ V_{+b} - v_b^+ V_{+a} + v_a^- V_{-b} - v_b^- V_{-a} + v_a^0 V_{0b} - v_b^0 V_{0a}, \\ \partial D &= -v_a^+ V_{+a} + v_a^- V_{-b} - \sigma^+ S_+ + \sigma^- S_-, \end{aligned} \quad (6)$$

and zero elsewhere, where we have omitted $\otimes$ in expressions such as $\sigma^+ \otimes S_+$, et cetera. The cochains do not have any legs on $\mathfrak{h}$ because we do not wish to deform any of the brackets involving J_{ab} or D . The $\mathfrak{h}$ -equivariance ensures that the Jacobi identity involving one generator in $\mathfrak{h}$ is satisfied.

Deformation theory only requires C^p for $p = 1, 2, 3$, of which the first two are

$$\begin{aligned}C^1 &= \text{span}_{\mathbb{R}}(v^+ V_+, v^- V_-, v^0 V_0, \sigma^+ S_+, \sigma^- S_-), \\ C^2 &= \text{span}_{\mathbb{R}}(v^+ v^- J, v^+ v^- D, \tfrac{1}{2} v^0 v^0 J, \sigma^+ \sigma^- D, v^0 v^+ S_+, v^0 v^- S_-, \sigma^+ v^- V_0, \sigma^- v^+ V_0, \sigma^+ v^0 V_+, \sigma^- v^0 V_-), \end{aligned} \quad (7)$$

where we have introduced shorthand notations such as

$$v^+ V_+ := v_a^+ \otimes V_{+a} \quad \text{and} \quad v^+ v^- J := v_a^+ \wedge v_b^- \otimes J_{ab}. \quad (8)$$

B. Infinitesimal deformations

Infinitesimal deformations are classified by H^2 , which, since $\partial: C^1 \rightarrow C^2$ is the zero map, is given by $H^2 = \ker(\partial: C^2 \rightarrow C^3)$. The differential $\partial: C^2 \rightarrow C^3$ has nonzero components

$$\begin{aligned}\partial(\sigma^+ \sigma^- D) &= \sigma^+ \sigma^- v^- V_- - \sigma^+ \sigma^- v^+ V_+, \\ \partial(v^+ v^- D) &= -v^+ v^- v^+ V_+ + v^+ v^- v^- V_- - \sigma^+ v^+ v^- S_+ + \sigma^- v^+ v^- S_-, \\ \partial(v^+ v^- J) &= v^+ v^- v^+ V_+ - v^+ v^- v^- V_- + v^0 v^+ v^- V_0 + v^0 v^- v^+ V_0, \\ \partial(\tfrac{1}{2} v^0 v^0 J) &= -v^0 v^+ v^0 V_+ - v^0 v^- v^0 V_-, \end{aligned} \quad (9)$$

where the abbreviated notation is such that $v^+ v^- v^+ V_+ := v_a^+ \wedge v_b^- \wedge v_c^+ \otimes V_{+b}$, et cetera. This implies that the infinitesimal (i.e., first order) deformations are classified by

$$H^2 = \text{span}_{\mathbb{R}}(v^0 v^+ S_+, v^0 v^- S_-, \sigma^+ v^- V_0, \sigma^- v^+ V_0, \sigma^+ v^0 V_+, \sigma^- v^0 V_-). \quad (10)$$

The most general infinitesimal deformation is therefore given by

$$\mu_1 = t_1^+ v^0 v^+ S_+ + t_1^- v^0 v^- S_- + t_2^+ \sigma^+ v^- V_0 + t_2^- \sigma^- v^+ V_0 + t_3^+ \sigma^+ v^0 V_+ + t_3^- \sigma^- v^0 V_-, \quad (11)$$

where we have introduced parameters $t_1^\pm, t_2^\pm, t_3^\pm \in \mathbb{R}$.

C. Obstructions

The obstructions to integrating the infinitesimal deformation μ_1 are classes in H^3 , the first of which is the class of $\frac{1}{2} \llbracket \mu_1, \mu_1 \rrbracket$, where $\llbracket -, - \rrbracket: C^p \times C^q \rightarrow C^{p+q-1}$ is the Nijenhuis–Richardson bracket,¹¹ which defines a graded Lie superalgebra structure on $A^\bullet = \oplus_{p \geq -1} A^p$, where $A^p = C^{p+1}$. If $\alpha \otimes X \in C^{p+1}$ and $\beta \otimes Y \in C^{q+1}$, then their Nijenhuis–Richardson bracket is given by a graded commutator

$$\llbracket \alpha \otimes X, \beta \otimes Y \rrbracket = (\alpha \otimes X) \bullet (\beta \otimes Y) - (-1)^{pq} (\beta \otimes Y) \bullet (\alpha \otimes X), \quad (12)$$

where the operation $\bullet$, which is not associative, is given by

$$(\alpha \otimes X) \bullet (\beta \otimes Y) = \alpha \wedge \iota_X \beta \otimes Y. \quad (13)$$

Table I lists the $\bullet$ product $\bullet: C^2 \times C^2 \rightarrow C^3$, from where we can read off the Nijenhuis–Richardson brackets. The shorthand notation in that table is such that, for example,

$$v^0 v^+ v^- V_0 := v_a^0 \wedge v_a^+ \wedge v_b^- \otimes V_{0b}, \quad (14)$$

et cetera.

Calculating $\frac{1}{2} \llbracket \mu_1, \mu_1 \rrbracket = \mu_1 \bullet \mu_1$, we find

$$\begin{aligned}\tfrac{1}{2} \llbracket \mu_1, \mu_1 \rrbracket &= t_1^+ t_2^+ v^0 v^+ v^- V_0 + t_1^+ t_3^+ v^0 v^+ v^0 V_+ + t_1^- t_2^- v^0 v^- v^+ V_0 + t_1^- t_3^- v^0 v^- v^0 V_- - t_1^+ t_2^+ \sigma^+ v^+ v^- S_+ \\ &\quad + t_1^- t_2^- \sigma^- v^- v^+ S_- - t_2^- t_3^+ \sigma^+ \sigma^- v^+ V_+ + t_2^+ t_3^- \sigma^- \sigma^+ v^- V_- + (t_2^- t_3^+ - t_2^+ t_3^-) \sigma^+ \sigma^- v^0 V_0. \end{aligned} \quad (15)$$

This has to be a coboundary and so equal to $\partial \mu_2$, where

TABLE I. Some components of the Nijenhuis–Richardson product $\bullet: \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}^3$.

$\bullet$	$v^0 v^+ S_+$	$v^0 v^- S_-$	$\sigma^+ v^- V_0$	$\sigma^- v^+ V_0$	$\sigma^+ v^0 V_+$	$\sigma^- v^0 V_-$
$v^0 v^+ S_+$			$v^0 v^+ v^- V_0$		$v^0 v^+ v^0 V_+$	
$v^0 v^- S_-$				$v^0 v^- v^+ V_0$		$v^0 v^- v^0 V_-$
$\sigma^+ v^- V_0$	$-\sigma^+ v^+ v^- S_+$					$\sigma^+ \sigma^- v^- V_-$
$\sigma^- v^+ V_0$		$\sigma^- v^+ v^- S_-$			$-\sigma^+ \sigma^- v^+ V_+$	
$\sigma^+ v^0 V_+$				$\sigma^+ \sigma^- v^0 V_0$		
$\sigma^- v^0 V_-$			$-\sigma^+ \sigma^- v^0 V_0$			

$$\mu_2 = u_1 \sigma^+ \sigma^- D + u_2 v^+ v^- D + u_3 v^+ v^- J + \frac{1}{2} u_4 v^0 v^0 J, \quad (16)$$

for some $u_1, u_2, u_3, u_4 \in \mathbb{R}$. From Eq. (9), we see that

$$\begin{aligned} \partial \mu_2 = & u_3 v^0 v^+ v^- V_0 - u_4 v^0 v^+ v^0 V_+ + u_3 v^0 v^- v^+ V_0 - u_4 v^0 v^- v^0 V_- - u_2 \sigma^+ v^+ v^- S_+ + u_2 \sigma^- v^+ v^- S_- \\ & - u_1 \sigma^+ \sigma^- v^+ V_+ + u_1 \sigma^+ \sigma^- v^- V_- + (u_2 - u_3)(-v^+ v^- v^+ V_+ + v^+ v^- v^- V_-). \end{aligned} \quad (17)$$

The first obstruction equation $\partial \mu_2 = \frac{1}{2} \llbracket \mu_1, \mu_1 \rrbracket$ has a solution provided that the following equations are satisfied:

$$\begin{aligned} u_1 &= t_2^+ t_3^- & t_1^+ t_2^+ &= t_1^- t_2^-, \\ u_2 &= u_3 = t_1^+ t_2^+ & \text{and} & \quad t_1^+ t_3^+ = t_1^- t_3^-, \\ u_4 &= -t_1^+ t_3^+ & t_2^+ t_3^- &= t_2^- t_3^+, \end{aligned} \quad (18)$$

in which case the infinitesimal deformation μ_1 integrates to second order with quadratic terms,

$$\mu_2 = t_2^+ t_3^- \sigma^+ \sigma^- D + t_1^+ t_2^+ (v^+ v^- D + v^+ v^- J) - \frac{1}{2} t_1^+ t_3^+ v^0 v^0 J. \quad (19)$$

To probe integrability to third order, we must solve $\partial \mu_3 = \llbracket \mu_1, \mu_2 \rrbracket$ for some μ_3 . Since there are no cochains with legs on D or J_{ab} in our complex, $\mu_2 \bullet \mu_1 = 0$ and hence $\llbracket \mu_1, \mu_2 \rrbracket = \mu_1 \bullet \mu_2$, which can be calculated making use of Table II. Using Eq. (18), one finds that $\mu_1 \bullet \mu_2 = 0$, so we can take $\mu_3 = 0$. We then notice that $\llbracket \mu_2, \mu_2 \rrbracket = 0$ identically so that $\mu_4 = 0$, and hence all higher deformations vanish.

In summary, the most general integrable deformation is given by

$$\begin{aligned} \mu = & t_1^+ v^0 v^+ S_+ + t_1^- v^0 v^- S_- + t_2^+ \sigma^+ v^- V_0 + t_2^- \sigma^- v^+ V_0 + t_3^+ \sigma^+ v^0 V_+ + t_3^- \sigma^- v^0 V_- \\ & + t_2^+ t_3^- \sigma^+ \sigma^- D + t_1^+ t_2^+ (v^+ v^- D + v^+ v^- J) - \frac{1}{2} t_1^+ t_3^+ v^0 v^0 J, \end{aligned} \quad (20)$$

subject to the following quadratic equations:

$$t_1^+ t_2^+ = t_1^- t_2^-, \quad t_1^+ t_3^+ = t_1^- t_3^-, \quad \text{and} \quad t_2^+ t_3^- = t_2^- t_3^+. \quad (21)$$

These equations are easy to interpret: they simply state that (t_1^-, t_1^+) , (t_2^+, t_2^-) , and (t_3^+, t_3^-) are collinear vectors in $\mathbb{R}^2$. In other words, there exists a nonzero vector $(x, y) \in \mathbb{R}^2$ and reals $\alpha_1, \alpha_2, \alpha_3$ so that

$$(t_1^-, t_1^+) = \alpha_1(x, y), \quad (t_2^+, t_2^-) = \alpha_2(x, y), \quad \text{and} \quad (t_3^+, t_3^-) = \alpha_3(x, y). \quad (22)$$

Then the deformation becomes

$$\begin{aligned} \mu = & \alpha_1 y v^0 v^+ S_+ + \alpha_1 x v^0 v^- S_- + \alpha_2 x \sigma^+ v^- V_0 + \alpha_2 y \sigma^- v^+ V_0 + \alpha_3 x \sigma^+ v^0 V_+ + \alpha_3 y \sigma^- v^0 V_- \\ & + \alpha_2 \alpha_3 x y \sigma^+ \sigma^- D + \alpha_1 \alpha_2 x y (v^+ v^- D + v^+ v^- J) - \frac{1}{2} \alpha_1 \alpha_3 x y v^0 v^0 J. \end{aligned} \quad (23)$$

TABLE II. More components of the Nijenhuis–Richardson product $\bullet: \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}^3$.

$\bullet$	$\sigma^+ \sigma^- D$	$v^+ v^- D$	$v^+ v^- J$	$\frac{1}{2} v^0 v^0 J$
$v^0 v^+ S_+$	$\sigma^- v^0 v^+ D$			
$v^0 v^- S_-$	$-\sigma^+ v^0 v^- D$			
$\sigma^+ v^- V_0$				$\sigma^+ v^0 v^- J$
$\sigma^- v^+ V_0$				$\sigma^- v^0 v^+ J$
$\sigma^+ v^0 V_+$		$\sigma^+ v^0 v^- D$	$\sigma^+ v^0 v^- J$	
$\sigma^- v^0 V_-$		$-\sigma^- v^0 v^+ D$	$\sigma^- v^0 v^+ J$	

This results in the following Lie brackets in addition to those in (3):

$$\begin{aligned} [S_+, V_{0a}] &= \alpha_3 x V_{+a} & [V_{0a}, V_{0b}] &= -\alpha_1 \alpha_3 xy J_{ab}, \\ [S_-, V_{0a}] &= \alpha_3 y V_{-a} & [V_{0a}, V_{+b}] &= \alpha_1 y \delta_{ab} S_+, \\ [S_+, V_{-a}] &= \alpha_2 x V_{0a} & [V_{0a}, V_{-b}] &= \alpha_1 x \delta_{ab} S_-, \\ [S_-, V_{+a}] &= \alpha_2 y V_{0a} & [V_{+a}, V_{-b}] &= \alpha_1 \alpha_2 xy (J_{ab} + \delta_{ab} D), \\ [S_+, S_-] &= \alpha_2 \alpha_3 xy D \end{aligned} \quad \text{and} \quad (24)$$

D. Isomorphism classes of deformations

Clearly the parameters in Eq. (24) are not effective. To bring these brackets to normal form, we need to identify Lie algebras which are related by automorphisms of the static graded conformal algebra, which is the “gauge” symmetry of the deformation complex. We will not need to determine the full automorphism group, but it will suffice to consider automorphisms of two types. One is a $\mathbb{Z}_2$ subgroup which acts by

$$J \mapsto J, \quad D \mapsto -D, \quad S_{\pm} \mapsto S_{\mp}, \quad V_{\pm} \mapsto V_{\mp}, \quad \text{and} \quad V_0 \mapsto V_0. \quad (25)$$

Its effect on the Lie brackets in (24) is to exchange x and y . The second type of automorphism acts by rescaling some of the generators,

$$J \mapsto J, \quad D \mapsto D, \quad S_{\pm} \mapsto \alpha_{\pm} S_{\pm}, \quad V_{\pm} \mapsto \beta_{\pm} V_{\pm}, \quad \text{and} \quad V_0 \mapsto \gamma V_0, \quad (26)$$

for some $\alpha_{\pm}, \beta_{\pm}, \gamma \in \mathbb{R}^{\times}$.

There are three cases to consider: $x = 0$, $y = 0$, and $xy \neq 0$. The first two are actually related by the automorphism (25), so this leaves two cases to consider: $y = 0$ (and hence $x \neq 0$) and $xy \neq 0$.

1. Branch $y = 0$ (and hence $x \neq 0$)

In this case, the nonzero brackets in addition to those in (3) are

$$\begin{aligned} [S_+, V_{0a}] &= \alpha_3 x V_{+a}, \\ [S_+, V_{-a}] &= \alpha_2 x V_{0a}, \\ [V_{0a}, V_{-b}] &= \alpha_1 x \delta_{ab} S_-. \end{aligned} \quad (27)$$

Via automorphism of type (26), we can bring any nonzero $\alpha_i x$ to 1. Therefore we have eight isomorphism classes of graded conformal algebras: the static graded conformal algebra and seven nontrivial deformations, depending on whether or not each of $\alpha_{1,2,3}$ vanishes. The isomorphism classes of such Lie algebras are tabulated in Table III.

2. Branch $xy \neq 0$

We can now use automorphisms of type (26) in order to rescale the generators and bring the structure constants to a normal form. Under the automorphism (26), we find

$$\begin{aligned} \alpha_1 x &\mapsto \frac{\alpha_-}{\gamma \beta_-} \alpha_1 x & \alpha_1 y &\mapsto \frac{\alpha_+}{\gamma \beta_+} \alpha_1 y, \\ \alpha_2 x &\mapsto \frac{\gamma}{\alpha_+ \beta_-} \alpha_2 x & \alpha_2 y &\mapsto \frac{\gamma}{\alpha_- \beta_+} \alpha_2 y, \\ \alpha_3 x &\mapsto \frac{\beta_+}{\gamma \alpha_+} \alpha_3 x & \alpha_3 y &\mapsto \frac{\beta_-}{\gamma \alpha_-} \alpha_3 y. \end{aligned} \quad (28)$$

TABLE III. Isomorphism classes of graded conformal algebras ($y = 0$).

α_1	α_2	α_3	Nonzero Lie brackets in addition to (3)		
0	0	0			
1	0	0	$[V_0, V_-] = S_-$		
0	1	0		$[S_+, V_-] = V_0$	
0	0	1			$[S_+, V_0] = V_+$
1	1	0	$[V_0, V_-] = S_-$	$[S_+, V_-] = V_0$	
1	0	1	$[V_0, V_-] = S_-$		$[S_+, V_0] = V_+$
0	1	1		$[S_+, V_-] = V_0$	$[S_+, V_0] = V_+$
1	1	1	$[V_0, V_-] = S_-$	$[S_+, V_-] = V_0$	$[S_+, V_0] = V_+$

TABLE IV. Normal forms for structure constants.

α_1	α_2	α_3	$\alpha_1 x$	$\alpha_1 y$	$\alpha_2 x$	$\alpha_2 y$	$\alpha_3 x$	$\alpha_3 y$
0	0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	0	0
0	1	0	0	0	1	1	0	0
0	0	1	0	0	0	0	1	1
1	1	0	1	1	1	1	0	0
1	0	1	ε	ε	0	0	1	1
0	1	1	0	0	1	1	1	1
1	1	1	ε	ε	1	1	1	1

Table IV shows to what normal form we can bring $\alpha_1 x$ and $\alpha_1 y$ depending on whether or not each α_i vanishes. The notation is such that ε is the sign of $\alpha_1 \alpha_3 xy$, which is an invariant: indeed, under the above rescalings $\alpha_1 \alpha_3 xy \mapsto \gamma^{-2} \alpha_1 \alpha_3 xy$, so the sign cannot change.

The resulting isomorphism classes of Lie algebras are tabulated in Table V, where we recognise the Lie algebras (4).

E. Deformations for $d = 4$

If $d = 4$, all that happens is that $\mathfrak{so}(4)$ is not simple, and hence we can decompose the rotation generators J_{ab} into their (anti)self-dual parts $J_{ab}^\pm$. (The sign has nothing to do with the D-weight, despite the notation.) Following the above calculation now in $d = 4$, we see that the infinitesimal deformations do not change so that μ_1 is formally as in $d \geq 5$ and hence so is $\frac{1}{2} [\mu_1, \mu_1]$. The cochain μ_2 changes, in principle, since now

$$\mu_2 = u_1 \sigma^+ \sigma^- D + u_2 v^+ v^- D + u_3^+ v^+ v^- J^+ + u_3^- v^+ v^- J^- + u_4^+ \frac{1}{2} v^0 v^0 J^+ + u_4^- \frac{1}{2} v^0 v^0 J^-. \quad (29)$$

However, in order to cancel $\frac{1}{2} [\mu_1, \mu_1]$, we must take $u_3^+ = u_3^- = u_3$ and $u_4^+ = u_4^- = u_4$, where u_3, u_4 are as in the $d \geq 5$ calculation, and thus μ_2 is formally equal to the one for $d \geq 5$. The rest of the calculation is formally identical to the $d \geq 5$ case, with identical results.

F. Summary

We can now summarise our results and list the isomorphism classes of graded conformal algebras with $d \geq 4$. These are displayed in Table VI. The first column is our label in this paper and should not be taken too seriously. The Lie algebra $GCA_{15}^{(\varepsilon)}$ is isomorphic to $\mathfrak{so}(d+1, 2)$ if $\varepsilon = -1$ and to $\mathfrak{so}(d+2, 1)$ if $\varepsilon = 1$.

G. Invariant inner products

Let us now consider whether any of the Lie algebras in Table VI admit an invariant inner product, that is, a non-degenerate symmetric bilinear form $\langle -, - \rangle$ which is “associative” in the sense that

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle \quad (30)$$

for all X, Y, Z in the Lie algebra. As we will see now, with the exception of the simple Lie algebras $GCA_{15}^{(\varepsilon)}$, none of the other Lie algebras in Table VI admit an invariant inner product.

Indeed, invariance under D and J already says that for a graded conformal algebra the only possible nonzero components of such an inner product are

$$\langle S_+, S_- \rangle, \quad \langle V_+, V_- \rangle, \quad \langle V_0, V_0 \rangle, \quad \langle D, D \rangle, \quad \text{and} \quad \langle J, J \rangle. \quad (31)$$

TABLE V. Isomorphism classes of graded conformal algebras ($xy \neq 0$).

