THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Frequency-dependent anisotropy in a partially saturated fractured rock

Citation for published version:

Jin, Z, Chapman, M & Papageorgiou, G 2018, 'Frequency-dependent anisotropy in a partially saturated fractured rock', *Geophysical Journal International*. <https://doi.org/10.1093/gji/ggy399>

Digital Object Identifier (DOI):

[10.1093/gji/ggy399](https://doi.org/10.1093/gji/ggy399)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Peer reviewed version

Published In:

Geophysical Journal International

Publisher Rights Statement:

© The Author(s) 2018. Published by Oxford University Press on behalf of The Royal Astronomical Society.

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

Frequency-dependent anisotropy in a partially saturated fractured rock

Zhaoyu Jin^{1,*}, Mark Chapman¹, Giorgos Papageorgiou^{1,2}

1. School of GeoSciences, The University of Edinburgh, Grant Institute, James Hutton Road, Edinburgh, EH9 3FE, UK.

2. Department of Geoscience, NTNU, S.P. Andersensvei 15A, 7491 Trondheim, Norway.

* E-mail: Zhaoyu.Jin@ed.ac.uk

Abbreviated title: Frequency-dependent anisotropy in partially saturated rocks

Summary

Seismic wave propagation through fractured rocks is greatly influenced by their fracture system and fluid content. In this paper we derive expressions for the anisotropic frequency-dependent elastic constants. These depend on the relative mobilities of the saturating fluids and the coupled impact of “squirt” and “patch” effects, which have typically been considered independently, on anisotropic seismic wave propagation. The effect of relative permeability is pronounced; fluid mobility can be lower in partially saturated rocks compared to the fully saturated case, and this can lead to a stiffening which dominates compressibility effects. This can result in unexpected non-monotonic relationships between moduli and water saturation, complicating attempts to invert saturation from seismic data. We use our model to explain laboratory measurements of seismic anisotropy in partially saturated fractured rocks, concluding that both squirt and patch mechanisms are significant.

Keywords: Permeability and porosity; Seismic anisotropy; Seismic attenuation; Fracture and flow; Wave propagation; Elasticity and anelasticity

Introduction

The presence of fractures in a reservoir significantly influences permeability and enhances the subsurface flow of fluid. Describing fracture systems is therefore a key problem in various applications including hydrocarbon production and CO₂ monitoring.

One of the most commonly used methods for fracture characterization is the use of S-wave anisotropy, from which the orientation and density of the fractures can be deduced. It has been reported that fractures often lead to frequency-dependent anisotropy (Marson-Pidgeon and Savage 1997; Chesnokov et al. 2001), and one important mechanism accounting for such velocity dispersion is “squirt flow” arising from pressure gradients within the pore space induced by passing seismic waves (Dvorkin et al. 1995). Increasing attention has recently been paid to the fluid effects in a fractured rock on frequency-dependent anisotropy that could potentially be analysed to provide information about fracture size, fluid saturation, and fluid mobility. This includes laboratory measurements of S-wave splitting (SWS) in fluid saturated rocks (Tillotson et al. 2011; Amalokwu et al. 2014), fracture and fluid discriminations from field data analysis (Maultzsch et al. 2003; Qian et al. 2007; Al-Harrasi et al. 2011), and theoretical descriptions of frequency-dependent anisotropic behaviour (Chapman 2003; Chapman et al. 2003; Carcione et al. 2013; Galvin and Gurevich 2015).

Most of current theories on frequency-dependent anisotropy have been limited to the single fluid assumption, despite the fact that almost all reservoirs are partially saturated. Recent laboratory study by Amalokwu et al. (2014) addressed this situation by measuring SWS in a synthetic rock with aligned fractures under partial saturation conditions. They showed a dependence of SWS on water saturation, and attempted to explain such saturation dependence by using an effective fluid approach combined with a single-fluid rock physics model. More recently, Papageorgiou and Chapman (2017) proposed an isotropic model describing squirt-flow behaviour in a porous rock saturated by two immiscible fluids, and

showed how P-wave dispersion and attenuation could be affected by the relative permeability of each saturating fluid. Within their approach, it is unclear how to account for fracture-induced anisotropy.