α_1	α_2	α_3	Nonzero Lie brackets in addition to (3)					
0	0	0						
1	0	0						$[V_0, V_\pm] = S_\pm$
0	1	0	$[S_\pm, V_\mp] = V_0$					
0	0	1		$[S_\pm, V_0] = V_\pm$				
1	1	0	$[S_\pm, V_\mp] = V_0$					$[V_0, V_\pm] = S_\pm$
1	0	1		$[S_\pm, V_0] = V_\pm$	$[V_0, V_0] = -\varepsilon J$		$[V_0, V_\pm] = \varepsilon S_\pm$	$[V_+, V_-] = (J + D)$
0	1	1	$[S_+, S_-] = D$	$[S_\pm, V_\mp] = V_0$	$[S_\pm, V_0] = V_\pm$			
1	1	1	$[S_+, S_-] = D$	$[S_\pm, V_\mp] = V_0$	$[S_\pm, V_0] = V_\pm$	$[V_0, V_0] = -\varepsilon J$	$[V_0, V_\pm] = \varepsilon S_\pm$	$[V_+, V_-] = \varepsilon(J + D)$

TABLE VI. Isomorphism classes of graded conformal algebras ($d \geq 4$).

Label	Nonzero Lie brackets in addition to (3)					
GCA ₁						$[V_0, V_-] = S_-$
GCA ₂						
GCA ₃	$[S_+, V_-] = V_0$					
GCA ₄		$[S_+, V_0] = V_+$				
GCA ₅	$[S_+, V_-] = V_0$				$[V_0, V_-] = S_-$	
GCA ₆		$[S_+, V_0] = V_+$			$[V_0, V_-] = S_-$	
GCA ₇	$[S_+, V_-] = V_0$	$[S_+, V_0] = V_+$				
GCA ₈	$[S_+, V_-] = V_0$	$[S_+, V_0] = V_+$			$[V_0, V_-] = S_-$	
GCA ₉					$[V_0, V_{\pm}] = S_{\pm}$	
GCA ₁₀	$[S_{\pm}, V_{\mp}] = V_0$					
GCA ₁₁		$[S_{\pm}, V_0] = V_{\pm}$				
GCA ₁₂	$[S_{\pm}, V_{\mp}] = V_0$				$[V_0, V_{\pm}] = S_{\pm}$	$[V_+, V_-] = J + D$
GCA ₁₃ ^($\epsilon = \pm 1$)		$[S_{\pm}, V_0] = V_{\pm}$	$[V_0, V_0] = -\epsilon J$		$[V_0, V_{\pm}] = \epsilon S_{\pm}$	
GCA ₁₄	$[S_+, S_-] = D$	$[S_{\pm}, V_{\mp}] = V_0$	$[S_{\pm}, V_0] = V_{\pm}$			
GCA ₁₅ ^($\epsilon = \pm 1$)	$[S_+, S_-] = D$	$[S_{\pm}, V_{\mp}] = V_0$	$[S_{\pm}, V_0] = V_{\pm}$	$[V_0, V_0] = -\epsilon J$	$[V_0, V_{\pm}] = \epsilon S_{\pm}$	$[V_+, V_-] = \epsilon(J + D)$

Concentrating on the first component and using (30), we find

$$\langle S_+, S_- \rangle = \langle [D, S_+], S_- \rangle = \langle D, [S_+, S_-] \rangle, \quad (32)$$

so that unless a D appears in $[S_+, S_-]$, the inner product $\langle S_+, S_- \rangle$ is zero and hence it is degenerate. Similarly,

$$\langle V_+, V_- \rangle = \langle [D, V_+], V_- \rangle = \langle D, [V_+, V_-] \rangle \quad (33)$$

so that again the inner product is degenerate unless D appears in $[V_+, V_-]$. Inspecting Table VI, we see that only GCA₁₅^(ϵ) satisfies these conditions. Since these Lie algebras are simple, the Killing form is non-degenerate and hence they are the only metric Lie algebras in Table VI.

H. Contractions

It is clear from expression (24) that every graded conformal algebra in $d \geq 4$ can be obtained as a contraction of GCA₁₅^(ϵ). Indeed, let $t \in (0, 1]$ and consider the following invertible linear transformation on the underlying vector space of the Lie algebra:

$$\varphi_t J = J, \quad \varphi_t D = D, \quad \varphi_t S_{\pm} = t^{a_{\pm}} S_{\pm}, \quad \varphi_t V_{\pm} = t^{b_{\pm}} V_{\pm}, \quad \text{and} \quad \varphi_t V_0 = t^c V_0 \quad (34)$$

for some real numbers $a_{\pm}$, $b_{\pm}$, c . We define new Lie brackets (which are isomorphic for all $t \in (0, 1]$)

$$[X, Y]_t = \varphi_t^{-1} [\varphi_t X, \varphi_t Y]. \quad (35)$$

If the limit $t \rightarrow 0$ of $[-, -]_t$ exists, then it defines a Lie algebra which is typically not isomorphic to the original Lie algebra at $t = 1$. The Lie algebra defined by $[-, -]_0$ is a contraction of the Lie algebra defined by $[-, -]_1$. If we start at $t = 1$ with GCA₁₅^(ϵ), then by choosing the weights $a_{\pm}$, $b_{\pm}$, c judiciously, we can arrive at all the other graded conformal algebras GCA₁ to GCA₁₄. Table VII

TABLE VII. Weights for contractions of simple conformal algebras.

Contraction	a_+	a_-	b_+	b_-	c
GCA ₁₅ $\rightarrow$ GCA ₁	1	1	1	1	1
GCA ₁₅ $\rightarrow$ GCA ₂	1	2	1	1	1
GCA ₁₅ $\rightarrow$ GCA ₃	3	2	2	0	3
GCA ₁₅ $\rightarrow$ GCA ₄	1	1	2	1	1
GCA ₁₅ $\rightarrow$ GCA ₅	1	1	1	0	1
GCA ₁₅ $\rightarrow$ GCA ₆	1	2	2	1	1
GCA ₁₅ $\rightarrow$ GCA ₇	1	1	3	1	2
GCA ₁₅ $\rightarrow$ GCA ₈	1	1	2	0	1
GCA ₁₅ $\rightarrow$ GCA ₉	2	1	1	0	1
GCA ₁₅ $\rightarrow$ GCA ₁₀	1	0	1	0	1
GCA ₁₅ $\rightarrow$ GCA ₁₁	1	0	2	1	1
GCA ₁₅ $\rightarrow$ GCA ₁₂	1	1	0	0	1
GCA ₁₅ $\rightarrow$ GCA ₁₃	1	0	1	0	0
GCA ₁₅ $\rightarrow$ GCA ₁₄	0	0	1	1	1

gives (non-unique) values for these weights for each of the non-simple graded conformal algebras. Of course, contracting with zero weights does not change the isomorphism class of the Lie algebra.

Contractions define a partial order in the set of isomorphism classes of conformal Lie algebras and, as any partial order, they have an associated Hasse diagram. This diagram appears in Fig. 1, which also takes into account the graded conformal algebras for $d = 3$ and $d = 2$. The red vertices depict the algebras unique to $d = 3$. Deleting these vertices and any edges from it, one recovers the Hasse diagram for graded conformal algebras in $d \geq 4$ and, as we shall see below, also for $d = 2$.

III. DEFORMATIONS FOR $d = 3$

We now extend the above classification to $d = 3$.

A. The deformation complex

The existence of the Levi-Civita symbol ϵ_{abc} now embiggens the deformation complex. There are now additional cochains in C^1 and C^2 ,

$$\begin{aligned} C^1 &= C_{d \geq 4}^1 \oplus \text{span}_{\mathbb{R}}(\tfrac{1}{2}v^0J), \\ C^2 &= C_{d \geq 4}^2 \oplus \text{span}_{\mathbb{R}}(v^+v^-V_0, v^0v^+V_+, v^0v^-V_-, \tfrac{1}{2}v^0v^0V_0, \tfrac{1}{2}\sigma^+v^-J, \tfrac{1}{2}\sigma^-v^+J), \end{aligned} \quad (36)$$

where $C_{d \geq 4}^{1,2}$ are given by Eq. (7) and where we continue using a shorthand notation where, for example,

$$v^0J := \epsilon_{abc}v_a^0 \otimes J_{bc}, \quad v^+v^-V_0 := \epsilon_{abc}v_a^+ \wedge v_b^- \otimes V_{0c}, \quad (37)$$

et cetera.

B. Infinitesimal deformations

The differential on generators is unchanged, but now $\partial: C^1 \rightarrow C^2$ is no longer the zero map since

$$\partial(\tfrac{1}{2}v^0J) = -v^0v^+V_+ - v^0v^-V_- - v^0v^0V_0, \quad (38)$$

whose right-hand side spans the space B^2 of 2-coboundaries. The 2-cocycles are now

$$Z^2 = \text{span}_{\mathbb{R}}(v^0v^+S_+, v^0v^-S_-, \sigma^+v^-V_0, \sigma^-v^+V_0, \sigma^+v^0V_+, \sigma^-v^0V_-, v^+v^-V_0, v^0v^+V_+, v^0v^-V_-, \tfrac{1}{2}v^0v^0V_0). \quad (39)$$

We may use the freedom to modify cocycles by coboundaries in order to “normalise” the cocycles and in this way choose a unique cocycle representative for each cohomology class. A convenient normalisation condition is to demand that the coefficient of $\tfrac{1}{2}v^0v^0V_0$ should be zero. This can always be achieved by adding a suitable multiple of $\partial(\tfrac{1}{2}v^0J)$. It is clear that every cohomology class has a unique normalised cocycle and hence, in summary,

$$H^2 \cong \text{span}_{\mathbb{R}}(v^0v^+S_+, v^0v^-S_-, \sigma^+v^-V_0, \sigma^-v^+V_0, \sigma^+v^0V_+, \sigma^-v^0V_-, v^+v^-V_0, v^0v^+V_+, v^0v^-V_-). \quad (40)$$

The most general infinitesimal deformation is then given by

$$\begin{aligned} \mu_1 &= t_1^+v^0v^+S_+ + t_1^-v^0v^-S_- + t_2^+\sigma^+v^-V_0 + t_2^-\sigma^-v^+V_0 + t_3^+\sigma^+v^0V_+ + t_3^-\sigma^-v^0V_- \\ &\quad + t_4v^+v^-V_0 + t_5^+v^0v^+V_+ + t_5^-v^0v^-V_- \end{aligned} \quad (41)$$

for some real parameters $t_1^\pm, t_2^\pm, t_3^\pm, t_4, t_5^\pm$.

C. Obstructions

The first obstruction to integrating μ_1 is given by $\tfrac{1}{2}[[\mu_1, \mu_1]]$. We can reuse some of the calculations in Sec. II C and embed Table I into Table VIII.

We find that $\tfrac{1}{2}[[\mu_1, \mu_1]] = \mu_1 \bullet \mu_1$ is given by

$$\begin{aligned} \tfrac{1}{2}[[\mu_1, \mu_1]] &= (t_1^+t_2^+ - t_4t_5^-)v^0v^+v^-V_0 + (t_1^+t_3^+ + (t_5^+)^2)v^0v^+v^0V_+ + (t_1^-t_2^- - t_4t_5^+)v^0v^-v^+V_0 \\ &\quad + (t_1^-t_3^- + (t_5^-)^2)v^0v^-v^0V_- - t_1^+t_2^+\sigma^+v^+v^-S_+ + t_1^-t_2^-\sigma^-v^+v^-S_- - t_2^+t_3^+\sigma^+\sigma^-v^+V_+ \\ &\quad + t_2^-t_3^-\sigma^-\sigma^-v^-V_- + (t_2^+t_3^+ - t_2^+t_3^-)\sigma^+\sigma^-v^0V_0 + t_4(t_5^+ + t_5^-)v^+v^-v^0V_0 + t_4t_5^+v^+v^-v^+V_+ \\ &\quad - t_4t_5^-v^+v^-v^-V_- + t_1^+t_4v^+v^+v^-S_+ + t_1^-t_4v^+v^-v^-S_- - t_1^+t_5^+v^0v^+v^+S_+ - t_1^-t_5^-v^0v^-v^-S_- \\ &\quad + (t_3^+t_4 - t_2^+t_5^-)\sigma^+v^0v^-V_0 + t_3^+t_5^+\sigma^+v^0v^0V_+ + t_2^+t_5^-\sigma^+v^-v^-V_- + (t_2^+t_5^+ - t_3^+t_4)\sigma^+v^+v^-V_+ \\ &\quad + (t_3^-t_4 - t_2^-t_5^+)\sigma^-v^0v^+V_0 + t_3^-t_5^-\sigma^-v^0v^0V_- + t_2^-t_5^+\sigma^-v^+v^+V_+ + (t_2^-t_5^- - t_3^-t_4)\sigma^-v^+v^-V_-, \end{aligned} \quad (42)$$

TABLE VIII. Some components of $\bullet : C^2 \times C^2 \rightarrow C^3$.

$\bullet$	$v^0 v^+ S_+$	$v^0 v^- S_-$	$\sigma^+ v^- V_0$	$\sigma^- v^+ V_0$	$\sigma^+ v^0 V_+$	$\sigma^- v^0 V_-$	$v^+ v^- V_0$	$v^0 v^+ V_+$	$v^0 v^- V_-$
$v^0 v^+ S_+$			$v^0 v^+ v^- V_0$		$v^0 v^+ v^0 V_+$				
$v^0 v^- S_-$				$v^0 v^- v^+ V_0$		$v^0 v^- v^0 V_-$			
$\sigma^+ v^- V_0$	$-\sigma^+ v^+ v^- S_+$					$\sigma^+ \sigma^- v^- V_-$		$\sigma^+ v^+ v^- V_+$	$\sigma^+ v^- v^- V_-$
$\sigma^- v^+ V_0$		$\sigma^- v^+ v^- S_-$			$-\sigma^+ \sigma^- v^+ V_+$			$\sigma^- v^+ v^+ V_+$	$\sigma^- v^+ v^- V_-$
$\sigma^+ v^0 V_+$				$\sigma^+ \sigma^- v^0 V_0$			$\sigma^+ v^0 v^- V_0$	$\sigma^+ v^0 v^0 V_+$	
$\sigma^- v^0 V_-$			$-\sigma^+ \sigma^- v^0 V_0$				$\sigma^- v^0 v^+ V_0$		$\sigma^- v^0 v^0 V_-$
$v^+ v^- V_0$	$v^+ v^+ v^- S_+$	$v^+ v^- v^- S_-$			$-\sigma^+ v^+ v^- V_+$	$-\sigma^- v^+ v^- V_-$		$v^+ v^- v^+ V_+$	$-v^+ v^- v^- V_-$
$v^0 v^+ V_+$	$-v^0 v^0 v^+ S_+$			$-\sigma^- v^0 v^+ V_0$			$v^+ v^- v^0 V_0$ $-v^0 v^- v^+ V_0$	$v^0 v^+ v^0 V_+$	
$v^0 v^- V_-$		$-v^0 v^0 v^- S_-$	$-\sigma^+ v^0 v^- V_0$				$v^+ v^- v^0 V_0$ $-v^0 v^+ v^- V_0$		$v^0 v^- v^0 V_-$

which has to equal $\partial\mu_2$ with

$$\mu_2 = u_1 \sigma^+ \sigma^- D + u_2 v^+ v^- D + u_3 v^+ v^- J + \frac{1}{2} u_4 v^0 v^0 J + \frac{1}{2} u_5^+ \sigma^+ v^- J + \frac{1}{2} u_5^- \sigma^- v^+ J. \quad (43)$$

We calculate

$$\begin{aligned} \partial\mu_2 = & u_3 v^0 v^+ v^- V_0 - u_4 v^0 v^+ v^0 V_+ + u_3 v^0 v^- v^+ V_0 - u_4 v^0 v^- v^0 V_- - u_2 \sigma^+ v^+ v^- S_+ + u_2 \sigma^- v^+ v^- S_- \\ & - u_1 \sigma^+ \sigma^- v^+ V_+ + u_1 \sigma^+ \sigma^- v^- V_- + (u_2 - u_3)(-v^+ v^- v^+ V_+ + v^+ v^- v^- V_-) \\ & + u_5^+ (\sigma^+ v^+ v^- V_+ + \sigma^+ v^- v^- V_- + \sigma^+ v^0 v^- V_0) + u_5^- (\sigma^- v^+ v^+ V_+ + \sigma^- v^+ v^- V_- + \sigma^- v^0 v^+ V_0). \end{aligned} \quad (44)$$

The equation $\partial\mu_2 = \frac{1}{2} \llbracket \mu_1, \mu_1 \rrbracket$ is equivalent to the following conditions:

$$\begin{aligned} u_1 &= t_2^+ t_3^- = t_2^- t_3^+ & t_1^\pm t_4 &= 0, \\ u_2 &= u_3 = t_1^+ t_2^+ = t_1^- t_2^- & t_1^\pm t_5^\pm &= 0, \\ u_4 &= -t_1^+ t_3^+ - (t_5^+)^2 = -t_1^- t_3^- - (t_5^-)^2 & t_3^\pm t_5^\pm &= 0, \\ u_5^\pm &= t_2^\pm t_5^\mp = t_2^\pm t_5^\pm - t_3^\pm t_4 = t_3^\pm t_4 - t_2^\pm t_5^\mp & t_4 t_5^\pm &= 0, \end{aligned} \quad \text{and} \quad (45)$$

after a little simplification.

To determine the obstruction to integrability to third order, we need to calculate $\llbracket \mu_1, \mu_2 \rrbracket$, for which we extend Table II to the larger Table IX which includes the new cochains. In arriving at some of the new entries, we have used the identity

$$J_{ab} = \frac{1}{2} \epsilon_{abc} \epsilon_{cde} J_{de}. \quad (46)$$

As before, $\llbracket \mu_1, \mu_2 \rrbracket = \mu_1 \bullet \mu_2$ since $\mu_2 \bullet \mu_1 = 0$. We calculate

TABLE IX. More components of $\bullet : C^2 \times C^2 \rightarrow C^3$.

$\bullet$	$\sigma^+ \sigma^- D$	$v^+ v^- D$	$v^+ v^- J$	$\frac{1}{2} v^0 v^0 J$	$\frac{1}{2} \sigma^+ v^- J$	$\frac{1}{2} \sigma^- v^+ J$
$v^0 v^+ S_+$	$\sigma^- v^0 v^+ D$				$\frac{1}{2} v^0 v^+ v^- J$	
$v^0 v^- S_-$	$-\sigma^+ v^0 v^- D$					$\frac{1}{2} v^0 v^- v^+ J$
$\sigma^+ v^- V_0$				$\sigma^+ v^0 v^- J$		
$\sigma^- v^+ V_0$				$\sigma^- v^0 v^+ J$		
$\sigma^+ v^0 V_+$		$\sigma^+ v^0 v^- D$	$\sigma^+ v^0 v^- J$			$\frac{1}{2} \sigma^+ \sigma^- v^0 J$
$\sigma^- v^0 V_-$		$-\sigma^- v^0 v^+ D$	$\sigma^- v^0 v^+ J$		$-\frac{1}{2} \sigma^+ \sigma^- v^0 J$	
$v^+ v^- V_0$				$\frac{1}{2} v^0 v^+ v^- J$ $+\frac{1}{2} v^0 v^- v^+ J$		
$v^0 v^+ V_+$		$v^0 v^+ v^- D$	$-\frac{1}{2} v^0 v^- v^+ J$ $-\frac{1}{2} v^+ v^- v^0 J$			$-\sigma^- v^0 v^+ J$
$v^0 v^- V_-$		$-v^0 v^+ v^- D$	$\frac{1}{2} v^+ v^- v^0 J$ $-\frac{1}{2} v^0 v^+ v^- J$		$-\sigma^+ v^0 v^- J$	

$$\begin{aligned} \llbracket \mu_1, \mu_2 \rrbracket = & (t_1^+ u_1 - t_3^- u_2) \sigma^- v^0 v^+ D + (t_3^+ u_2 - t_1^- u_1) \sigma^+ v^0 v^- D + (t_5^+ - t_5^-) u_2 v^0 v^+ v^- D \\ & + (t_2^+ u_4 + t_3^+ u_3 - t_5^- u_5^+) \sigma^+ v^0 v^- J + (t_2^- u_4 + t_3^- u_3 - t_5^+ u_5^-) \sigma^- v^0 v^+ J \\ & + \frac{1}{2} (t_4 u_4 - t_5^- u_3 + t_1^+ u_5^+) v^0 v^+ v^- J + \frac{1}{2} (t_4 u_4 - t_5^+ u_3 + t_1^- u_5^-) v^0 v^- v^+ J \\ & + \frac{1}{2} (t_5^- - t_5^+) u_3 v^+ v^- v^0 J + \frac{1}{2} (t_3^+ u_5^- - t_3^- u_5^+) \sigma^+ \sigma^- v^0 J, \end{aligned} \quad (47)$$

which, after using conditions (45), reduces to

$$\llbracket \mu_1, \mu_2 \rrbracket = -t_2^+ \left((t_5^+)^2 + (t_5^-)^2 \right) \sigma^+ v^0 v^- J - t_2^- \left((t_5^+)^2 + (t_5^-)^2 \right) \sigma^- v^0 v^+ J. \quad (48)$$

This cannot be canceled with $\partial \mu_3$ unless it actually vanishes, which implies the extra conditions

$$t_2^\pm \left((t_5^+)^2 + (t_5^-)^2 \right) = 0. \quad (49)$$

This being the case, the obstruction is overcome with $\mu_3 = 0$ and since $\llbracket \mu_2, \mu_2 \rrbracket$ is identically zero, also all higher μ_i vanish. In summary, the most general integrable deformation is given by

$$\begin{aligned} \mu = & t_1^+ v^0 v^+ S_+ + t_1^- v^0 v^- S_- + t_2^+ \sigma^+ v^- V_0 + t_2^- \sigma^- v^+ V_0 + t_3^+ \sigma^+ v^0 V_+ + t_3^- \sigma^- v^0 V_- + t_4 v^+ v^- V_0 \\ & + t_5^+ v^0 v^+ V_+ + t_5^- v^0 v^- V_- + t_2^+ t_3^- \sigma^+ \sigma^- D + t_1^+ t_2^+ (v^+ v^- D + v^+ v^- J) \\ & - \frac{1}{2} \left(t_1^+ t_3^+ + (t_5^+)^2 \right) v^0 v^0 J + \frac{1}{2} t_2^+ t_5^- \sigma^+ v^- J + \frac{1}{2} t_2^- t_5^+ \sigma^- v^+ J, \end{aligned} \quad (50)$$

subject to the conditions

$$\begin{aligned} t_2^+ t_3^- &= t_2^- t_3^+ & t_2^\pm t_5^\pm &= 3t_2^\pm t_5^\mp, \\ t_1^+ t_2^+ &= t_1^- t_2^- & t_1^\pm t_4 &= 0, \\ t_1^+ t_3^- - t_1^- t_3^+ &= (t_5^- - t_5^+) (t_5^- + t_5^+) & t_1^\pm t_5^\pm &= 0, \\ t_2^\pm \left((t_5^+)^2 + (t_5^-)^2 \right) &= 0 & t_3^\pm t_5^\pm &= 0, \\ t_3^\pm t_4 &= 2t_2^\pm t_5^\mp & t_4 t_5^\pm &= 0. \end{aligned} \quad (51)$$

It follows from these conditions that

$$t_2^\pm t_5^\pm = 0, \quad t_2^\pm t_5^\mp = 0, \quad \text{and} \quad t_3^\pm t_4 = 0, \quad (52)$$

and hence that $u_5^\pm = 0$. We may therefore summarise again the discussion by saying that the most general integrable deformation is

$$\begin{aligned} \mu = & t_1^+ v^0 v^+ S_+ + t_1^- v^0 v^- S_- + t_2^+ \sigma^+ v^- V_0 + t_2^- \sigma^- v^+ V_0 + t_3^+ \sigma^+ v^0 V_+ + t_3^- \sigma^- v^0 V_- + t_4 v^+ v^- V_0 \\ & + t_5^+ v^0 v^+ V_+ + t_5^- v^0 v^- V_- + t_2^+ t_3^- \sigma^+ \sigma^- D + t_1^+ t_2^+ (v^+ v^- D + v^+ v^- J) - \frac{1}{2} \left(t_1^+ t_3^+ + (t_5^+)^2 \right) v^0 v^0 J, \end{aligned} \quad (53)$$

subject to the conditions

$$\begin{aligned} t_2^+ t_3^- &= t_2^- t_3^+ & t_1^\pm t_5^\pm &= 0 & t_4 t_1^\pm &= 0, \\ t_1^+ t_2^+ &= t_1^- t_2^- & t_2^\pm t_5^\pm &= t_2^\mp t_5^\pm = 0 & t_4 t_3^\pm &= 0, \\ t_1^+ t_3^- - t_1^- t_3^+ &= (t_5^- - t_5^+) (t_5^- + t_5^+) & t_3^\pm t_5^\pm &= 0 & t_4 t_5^\pm &= 0. \end{aligned} \quad (54)$$

The case $t_4 = t_5^\pm = 0$ was already treated in Sec. II C, and proceeding in the same manner, we arrive at the $d = 3$ version of Table VI. Hence those Lie algebras exist for all $d \geq 3$. It remains to classify graded conformal algebras which are unique to $d = 3$, and to this end, we will assume from now on that at least one of the t_4 , t_5^+ , and t_5^- is different from zero.

The last equation ($t_4 t_5^\pm = 0$) in (54) implies that if $t_4 \neq 0$ then $t_5^\pm = 0$ and, viceversa, if at least one of the $t_5^\pm$ is different from zero, then $t_4 = 0$. This breaks up the problem into several branches:

- (1) $t_4 \neq 0$;
- (2) $t_5^\pm \neq 0$;
- (3) $t_5^+ \neq 0$ and $t_5^- = 0$; and
- (4) $t_5^- \neq 0$ and $t_5^+ = 0$.

Since the last two branches are related by the automorphism (25), it is enough to consider one of them.

TABLE X. Isomorphism classes of graded conformal algebras ($d = 3$, $t_4 \neq 0$).

t_2^+	t_2^-	Nonzero Lie brackets in addition to (3)		
0	0			$[V_+, V_-] = V_0$
1	0	$[S_+, V_-] = V_0$		$[V_+, V_-] = V_0$
1	1	$[S_+, V_-] = V_0$	$[S_-, V_+] = V_0$	$[V_+, V_-] = V_0$

1. The branch $t_4 \neq 0$

In this case, $t_1^\pm = t_3^\pm = t_5^\pm = 0$ and the deformation is

$$\mu = t_2^+ \sigma^+ v^- V_0 + t_2^- \sigma^- v^+ V_0 + t_4 v^+ v^- V_0, \quad (55)$$

with Lie brackets

$$[S_+, V_{-a}] = t_2^+ V_{0a}, \quad [S_-, V_{+a}] = t_2^- V_{0a}, \quad \text{and} \quad [V_{+a}, V_{-b}] = t_4 \epsilon_{abc} V_{0c}, \quad (56)$$

for any $t_2^\pm$ and $t_4 \neq 0$. We can redefine V_0 and, without loss of generality, assume that $t_4 = 1$. If nonzero, the other two parameters $t_2^\pm$ can also be set to 1 by redefining $S_\pm$, respectively. The cases $(t_2^+, t_2^-) = (1, 0)$ and $(t_2^+, t_2^-) = (0, 1)$ are related by the automorphism (25), so they give isomorphic Lie algebras. In summary, we have three isomorphism classes of graded conformal algebras in this branch, which are listed in Table X.

2. The branch $t_4 = 0$ and $t_5^+ t_5^- \neq 0$

In this case, $t_1^\pm = t_2^\pm = t_3^\pm = t_4 = 0$ and also $(t_5^+)^2 = (t_5^-)^2 \neq 0$. The deformation becomes

$$\mu = t_5^+ v^0 v^+ V_+ + t_5^- v^0 v^- V_- - \frac{1}{2} (t_5^+)^2 v^0 v^0 J. \quad (57)$$

If $t_5^+ = t_5^-$, then we can rescale V_0 to arrive at

$$\mu = v^0 v^+ V_+ + v^0 v^- V_- - \frac{1}{2} v^0 v^0 J, \quad (58)$$

whereas if $t_5^- = -t_5^+$, then under the same rescaling,

$$\mu = v^0 v^+ V_+ - v^0 v^- V_- - \frac{1}{2} v^0 v^0 J. \quad (59)$$

In summary, we have two isomorphism classes of Lie algebras

$$[V_{0a}, V_{+b}] = \epsilon_{abc} V_{+c}, \quad [V_{0a}, V_{-b}] = \epsilon \epsilon_{abc} V_{-c}, \quad \text{and} \quad [V_{0a}, V_{0b}] = -J_{ab}, \quad (60)$$

where $\epsilon = \pm 1$.

3. The branch $t_4 = 0$ and $t_5^- = 0$

In this case, $t_5^+ \neq 0$ and hence $t_1^+ = t_2^+ = t_3^+ = 0$. The remaining integrability condition is

$$t_1^- t_3^- = (t_5^+)^2. \quad (61)$$

Therefore the deformation becomes

$$\mu = t_1^- v^0 v^- S_- + t_3^- \sigma^- v^0 V_- + t_5^+ v^0 v^+ V_+ - \frac{1}{2} t_1^- t_3^- v^0 v^0 J, \quad (62)$$

where $t_1^- t_3^- = (t_5^+)^2 \neq 0$. The Lie brackets are

$$\begin{aligned} [V_{0a}, V_{-b}] &= t_1^- \delta_{ab} S_-, \\ [S_-, V_{0a}] &= t_3^- V_{-a}, \\ [V_{0a}, V_{+b}] &= t_5^+ \epsilon_{abc} V_{+c}, \\ [V_{0a}, V_{0b}] &= -t_1^- t_3^- J_{ab}. \end{aligned} \quad (63)$$

We can rescale generators $V_0 \mapsto \frac{1}{t_5^+} V_0$, $S_- \mapsto \frac{t_1^-}{t_5^+} S_-$ and arrive at

$$\begin{aligned} [V_{0a}, V_{-b}] &= \delta_{ab} S_-, \\ [S_-, V_{0a}] &= V_{-a}, \\ [V_{0a}, V_{+b}] &= \epsilon_{abc} V_{+c}, \\ [V_{0a}, V_{0b}] &= -J_{ab}. \end{aligned} \quad (64)$$

TABLE XI. Isomorphism classes of graded conformal algebras unique to $d = 3$.

Label	Nonzero Lie brackets in addition to (3)				
GCA ₁₆				$[V_+, V_-] = V_0$	
GCA ₁₇	$[S_+, V_-] = V_0$			$[V_+, V_-] = V_0$	
GCA ₁₈	$[S_\pm, V_\mp] = V_0$			$[V_+, V_-] = V_0$	
GCA ₁₉	$[S_-, V_0] = V_-$	$[V_0, V_+] = V_+$	$[V_0, V_-] = S_-$	$[V_+, V_-] = V_0$	$[V_0, V_0] = -J$
GCA ₂₀ ^($\epsilon = \pm 1$)		$[V_0, V_+] = V_+$	$[V_0, V_-] = \epsilon V_-$		$[V_0, V_0] = -J$

D. Isomorphism classes of deformations

We summarise the isomorphism classes of graded conformal algebras which are unique to $d = 3$: they all involve the ϵ_{abc} symbol in the Lie brackets. They are listed in Table XI, where the first column is the label used in this paper and which continues from Table VI.

It is not hard to see that none of these Lie algebras admit an invariant inner product. Although now for $d = 3$ invariance under J and D allows for a component $\langle V_0, J \rangle$ in addition to those in Eq. (31), the argument in Sec. II G still holds and non-degeneracy requires D appearing in $[S_+, S_-]$ and $[V_+, V_-]$, which is not the case for any of the Lie algebras in Table XI.

E. Contractions

No Lie algebra in Table VI can contract to a Lie algebra in Table XI because all Lie algebras in the latter table contain the Levi-Civita symbol ϵ_{abc} in at least one bracket, whereas no Lie algebra in the former table does. Any new contraction must therefore be from some Lie algebras in Table XI to a Lie algebra in either table. All Lie algebras in Table XI contract to the static graded conformal algebra GCA₁. One can work out the contractions relating the different algebras and display the result as a directed graph, as in Fig. 1. Contractions can be composed, so the actual graph of contractions is the transitive closure of the graph in the figure. In Table XII, we give the values of the weights $a_\pm$, $b_\pm$, c of $S_\pm$, $V_\pm$, and V_0 which are responsible for the different contractions, in the language of the discussion in Sec. II H. In the right-most part of the table, the contractions below the horizontal line involve graded conformal algebras unique to $d = 3$.

The simple conformal algebras $\mathfrak{so}(d+1, 2)$ and $\mathfrak{so}(d+2, 1)$ in dimension $d + 1$ are isomorphic to the Lie algebra of isometries of the de Sitter spacetimes in dimension $d + 2$. In other words, they are kinematical Lie algebras with $(d + 1)$ -dimensional space isotropy. The kinematical contractions of these algebras are contained among the kinematical Lie algebras in Ref. 2 (since $d + 1 \geq 4$ for $d \geq 3$). It is therefore a natural question to ask whether any of the other Lie algebras in Table VI can be interpreted as a kinematical Lie algebra in $d + 1$ although perhaps with $\mathfrak{so}(d, 1)$ isotropy as opposed to $\mathfrak{so}(d+1)$. Let $\mu = (a, b)$, where $a = 1, \dots, d$, and let $J_{\mu\nu}$, B_μ , P_μ , and H generate a kinematical Lie algebra. This means that, in particular, the following brackets exist:

$$\begin{aligned}
 [J_{\mu\nu}, J_{\rho\sigma}] &= \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\nu\rho}, \\
 [J_{\mu\nu}, B_\rho] &= \eta_{\nu\rho} B_\mu - \eta_{\mu\rho} B_\nu, \\
 [J_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu,
 \end{aligned} \tag{65}$$

where $\eta_{ab} = \delta_{ab}$ and $\eta_{\hat{a}\hat{b}} = \pm 1$. Breaking the symmetry down to $\mathfrak{so}(d)$, we find that

- (1) J_{ab} generate an $\mathfrak{so}(d)$ subalgebra;
- (2) $J_{a\hat{b}}$, B_a , and P_a are vectors;
- (3) $B_{\hat{b}}$, $P_{\hat{b}}$, and H are scalars;
- (4) and we have the following additional Lie brackets:

$$\begin{aligned}
 [J_{a\hat{b}}, J_{b\hat{c}}] &= -\eta_{\hat{b}\hat{c}} J_{ab}, \\
 [J_{a\hat{b}}, B_{\hat{b}}] &= -\delta_{ab} B_{\hat{b}}, \\
 [J_{a\hat{b}}, B_{\hat{c}}] &= \eta_{\hat{b}\hat{c}} B_a, \\
 [J_{a\hat{b}}, P_{\hat{b}}] &= -\delta_{ab} P_{\hat{b}}, \\
 [J_{a\hat{b}}, P_{\hat{c}}] &= \eta_{\hat{b}\hat{c}} P_a.
 \end{aligned} \tag{66}$$

In summary, there is a vector V such that (ignoring signs, ...) $[V, V] = J$ and there are additional vectors V' , V'' and scalars S' , S'' such that $[V, V'] = S'$, $[V, V''] = S''$, $[V, S'] = V'$, and $[V, S''] = V''$. Inspecting Tables VI and XI, we see that apart from the de Sitter algebras, the only other Lie algebras satisfying this condition are GCA₁₃^(ϵ). To identify this Lie algebra, we let $J_{a\hat{b}} := V_0$, $B_a := V_+$, $P_a := V_-$, $B_{\hat{b}} := -\epsilon S^+$, $P_{\hat{b}} := -\epsilon S^-$, and $H := D$, and we see that this is kinematical with $\eta_{\hat{b}\hat{c}} = \epsilon$. The only nonzero brackets in addition to the kinematical ones are

$$[H, B] = B \quad \text{and} \quad [H, P] = -P, \tag{67}$$

TABLE XII. Weights for contractions between graded conformal algebras.

Contraction	a_+	a_-	b_+	b_-	c
$GCA_{15} \rightarrow GCA_{14}$	0	0	1	1	1
$GCA_{15} \rightarrow GCA_{13}$	1	0	1	0	0
$GCA_{15} \rightarrow GCA_{12}$	1	1	0	0	1
$GCA_{15} \rightarrow GCA_8$	1	1	2	0	1
$GCA_{14} \rightarrow GCA_{11}$	1	1	1	1	0
$GCA_{14} \rightarrow GCA_{10}$	1	1	0	0	1
$GCA_{13} \rightarrow GCA_{11}$	0	0	1	1	1
$GCA_{13} \rightarrow GCA_9$	1	1	0	0	1
$GCA_{12} \rightarrow GCA_{10}$	0	0	1	1	1
$GCA_{12} \rightarrow GCA_9$	1	1	1	1	0
$GCA_{11} \rightarrow GCA_4$	0	1	0	0	0
$GCA_{10} \rightarrow GCA_3$	0	1	0	0	0
$GCA_9 \rightarrow GCA_2$	0	0	1	0	0
$GCA_8 \rightarrow GCA_7$	1	0	2	0	1
$GCA_8 \rightarrow GCA_6$	0	1	0	1	0
$GCA_8 \rightarrow GCA_5$	0	2	0	1	1
$GCA_7 \rightarrow GCA_4$	0	0	0	1	0
$GCA_7 \rightarrow GCA_3$	1	0	0	0	1
$GCA_6 \rightarrow GCA_4$	0	0	1	0	1
$GCA_6 \rightarrow GCA_2$	0	1	0	0	1
$GCA_5 \rightarrow GCA_3$	1	0	0	0	1
$GCA_5 \rightarrow GCA_2$	0	1	0	1	0
$GCA_4 \rightarrow GCA_1$	1	0	0	0	0
$GCA_3 \rightarrow GCA_1$	1	0	0	0	0
$GCA_2 \rightarrow GCA_1$	0	0	0	0	1
$GCA_{20} \rightarrow GCA_1$	0	0	0	0	1
$GCA_{19} \rightarrow GCA_4$	0	0	0	1	1
$GCA_{19} \rightarrow GCA_2$	0	1	0	0	1
$GCA_{18} \rightarrow GCA_{17}$	0	1	0	0	0
$GCA_{18} \rightarrow GCA_{10}$	0	0	1	1	1
$GCA_{17} \rightarrow GCA_{16}$	1	0	0	0	0
$GCA_{17} \rightarrow GCA_3$	0	0	1	0	0
$GCA_{16} \rightarrow GCA_1$	0	0	1	0	0

which means that it is isomorphic to a Newton-Hooke algebra (but in $d + 2$ dimensions) if $\varepsilon = 1$ and to a pseudo-Newton-Hooke algebra if $\varepsilon = -1$.

IV. DEFORMATIONS FOR $d = 2$

As explained in the context of kinematical Lie algebras in Ref. 3, when $d = 2$, it is convenient to work with the complexified Lie algebra. The reason is two-fold: first of all, the vector representation of $\mathfrak{so}(2)$ has a larger endomorphism ring than that of $\mathfrak{so}(d)$ for any $d > 2$. This is because $\mathfrak{so}(2)$ is abelian and extends the endomorphism ring from $\mathbb{R}$ to $\mathbb{C}$. It is convenient to complexify so that $\mathbb{C}$ acts by complex multiplication. The second reason, which is the same reason in disguise, is that over the complex numbers we can diagonalise the action of J .

A. The complex Lie algebra

Let $\mathfrak{g}_{\mathbb{C}}$ be the complex Lie algebra spanned by $J, D, \mathbb{V}_0 := V_{01} + iV_{02}, \bar{\mathbb{V}}_0 := V_{01} - iV_{02}, \mathbb{V}_{\pm} := V_{\pm 1} + iV_{\pm 2}, \bar{\mathbb{V}}_{\pm} := V_{\pm 1} - iV_{\pm 2}, S_{\pm}$, subject to the Lie brackets

$$\begin{aligned}
 [J, \mathbb{V}_{\pm}] &= -i\mathbb{V}_{\pm} & [D, \mathbb{V}_{\pm}] &= \pm\mathbb{V}_{\pm}, \\
 [J, \mathbb{V}_0] &= -i\mathbb{V}_0 & [D, \bar{\mathbb{V}}_{\pm}] &= \pm\bar{\mathbb{V}}_{\pm}, \\
 [J, \bar{\mathbb{V}}_{\pm}] &= i\bar{\mathbb{V}}_{\pm} & [D, S_{\pm}] &= \pm S_{\pm}, \\
 [J, \bar{\mathbb{V}}_0] &= i\bar{\mathbb{V}}_0
 \end{aligned}
 \quad \text{and} \quad (68)$$

The complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$ which is the real Lie algebra fixed under the antilinear involutive homomorphism $\star$ defined by

$$\star J = J, \quad \star D = D, \quad \star \mathbb{V}_0 = \bar{\mathbb{V}}_0, \quad \star \mathbb{V}_{\pm} = \bar{\mathbb{V}}_{\pm}, \quad \text{and} \quad \star S_{\pm} = S_{\pm}. \quad (69)$$

B. The deformation complex

The deformation complex for $\mathfrak{g}_{\mathbb{C}}$ has cochains

$$C_{\mathbb{C}}^p := \text{Hom}(\wedge^p \mathbb{W}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}}, \quad (70)$$

where $\mathbb{W}_{\mathbb{C}}$ is the complex span of $\mathbb{V}_0, \bar{\mathbb{V}}_0, \mathbb{V}_{\pm}, \bar{\mathbb{V}}_{\pm}, S_{\pm}$ and $\mathfrak{h}_{\mathbb{C}}$ is the complex abelian Lie algebra spanned by J, D . The differential $\partial : C_{\mathbb{C}}^p \rightarrow C_{\mathbb{C}}^{p+1}$ is defined on generators by

$$\begin{aligned} \partial J &= i\nu^+ \mathbb{V}_+ - i\bar{\nu}^+ \bar{\mathbb{V}}_+ + i\nu^- \mathbb{V}_- - i\bar{\nu}^- \bar{\mathbb{V}}_- + i\nu^0 \mathbb{V}_0 - i\bar{\nu}^0 \bar{\mathbb{V}}_0, \\ \partial D &= -\nu^+ \mathbb{V}_+ - \bar{\nu}^+ \bar{\mathbb{V}}_+ + \nu^- \mathbb{V}_- + \bar{\nu}^- \bar{\mathbb{V}}_- - \sigma^+ S_+ + \sigma^- S_-, \end{aligned} \quad (71)$$

and zero on other generators, where $\nu^0, \bar{\nu}^0, \nu^{\pm}, \bar{\nu}^{\pm}, \sigma^{\pm}$ are the canonical dual basis for $\mathbb{W}_{\mathbb{C}}^*$.