The purpose of this paper is to develop a model that describes frequency-dependent anisotropy in a fractured, partially saturated rock. We do this by extending the theory of Papageorgiou and Chapman (2017) to the anisotropic case in which the fluid flow is described by an effective fluid mobility that is affected by the relative permeabilities of the saturating fluids. “Patch” effects arising from unequilibrated fluid pressures are also described by a non-dimensional parameter which captures pore scale capillary effects. We show that the effects of relative permeability give rise to observable non-monotonic variations of SWS with changing water saturation. Patchy saturation tends to reduce SWS at a given frequency. We also present an interpretation of experimental measurements of SWS by Amalokwu et al. (2014) in terms of coupled squirt and patch effects.

We organize the paper as follows. We first review a previous single-fluid approach to describing frequency-dependent anisotropy. We then show how partial saturation could be addressed by deriving the elastic constants of a fractured rock saturated by two immiscible fluids. Based on our model, we carry out a numerical study to demonstrate the multi-fluid effects on the variation of frequency-dependent anisotropy with water saturation. Finally, we use our model to explain laboratory data, and emphasise the requirement to include coupled squirt and patch mechanisms for the interpretation of saturation effects on SWS.

Theory and methods

Chapman (2003) proposed a squirt model that assumes a pore space consisting of an isotropic collection of grain-scale pores and cracks and a set of aligned meso-scale fractures. The radii of the fractures are larger than the grain size but smaller than the seismic wavelength. Assuming that the fracture normal direction is aligned with x_3 -axis in the Cartesian

coordinates, the resulting medium is transversely isotropic with vertical axis of symmetry. The theory assumes that the fractured rock is fully saturated with a single type of fluid. During the passage of a seismic wave, squirt flow arises from fluid pressure gradients between the inclusions with different compliances, leading to dispersive effects on anisotropy. With the adoption of the Voigt notation, the frequency-dependent elastic constants are of the form

$$C_{IJ}(\omega) = C_{IJ}^0 - \phi_0^\ominus C_{IJ}^1(\omega) - \varepsilon_0^\ominus C_{IJ}^2(\omega) - \varepsilon_0^\dagger C_{IJ}^3(\omega) \quad (1)$$

where ω is the angular frequency, C_{IJ}^0 is the isotropic elastic constant of the matrix with Lamé parameters λ and μ , C_{IJ}^1 , C_{IJ}^2 , and C_{IJ}^3 are corrections associated with pores, microcracks and fractures, respectively, scaled by the porosity $\phi_0^\ominus$, the crack density $\varepsilon_0^\ominus$ and the fracture density $\varepsilon_0^\dagger$.

In this model, the fluid and fracture properties are described by the frequency-dependent constants C_{IJ}^1 , C_{IJ}^2 and C_{IJ}^3 , which are functions of the fluid compressibility and two characteristic frequencies associated with fluid flows taking place at grain and fracture scales respectively. The characteristic frequency (or relaxation time), which describes the frequency regime where dispersion occurs, is controlled by the length scale of the pore network and the fluid mobility that is defined as the ratio of rock permeability to fluid viscosity.

The description of fluid flow in this theory is based on Darcy's law in which the fluid mobility proportionally relates the volumetric flux with the pressure gradient driving the flux. When the porous rock is saturated by two immiscible fluids, the fluid flow of each fluid would depend on the effective mobility and pressure of the corresponding fluid. According to multi-fluid Darcy's law, the mobility of each fluid is scaled by the corresponding relative permeability via the following equations

$$Q_w = -A\kappa_w \frac{\kappa}{\eta_w} \partial_x P_w; \quad Q_g = -A\kappa_g \frac{\kappa}{\eta_g} \partial_x P_g \quad (2)$$

where Q is the volumetric flow rate through cross-sectional area A , $\partial_x P$ is the fluid pressure gradient along flux direction x , η is the viscosity, κ is the absolute permeability, and κ_w and κ_g are the relative permeabilities of water and gas. We use subscripts w and g to label the fluid types as water-gas only for simplicity, as the theory is applicable to any two-fluid system. In the Appendix, we give implementation of these equations and show how an effective mobility of the multi-fluid mixture can be accomplished.