We shall now enumerate the first few spaces of cochains. To ensure that we do not leave any cochains out, we can use character theory to calculate the dimension of these spaces. The $\mathfrak{h}_{\mathbb{C}}$ character of $\mathbb{W}_{\mathbb{C}}$ is given by

$$\chi_{\mathbb{W}_{\mathbb{C}}}(q, \tau) = \underbrace{(\tau + \tau^{-1})}_{S_{\pm}} + \underbrace{(\tau + \tau^{-1})(q + q^{-1})}_{\mathbb{V}_{\pm}, \bar{\mathbb{V}}_{\pm}} + \underbrace{(q + q^{-1})}_{\mathbb{V}_0, \bar{\mathbb{V}}_0}, \quad (72)$$

and that of $\mathfrak{g}_{\mathbb{C}}$ is given by

$$\chi_{\mathfrak{g}_{\mathbb{C}}}(q, \tau) = 2 + (\tau + \tau^{-1}) + (\tau + \tau^{-1})(q + q^{-1}) + (q + q^{-1}), \quad (73)$$

where the 2 is due to J and D . The dimension of $C_{\mathbb{C}}^p$ is the constant term in

$$\chi_{\wedge^p \mathbb{W}_{\mathbb{C}}^* \otimes \mathfrak{g}_{\mathbb{C}}}(q, \tau) = \chi_{\wedge^p \mathbb{W}_{\mathbb{C}}^*}(q, \tau) \chi_{\mathfrak{g}_{\mathbb{C}}}(q, \tau). \quad (74)$$

To calculate this, we observe that $\chi_{\wedge^p \mathbb{W}_{\mathbb{C}}^*}(q, \tau) = \chi_{\wedge^p \mathbb{W}_{\mathbb{C}}}(q^{-1}, \tau^{-1}) = \chi_{\wedge^p \mathbb{W}_{\mathbb{C}}}(q, \tau)$, where the last equality follows by inspection. The generating function for the characters of the exterior powers is given by

$$\sum_{p=0}^{\infty} t^p \chi_{\wedge^p \mathbb{W}_{\mathbb{C}}}(q, \tau) = \exp \left(- \sum_{\ell=1}^{\infty} \frac{(-t)^{\ell}}{\ell} \chi_{\mathbb{W}_{\mathbb{C}}}(q^{\ell}, \tau^{\ell}) \right). \quad (75)$$

Expanding and collecting terms, we find the following for the first few spaces of cochains:

$$\dim C_{\mathbb{C}}^0 = 2, \quad \dim C_{\mathbb{C}}^1 = 8, \quad \dim C_{\mathbb{C}}^2 = 20, \quad \text{and} \quad \dim C_{\mathbb{C}}^3 = 36, \quad (76)$$

which we proceed to enumerate

$$\begin{aligned} C_{\mathbb{C}}^0 &= \text{span}_{\mathbb{C}}(J, D), \\ C_{\mathbb{C}}^1 &= \text{span}_{\mathbb{C}}(\nu^+ \mathbb{V}_+, \bar{\nu}^+ \bar{\mathbb{V}}_+, \nu^- \mathbb{V}_-, \bar{\nu}^- \bar{\mathbb{V}}_-, \nu^0 \mathbb{V}_0, \bar{\nu}^0 \bar{\mathbb{V}}_0, \sigma^+ S_+, \sigma^- S_-), \\ C_{\mathbb{C}}^2 &= \text{span}_{\mathbb{C}}(\mathbb{c}_1, \bar{\mathbb{c}}_1, \mathbb{c}_2, \bar{\mathbb{c}}_2, \dots, \mathbb{c}_8, \bar{\mathbb{c}}_8, \mathbb{c}_9, \mathbb{c}_{10}, \mathbb{c}_{11}, \mathbb{c}_{12}), \\ C_{\mathbb{C}}^3 &= \text{span}_{\mathbb{C}}(\mathbb{b}_1, \bar{\mathbb{b}}_1, \mathbb{b}_2, \bar{\mathbb{b}}_2, \dots, \mathbb{b}_{16}, \bar{\mathbb{b}}_{16}, \mathbb{b}_{17}, \mathbb{b}_{18}, \mathbb{b}_{19}, \mathbb{b}_{20}), \end{aligned} \quad (77)$$

where (omitting $\otimes$ and $\wedge$)

$$\begin{aligned} \mathbb{c}_1 &= \nu^0 \bar{\nu}^+ S_+, & \mathbb{c}_4 &= \sigma^- \nu^+ \mathbb{V}_0, & \mathbb{c}_7 &= \nu^+ \bar{\nu}^- J, & \mathbb{c}_{10} &= \sigma^+ \sigma^- D, \\ \mathbb{c}_2 &= \nu^0 \bar{\nu}^- S_-, & \mathbb{c}_5 &= \sigma^+ \nu^0 \mathbb{V}_+, & \mathbb{c}_8 &= \nu^+ \bar{\nu}^- D, & \mathbb{c}_{11} &= i\nu^0 \bar{\nu}^0 J, \\ \mathbb{c}_3 &= \sigma^+ \nu^- \mathbb{V}_0, & \mathbb{c}_6 &= \sigma^- \nu^0 \mathbb{V}_-, & \mathbb{c}_9 &= \sigma^+ \sigma^- J, & \mathbb{c}_{12} &= i\nu^0 \bar{\nu}^0 D \end{aligned} \quad (78)$$

and

$$\begin{aligned} \mathbb{b}_1 &= \nu^+ \nu^- \bar{\nu}^+ \mathbb{V}_+, & \mathbb{b}_6 &= \nu^0 \nu^- \bar{\nu}^+ \mathbb{V}_0, & \mathbb{b}_{11} &= \sigma^+ \nu^+ \bar{\nu}^- S_+, & \mathbb{b}_{16} &= \sigma^- \nu^0 \bar{\nu}^+ D, \\ \mathbb{b}_2 &= \nu^+ \nu^- \bar{\nu}^- \mathbb{V}_-, & \mathbb{b}_7 &= \nu^0 \nu^- \bar{\nu}^0 \mathbb{V}_-, & \mathbb{b}_{12} &= \sigma^- \nu^- \bar{\nu}^+ S_-, & \mathbb{b}_{17} &= i\sigma^+ \nu^0 \bar{\nu}^0 S_+, \\ \mathbb{b}_3 &= \nu^+ \nu^- \bar{\nu}^0 \mathbb{V}_0, & \mathbb{b}_8 &= \sigma^+ \sigma^- \nu^+ \mathbb{V}_+, & \mathbb{b}_{13} &= \sigma^+ \nu^0 \bar{\nu}^- J, & \mathbb{b}_{18} &= i\sigma^- \nu^0 \bar{\nu}^0 S_-, \\ \mathbb{b}_4 &= \nu^0 \nu^+ \bar{\nu}^- \mathbb{V}_0, & \mathbb{b}_9 &= \sigma^+ \sigma^- \nu^- \mathbb{V}_-, & \mathbb{b}_{14} &= \sigma^+ \nu^0 \bar{\nu}^- D, & \mathbb{b}_{19} &= i\sigma^+ \nu^- \bar{\nu}^- S_-, \\ \mathbb{b}_5 &= \nu^0 \nu^+ \bar{\nu}^0 \mathbb{V}_+, & \mathbb{b}_{10} &= \sigma^+ \sigma^- \nu^0 \mathbb{V}_0, & \mathbb{b}_{15} &= \sigma^- \nu^0 \bar{\nu}^+ J, & \mathbb{b}_{20} &= i\sigma^- \nu^+ \bar{\nu}^+ S_+. \end{aligned} \quad (79)$$

Here $\bar{\mathbb{c}}_i = \mathbb{c}_i$ for $i = 9, 10, 11, 12$ and $\bar{\mathbb{b}}_i = \mathbb{b}_i$ for $i = 17, 18, 19, 20$. For the other cochains, complex conjugation is as expected, e.g., $\bar{\mathbb{c}}_1 = \bar{\nu}^0 \nu^+ S_+ = -\nu^+ \bar{\nu}^0 S_+$, et cetera.

C. Infinitesimal deformations

The subspace of real cochains is a subcomplex, and we are interested in its cohomology in degree 2. We observe that $\partial : C_{\mathbb{C}}^1 \rightarrow C_{\mathbb{C}}^2$ is the zero map so that the cohomology $H_{\mathbb{C}}^2$ coincides with the space $Z_{\mathbb{C}}^2$ of cocycles. The map $\partial : C_{\mathbb{C}}^2 \rightarrow C_{\mathbb{C}}^3$ is such that $c_1, \bar{c}_1, \dots, c_6, \bar{c}_6$ are cocycles and

$$\begin{aligned}\partial c_7 &= i(\bar{b}_1 - b_2 + b_4 + \bar{b}_6), & \partial c_{10} &= -(b_8 + \bar{b}_8) + (b_9 + \bar{b}_9), \\ \partial c_8 &= \bar{b}_1 - b_2 - b_{11} - \bar{b}_{12}, & \partial c_{11} &= (b_5 + \bar{b}_5) + (b_7 + \bar{b}_7), \\ \partial c_9 &= i(b_8 - \bar{b}_8) + i(b_9 - \bar{b}_9) + i(b_{10} - \bar{b}_{10}), & \partial c_{12} &= i(b_5 - \bar{b}_5) - i(b_7 - \bar{b}_7) - b_{17} + b_{18},\end{aligned}\quad (80)$$

which spans the space $B_{\mathbb{C}}^3$ of 3-coboundaries. For future use, we notice that there are some cochains in $C_{\mathbb{C}}^3$ which do not appear as components of any coboundary in $B_{\mathbb{C}}^3$: $b_3, b_{13}, b_{14}, b_{15}, b_{16}$ (and their complex conjugates) and also $_{19,20}$.

The space of infinitesimal deformations is

$$H_{\mathbb{C}}^2 \cong Z_{\mathbb{C}}^2 = \text{span}_{\mathbb{C}}(c_1, \bar{c}_1, \dots, c_6, \bar{c}_6), \quad (81)$$

and hence the most general (real) infinitesimal deformation is given by

$$\varphi^{(1)} = \sum_{i=1}^6 t_i c_i + \sum_{i=1}^6 \bar{t}_i \bar{c}_i =: \psi + \bar{\psi} \quad (82)$$

for some $t_i \in \mathbb{C}$.

D. Obstructions

The first obstruction to integrability of the infinitesimal deformation $\varphi^{(1)}$ is given by the cohomology class of $\varphi^{(1)} \bullet \varphi^{(1)}$ in $H_{\mathbb{C}}^3$, where $\bullet$ is the complex-linear extension of the Lie-admissible product introduced in Sec. II C. We tabulate this product on the space of cocycles in Table XIII.

Calculating the product $\varphi^{(1)} \bullet \varphi^{(1)}$, we can use

$$\varphi^{(1)} \bullet \varphi^{(1)} = (\psi + \bar{\psi}) \bullet (\psi + \bar{\psi}) = \psi \bullet \psi + \psi \bullet \bar{\psi} + \text{c.c.} \quad (83)$$

to arrive at

$$\begin{aligned}\varphi^{(1)} \bullet \varphi^{(1)} &= (t_3 \bar{t}_1 - \bar{t}_2 t_4) b_3 - t_2 t_4 b_4 - \bar{t}_1 t_5 b_5 - t_1 t_3 b_6 - \bar{t}_2 t_6 b_7 - t_4 t_5 b_8 + t_3 t_6 b_9 \\ &\quad + (t_4 t_5 - t_3 t_6) b_{10} - \bar{t}_1 \bar{t}_3 b_{11} + t_2 t_4 b_{12} + i \bar{t}_1 t_5 b_{17} + i \bar{t}_2 t_6 b_{18} - i t_2 t_3 b_{19} - i t_1 t_4 b_{20} + \text{c.c.}\end{aligned}\quad (84)$$

We need to cancel this with a coboundary $\partial \varphi^{(2)}$, where

$$\varphi^{(2)} = u_1 c_7 + \bar{u}_1 \bar{c}_7 + u_2 c_8 + \bar{u}_2 \bar{c}_8 + u_3 c_9 + u_4 c_{10} + u_5 c_{11} + u_6 c_{12}, \quad (85)$$

with $u_1, u_2 \in \mathbb{C}$ and $u_3, \dots, u_6 \in \mathbb{R}$. Calculating $\partial \varphi^{(2)}$, we find

$$\begin{aligned}\partial \varphi^{(2)} &= (\bar{u}_2 - i \bar{u}_1) b_1 - (u_2 + i u_1) b_2 + i u_1 b_4 + (u_5 + i u_6) b_5 - i \bar{u}_1 b_6 + (u_5 - i u_6) b_7 - (u_4 - i u_3) b_8 \\ &\quad + (u_4 + i u_3) b_9 + i u_3 b_{10} - u_2 b_{11} + \bar{u}_2 b_{12} - \frac{1}{2} u_6 b_{17} + \frac{1}{2} u_6 b_{18} + \text{c.c.}\end{aligned}\quad (86)$$

TABLE XIII. Some components of the Nijenhuis–Richardson product $\bullet : C_{\mathbb{C}}^2 \times C_{\mathbb{C}}^2 \rightarrow C_{\mathbb{C}}^3$.

$\bullet$	c_1	$\bar{c}_1$	c_2	$\bar{c}_2$	c_3	$\bar{c}_3$	c_4	$\bar{c}_4$	c_5	$\bar{c}_5$	c_6	$\bar{c}_6$
c_1					$-b_6$	$\bar{b}_3$				$-\bar{b}_5$		
$\bar{c}_1$					b_3	$-\bar{b}_6$			$-b_5$			
c_2							$-b_4$	$-\bar{b}_3$				$-\bar{b}_7$
$\bar{c}_2$							$-b_3$	$-\bar{b}_4$			$-b_7$	$\bar{b}_9$
c_3	$-\bar{b}_{11}$		$-i b_{19}$									
$\bar{c}_3$		$-b_{11}$		$i b_{19}$								
c_4	$-i b_{20}$		b_{12}	$\bar{b}_{12}$					$-b_8$	$-\bar{b}_8$		
$\bar{c}_4$		$i b_{20}$		$\bar{b}_{12}$								
c_5		$i b_{17}$					b_{10}	$\bar{b}_{10}$				
$\bar{c}_5$	$-i b_{17}$											
c_6					$-b_{10}$							
$\bar{c}_6$			$-i b_{18}$			$-\bar{b}_{10}$						

TABLE XIV. Further components of the Nijenhuis–Richardson product $\bullet : C_{\mathbb{C}}^2 \times C_{\mathbb{C}}^2 \rightarrow C_{\mathbb{C}}^3$.

$\bullet$	c_7	$\bar{c}_7$	c_8	$\bar{c}_8$	c_9	c_{10}	c_{11}	c_{12}
c_1					b_{15}	b_{16}		
$\bar{c}_1$					$\bar{b}_{15}$	$\bar{b}_{16}$		
c_2					$-b_{13}$	$-b_{14}$		
$\bar{c}_2$					$-b_{13}$	$-b_{14}$		
c_3							$-ib_{13}$	$-ib_{14}$
$\bar{c}_3$							ib_{13}	ib_{14}
c_4							$-ib_{15}$	$-ib_{16}$
$\bar{c}_4$							ib_{15}	ib_{16}
c_5	b_{13}		b_{14}					
$\bar{c}_5$		$\bar{b}_{13}$		$\bar{b}_{14}$				
c_6		$-b_{15}$		$-b_{16}$				
$\bar{c}_6$	$-b_{15}$							

The first obstruction equation $\partial\varphi^{(2)} = \varphi^{(1)} \bullet \varphi^{(1)}$ implies

$$u_1 = it_2 t_4, \quad u_2 = t_2 t_4, \quad u_3 = 0, \quad u_4 = t_3 t_6, \quad u_5 = -\bar{t}_1 t_5, \quad \text{and} \quad u_6 = 0 \quad (87)$$

and results in the following relations between the t_i :

$$t_3 t_6 = t_4 t_5 \in \mathbb{R}, \quad \bar{t}_1 t_5 = \bar{t}_2 t_6 \in \mathbb{R}, \quad \bar{t}_1 t_3 = \bar{t}_2 t_4, \quad \bar{t}_1 \bar{t}_3 = t_2 t_4, \quad t_1 t_4 \in \mathbb{R}, \quad \text{and} \quad t_2 t_3 \in \mathbb{R}. \quad (88)$$

The second obstruction equation is $\partial\varphi^{(3)} = \varphi^{(1)} \bullet \varphi^{(2)} + \varphi^{(2)} \bullet \varphi^{(1)}$, for which we need further components of the Nijenhuis–Richardson Lie-admissible product. Notice that $\varphi^{(2)} \bullet \varphi^{(1)}$ since there are no generators dual to J or D so that we only need $\varphi^{(1)} \bullet \varphi^{(2)}$. The relevant products are tabulated in Table XIV.

Using that $u_3 = u_6 = 0$, we find that

$$\varphi^{(1)} \bullet \varphi^{(2)} = (u_1 t_5 + i\bar{t}_3 u_5) b_{13} + (t_5 u_2 - t_2 u_4) b_{14} + (i\bar{t}_4 u_5 - t_6 \bar{u}_1) b_{15} + (t_1 u_4 - t_6 \bar{u}_2) b_{16} + \text{c.c.} \quad (89)$$

Using Eq. (88), we see that $\varphi^{(1)} \bullet \varphi^{(2)} = 0$ identically. Therefore we can take $\varphi^{(3)} = 0$. Since $\varphi^{(2)} \bullet \varphi^{(2)} = 0$, we also have $\varphi^{(4)} = 0$ and indeed all higher $\varphi^{(i)} = 0$ as well.

In summary, the most general deformation is

$$\varphi = \sum_{i=1}^6 t_i c_i + \sum_{i=1}^6 \bar{t}_i \bar{c}_i + it_2 t_4 c_7 - i\bar{t}_2 \bar{t}_4 \bar{c}_7 + t_2 t_4 c_8 + \bar{t}_2 \bar{t}_4 \bar{c}_8 + t_3 t_6 c_{10} - \bar{t}_1 t_5 c_{11}, \quad (90)$$

subject to the integrability conditions in Eq. (88).

E. Isomorphism classes of deformations

The first three quadratic equations in (88) are equivalent to the vanishing of the following determinants:

$$\begin{vmatrix} t_4 & t_6 \\ t_3 & t_5 \end{vmatrix}, \quad \begin{vmatrix} \bar{t}_1 & t_6 \\ \bar{t}_2 & t_5 \end{vmatrix}, \quad \begin{vmatrix} \bar{t}_1 & t_4 \\ \bar{t}_2 & t_3 \end{vmatrix}, \quad (91)$$

which is equivalent to the columns being collinear. In other words, these equations imply that there exists $(x, y) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ such that

$$(\bar{t}_1, \bar{t}_2) = \alpha_1(x, y), \quad (t_4, t_3) = \alpha_2(x, y), \quad \text{and} \quad (t_6, t_5) = \alpha_3(x, y). \quad (92)$$

The fourth quadratic equation in (88) says that

$$t_1 t_3 = \bar{t}_2 \bar{t}_4 \iff (\bar{\alpha}_1 \alpha_2 - \alpha_1 \bar{\alpha}_2) \bar{x} y = 0. \quad (93)$$

The reality conditions in (88) say that the following are real:

$$\alpha_2 \alpha_3 x y, \quad \alpha_1 \alpha_3 x y, \quad \bar{\alpha}_1 \alpha_2 |x|^2, \quad \text{and} \quad \bar{\alpha}_1 \alpha_2 |y|^2. \quad (94)$$

Since $(x, y) \neq (0, 0)$, $|x|^2 + |y|^2 \neq 0$, and therefore $\bar{\alpha}_1 \alpha_2 \in \mathbb{R}$, which implies Eq. (93).

In summary, the integrability conditions are equivalent to Eq. (92) and the reality conditions

$$\bar{\alpha}_1 \alpha_2 \in \mathbb{R}, \quad \alpha_2 \alpha_3 x y \in \mathbb{R}, \quad \text{and} \quad \alpha_1 \alpha_3 x y \in \mathbb{R}. \quad (95)$$

We will now analyze the different branches of solutions of these equations.

Since $(x, y) \neq (0, 0)$, we have three possibilities:

- (1) $y = 0$ (and hence $x \neq 0$),
- (2) $x = 0$ (and hence $y \neq 0$), and
- (3) $xy \neq 0$.

The first two are related by the involutive automorphism τ of $\mathfrak{g}_{\mathbb{C}}$, defined by

$$\tau : (J, D, \mathbb{V}_0, \mathbb{V}_{\pm}, S_{\pm}) \mapsto (J, -D, \mathbb{V}_0, \mathbb{V}_{\mp}, S_{\mp}), \quad (96)$$

whose effect on the deformations is to exchange $x \leftrightarrow y$. Indeed, notice that on cochains τ exchanges

$$c_1 \leftrightarrow c_2, \quad c_3 \leftrightarrow c_4, \quad \text{and} \quad c_5 \leftrightarrow c_6, \quad (97)$$

which, at the level of the parameters in the infinitesimal deformation, is equivalent to exchanging

$$t_1 \leftrightarrow t_2, \quad t_3 \leftrightarrow t_4, \quad \text{and} \quad t_5 \leftrightarrow t_6, \quad (98)$$

which, from Eq. (92), can be seen to be equivalent to $x \leftrightarrow y$.

This leaves two branches, corresponding to (1) and (3).

1. Branch $y = 0$ and $x \neq 0$

Here $t_2 = t_3 = t_5 = 0$ and $t_1 = \bar{\alpha}_1 \bar{x}$, $t_4 = \alpha_2 x$, and $t_6 = \alpha_3 x$, with $\bar{\alpha}_1 \alpha_2 \in \mathbb{R}$. The corresponding deformation is

$$\varphi = \bar{\alpha}_1 \bar{x} c_1 + \alpha_2 x c_4 + \alpha_3 x c_6 + c.c. \quad (99)$$

We still have the possibility of acting with automorphisms

$$\mathbb{V}_{\pm} \mapsto \lambda_{\pm} \mathbb{V}_{\pm}, \quad \mathbb{V}_0 \mapsto \mu \mathbb{V}_0, \quad \text{and} \quad S_{\pm} \mapsto \xi_{\pm} S_{\pm}, \quad (100)$$

where $\lambda_{\pm}, \mu \in \mathbb{C}^{\times}$ and $\xi_{\pm} \in \mathbb{R}^{\times}$. The effect on cochains is

$$c_1 \mapsto \frac{\xi_+}{\mu \bar{\lambda}_+} c_1, \quad c_4 \mapsto \frac{\mu}{\xi_- \bar{\lambda}_+} c_4, \quad \text{and} \quad c_6 \mapsto \frac{\lambda_-}{\xi_- \mu} c_6, \quad (101)$$

and hence on parameters

$$t_1 \mapsto \frac{\mu \bar{\lambda}_+}{\xi_+} t_1, \quad t_4 \mapsto \frac{\xi_- \bar{\lambda}_+}{\mu} t_4, \quad \text{and} \quad t_6 \mapsto \frac{\xi_- \mu}{\lambda_-} t_6. \quad (102)$$

We now claim that if $\alpha_1 x$ is different from zero, we may bring it to 1. Indeed, suppose that $\alpha_1 \neq 0$. Then let $\mu = \frac{\xi_+}{\lambda_+ \bar{\alpha}_1 \bar{x}}$. If $\alpha_1 = 0$, then μ remains unconstrained. Suppose that $\alpha_2 \neq 0$, then if $\alpha_1 = 0$ we choose $\mu = \xi_- \lambda_+ \alpha_2 x$, whereas if $\alpha_1 \neq 0$, then choose $\xi_- = \frac{\xi_+}{|\lambda_+|^2 |x|^2 \bar{\alpha}_1 \alpha_2}$, which is indeed real. Finally, if $\alpha_3 \neq 0$, then choose $\lambda_- = \xi_- \mu \alpha_3 x$.

In summary, we arrive at eight isomorphism classes of deformations in this branch summarised in Table XV.

Working out the corresponding brackets, we find [up to the occasional rescaling and after applying the automorphism τ in (25)] the same list of Lie algebras as in Table III. In other words, for this branch at least, the classification of $d = 2$ agrees with that of $d > 3$.

2. Branch $xy \neq 0$

In this branch, we have

$$\begin{array}{llll} t_1 = \bar{\alpha}_1 \bar{x} & t_2 = \bar{\alpha}_1 \bar{y} & & \bar{\alpha}_1 \alpha_2 \in \mathbb{R}, \\ t_3 = \alpha_2 y & t_4 = \alpha_2 x & \text{subject to} & \alpha_2 \alpha_3 xy \in \mathbb{R}, \\ t_5 = \alpha_3 y & t_6 = \alpha_3 x & & \alpha_1 \alpha_3 xy \in \mathbb{R}, \end{array} \quad (103)$$

TABLE XV. Isomorphism classes of deformations ($d = 2$, $y = 0$).

α_1	α_2	α_3	Deformation φ
0	0	0	0
1	0	0	$\mathbb{V}^0 \bar{\mathbb{V}}^+ S_+ - \mathbb{V}^+ \bar{\mathbb{V}}^0 S_+$
0	1	0	$\sigma^- \mathbb{V}^+ \bar{\mathbb{V}}_0 + \sigma^- \bar{\mathbb{V}}^+ \bar{\mathbb{V}}_0$
0	0	1	$\sigma^- \mathbb{V}^0 \bar{\mathbb{V}}_- + \sigma^- \bar{\mathbb{V}}^0 \bar{\mathbb{V}}_-$
1	1	0	$\mathbb{V}^0 \bar{\mathbb{V}}^+ S_+ + \sigma^- \mathbb{V}^+ \bar{\mathbb{V}}_0 - \mathbb{V}^+ \bar{\mathbb{V}}^0 S_+ + \sigma^- \bar{\mathbb{V}}^+ \bar{\mathbb{V}}_0$
1	0	1	$\mathbb{V}^0 \bar{\mathbb{V}}^+ S_+ + \sigma^- \mathbb{V}^0 \bar{\mathbb{V}}_- - \mathbb{V}^+ \bar{\mathbb{V}}^0 S_+ + \sigma^- \bar{\mathbb{V}}^0 \bar{\mathbb{V}}_-$
0	1	1	$\sigma^- \mathbb{V}^+ \bar{\mathbb{V}}_0 + \sigma^- \mathbb{V}^0 \bar{\mathbb{V}}_- + \sigma^- \bar{\mathbb{V}}^+ \bar{\mathbb{V}}_0 + \sigma^- \bar{\mathbb{V}}^0 \bar{\mathbb{V}}_-$
1	1	1	$\mathbb{V}^0 \bar{\mathbb{V}}^+ S_+ + \sigma^- \mathbb{V}^+ \bar{\mathbb{V}}_0 + \sigma^- \mathbb{V}^0 \bar{\mathbb{V}}_- - \mathbb{V}^+ \bar{\mathbb{V}}^0 S_+ + \sigma^- \bar{\mathbb{V}}^+ \bar{\mathbb{V}}_0 + \sigma^- \bar{\mathbb{V}}^0 \bar{\mathbb{V}}_-$

TABLE XVI. Isomorphism classes of deformations ($d = 2$, $xy \neq 0$).

α_1	α_2	α_3	Deformation φ
0	0	0	0
*	0	0	$\mathfrak{c}_1 + \mathfrak{c}_2 + \text{C.C.}$
0	*	0	$\mathfrak{c}_3 + \mathfrak{c}_4 + \text{C.C.}$
0	0	*	$\mathfrak{c}_5 + \mathfrak{c}_6 + \text{C.C.}$
*	*	0	$\mathfrak{c}_1 + \mathfrak{c}_2 + \mathfrak{c}_3 + \mathfrak{c}_4 + i\mathfrak{c}_7 + \mathfrak{c}_8 + \text{C.C.}$
*	0	*	$(\mathfrak{c}_1 + \mathfrak{c}_2 + \varepsilon\mathfrak{c}_5 + \varepsilon\mathfrak{c}_6 + \text{C.C.}) - \varepsilon\mathfrak{c}_{11}$
0	*	*	$(\mathfrak{c}_3 + \mathfrak{c}_4 + \mathfrak{c}_5 + \mathfrak{c}_6 + \text{C.C.}) + \mathfrak{c}_{10}$
*	*	*	$(\varepsilon\mathfrak{c}_1 + \varepsilon\mathfrak{c}_2 + \mathfrak{c}_3 + \mathfrak{c}_4 + \mathfrak{c}_5 + \mathfrak{c}_6 + i\varepsilon\mathfrak{c}_7 + \varepsilon\mathfrak{c}_8 + \text{C.C.}) + \mathfrak{c}_{10} - \varepsilon\mathfrak{c}_{11}$

and we have the possibility of applying the automorphisms in (100), whose effect on t_i is as follows:

$$\begin{aligned}
 t_1 &\mapsto \frac{\mu\bar{\lambda}_+}{\xi_+} t_1 & t_2 &\mapsto \frac{\mu\bar{\lambda}_-}{\xi_-} t_2, \\
 t_3 &\mapsto \frac{\xi_+\lambda_-}{\mu} t_3 & \text{and} & & t_4 &\mapsto \frac{\xi_-\lambda_+}{\mu} t_4, \\
 t_5 &\mapsto \frac{\xi_+\mu}{\lambda_+} t_5 & t_6 &\mapsto \frac{\xi_-\mu}{\lambda_-} t_6.
 \end{aligned} \tag{104}$$

It is then straightforward to go through the eight possibilities, according to whether or not $\alpha_i = 0$ and bring these to normal forms which are tabulated in Table XVI, where an asterisk in the α_i column means that it is not zero. As before, we have introduced $\varepsilon = \pm 1$, which arises whenever $\alpha_1\alpha_3 \neq 0$ and corresponds to the sign of the nonzero (real) number $\alpha_1\alpha_3xy$.

Substituting the definitions (78) for the cochains and working out the Lie brackets, we find (up to the occasional rescaling) the same list of Lie algebras as in Table V. In other words, the deformation problem for $d = 2$ has the same solution *mutatis mutandis* as in $d > 3$, resulting in the same list of isomorphism classes of Lie algebras in Table VI.

V. CENTRAL EXTENSIONS

It is a natural question to ask whether a given Lie algebra admits central extensions, given the important rôle they play in applications. Central extensions of a Lie algebra²⁷ $\mathfrak{g}$ are classified by the second cohomology group $H^2(\mathfrak{g})$ with values in the trivial one-dimensional representation.

A. Central extensions for $d \geq 3$

Let $d \geq 3$. All graded conformal algebras have a semisimple subalgebra $\mathfrak{h}$ isomorphic to $\mathfrak{so}(d)$, namely, the span of J_{ab} . The factorisation theorem of Hochschild and Serre¹² implies that $H^2(\mathfrak{g}) \cong H^2(\mathfrak{g}, \mathfrak{h})$, the relative cohomology group computed from the complex of $\mathfrak{h}$ -invariant cochains with no legs along $\mathfrak{h}$. Since $\mathfrak{h}$ lies in degree 0 in every graded conformal algebra $\mathfrak{g}$, the complex $C^*(\mathfrak{g}, \mathfrak{h})$ breaks up into the direct sum of subcomplexes of a fixed degree. Since the grading element D acts trivially on cohomology and acts reducibly in the complex, we have further that $H^2(\mathfrak{g}, \mathfrak{h}) \cong H^{0,2}(\mathfrak{g}, \mathfrak{h})$, which can be computed from the degree-0 piece of the complex. This complex is

$$C^{0,p}(\mathfrak{g}, \mathfrak{h}) \cong (\wedge^p(\mathfrak{g}/\mathfrak{h})^*)^{\deg 0}, \tag{105}$$

with the differential induced from that of the Chevalley–Eilenberg complex of $\mathfrak{g}$.

For the graded conformal algebras under discussion and letting δ denote the dual to D , we have that

$$C^{0,1}(\mathfrak{g}, \mathfrak{h}) = \text{span}_{\mathbb{R}}(\delta) \quad \text{and} \quad C^{0,2}(\mathfrak{g}, \mathfrak{h}) = \text{span}_{\mathbb{R}}(\sigma^+ \sigma^- := \sigma^+ \wedge \sigma^-, v^+ v^- := v_a^+ \wedge v_a^-). \tag{106}$$

It is then a simple matter to go down Table VI and calculate $\partial\delta$, $\partial(\sigma^+ \sigma^-)$, and $\partial(v^+ v^-)$ to determine $H^2(\mathfrak{g})$ and hence the possible central extensions.

1. GCA_7 , GCA_3 , GCA_{10} , GCA_{16} , GCA_{17} , and GCA_{18}

These share the same differential,

$$\partial\delta = 0, \quad \partial\sigma^\pm = \mp\delta\sigma^\pm, \quad \text{and} \quad \partial v^\pm = \mp\delta v^\pm, \tag{107}$$

which implies

$$\partial(\sigma^+ \sigma^-) = 0 \quad \text{and} \quad \partial(v^+ v^-) = 0, \tag{108}$$

and hence

$$H^2 \cong \text{span}_{\mathbb{R}}(\sigma^+ \sigma^-, v^+ v^-). \tag{109}$$

This results in the additional brackets

$$[S_+, S_-] = Z_1 \quad \text{and} \quad [V_+, V_-] = Z_2, \quad (110)$$

where we have introduced two central generators Z_1 and Z_2 .

2. GCA_2 and GCA_5

These share the same differential,

$$\partial \delta = 0, \quad \partial \sigma^+ = -\delta \sigma^+, \quad \partial \sigma^- = \delta \sigma^- - v^0 v^-, \quad \text{and} \quad \partial v^\pm = \mp \delta v^\pm, \quad (111)$$

so that

$$\partial(\sigma^+ \sigma^+) = \sigma^+ v^0 v^- \quad \text{and} \quad \partial(v^+ v^-) = 0, \quad (112)$$

and hence

$$H^2 \cong \text{span}_{\mathbb{R}}(v^+ v^-), \quad (113)$$

with additional Lie brackets

$$[V_+, V_-] = Z. \quad (114)$$

3. GCA_4 and GCA_7

These share the same differential,

$$\partial \delta = 0, \quad \partial \sigma^\pm = \mp \delta \sigma^\pm, \quad \partial v^- = \delta v^-, \quad \text{and} \quad \partial v^+ = -\delta v^+ - \sigma^+ v^0, \quad (115)$$

so that

$$\partial(\sigma^+ \sigma^-) = 0 \quad \text{and} \quad \partial(v^+ v^-) = -\sigma^+ v^0 v^- \quad (116)$$

so that

$$H^2 \cong \text{span}_{\mathbb{R}}(\sigma^+ \sigma^-), \quad (117)$$

with additional brackets

$$[S_+, S_-] = Z. \quad (118)$$

4. GCA_6 and GCA_8

These share the same differential,

$$\partial \delta = 0, \quad \partial \sigma^+ = -\delta \sigma^+, \quad \partial \sigma^- = \delta \sigma^- - v^0 v^-, \quad \partial v^+ = -\delta v^+ - \sigma^- v^0, \quad \text{and} \quad \partial v^- = \delta v^-, \quad (119)$$

so that

$$\partial(\sigma^+ \sigma^-) = \sigma^+ v^0 v^- \quad \text{and} \quad \partial(v^+ v^-) = -\sigma^+ v^0 v^- \quad (120)$$

so that

$$H^2 \cong \text{span}_{\mathbb{R}}(\sigma^+ \sigma^- + v^+ v^-), \quad (121)$$

with additional brackets

$$[S_+, S_-] = Z \quad \text{and} \quad [V_+, V_-] = Z. \quad (122)$$

5. GCA_9

Here the differential is given by

$$\partial \delta = 0, \quad \partial \sigma^\pm = \mp \delta \sigma^\pm - v^0 v^\pm, \quad \text{and} \quad \partial v^\pm = \mp \delta v^\pm, \quad (123)$$

so that

$$\partial(\sigma^+ \sigma^-) = -\sigma^- v^0 v^+ + \sigma^+ v^0 v^-, \quad \text{and} \quad \partial(v^+ v^-) = 0 \quad (124)$$

so that

$$H^2 \cong \text{span}_{\mathbb{R}}(v^+ v^-), \quad (125)$$

with additional brackets

$$[V_+, V_-] = Z. \quad (126)$$

6. GCA_{11}

Here the differential is given by

$$\partial \delta = 0, \quad \partial \sigma^\pm = \mp \delta \sigma^\pm, \quad \text{and} \quad \partial v^\pm = \mp \delta v^\pm - \sigma^\pm v^0 \quad (127)$$

so that

$$\partial(\sigma^+ \sigma^-) = 0 \quad \text{and} \quad \partial(v^+ v^-) = -\sigma^+ v^0 v^- + \sigma^- v^0 v^+ \quad (128)$$

so that

$$H^2 \cong \text{span}_{\mathbb{R}}(\sigma^+ \sigma^-), \quad (129)$$

with additional brackets

$$[S_+, S_-] = Z. \quad (130)$$

7. GCA_{12}

Here the differential is given by

$$\partial \delta = -v^+ v^-, \quad \partial \sigma^\pm = \mp \delta \sigma^\pm - v^0 v^\pm, \quad \text{and} \quad \partial v^\pm = \mp \delta v^\pm \quad (131)$$

so that

$$\partial(\sigma^+ \sigma^-) = -\sigma^- v^0 v^+ + \sigma^+ v^0 v^- \quad \text{and} \quad \partial(v^+ v^-) = 0 \quad (132)$$

so that the only 2-cocycle is $v^+ v^-$, which is also a coboundary, so that $H^2 = 0$ and GCA_{12} admits no (nontrivial) central extensions.

8. $GCA_{13}^{(\varepsilon)}$

The differential is given by

$$\partial \delta = 0, \quad \partial \sigma^\pm = \mp \delta \sigma^\pm - \varepsilon v^0 v^\pm, \quad \text{and} \quad \partial v^\pm = \mp \delta v^\pm - \sigma^\pm v^0 \quad (133)$$

so that

$$\partial(\sigma^+ \sigma^-) = \varepsilon \sigma^+ v^0 v^- - \varepsilon \sigma^- v^0 v^+ \quad \text{and} \quad \partial(v^+ v^-) = -\sigma^+ v^0 v^- + \sigma^- v^0 v^+, \quad (134)$$

and hence

$$H^2 \cong \text{span}_{\mathbb{R}}(\sigma^+ \sigma^- + \varepsilon v^+ v^-), \quad (135)$$

with additional brackets

$$[S_+, S_-] = Z \quad \text{and} \quad [V_+, V_-] = \varepsilon Z. \quad (136)$$

This central extension is isomorphic to the “pseudo-Bargmann” Lie algebra which has recently appeared in Ref. 13, where it is exhibited as a contraction of a trivial central extension of the de Sitter algebras in $d + 2$ dimensions.

9. GCA_{14}

Here the differential is given by

$$\partial \delta = -\sigma^+ \sigma^-, \quad \partial \sigma^\pm = \mp \delta \sigma^\pm, \quad \text{and} \quad \partial v^\pm = \mp \delta v^\pm - \sigma^\pm v^0 \quad (137)$$

so that

$$\partial(\sigma^+ \sigma^-) = 0 \quad \text{and} \quad \partial(v^+ v^-) = -\sigma^+ v^0 v^- + \sigma^- v^0 v^+ \quad (138)$$

so that the only 2-cocycle is $\sigma^+ \sigma^-$, which is also a coboundary, so that $H^2 = 0$ and GCA_{14} admits no (nontrivial) central extensions.

Lie algebra GCA_{14} is isomorphic to the Galilean conformal algebra of Ref. 14, which was known not to admit a central extension. The construction in Ref. 14 is precisely the contraction described above from the de Sitter algebras.

10. $GCA_{15}^{(\varepsilon)}$

These are semisimple Lie algebras and, by the second Whitehead Lemma, the second cohomology $H^2 = 0$.

11. GCA_{19}

Here the differential is given by

$$\begin{aligned} \partial \delta &= 0, \\ \partial \sigma^+ &= -\delta \sigma^+, \\ \partial \sigma^- &= \delta \sigma^- - v^0 v^-, \\ \partial v_a^+ &= -\delta v_a^+ - \epsilon_{abc} v_b^0 v_c^+, \\ \partial v^- &= \delta v^- - \sigma^- v^0 \end{aligned} \quad (139)$$

so that

$$\partial(\sigma^+ \sigma^-) = \sigma^+ v^0 v^- \quad \text{and} \quad \partial(v^+ v^-) = \sigma^- v^0 v^+ - v^0 v^+ v^- \quad (140)$$

so that $H^2 = 0$, and this conformal algebra admits no (nontrivial) central extensions. This result also follows with little or no calculation by using Hochschild–Serre relative to the semisimple subalgebra spanned by J and V_0 which is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$. In this case, the relative complex has no cochains in degree 0.

TABLE XVII. Central extensions of graded conformal algebras ($d \geq 3$).

Label	Additional central brackets		$\dim H^2$
GCA ₁	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA ₂	$[V_+, V_-] = Z$		1
GCA ₃	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA ₄		$[S_+, S_-] = Z$	1
GCA ₅	$[V_+, V_-] = Z$		1
GCA ₆	$[V_+, V_-] = Z$	$[S_+, S_-] = Z$	1
GCA ₇		$[S_+, S_-] = Z$	1
GCA ₈	$[V_+, V_-] = Z$	$[S_+, S_-] = Z$	1
GCA ₉	$[V_+, V_-] = Z$		1
GCA ₁₀	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA ₁₁		$[S_+, S_-] = Z$	1
GCA ₁₂			0
GCA ₁₃ ^($\varepsilon = \pm 1$)	$[V_+, V_-] = \varepsilon Z$	$[S_+, S_-] = Z$	1
GCA ₁₄			0
GCA ₁₅ ^($\varepsilon = \pm 1$)			0
GCA ₁₆	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA ₁₇	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA ₁₈	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA ₁₉			0
GCA ₂₀ ^($\varepsilon = +1$)	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA ₂₀ ^($\varepsilon = -1$)		$[S_+, S_-] = Z$	1

12. GCA₂₀^(ε)

The differential is given by

$$\begin{aligned}
 \partial \delta &= 0, \\
 \partial \sigma^\pm &= \mp \delta \sigma^\pm, \\
 \partial v_a^+ &= -\delta v_a^+ - \epsilon_{abc} v_b^0 v_c^+, \\
 \partial v_a^- &= \delta v_a^- = \delta v_a^- - \varepsilon \epsilon_{abc} v_b^0 v_c^-
 \end{aligned} \tag{141}$$

so that

$$\partial(\sigma^+ \sigma^-) = 0 \quad \text{and} \quad \partial(v^+ v^-) = (\varepsilon - 1) v^0 v^+ v^- \tag{142}$$

and hence

$$H^2 \cong \begin{cases} \text{span}_{\mathbb{R}}(\sigma^+ \sigma^-, v^+ v^-) & \varepsilon = 1 \\ \text{span}_{\mathbb{R}}(\sigma^+ \sigma^-) & \varepsilon = -1, \end{cases} \tag{143}$$

with additional brackets

$$\begin{cases} [S_+, S_-] = Z_1 & \text{and} & [V_+, V_-] = Z_2, & \varepsilon = 1 \\ [S_+, S_-] = Z, & & & \varepsilon = -1. \end{cases} \tag{144}$$

The results are summarised in [Table XVII](#).

B. Invariant inner products

It often happens that a Lie algebra does not admit an invariant inner product, yet a central extension of it does. For this to happen, the degeneracy of the invariant inner product in the original Lie algebra should be curable by adding central elements. For the kind of Lie algebras under consideration, this happens if and only if the original Lie algebra has an invariant symmetric bilinear form with a one-dimensional kernel spanned by D . It will turn out that there is only one additional metric Lie algebra: a one-dimensional central extension of GCA₁₆, which only exists for $d = 3$. In this section, we provide the details.

First of all, we notice that (suppressing indices)

$$\langle V_+, V_- \rangle = \langle V_+, [V_-, J] \rangle = \langle [V_+, V_-], J \rangle \tag{145}$$

so that unless $\langle [V_+, V_-], J \rangle \neq 0$, any invariant bilinear form is degenerate. Clearly no central term in $[V_+, V_-]$ contributes to $\langle [V_+, V_-], J \rangle$, so the only way $\langle [V_+, V_-], J \rangle$ can be nonzero is if $[V_+, V_-] = J + \dots$ or, if $d = 3$, $[V_+, V_-] = V_0 + \dots$. This occurs only for GCA₁₂, GCA₁₅^(ε), GCA₁₆, GCA₁₇, and GCA₁₈. The Lie algebras GCA₁₂ and GCA₁₅^(ε) do not admit central extensions, so we concentrate on the rest.