We also assume that the wave-induced fluid pressure may differ between the fluids in each inclusion due to capillary effects or saturation heterogeneities. Following Papageorgiou and Chapman (2017), we allow uneven fluid pressures in each inclusion. The pressures are related by a non-dimensional parameter q that proportionally relates the induced pressure of one fluid to the other, i.e.

$$P_g = qP_w \quad (3)$$

where P_w and P_g represent pressures of these two fluids. It has been discussed by Papageorgiou and Chapman (2017) that q acts as a parameter that quantifies the variation of capillary pressure and should lie within $[q_0, 1]$ where $q_0 = \frac{K_g}{K_w}$ and the fluid bulk modulus K_g is smaller than K_w . $q = 1$ corresponds to the iso-stress condition in which the two fluids are mixed at a fine scale. Conditions where this iso-stress average is appropriate are generally referred to as “uniform saturation” (Mavko and Mukerji 1998). Values less than 1 represent unequilibrated pressures between the fluids which could give rise to the so-called “patchy saturation”.

Incorporating these effects into the current theory, we derive expressions of the constants C_{IJ}^1 , C_{IJ}^2 and C_{IJ}^3 in the Appendix, and show that the effective fluid bulk modulus K_f and the effective fluid mobility M_f now depend on both fluids through the following equations

$$\frac{1}{K_f} = \frac{S_w}{\tilde{q}K_w} + \frac{(1 - S_w)q}{\tilde{q}K_g} \quad (4)$$

$$M_f = \frac{\kappa_w}{\tilde{q}} M_w + \frac{q\kappa_g}{\tilde{q}} M_g \quad (5)$$

where

$$\tilde{q} = S_w + q(1 - S_w); \quad M_w = \frac{\kappa}{\eta_w}; \quad M_g = \frac{\kappa}{\eta_g} \quad (6)$$

S_w is the water saturation, and M_w and M_g are the mobilities of water and gas.

Equation (4) suggests that K_f is a Reuss average of the two fluid bulk moduli weighted by the patch parameter q , the value of which would therefore affect the stiffness of the effective fluid. The effective mobility M_f of the fluid mixture, given by equation (5), is a weighted average of the two fluid mobilities influenced by both q and the relative permeability. As a result, the characteristic frequencies will not only depend on the fracture size, but also the effects of the relative permeability and uneven fluid pressures.

Our calculation shows that, in a partially saturated fractured rock, the grain-scale squirt frequency ω_m and the fracture-scale characteristic frequency ω_f are given by

$$\omega_m = \frac{\omega_0 M_f}{M_w}; \quad \omega_f = \frac{\omega'_0 M_f}{M_w} \quad (7)$$

where ω_0 and ω'_0 are the values of ω_m and ω_f at full water saturation, and they are related via

$$\omega'_0 = \frac{\varsigma}{a_f} \omega_0 \quad (8)$$

where ς is the grain size that is assumed to be identified with the radii of the pore and cracks, and a_f is the fracture length.

Equation (7) allows us to relate the multi-fluid characteristic frequencies with their values at full saturation. It is clear that the main difference between partial and full saturation cases comes from the effective mobility of the two-fluid mixture. A lower mobility leads to a lower characteristic frequency. This can be understood that fluids with lower mobility flow more slowly and would therefore require more time for the pressure gradients to relax, which in turn results in a lower characteristic frequency. Since the effective mobility depends on the relative permeability and the uneven fluid pressures, we should expect a more complicated squirt flow behaviour than the single fluid scenario.