No central extension of the Lie algebras GCA_{17} and GCA_{18} admit an invariant inner product. Indeed, omitting indices,

$$\langle V_0, V_0 \rangle = \langle V_0, [V_0, J] \rangle = \langle [V_0, V_0], J \rangle = 0 \quad (146)$$

and

$$\langle J, V_0 \rangle = \langle J, [S_+, V_-] \rangle = \langle [J, S_+], V_- \rangle = 0. \quad (147)$$

Therefore $\langle V_0, - \rangle = 0$, and any invariant symmetric bilinear form is degenerate.

Finally, let us consider central extensions of GCA_{16} , whose Lie brackets, in addition to (3), are (omitting indices)

$$[V_+, V_-] = V_0 + Z_1 \quad \text{and} \quad [S_+, S_-] = Z_2. \quad (148)$$

The two-dimensional central extension does not admit an invariant inner product, but imposing a linear relation between Z_1 and Z_2 results in a metric Lie algebra. We may describe it as a double extension^{15,16} of the abelian Lie algebra $\mathfrak{a}$ with generators V_+ , V_- , S_+ , S_- with (trivially invariant) inner product

$$\langle V_+, V_- \rangle = \alpha \delta_{ab} \quad \text{and} \quad \langle S_+, S_- \rangle = \beta, \quad (149)$$

for $\alpha, \beta \neq 0$. The Lie algebra $\mathfrak{g} \cong \mathfrak{co}(3)$ spanned by $R_a = -\frac{1}{2}\epsilon_{abc}J_{bc}$ and D acts on $\mathfrak{a}$ via skew-symmetric derivations,

$$[R_a, V_{\pm b}] = \epsilon_{abc}V_{\pm c}, \quad [D, V_{\pm a}] = \pm V_{\pm a}, \quad \text{and} \quad [D, S_{\pm}] = \pm S_{\pm}. \quad (150)$$

To construct the double extension, we add generators V_0 dual to R_a and Z dual to D , together with the dual pairings

$$\langle R_a, V_{0b} \rangle = \delta_{ab} \quad \text{and} \quad \langle D, Z \rangle = 1 \quad (151)$$

and extend the brackets of $\mathfrak{a}$ by

$$[V_+, V_-] = \alpha \epsilon_{abc}V_{0c} + \alpha \delta_{ab}Z \quad \text{and} \quad [S_+, S_-] = \beta Z. \quad (152)$$

Choosing $\alpha = 1$, we see that this is a quotient of the universal central extension of GCA_{16} where $Z_1 = Z$ and $Z_2 = \beta Z$. The most general invariant inner product is given by

$$\begin{aligned} \langle V_+, V_- \rangle &= \delta_{ab} & \langle D, Z \rangle &= 1, \\ \langle S_+, S_- \rangle &= \beta & \text{and} & \quad \langle D, D \rangle = \lambda, \\ \langle R_a, V_{0b} \rangle &= \delta_{ab} & \langle R_a, R_b \rangle &= \mu \delta_{ab}, \end{aligned} \quad (153)$$

for all $\lambda, \mu \in \mathbb{R}$ and $\beta \in \mathbb{R}^\times$.

C. Central extensions for $d = 2$

In $d = 2$, the calculation differs because $\mathfrak{so}(2)$ is not semisimple and the factorisation theorem does not apply. Nevertheless the two-dimensional abelian subalgebra $\mathfrak{r} = \text{span}_{\mathbb{R}}(J, D)$ acts reducibly and trivially on cohomology, and therefore we may work with the subcomplex of $\mathfrak{r}$ -invariant cochains. The 1- and 2-cochains are as follows:

$$\begin{aligned} C^1 &= \text{span}_{\mathbb{R}}(\rho, \delta), \\ C^2 &= \text{span}_{\mathbb{R}}(\rho\delta, \delta_{ab}v_a^+v_b^-, \epsilon_{ab}v_a^+v_b^-, \epsilon_{ab}v_a^0v_b^0, \sigma^+\sigma^-). \end{aligned} \quad (154)$$

The calculations are routine, and we only list the result, which is summarised in Table XVIII.

D. Invariant inner products

A natural question is whether there are any centrally extended Lie algebras for $d = 2$ which admit an invariant inner product. Again, there are some Lie algebras which do not admit central extensions: GCA_{12} , $GCA_{13}^{(\epsilon)}$, and $GCA_{15}^{(\epsilon)}$, and hence we will not consider them further. In $d = 2$, an invariant inner product must have $\langle V_0, V_0 \rangle$ and $\langle V_+, V_- \rangle$ nonzero, but (omitting indices)

$$\langle V_+, V_- \rangle = \langle V_+, [V_-, J] \rangle = \langle [V_+, V_-], J \rangle \quad (155)$$

and, similarly,

$$\langle V_0, V_0 \rangle = \langle V_0, [V_0, J] \rangle = \langle [V_0, V_0], J \rangle, \quad (156)$$

where now any of J, D, Z_i can have nonzero inner product with J .

Every central extension of GCA_4 and GCA_{11} has $[V_+, V_-] = 0$, and every central extension of GCA_{10} has $[V_0, V_0] = 0$; hence, they cannot be metric. Neither can any central extension of GCA_7 or GCA_{14} because (omitting indices)

TABLE XVIII. Central extensions of graded conformal algebras ($d = 2$).

Label	Additional central brackets				dim H^2
GCA ₁	$[J, D] = Z_1$	$[V_+ a, V_- b] = \delta_{ab} Z_2 + \epsilon_{ab} Z_3$	$[V_0 a, V_0 b] = \epsilon_{ab} Z_4$	$[S_+, S_-] = Z_5$	5
GCA ₂	$[J, D] = Z_1$	$[V_+ a, V_- b] = \delta_{ab} Z_2 + \epsilon_{ab} Z_3$	$[V_0 a, V_0 b] = \epsilon_{ab} Z_4$		4
GCA ₃	$[J, D] = Z_1$	$[V_+ a, V_- b] = \delta_{ab} Z_2 + \epsilon_{ab} Z_3$		$[S_+, S_-] = Z_4$	4
GCA ₄	$[J, D] = Z_1$		$[V_0 a, V_0 b] = \epsilon_{ab} Z_2$	$[S_+, S_-] = Z_3$	3
GCA ₅	$[J, D] = Z_1$	$[V_+ a, V_- b] = \delta_{ab} Z_2 + \epsilon_{ab} Z_3$			3
GCA ₆	$[J, D] = Z_1$	$[V_+ a, V_- b] = \delta_{ab} Z_2$	$[V_0 a, V_0 b] = \epsilon_{ab} Z_3$	$[S_+, S_-] = -Z_2$	3
GCA ₇	$[J, D] = Z_1$	$[V_+ a, V_- b] = \epsilon_{ab} Z_2$	$[V_0 a, V_0 b] = -\epsilon_{ab} Z_2$	$[S_+, S_-] = Z_3$	3
GCA ₈	$[J, D] = Z_1$	$[V_+ a, V_- b] = \delta_{ab} Z_2 + \epsilon_{ab} Z_3$	$[V_0 a, V_0 b] = -\epsilon_{ab} Z_3$	$[S_+, S_-] = Z_2$	3
GCA ₉	$[J, D] = Z_1$	$[V_+ a, V_- b] = \delta_{ab} Z_2 + \epsilon_{ab} Z_3$	$[V_0 a, V_0 b] = \epsilon_{ab} Z_4$		4
GCA ₁₀	$[J, D] = Z_1$	$[V_+ a, V_- b] = \delta_{ab} Z_2 + \epsilon_{ab} Z_3$		$[S_+, S_-] = Z_4$	4
GCA ₁₁	$[J, D] = Z_1$		$[V_0 a, V_0 b] = \epsilon_{ab} Z_2$	$[S_+, S_-] = Z_3$	3
GCA ₁₂					0
GCA ₁₃ ^($\epsilon = \pm 1$)					0
GCA ₁₄		$[V_+ a, V_- b] = \epsilon_{ab} Z_1$	$[V_0 a, V_0 b] = -\epsilon_{ab} Z_1$	$[S_+, S_-] = Z_2$	2
GCA ₁₅ ^($\epsilon = \pm 1$)					0

$$\langle V_0, V_0 \rangle = \langle V_0, [S_+, V_-] \rangle = \langle [V_-, V_0], S_+ \rangle = 0. \quad (157)$$

Similarly, no central extension of GCA₃ or GCA₅ can be metric because

$$\langle V_0, V_0 \rangle = \langle V_0, [S_+, V_-] \rangle = \langle [V_0, S_+], V_- \rangle = 0. \quad (158)$$

No central extension of GCA₂ can be metric,

$$\langle S_+, S_- \rangle = \langle S_+, [V_0, V_-] \rangle = \langle [S_+, V_0], V_- \rangle = 0, \quad (159)$$

and neither can any central extension of GCA₆,

$$\langle S_+, S_- \rangle = \langle S_+, [V_0, V_-] \rangle = \langle [V_-, S_+], V_0 \rangle = 0, \quad (160)$$

or of GCA₉,

$$\langle S_+, S_- \rangle = \langle [V_0, V_+], S_- \rangle = \langle V_0, [V_+, S_0] \rangle = 0. \quad (161)$$

This leaves only GCA₁ and GCA₈ to consider. We will see that in each case there is a quotient of the universal central extension which is metric. We will do this by exhibiting them as suitable double extensions.^{15,16}

Let $\mathfrak{a}$ denote the abelian Lie algebra spanned by $S_\pm, V_\pm, V_0$ and with inner product

$$\langle S_+, S_- \rangle = \alpha, \quad \langle V_+ a, V_- b \rangle = \beta \delta_{ab}, \quad \text{and} \quad \langle V_0 a, V_0 b \rangle = \gamma \delta_{ab}, \quad (162)$$

for some $\alpha, \beta, \gamma \in \mathbb{R}^\times$. Then J and D act on $\mathfrak{a}$ via (3) preserving the inner product. Introduce central elements Z and Z' dual to J and D , respectively, so that

$$\langle Z, J \rangle = 1 \quad \text{and} \quad \langle Z', D \rangle = 1, \quad (163)$$

and centrally extend $\mathfrak{a}$ as follows:

$$[V_0 a, V_0 b] = \gamma \epsilon_{ab} Z, \quad [V_+ a, V_- b] = \beta \epsilon_{ab} Z + \beta \delta_{ab} Z', \quad \text{and} \quad [S_+, S_-] = \alpha Z'. \quad (164)$$

If we normalise the inner product by setting $\beta = 1$, then we see that this metric Lie algebra corresponds to the quotient of the universal central extension of GCA₁ in Table XVIII where $Z_1 = 0$, $Z_2 = Z'$, $Z_3 = Z$, $Z_4 = \gamma Z$, and $Z_5 = \alpha Z'$. The most general invariant inner product (up to scale) is given by Eqs. (162) and (163) and any symmetric bilinear form on the span of J and D .

Finally, let $\mathfrak{g}$ denote the Lie algebra spanned by $S_\pm, V_\pm, V_0$ subject to the brackets (omitting indices)

$$[S_+, V_-] = V_0, \quad [S_+, V_0] = V_+, \quad \text{and} \quad [V_0, V_-] = S_-. \quad (165)$$

The Lie algebra $\mathfrak{g}$ is metric relative to the following inner product:

$$\langle S_+, S_- \rangle = 1, \quad \langle V_0 a, V_0 b \rangle = -\delta_{ab}, \quad \text{and} \quad \langle V_+ a, V_+ b \rangle = \delta_{ab}. \quad (166)$$

The generators J and D act on $\mathfrak{g}$ via (3) preserving both the brackets and the inner product. Therefore we can introduce dual generators Z and Z' to J and D , respectively, with the corresponding dual pairing,

$$\langle J, Z \rangle = 1 \quad \text{and} \quad \langle D, Z' \rangle = 1, \quad (167)$$

and centrally extend $\mathfrak{g}$ via

$$[V_{0a}, V_{0b}] = -\epsilon_{ab}Z, \quad [V_{+a}, V_{-b}] = \epsilon_{ab}Z + \delta_{ab}Z', \quad \text{and} \quad [S_+, S_-] = Z'. \quad (168)$$

The resulting Lie algebra is metric relative to the inner product given by (166) and (167) and any symmetric bilinear form in the span of J and D . Comparing with Table XVIII, we see that this is the quotient of the universal central extension of GCA_8 where $Z_1 = 0$, $Z_2 = Z'$, and $Z_3 = Z$.

VI. GENERALISED CONFORMAL ALGEBRAS

A more general notion of conformal algebra results from dropping the requirement that the Lie algebra be graded by the action of D . In other words, a modified definition of conformal algebra might be the following:

Definition 2. By a **generalised conformal Lie algebra** with d -dimensional space isotropy, we mean a real $\frac{1}{2}(d+2)(d+3)$ -dimensional Lie algebra with generators $J_{ab} = -J_{ba}$, with $1 \leq a, b \leq d$, spanning a Lie subalgebra $\mathfrak{r} \cong \mathfrak{so}(d)$, that is,

$$[J_{ab}, J_{cd}] = \delta_{bc}J_{ad} - \delta_{ac}J_{bd} - \delta_{bd}J_{ac} + \delta_{ad}J_{bc}, \quad (169)$$

and $3d+3$ generators: rotational vectors V_a^i , for $i = 1, 2, 3$, and rotational scalars S^A , for $A = 1, 2, 3$. In other words, under $\mathfrak{so}(d)$, they transform like

$$[J_{ab}, V_c^i] = \delta_{bc}V_a^i - \delta_{ac}V_b^i \quad \text{and} \quad [J_{ab}, S^A] = 0. \quad (170)$$

The rest of the brackets between V_a^i and S^A are only subject to the Jacobi identity: in particular, they must be $\mathfrak{r}$ -equivariant.

In this definition, we have dropped the condition that D is a grading element. It is not clear why one should call such an algebra conformal since all it shares with the simple conformal algebras is the existence of an $\mathfrak{so}(d)$ subalgebra which acts in the same way on the additional generators. But these algebras form a large class of Lie algebras with relation to the other candidates for the conformal fellowship.

The classification problem for generalised conformal algebras is much more involved than for the graded conformal algebras treated in the bulk of this paper, and we will not solve it here. Nevertheless let us make a few comments. If $d = 0$, a conformal Lie algebra has dimension 3 and hence this agrees with the Bianchi classification of real three-dimensional Lie algebras¹⁷ (see Ref. 18 for an English translation). By contrast, the graded conformal algebras for $d = 0$ are Bianchi VI₀ and Bianchi VIII. If $d = 1$, there are no rotations and any real Lie algebra of dimension 6 is conformal. Six-dimensional real Lie algebras have been classified in a series of papers starting with Morozov,¹⁹ who classified nilpotent six-dimensional Lie algebras, then Mubarakzjanov,²⁰ who classified solvable six-dimensional solvable Lie algebras with five-dimensional nil-radical, and Turkowski,²¹ who classified six-dimensional solvable Lie algebras with a four-dimensional nil-radical. Having thus classified the solvable six-dimensional Lie algebras, it remains to classify the non-solvable ones, which can be performed in principle via the Levi decomposition, although I am not aware of any published list. The case of the graded conformal algebras for $d = 1$ has not been studied.

A. The deformation complex for $d \geq 5$

In this section, we will take $d \geq 5$ that being the generic range of dimensions. As before, the deformation theory approach to the classification of generalised conformal Lie algebras consists in a perturbative approach to the solution of the corresponding Maurer–Cartan equation for the difference between the Lie brackets of the Lie algebras we want to classify and the ones of the static Lie algebra, here denoted $\mathfrak{g}$. This approach consists in the first instance in the calculation of the relative Lie algebra cohomology group $H^2(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$, with $\mathfrak{r}$ being the rotational subalgebra, which classifies the infinitesimal deformations. The obstructions to integrating infinitesimal deformations live in $H^3(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$ although it is seldom necessary or convenient to calculate that cohomology group. What we will need to compute is the Nijenhuis–Richardson bracket on the cochains

$$[[-, -]] : C^2(\mathfrak{g}, \mathfrak{r}; \mathfrak{g}) \times C^2(\mathfrak{g}, \mathfrak{r}; \mathfrak{g}) \rightarrow C^3(\mathfrak{g}, \mathfrak{r}; \mathfrak{g}), \quad (171)$$

which is symmetric in this degree.

In this section, we will determine the first few spaces $C^\bullet(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$ of cochains, the action of the differential and the Nijenhuis–Richardson bracket.

We start by introducing a convenient notation to perform the necessary calculations. We will let V_a^i , for $i = 1, 2$, and 3 , stand for B_a , P_a , and K_a , respectively. Similarly we will let S^A , for $A = 1, 2$, and 3 , stand for D , H , and L , respectively. In this notation, the static conformal Lie algebra $\mathfrak{g}$ is the real span of J_{ab} , V_a^i , and S^A and subject to the following brackets:

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac} + \delta_{ad} J_{bc}, \\
[J_{ab}, V_c^i] &= \delta_{bc} V_a^i - \delta_{ac} V_b^i, \\
[J_{ab}, S^A] &= 0,
\end{aligned} \tag{172}$$

with all other brackets vanishing.

We will let $\mathbb{W}$ denote the real vector space spanned by V_a^i and S^A . The canonical dual basis for $\mathbb{W}^*$ is denoted v_{ai} and σ_A , respectively. We may use the rotational invariant δ_{ab} to raise and lower the vector $\mathfrak{so}(d)$ indices with impunity, but the automorphism group of the deformation complex, which is isomorphic to $GL(3, \mathbb{R}) \times GL(3, \mathbb{R})$ acting in the natural way on the three copies of the vector representation of $\mathfrak{so}(d)$ and on the three copies of the scalar representation of $\mathfrak{so}(d)$, does distinguish between these representations and their duals, hence the need to be careful with where we position the indices.

The cochain complex $C^\bullet := C^\bullet(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$ is isomorphic to $(\wedge^\bullet \mathbb{W}^* \otimes \mathfrak{g})^\mathfrak{r}$, and we proceed to enumerate the first few spaces. Omitting the $\wedge$ and $\otimes$ products, we see that

- (0) C^0 is spanned by the three cochains S^A , for $A = 1, 2, 3$;
- (1) C^1 is spanned by the $18 = 9 + 9$ cochains $\sigma_A S^B$ and $v_{ai} V_a^j$, where we are summing over a in order to arrive at a rotational scalar;
- (2) C^2 is spanned by the $51 = 9 + 27 + 6 + 9$ cochains $\sigma_A \sigma_B S^C$, $\sigma_A v_{ai} V_a^j$, $v_{ai} v_{bj} J_{ab}$, and $v_{ai} v_{aj} S^A$; and
- (3) C^3 is spanned by the $102 = 3 + 27 + 18 + 27 + 27$ cochains $\sigma_A \sigma_B \sigma_C S^D$, $\sigma_A \sigma_B v_{ai} V_a^j$, $\sigma_A v_{ai} v_{bj} J_{ab}$, $\sigma_A v_{ai} v_{aj} S^B$, and $v_{ai} v_{aj} v_{bk} V_b^l$.

The differential $\partial: C^\bullet \rightarrow C^{\bullet+1}$ is uniquely defined by its action on the generators,

$$\partial S^A = \partial V_a^i = \partial \sigma_A = \partial v_{ai} = 0 \quad \text{and} \quad \partial J_{ab} = v_{ai} V_b^i - v_{bi} V_a^i. \tag{173}$$

B. Infinitesimal deformations

It follows from the above expression for the differential that $\partial: C^0 \rightarrow C^1$ and $\partial: C^1 \rightarrow C^2$ are the zero maps, and thus the space of infinitesimal deformations is naturally isomorphic to the kernel of $\partial: C^2 \rightarrow C^3$. Computing the differential on the above basis for C^2 , we find

$$\partial(\sigma_A \sigma_B S^C) = 0, \quad \partial(\sigma_A v_{ai} V_a^j) = 0, \quad \partial(v_{ai} v_{bj} J_{ab}) = v_{ak} (v_{ai} v_{bj} + v_{aj} v_{bi}) V_b^k, \quad \partial(v_{ai} v_{aj} S^A) = 0. \tag{174}$$

Therefore $H^2(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$ is a real 45-dimensional vector space spanned by

$$\sigma_A \sigma_B S^C, \quad \sigma_A v_{ai} V_a^j, \quad \text{and} \quad v_{ai} v_{aj} S^A. \tag{175}$$

The general infinitesimal deformation is given by

$$\varphi_1 = t_C^{AB} \sigma_A \sigma_B S^C + t_j^{Ai} \sigma_A v_{ai} V_a^j + t_A^{ij} v_{ai} v_{aj} S^A, \tag{176}$$

for real parameters $t_C^{AB} = -t_C^{BA}$, t_j^{Ai} , and $t_A^{ij} = -t_A^{ji}$.

C. Obstructions to integrability

In order to explore the integrability properties of the infinitesimal deformations, we will need to calculate the Nijenhuis–Richardson bracket on C^2 . This is tabulated below in [Table XIX](#), where we have abbreviated the notation by omitting the $\wedge$ and $\otimes$ products.

TABLE XIX. Nijenhuis–Richardson •.

•	$\sigma_D \sigma_E S^F$	$\sigma_D v_{ck} V_c^l$	$v_{ck} v_{cl} J_{cd}$	$v_{ck} v_{cl} S^D$
$\sigma_A \sigma_B S^C$	$\sigma_A \sigma_B (\delta_D^C \sigma_E - \delta_E^C \sigma_D) S^F$	$\delta_D^C \sigma_A \sigma_B v_{ck} V_c^l$	0	0
$\sigma_A v_{ai} V_a^j$	0	$\delta_k^j \sigma_A \sigma_D v_{ai} V_a^l$	$\sigma_A v_{ai} (\delta_k^j v_{cl} + \delta_l^j v_{ck}) J_{ac}$	$\sigma_A v_{ai} (\delta_k^j v_{al} - \delta_l^j v_{ak}) S^D$
$v_{ai} v_{bj} J_{ab}$	0	0	0	0
$v_{ai} v_{bj} S^A$	$v_{ai} v_{aj} (\delta_D^A \sigma_E - \delta_E^A \sigma_D) S^F$	$\delta_D^A v_{ai} v_{aj} v_{ck} V_c^l$	0	0

From Table XIX, it is easy to write down the Nijenhuis–Richardson bracket, which in this degree is the anticommutator of the (non-associative) $\bullet$ product. The nonzero brackets are given by

$$\begin{aligned} \llbracket \sigma_A \sigma_B S^C, \sigma_D \sigma_E S^F \rrbracket &= \sigma_A \sigma_B (\delta_D^C \sigma_E - \delta_E^C \sigma_D) S^F + \sigma_D \sigma_E (\delta_A^F \sigma_B - \delta_B^F \sigma_A) S^C, \\ \llbracket \sigma_A \sigma_B S^C, \sigma_D v_{ck} V_c^\ell \rrbracket &= \delta_D^C \sigma_A \sigma_B v_{ck} V_c^\ell, \\ \llbracket \sigma_A \sigma_B S^C, v_{ck} v_{cl} S^D \rrbracket &= v_{ck} v_{cl} (\delta_A^D \sigma_B - \delta_B^D \sigma_A) S^C, \\ \llbracket \sigma_A v_{ai} V_a^j, \sigma_D v_{ck} V_c^\ell \rrbracket &= \sigma_A \sigma_D (\delta_k^j v_{ai} V_a^\ell - \delta_i^\ell v_{ak} V_a^j), \\ \llbracket \sigma_A v_{ai} V_a^j, v_{ck} v_{cl} J_{cd} \rrbracket &= \sigma_A v_{ai} (\delta_k^j v_{cl} + \delta_l^j v_{ck}) J_{ac}, \\ \llbracket \sigma_A v_{ai} V_a^j, v_{ck} v_{cl} S^D \rrbracket &= \sigma_A v_{ai} (\delta_k^j v_{al} - \delta_l^j v_{ak}) S^D + \delta_A^D v_{ck} v_{cl} v_{ai} V_a^j \end{aligned} \quad (177)$$

Let φ_1 be the general infinitesimal deformation in Eq. (176). The first obstruction to integrating φ_1 is the cohomology class of $\frac{1}{2} \llbracket \varphi_1, \varphi_1 \rrbracket$ in $H^3(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$. From Eq. (177), we calculate

$$\begin{aligned} \frac{1}{2} \llbracket \varphi_1, \varphi_1 \rrbracket &= 2t_C^{AB} t_F^{CE} \sigma_A \sigma_B \sigma_E S^F + \left(t_C^{AB} t_\ell^{Ck} + \frac{1}{2} t_j^{Ak} t_\ell^{Bj} - \frac{1}{2} t_j^{Bk} t_\ell^{Aj} \right) \sigma_A \sigma_B v_{ak} V_a^\ell \\ &\quad + 2 \left(t_C^{AB} t_A^{k\ell} + \frac{1}{2} t_i^{Bk} t_C^{i\ell} - \frac{1}{2} t_i^{B\ell} t_C^{ik} \right) \sigma_B v_{bk} v_{bl} S^C + t_j^{Ai} t_A^{k\ell} v_{ai} v_{bk} v_{bl} V_a^j. \end{aligned} \quad (178)$$

From Eq. (174), the only way that this can be $\partial\varphi_2$ for some $\varphi_2 \in C^2$ is if the following equations are satisfied:

$$\begin{aligned} t_C^{[AB]E} t_F^C &= 0, \\ t_j^{Ak} t_\ell^{Bj} - t_j^{Bk} t_\ell^{Aj} &= -2t_C^{AB} t_\ell^{Ck}, \\ t_i^{Bk} t_C^{i\ell} - t_i^{B\ell} t_C^{ik} &= -2t_C^{AB} t_A^{k\ell}, \\ 2t_j^{Ai} t_A^{k\ell} &= u^{li} \delta_j^k - u^{ki} \delta_j^\ell \quad \exists u^{ij} = u^{ji}, \end{aligned} \quad (179)$$

where $\varphi_2 = \frac{1}{2} u^{ij} v_{ai} v_{bj} J_{ab}$. Assuming these equations, the next obstruction is the class in $H^3(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$ of

$$\llbracket \varphi_1, \varphi_2 \rrbracket = t_j^{Ai} u^{j\ell} \sigma_A v_{ai} v_{cl} J_{ac}. \quad (180)$$

The only way that this can be a coboundary is if it vanishes identically, which becomes the second-order obstruction

$$u^{j\ell} t_j^{Ai} + u^{ji} t_j^{A\ell} = 0. \quad (181)$$

If this is the case, then we can take $\varphi_3 = 0$ and since $\llbracket \varphi_2, \varphi_2 \rrbracket = 0$ identically, we can also take $\varphi_{n>3} = 0$ and the deformation integrates. In summary, the integrability domain is the locus of Eqs. (179) and (181).

Contracting the last equation in (179) with δ_k^j , we can solve

$$u^{li} = t_k^{Ai} t_A^{k\ell}, \quad (182)$$

which implies for consistency the symmetry in $\ell \leftrightarrow i$ of the RHS,

$$t_k^{Ai} t_A^{k\ell} = t_k^{A\ell} t_A^{ki}. \quad (183)$$

In summary, the most general deformation is given by

$$\varphi = t_C^{AB} \sigma_A \sigma_B S^C + t_j^{Ai} \sigma_A v_{ai} V_a^j + t_A^{ij} v_{ai} v_{aj} S^A + \frac{1}{2} t_k^{Aj} t_A^{ki} v_{ai} v_{bj} J_{ab}, \quad (184)$$

for some parameters t_C^{AB} , t_A^{ij} , and t_j^{Ai} and where we have omitted $\otimes$ and $\wedge$. This deformation is integrable provided the following polynomial equations of second and third degree are satisfied:

$$\begin{aligned} t_C^{[AB]E} t_F^C &= 0, \\ t_j^{Ak} t_\ell^{Bj} - t_j^{Bk} t_\ell^{Aj} &= -2t_C^{AB} t_\ell^{Ck}, \\ t_i^{Bk} t_C^{i\ell} - t_i^{B\ell} t_C^{ik} &= -2t_C^{AB} t_A^{k\ell}, \\ t_j^{Ai} t_A^{j\ell} &= t_j^{A\ell} t_A^{ji}, \\ 2t_j^{Ai} t_A^{k\ell} &= t_m^{Ai} t_A^{m\ell} \delta_j^k - t_m^{Ai} t_A^{mk} \delta_j^\ell, \\ t_j^{Ai} t_k^{Bj} t_B^{k\ell} + t_j^{A\ell} t_k^{Bj} t_B^{ki} &= 0. \end{aligned} \quad (185)$$

The integrability locus is the common zero of these equations, and the moduli space of conformal algebras (for $d \geq 4$) is the quotient of the integrability locus by the natural action of $GL(3, \mathbb{R}) \times GL(3, \mathbb{R})$ on the lower and upper case indices.

Some of the equations in (185) admit a natural Lie-theoretical interpretation. For example, the first equation says that t_C^{AB} are the structure constants of a real three-dimensional Lie algebra $\mathfrak{b}$. The second equation says that $-\frac{1}{2}t_A^{ij}$ define a three-dimensional real representation E , say, of $\mathfrak{b}$. The third equation says that t_A^{ij} defines a $\mathfrak{b}$ -equivariant map $\wedge^2 E \rightarrow \mathfrak{b}$. The fourth equation can be rewritten as

$$0 = t_A^{Ai} t_A^{j\ell} - t_A^{\ell j} t_A^{Ai} = \delta_B^A (t_B^{Bi} t_A^{j\ell} - t_B^{\ell j} t_A^{Ai}) = \delta_B^A (t_B^{Bi} t_A^{j\ell} + t_B^{\ell j} t_A^{Ai}) = \delta_B^A (2t_A^{BC} t_C^{i\ell}) = 2t_A^{AC} t_C^{i\ell}, \quad (186)$$

where in the penultimate equality we used the equivariance of t_A^{ij} under $\mathfrak{b}$. Notice that this equation is identically zero if $\mathfrak{b}$ is unimodular. We therefore see that solving the obstruction conditions requires, as a first step, classifying the three-dimensional representations of all the Bianchi Lie algebras.

As with deformations of kinematical and graded conformal Lie algebras, these are also the results for $d = 4$. In $d = 4$, we find that $\mathfrak{so}(4)$ is not simple but only semisimple, and hence we may split the rotational generators J_{ab} into their self-dual and anti-self-dual components. This does not alter the calculations of the infinitesimal deformations, and the additional freedom it gives in the form of φ_2 is not used in overcoming the obstruction to integrability to second-order. Therefore the results above for $d \geq 5$ actually apply to all $d \geq 4$.

Finally, we end this section with the observation that holographic conformal algebras (defined in the introduction) are special cases of the generalised conformal algebras in Definition 2. Indeed, a generalised conformal algebra is holographic if and only if it contains an $\mathfrak{so}(d+1)$ -subalgebra so that as a vector space, a holographic conformal algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \underbrace{\mathfrak{so}(d)}_{\cong \mathfrak{so}(d+1)} \oplus \underbrace{\mathbb{V}}_{\mathfrak{so}(d+1)\text{-vector}} \oplus \underbrace{(\mathbb{V} \oplus S)}_{\mathfrak{so}(d+1)\text{-vector}} \oplus \underbrace{(\mathbb{V} \oplus S)}_{\mathfrak{so}(d+1)\text{-vector}} \oplus \underbrace{S}_{\mathfrak{so}(d+1)\text{-scalar}}. \quad (187)$$

As shown in Sec. III E, the only graded conformal algebras which are holographic are the de Sitter algebras ($GCA_{15}^{(\varepsilon)}$) and the Newton–Hooke algebra ($GCA_{13}^{(\varepsilon=1)}$).

VII. GENERALISED LIFSHITZ ALGEBRAS

We shall now consider a generalisation of the graded conformal algebras which results from demanding the existence of a grading but not necessarily with the same weights as in Definition 1. This generalises the Lifshitz algebra extended by boosts (see, e.g., Ref. 22).

Definition 3. A **generalised Lifshitz algebra** (with d -dimensional space isotropy) is a real Lie algebra $\mathfrak{g}$ of dimension $\frac{1}{2}(d+1)(d+2)+1$ satisfying the following properties:

- (1) $\mathfrak{g}$ has a Lie subalgebra $\mathfrak{h} \cong \mathfrak{so}(d)$ and
- (2) as a vector space, $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{V}_\alpha \oplus \mathbb{V}_\beta \oplus S_\gamma$, where $\mathbb{V}_{\alpha,\beta}$ are copies of the d -dimensional vector representation of $\mathfrak{so}(d)$ with weights α, β relative to D and S_γ is a copy of the scalar representation of $\mathfrak{so}(d)$ and weight γ relative to D .

It follows from the definition that a generalised Lifshitz algebra is nothing but a kinematical Lie algebra $\mathfrak{k}$ (with d -dimensional space isotropy) extended by a grading element D which commutes with the rotations. Therefore to classify generalised Lifshitz algebras, we only need to classify possible gradings of kinematical Lie algebras where $\mathfrak{so}(d)$ lies in degree 0. Every such grading defines a derivation on $\mathfrak{k}$ which commutes with the rotations and which is diagonalisable over $\mathbb{R}$. Every such derivation integrates to an automorphism of $\mathfrak{k}$ which acts like the identity on the rotational subalgebra and lies in the identity component of the group of automorphisms. Automorphisms of kinematical Lie algebras have been discussed (except for a few examples where it was not then necessary) in Ref. 9 in the process of classifying simply connected homogeneous kinematical spacetimes. It is then a matter mostly of recontextualising those calculations in Ref. 9 to arrive at the results summarised in Table XXI.

In the notation of Table XXI, we do not write the $\mathfrak{so}(d)$ indices explicitly. We write $\mathbf{J}$, $\mathbf{B}$, $\mathbf{P}$, and H for the generators of the kinematical Lie algebra $\mathfrak{k}$ and write the kinematical Lie brackets as

$$[\mathbf{J}, \mathbf{J}] = \mathbf{J}, \quad [\mathbf{J}, \mathbf{B}] = \mathbf{B}, \quad [\mathbf{J}, \mathbf{P}] = \mathbf{P}, \quad \text{and} \quad [\mathbf{J}, H] = 0. \quad (188)$$

For $d \neq 2$, any other brackets can be reconstructed unambiguously from the abbreviated expression since there is only one way to reintroduce indices using only the $\mathfrak{so}(d)$ -invariant tensor δ_{ab} and when $d = 3$ also ϵ_{abc} on the right hand side of the brackets. For example,

TABLE XX. Notation for Lie algebras.

Notation	Name
$\mathfrak{s}$	Static
$\mathfrak{n}_+$	(Elliptic) Newton
$\mathfrak{n}_-$	(Hyperbolic) Newton
$\mathfrak{e}$	Euclidean
$\mathfrak{p}$	Poincaré
$\mathfrak{so}$	Orthogonal
$\mathfrak{g}$	Galilean
$\mathfrak{c}$	Carroll

$$[\mathbf{H}, \mathbf{B}] = \mathbf{P} \quad \text{stands for} \quad [\mathbf{H}, B_a] = P_a \quad \text{and} \quad [\mathbf{B}, \mathbf{P}] = \mathbf{H} + \mathbf{J} \quad \text{for} \quad [B_a, P_b] = \delta_{ab}H + J_{ab}. \quad (189)$$

In $d = 3$, we may also have brackets of the form

$$[\mathbf{P}, \mathbf{P}] = \mathbf{P} \quad \text{which we take to mean} \quad [P_a, P_b] = \epsilon_{abc}P_c. \quad (190)$$

If $d = 2$, then ϵ_{ab} is rotationally invariant and can appear in Lie brackets. So we will write, e.g.,

$$[\mathbf{H}, \mathbf{B}] = \mathbf{B} + \tilde{\mathbf{P}} \quad \text{for} \quad [\mathbf{H}, B_a] = B_a + \epsilon_{ab}P_b \quad (191)$$

and

$$[\mathbf{B}, \mathbf{B}] = \tilde{\mathbf{H}} \quad \text{for} \quad [B_a, B_b] = \epsilon_{ab}H, \quad (192)$$

et cetera. Also, whenever $\mathbf{J}$ appears, it denotes J_{ab} . In $d = 1$, it is tacitly assumed that we set any $\mathbf{J}$ to zero.

In Table XXI, we write the isomorphism class of the Lie algebra (if known) using the notation in Table XX.

Table XXI is subdivided into three sections separated by horizontal rules: kinematical Lie algebras which exist for generic d , those which are unique to $d = 3$ and those which are unique to $d = 2$. In $d = 1$, there are accidental isomorphisms: $\mathfrak{c} \cong \mathfrak{g}$,

TABLE XXI. Generalised Lifshitz algebras with d -dimensional space isotropy.

GLA#	d	$\cong$	Nonzero Lie brackets in addition to $[\mathbf{J}, \mathbf{J}] = \mathbf{J}$, $[\mathbf{J}, \mathbf{B}] = \mathbf{B}$ and $[\mathbf{J}, \mathbf{P}] = \mathbf{P}$				w_B	w_P	w_H
GLA ₁	≥ 0	$\mathfrak{s}$					α	β	γ
GLA ₂	≥ 1	$\mathfrak{g}$	$[\mathbf{H}, \mathbf{B}] = -\mathbf{P}$				α	$\alpha + \gamma$	γ
GLA ₃ ^{($\chi \in (-1, 1)$)}	≥ 1		$[\mathbf{H}, \mathbf{B}] = \chi \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$			α	β	0
GLA ₄	≥ 1		$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$			α	β	0
GLA ₅	≥ 1	$\mathfrak{n}_-$	$[\mathbf{H}, \mathbf{B}] = -\mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$			α	β	0
GLA ₆	≥ 1		$[\mathbf{H}, \mathbf{B}] = -\mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = \mathbf{B} + 2\mathbf{P}$			α	α	0
GLA ₇ ^($\chi > 0$)	≥ 1		$[\mathbf{H}, \mathbf{B}] = \chi \mathbf{B} + \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = \chi \mathbf{P} - \mathbf{B}$			α	α	0
GLA ₈	$\geq 1(\neq 2)$	$\mathfrak{n}_+$	$[\mathbf{H}, \mathbf{B}] = \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = -\mathbf{B}$			α	α	0
GLA ₉	≥ 2	$\mathfrak{c}$			$[\mathbf{B}, \mathbf{P}] = \mathbf{H}$		α	β	$\alpha + \beta$
GLA ₁₀ ^($\epsilon = \pm 1$)	≥ 2	$\mathfrak{p}$	$[\mathbf{H}, \mathbf{B}] = -\epsilon \mathbf{P}$		$[\mathbf{B}, \mathbf{B}] = \epsilon \mathbf{J}$	$[\mathbf{B}, \mathbf{P}] = \mathbf{H}$	0	β	β
GLA ₁₁	≥ 2	$\mathfrak{so}(d+1, 1)$	$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = -\mathbf{P}$		$[\mathbf{B}, \mathbf{P}] = \mathbf{H} + \mathbf{J}$	α	$-\alpha$	0
GLA ₁₂ ^($\epsilon = \pm 1$)	≥ 1	$\mathfrak{so}(d, 2)$ $\mathfrak{so}(d+2)$	$[\mathbf{H}, \mathbf{B}] = -\epsilon \mathbf{P}$	$[\mathbf{H}, \mathbf{P}] = \epsilon \mathbf{B}$	$[\mathbf{B}, \mathbf{B}] = \epsilon \mathbf{J}$	$[\mathbf{B}, \mathbf{P}] = \mathbf{H}$	0	0	0
GLA ₁₃ ^($\epsilon = \pm 1$)	3				$[\mathbf{B}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{P}, \mathbf{P}] = \epsilon(\mathbf{B} - \mathbf{J})$	0	0	γ
GLA ₁₄	3				$[\mathbf{B}, \mathbf{B}] = \mathbf{B}$		0	β	γ
GLA ₁₅	3				$[\mathbf{B}, \mathbf{B}] = \mathbf{P}$		α	2α	γ
GLA ₁₆	3			$[\mathbf{H}, \mathbf{P}] = \mathbf{P}$	$[\mathbf{B}, \mathbf{B}] = \mathbf{B}$		0	β	0
GLA ₁₇	3		$[\mathbf{H}, \mathbf{B}] = -\mathbf{P}$		$[\mathbf{B}, \mathbf{B}] = \mathbf{P}$		α	2α	α
GLA ₁₈	3		$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = 2\mathbf{P}$	$[\mathbf{B}, \mathbf{B}] = \mathbf{P}$		α	2α	0
GLA ₁₉ ^(ξ, χ)	2	$\mathfrak{n}_+$	$[\mathbf{H}, \mathbf{B}] = \mathbf{B}$	$[\mathbf{H}, \mathbf{P}] = \xi \mathbf{P} - \chi \tilde{\mathbf{P}}$		$(\xi \in [-1, 1], \chi > 0)$	α	β	0
GLA ₂₀	2		$[\mathbf{H}, \mathbf{B}] = \tilde{\mathbf{B}}$				α	β	0
GLA ₂₁	2				$[\mathbf{B}, \mathbf{B}] = \tilde{\mathbf{H}}$	$[\mathbf{P}, \mathbf{P}] = \tilde{\mathbf{H}}$	α	α	2α
GLA ₂₂	2		$[\mathbf{H}, \mathbf{B}] = \tilde{\mathbf{B}}$		$[\mathbf{B}, \mathbf{B}] = \tilde{\mathbf{H}}$	$[\mathbf{P}, \mathbf{P}] = \mathbf{J} + \tilde{\mathbf{H}}$	0	0	0
GLA ₂₃	2				$[\mathbf{B}, \mathbf{B}] = \tilde{\mathbf{H}}$		α	β	2α
GLA ₂₄	2		$[\mathbf{H}, \mathbf{B}] = \mathbf{P}$		$[\mathbf{B}, \mathbf{B}] = \tilde{\mathbf{H}}$		α	3α	2α
GLA ₂₅ ^($\epsilon = \pm 1$)	2		$[\mathbf{H}, \mathbf{B}] = \epsilon \tilde{\mathbf{B}}$		$[\mathbf{B}, \mathbf{B}] = \tilde{\mathbf{H}}$		0	β	0

$\mathfrak{so}(2,1) \cong \mathfrak{so}(1,2)$, $\mathfrak{e} \cong \mathfrak{n}_+$, and $\mathfrak{p} \cong \mathfrak{n}_-$. In addition, for $d = 2$, $\mathfrak{n}_+$ admits more gradings than for $d \neq 2$, which is why we have listed it separately in $d = 2$. The notation for the grading is such that w_X denotes the degree (or “weight”) of the generator X and $\alpha, \beta, \gamma \in \mathbb{R}$.

To each row in Table XXI, there corresponds a generalised Lifshitz algebra with additional generator D satisfying

$$[D, \mathbf{J}] = 0, \quad [D, \mathbf{B}] = w_B \mathbf{B}, \quad [D, \mathbf{P}] = w_P \mathbf{P}, \quad \text{and} \quad [D, H] = w_H H, \quad (193)$$

with the given w_B , w_P , and w_H . We may rescale D so that one of the nonzero w_B , w_P , and w_H is equal to 1.