The above equations also suggest that the ratio of these two frequencies ω_m/ω_f stands for the fracture length normalized by the grain size, i.e. $\frac{a_f}{\zeta}$. Since a_f is assumed to be larger than ζ , the fracture related ω_f should be smaller than the microcrack related ω_m . Possible physical explanation behind this is that for a larger fracture scale, we would expect relatively more fluid volume to move in and out of the fracture to reach pressure equilibration, which requires more time and therefore leads to lower characteristic frequency. In our model, both the two-fluid mixture and the fractures play a role in lowering the characteristic frequency, which could potentially lead to frequency-dependent anisotropy in the seismic frequency band.

Our model can be considered as an extension of previous theory to the anisotropic case, and should contain it as appropriate limits. In the absence of meso-scale fractures, fluid exchange only occurs between pores and cracks, reducing our model to the isotropic limit that can be specified by the bulk and shear moduli. In that case, we would expect no contribution from C_{IJ}^3 in equation (1) by setting the fracture density $\varepsilon_0^\dagger$ to zero. Our calculation in the Appendix shows that the bulk modulus $K_{eff}(\omega)$ is consistent with the one proposed by Papageorgiou and Chapman (2017). The effective shear modulus $\mu_{eff}(\omega)$ is given by C_{44} , which is of the form

$$\mu_{eff}(\omega) = \mu - \frac{4}{15} \phi_0^\ominus \frac{1}{1 + K_c} \frac{\mu^2}{\sigma_c} \left(K_c + \frac{1}{1 + \frac{i\omega M_w}{\omega_0 M_f}} \right) - \frac{8}{5} \phi_0^\ominus \frac{(1 - \nu)\mu}{(2 - \nu)\pi r} \quad (9)$$

$$- 15\mu\phi_0^\ominus \frac{1 - \nu}{7 - 5\nu}$$

where μ is the mineral shear modulus, ν is the Poisson's ratio, r is the aspect ratio of the crack. K_c is a non-dimensional parameter that represents the compressibility of a fluid saturated crack, as discussed by Zatsepin and Crampin (1997). It is related to K_f via equation

$$K_c = \frac{\sigma_c}{K_f}; \quad \sigma_c = \frac{\pi\mu r}{2(1-\nu)} \quad (10)$$

The fluid dependence of shear modulus therefore relies on the effective fluid mobility M_f and the effective fluid bulk modulus K_f . In the case of zero frequency, K_c and M_f cancel out and the result agrees with Gassmann's theory in which shear modulus is independent of fluid. At non-zero frequency, shear modulus would depend on both fluids through combined effects of the squirt flow mechanism, the relative permeability and the uneven fluid pressures.

Numerical example

Table 1, Parameters of a porous rock saturated by water and supercritical CO₂.

Dry frame bulk modulus (Pa)	1.40×10^{10}	Porosity	10%
Dry frame shear modulus (Pa)	7.29×10^9	Crack density	0.02
Mineral bulk modulus (Pa)	2.20×10^{10}	Aspect ratio	1.0×10^{-4}
Mineral shear modulus (Pa)	9.31×10^9	Solid density (Kg/m ³)	2650
Water viscosity (Pa.s)	6.0×10^{-4}	CO ₂ viscosity (Pa.s)	2.1×10^{-5}
Water density (Kg/m ³)	1000	CO ₂ density (Kg/m ³)	240
Water modulus (Pa)	2.4×10^9	CO ₂ modulus (Pa)	1.1×10^7

We study the effect of patchy saturation and squirt dispersion on S-waves by considering a porous rock saturated by water and supercritical CO₂. Table 1 shows the parameters. For each fluid phase, the relative permeability is often assumed as an increasing function of its saturation. In this study, we approximate the model of relative permeability as

$$\kappa_w = S_w^3; \kappa_g = (1 - S_w)^2 \quad (11)$$

which largely fits the curves given by Benson et al. (2013) for multi-phase flow in CO₂ storage reservoirs.

In the absence of fractures, we use equation (9) for the calculation of shear modulus $\mu_{eff}(\omega)$, based on which the S-wave velocity V_s and attenuation $1/Q_s$ at various frequencies ω relative to ω_0 can be derived from

$$\frac{\rho}{\mu_{eff}(\omega)} = \frac{1}{V_s^2} + i \frac{1}{Q_s V_s^2} \quad (12)$$

where ρ is the density of the rock.