In a generalised Lifshitz algebra, the grading element is an (outer) derivation of a kinematical Lie algebra. To be able to interpret the grading element as a dilatation, it seems reasonable to require that the generalised Lifshitz algebra $\mathfrak{g}$ can be “extended” (not in the algebraic sense of an extension) by additional generators, namely, one $\mathfrak{so}(d)$ scalar and/or one $\mathfrak{so}(d)$ vector, in such a way that the resulting algebra is still graded by the adjoint action of D but such that now D appears in the right-hand side of a Lie bracket. This is an interesting problem which is beyond the scope of this paper.

VIII. GENERALISED SCHRÖDINGER ALGEBRAS

As advocated in several places,^{23,24,14} it is worthwhile to think of the Schrödinger algebra as (the central extension of) a conformal algebra. The defining feature of Lie algebras such as the Schrödinger algebra is that it should be the central extension of a Lie algebra possessing an $\mathfrak{so}(d) \oplus \mathfrak{sl}(2, \mathbb{R})$ subalgebra and, in addition, two (or perhaps three) copies of the $\mathfrak{so}(d)$ vector representation. In other words, comparing to the generalised conformal algebra in Definition 2, one of the vector representations might be missing (typically the one which can be interpreted as spatial special conformal transformations) and the Bianchi Lie algebra spanned by the three scalars is isomorphic to Bianchi VIII. Of the graded conformal algebras classified in this paper, the only ones where the scalars span an $\mathfrak{sl}(2, \mathbb{R})$ subalgebra are the galilean conformal algebra (GCA_{14}) and the simple conformal algebras ($\text{GCA}_{15}^{(\epsilon)}$).

Definition 4. A **generalised Schrödinger algebra** (with d -dimensional space isotropy) is a real Lie algebra $\mathfrak{g}$ of dimension $\frac{1}{2}d(d+3)+4$ satisfying the following properties:

- (1) $\mathfrak{g}$ has a Lie subalgebra $\mathfrak{h} \cong \mathfrak{so}(d) \oplus \mathfrak{sl}(2, \mathbb{R})$ and
- (2) as a vector space, $\mathfrak{g} = \mathfrak{h} \oplus (\mathbb{V} \otimes E) \oplus \mathbb{R}Z$, where $\mathbb{V}$ is a copy of the d -dimensional vector representation of $\mathfrak{so}(d)$, E is a representation of $\mathfrak{sl}(2, \mathbb{R})$ of dimension 2 or 3, and Z is a central element.

A. The case $\dim E = 2$

We shall start with the case where E is two-dimensional. Observe that $\mathfrak{sl}(2, \mathbb{R})$ has precisely two inequivalent two-dimensional representations: the fundamental representation and the trivial representation. This gives rise to two classes of generalised Schrödinger algebras with $\dim E = 2$.

In more concrete terms, a generalised Schrödinger algebra (with $\dim E = 2$) admits a basis (J_{ab}, S^A, V_a^i, Z) , where J_{ab} span an $\mathfrak{so}(d)$ subalgebra, S^A are rotational scalars spanning an $\mathfrak{sl}(2, \mathbb{R})$ subalgebra, V_a^i , for $i = 1, 2$, are rotational vectors, and the rotational scalar Z is central. The (potentially) nonzero brackets (for $d \geq 4$) are given by

$$\begin{aligned} [J_{ab}, J_{cd}] &= \delta_{bc}J_{ad} - \delta_{ac}J_{bd} - \delta_{bd}J_{ac} + \delta_{ad}J_{bc}, \\ [J_{ab}, V_c^i] &= \delta_{bc}V_a^i - \delta_{ac}V_b^i, \\ [S^A, S^B] &= f^{AB}_C S^C, \\ [S^A, V_a^i] &= t^{Ai}_j V_a^j, \end{aligned} \quad (194)$$

where f^{AB}_C are the structure constants of $\mathfrak{sl}(2, \mathbb{R})$ and where $S^A \mapsto t^A i_j$ is a two-dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$, and in addition,

$$[V_a^i, V_b^j] = \delta_{ab}\epsilon_{ij}Z + u^{ij}J_{ab}, \quad (195)$$

where $u^{ij} = u^{ji}$. Notice that we cannot have S^A appearing in the $[V, V]$ brackets, for that would require the existence of an $\mathfrak{sl}(2, \mathbb{R})$ -equivariant map $f: \Lambda^2 E \rightarrow \mathfrak{sl}(2, \mathbb{R})$, but since $\Lambda^2 E$ is always a trivial representation (whether or not E is trivial), its image under f would have to be central in $\mathfrak{sl}(2, \mathbb{R})$, and $\mathfrak{sl}(2, \mathbb{R})$, being simple, has trivial centre.

In the first class of algebras, E is the trivial representation and hence $t^A i_j = 0$. Subjecting the Lie bracket (195) to the $[V, V, V]$ Jacobi identity, we find that

$$(u^{ij}\delta_\ell^k - u^{ki}\delta_\ell^j)\delta_{bc}\delta_{ae} + (u^{jk}\delta_\ell^i - u^{ij}\delta_\ell^k)\delta_{ca}\delta_{be} + (u^{ki}\delta_\ell^j - u^{jk}\delta_\ell^i)\delta_{ab}\delta_{ce} = 0 \quad (196)$$

for all $i, j, k, \ell = 1, 2$ and $a, b, c, e = 1, \dots, d$, where $d \geq 4$. Taking $a = b \neq c = e$, we are left with

$$u^{ki}\delta_\ell^j = u^{jk}\delta_\ell^i \quad (197)$$

for all $i, j, k, \ell = 1, 2$. Put $i \neq \ell = j$, and we see that $u^{ki} = 0$ for all k and all i . Therefore,

$$[V_a^i, V_b^j] = \delta_{ab}\epsilon_{ij}Z. \quad (198)$$

In the second class of algebras, E is the fundamental representation of $\mathfrak{sl}(2, \mathbb{R})$. Then the Jacobi identity implies, in particular, that u^{ij} is $\mathfrak{sl}(2, \mathbb{R})$ -invariant. But $u \in \odot^2 E$, which is a non-trivial irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$, and hence the only invariant is $u = 0$. Therefore, the additional brackets are those in (195) with $u^{ij} = 0$ and $t^a{}_i$ as the fundamental representation of $\mathfrak{sl}(2, \mathbb{R})$. This is (the $d \geq 4$ avatar of) the Schrödinger algebra of Refs. 25 and 26.

The situation for $d = 3$ is slightly more involved for the first type of algebras where E is a trivial representation of $\mathfrak{sl}(2, \mathbb{R})$ but not for the second type of algebras where E is the fundamental representation. Treating the problem using the methods of deformation theory, we look to classify the deformations of the Lie algebra $\mathfrak{g}$ with all brackets zero except for those in (194). The deformation complex, by Hochschild–Serre, is the Chevalley–Eilenberg complex of $\mathfrak{g}$ with values in the adjoint representation but relative to the semisimple subalgebra $\mathfrak{h} \cong \mathfrak{so}(3) \oplus \mathfrak{sl}(2, \mathbb{R})$. If we write $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{W}$, where $\mathbb{W}$ is the span of the V_a^i , then the cochains in the deformation complex are $C^p := (\wedge^p \mathbb{W}^* \otimes \mathfrak{g})^{\mathfrak{h}}$. The differential in this complex is zero, so $H^p = C^p$ for all p . If E is the fundamental representation of $\mathfrak{sl}(2, \mathbb{R})$, it is an easy calculation using the representation theory of $\mathfrak{so}(3)$ and $\mathfrak{sl}(2, \mathbb{R})$ that $C^2 = 0$. Therefore for the fundamental representation E , there are no deformations of $\mathfrak{g}$ and hence its central extension is again the Schrödinger algebra.

For the trivial representation E , however, Eq. (195) is replaced with

$$[V_a^i, V_b^j] = \delta_{ab}\epsilon_{ij}Z + u^{ij}J_{ab} + t_k^{ij}\epsilon_{abc}V_c^k, \quad (199)$$

subject to the $[V, V, V]$ Jacobi identity. There are three components to the Jacobi identity: the one along Z , the one along J , and the one along V itself. The component along Z says

$$t_\ell^{ij}\epsilon^{\ell k} + t_\ell^{jk}\epsilon^{\ell i} + t_\ell^{ki}\epsilon^{\ell j} = 0, \quad (200)$$

which translates into

$$t_2^{11} = t_1^{22} = 0, \quad t_1^{11} = 2t_2^{12}, \quad \text{and} \quad t_2^{22} = 2t_1^{12}. \quad (201)$$

The component along J gives

$$t_\ell^{ij}u^{\ell k}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + t_\ell^{jk}u^{\ell i}(\delta_{ba}\delta_{cd} - \delta_{bd}\delta_{ca}) + t_\ell^{ki}u^{\ell j}(\delta_{cb}\delta_{ad} - \delta_{cd}\delta_{ab}) = 0, \quad (202)$$

which taking (201) into account gives two relations,

$$2t_1^{12}u^{11} = t_1^{11}u^{12} \quad \text{and} \quad 2t_1^{12}u^{12} = t_1^{11}u^{22}. \quad (203)$$

Finally, the component of the $[V, V, V]$ Jacobi identity along V gives

$$(u^{ij}\delta_\ell^k - t_m^{ij}t_\ell^{mk})(\delta_{bc}\delta_{ad} - \delta_{ac}\delta_{bd}) + (u^{jk}\delta_\ell^i - t_m^{jk}t_\ell^{mi})(\delta_{ca}\delta_{bd} - \delta_{ba}\delta_{cd}) + (u^{ki}\delta_\ell^j - t_m^{ki}t_\ell^{mj})(\delta_{ab}\delta_{cd} - \delta_{cb}\delta_{ad}) = 0, \quad (204)$$

which taking (201) into account simply allows us to solve for u^{ij} in terms of the t_k^{ij} ,

$$u^{11} = (t_2^{12})^2, \quad u^{12} = t_1^{12}t_2^{12}, \quad \text{and} \quad u^{22} = (t_1^{12})^2. \quad (205)$$

Inserting these back into Eq. (203), we see that the two equations are identically satisfied.

Letting $t_1^{12} =: \lambda$ and $t_2^{12} =: \mu$, the $[V, V]$ bracket becomes (in shorthand notation)

$$[V^1, V^1] = \mu^2 J + 2\mu V^1, \quad [V^1, V^2] = Z + \lambda\mu J + \lambda V^1 + \mu V^2, \quad \text{and} \quad [V^2, V^2] = \lambda^2 J + 2\lambda V^2. \quad (206)$$

We have four cases depending on whether or not λ and μ are zero:

- (1) $\lambda = \mu = 0$: the only nonzero bracket is

$$[V_a^1, V_b^2] = \delta_{ab}Z; \quad (207)$$

- (2) $\lambda = 0$ and $\mu \neq 0$: the nonzero brackets are now (after rescaling V^1 by μ^{-1})

$$[V_a^1, V_b^1] = J_{ab} + 2\epsilon_{abc}V_c^1 \quad \text{and} \quad [V_a^1, V_b^2] = \delta_{ab}Z + \epsilon_{abc}V_c^2, \quad (208)$$

but changing basis to $V_a^1 \mapsto V_a^1 + \frac{1}{2}\epsilon_{abc}J_{bc}$, we arrive at only one nonzero bracket, namely,

$$[V_a^1, V_b^2] = \delta_{ab}Z; \quad (209)$$

- (3) $\lambda \neq 0$ and $\mu = 0$ are isomorphic to the previous case via the change in basis $(J, V^1, V^2, Z) \mapsto (J, V^2, V^1, -Z)$; and
 (4) $\lambda\mu \neq 0$, where the nonzero brackets are now (after rescaling V^1 by μ^{-1} and V^2 by λ^{-1})

$$\begin{aligned} [V_a^1, V_b^1] &= J_{ab} + 2\epsilon_{abc}V_c^1, \\ [V_a^2, V_b^2] &= J_{ab} + 2\epsilon_{abc}V_c^2, \\ [V_a^1, V_b^2] &= \delta_{ab}Z + J_{ab} + \epsilon_{abc}(V_c^1 + V_c^2), \end{aligned} \quad (210)$$

which, changing basis to $V^\pm := \frac{1}{2}(V^1 \pm V^2)$, simplifies to the following non-zero brackets:

$$[V_a^+, V_b^+] = J_{ab} + 2\epsilon_{abc}V_c^+ \quad \text{and} \quad [V_a^+, V_b^-] = -\frac{1}{2}\delta_{ab}Z + \epsilon_{abc}V_c^-, \quad (211)$$

and, finally, changing basis $V_a^+ \mapsto V_a^+ + \frac{1}{2}\epsilon_{abc}J_{bc}$ and redefining Z , we find that the only nonzero bracket remains

$$[V_a^+, V_b^-] = \delta_{ab}Z. \quad (212)$$

In summary, we find again that just as for $d \geq 4$, there is only one isomorphism class of Lie algebras, with an additional bracket given by Eq. (198).

B. The case $\dim E = 3$

Consider now the case where E is a three-dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$. Then we are essentially in the special case of Definition 2 where the Lie algebra spanned by the scalars is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ or, equivalently, Bianchi VIII. The representation E is isomorphic to one of the following three representations:

- (1) the trivial three-dimensional representation $\mathbb{R}^3$;
- (2) the direct sum $F \oplus \mathbb{R}$, where $\mathbb{R}$ is the trivial one-dimensional representation and F is the fundamental two-dimensional representation; or
- (3) the adjoint representation.

Restricting to $d \geq 4$, we have that the most general deformation is given by Eq. (184) and the obstructions to integrating it are given by Eq. (185). The first three equations say that t_C^{AB} are the structure constants of $\mathfrak{sl}(2, \mathbb{R})$ relative to some basis, $-\frac{1}{2}t_j^{Ai}$ define the three-dimensional representation E , and t_A^{ij} defines an $\mathfrak{sl}(2, \mathbb{R})$ -equivariant linear map $\wedge^2 E \rightarrow \mathfrak{sl}(2, \mathbb{R})$. The fourth obstruction equation vanishes because $\mathfrak{sl}(2, \mathbb{R})$ is unimodular, as shown by Eq. (186).

If E is the trivial representation, then $t_j^{Ai} = 0$, and since $\mathfrak{sl}(2, \mathbb{R})$ has no centre, $t_A^{ij} = 0$ as well. The remaining two equations in (185) are automatically satisfied. Hence the algebra is given by the common Lie brackets (169) and (170) and in addition $[S^A, S^B] = t_C^{AB}S^C$ defining $\mathfrak{sl}(2, \mathbb{R})$. We may centrally extend this algebra by generators $Z^{ij} = -Z^{ji}$ and brackets

$$[V_a^i, V_b^j] = \delta_{ab}Z^{ij}. \quad (213)$$

If $E = F \oplus \mathbb{R}^2$, then $\wedge^2 E \cong E$ as $\mathfrak{sl}(2, \mathbb{R})$ -representations, and since there is no $\mathfrak{sl}(2, \mathbb{R})$ -equivariant map $E \rightarrow \mathfrak{sl}(2, \mathbb{R})$, we see that $t_A^{ij} = 0$. The last two equations in (185) are automatically satisfied. Therefore the Lie algebra is spanned by $J, V^0, V^+, V^-, S^0, S^+, S^-$, with Lie brackets given by (169) and (170) and in addition

$$[S^\pm, V^\mp] = V^\pm, \quad [S^0, V^\pm] = \pm V^\pm, \quad [S^0, S^\pm] = \pm 2S^\pm, \quad \text{and} \quad [S^+, S^-] = S^0. \quad (214)$$

The Lie algebra admits a central extension with generator Z and Lie brackets

$$[V_a^+, V_b^-] = \delta_{ab}Z. \quad (215)$$

Notice that the subspace spanned by V_0 is an ideal, and quotienting by this ideal gives again the Schrödinger algebra. This Lie algebra is graded, but the weights are not those of a graded conformal algebra as in Definition 1. This suggests relaxing the definition of a graded conformal algebra by allowing arbitrary conformal weights. We have discussed this in Sec. VII, in the context of generalised Lifshitz algebras.

Finally, if E is the adjoint representation, then now t_A^{ij} is a constant multiple of the isomorphism $\wedge^2 E \rightarrow E \cong \mathfrak{sl}(2, \mathbb{R})$. For any value of this constant, one can check explicitly that the last two equations in (185) are satisfied. Therefore the resulting Lie algebra is isomorphic to one of the simple conformal algebras $GCA_{15}^{(\epsilon)}$. These algebras do not admit any nontrivial central extensions.

IX. CONCLUSIONS AND OPEN PROBLEMS

This paper is devoted to the slippery notion of a conformal algebra. We have focussed primarily on the notion of a graded conformal algebra with d -dimensional space isotropy (see Definition 1), and we have classified them up to isomorphism for all $d \geq 2$. We have performed this by classifying deformations of the static graded conformal algebra defined in the Introduction.

For each such $d \neq 3$, there are 17 isomorphism classes, which are listed in Table VI, which also applies when $d = 2$. For $d = 3$, there are in addition another 6 isomorphism classes, which are listed in Table XI. Some of these Lie algebras are related by contraction, and this defines a partial order in the set of isomorphism classes whose Hasse diagram is depicted in Fig. 1, which also applies to $d = 2$. With the exception of the simple Lie algebras $\text{GCA}_{15}^{(\varepsilon)}$ (isomorphic to $\mathfrak{so}(d+2, 1)$ or $\mathfrak{so}(d+1, 2)$), none of the other graded conformal algebras admit an invariant inner product. We then classified the central extensions of these Lie algebras. The situation was different in $d \geq 3$ and $d = 2$ and is summarised in Tables XVII and XVIII, respectively. We then investigated whether any central extended graded conformal algebras admit an invariant inner product, and we found that this was the case for GCA_{16} (which exists only for $d = 3$) and, provided that $d = 2$, also for GCA_1 and GCA_8 .

We then discussed some other notions of conformal algebras obtained by relaxing and/or modifying some of the properties of the graded conformal algebras.

In Sec. VI, we discussed a class of Lie algebras defined by dropping the condition that D is a grading element in Definition 1. We call the resulting Lie algebras *generalised conformal algebras* (see Definition 2), but it is questionable whether all such Lie algebras are in any way conformal since all they share with the ur-example of conformal algebra (the algebra of conformal symmetries of Minkowski spacetime) is that they have the generators transform in the same way under the $\mathfrak{so}(d)$ subalgebra of rotations. Nevertheless we present some preliminary results about their classification via deformation theory for $d \geq 4$. We will use some of these results below in a restricted context.

In Sec. VII, we discuss what we call *generalised Lifshitz algebras* (see Definition 3), which consist of a graded kinematical Lie algebra extended by the grading element but not requiring that the degrees are those of the graded conformal algebras treated in the bulk of this paper. The results here are preliminary, but we present a classification of the possible $\mathbb{Z}$ -gradings of all kinematical Lie algebras, which is contained in Table XXI.

In Sec. VIII, we discussed a class of conformal Lie algebras related to the Schrödinger algebra. The Schrödinger algebra can be understood as a central extension of a conformal algebra, and this suggests the definition of a *generalised Schrödinger algebra* (see Definition 4), whose main property is the existence of an $\mathfrak{so}(d) \oplus \mathfrak{sl}(2, \mathbb{R})$ subalgebra under which the remaining (non-central) generators transform according to $V \otimes E$, where V is the vector representation of $\mathfrak{so}(d)$ and E is a representation of $\mathfrak{sl}(2, \mathbb{R})$ of dimension 2 or 3. We classified the isomorphism classes of generalised Schrödinger algebras with $\dim E = 2$ for $d \geq 3$ and with $\dim E = 3$ for $d \geq 4$. Apart from the Schrödinger algebra, we find a (non-central) extension by a vector representation of $\mathfrak{so}(d)$ (trivial under $\mathfrak{sl}(2, \mathbb{R})$) and centrally extended algebras where E is the trivial representation of $\mathfrak{sl}(2, \mathbb{R})$.

There are a number of open problems remaining to complete the results in Secs. VI–VIII, among which we list the following in no particular order:

- Classify the generalised conformal algebras of Sec. VI. For $d \geq 4$, we only need to solve Eq. (185) and the quotient by the action of the automorphisms of the static algebra. For $d \leq 3$, the deformation problem has to be looked anew.
- Classify the generalised Schrödinger algebras of Sec. VIII for $d \leq 3$ (for $\dim E = 3$) and for $d = 2$ (for $\dim E = 2$).
- Study the addition of generators to the generalised Lifshitz algebras of Sec. VII. This is not unrelated to the classification of the generalised conformal algebras where one of the scalar generators is a grading element.
- Classify filtered subdeformations of $\mathfrak{so}(d+1, 2)$ or $\mathfrak{so}(d+2, 1)$ containing an $\mathfrak{so}(d)$ subalgebra.

A possible extension of this work is to determine the $(d+1)$ -dimensional homogeneous manifolds of the corresponding groups, as was performed in Ref. 9. This might give a rich class of “spacetimes” to discuss conformal-like theories. Of course, of the graded conformal algebras treated in the bulk of the paper, only $\text{GCA}_{15}^{(\varepsilon)}$ can act on a $(d+1)$ -dimensional (Riemannian or Lorentzian) manifold via conformal Killing vectors: their dimension would mean that the manifold is conformally flat and, hence, any such conformal algebra would be isomorphic to $\text{GCA}_{15}^{(\varepsilon)}$ for some value of ε .

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- ²⁷In this section, and for psychological reasons, $\mathfrak{g}$ shall denote a general Lie algebra, not necessarily the static graded conformal Lie algebra as it did in the sections where we discussed deformations.