We first investigate the variation of shear modulus with water saturation for a range of the patch parameters q in Figure 1. In the case of zero frequency, the model reduces to Gassmann's theory which shows no dependence of shear modulus on fluid. At the reference frequency ω_0 , shear modulus depends on fluid saturation and the patch parameter q . Since water has a higher bulk modulus and viscosity than gas, the effective fluid approach might expect increasing water saturation to lead to a stiffening effect for water- CO₂ mixtures. Our results in Figure 1 suggest that this may not be true as we predict a non-monotonic variation of shear modulus with water saturation. This is due to the relative permeability effect that leads to a decrease in fluid mobility and a higher shear modulus for intermediate water saturations.

Figure 1. The variation of shear modulus with water saturation for a range of q values at the reference frequency ω_0 (red curves). At zero frequency (black curve), the model reduces to Gassmann's theory where there is no dependence on fluid saturation or the uneven fluid pressure.

Figure 2 shows the variation of S-wave velocities and attenuation with water saturation at different frequencies and q values. S-wave velocities are seen to increase with frequency, while the maximum attenuation could occur at intermediate frequency. At zero frequency, the model reduces to Gassmann's theory where S-wave velocity only decreases with water saturation due to the density effect and no attenuation is predicted. This is consistent with Mavko and Mukerji (1998) where patchy saturation shows no effect on S-waves at low frequencies. At non-zero frequencies, patchy saturation could become important. For high (e.g. $100\omega_0$) frequencies, S-wave velocities at various values of q are slightly separated. For intermediate frequencies (e.g. ω_0), the fluid dependence of S-wave velocity is clearly affected by q , and multiple attenuation peaks can be observed at various water saturations. We notice that the variations of shear modulus, S-wave velocity and attenuation with water

saturation are overall much weaker than that of the bulk modulus and P-wave properties compared to the values shown by Papageorgiou and Chapman (2017) and Papageorgiou et al. (2018).

Figure 2. The variation of S-wave velocities and attenuation with water saturation at different frequencies and patch parameters q .

In the presence of aligned mesoscale fractures, the medium is expected to show frequency-dependent anisotropy. We now introduce a set of fractures with density 0.05 and length 10 cm. According to equation (8), the fracture related reference frequency ω'_0 is much lower than ω_0 since the fracture length a_f is supposed to be much larger than the grain size ς . Therefore, in the anisotropic case we proceed our numerical analysis by using frequencies relative to ω'_0 . We calculate SWS by using $(S_1 - S_2)/S_1$ where S_1 and S_2 are faster and slower velocities of the quasi S-wave V_{SV} and pure S-wave V_{SH} (Mavko et al. 2009):

$$V_{SV} = \left(\frac{C_{11} \sin^2 \theta + C_{33} \cos^2 \theta + C_{44} - \sqrt{H}}{2\rho} \right)^{1/2} \quad (13)$$

$$V_{SH} = \left(\frac{C_{66} \sin^2 \theta + C_{44} \cos^2 \theta}{\rho} \right)^{1/2} \quad (14)$$

where

$$H = [(C_{11} - C_{44}) \sin^2 \theta - (C_{33} - C_{44}) \cos^2 \theta]^2 + (C_{13} + C_{44})^2 \sin^2 2\theta \quad (15)$$

θ is the angle between wave propagation and the fracture normal (x_3 -axis). The explicit expressions of the elastic constants C_{IJ} are given in the Appendix.

Figure 3 shows the variation of SWS with water saturation for a range of parameter q and frequencies. Propagation is at 70 degrees to the fracture normal. In the isotropic case (Figure 1), we show that the shear modulus is unaffected by fluid at zero frequency. This is no longer the case for the anisotropic case where evident dependence of SWS on water saturation and the patch parameter q can be observed even at zero frequency. Such fluid dependence arises from the compressibility effect of the water-CO₂ mixture as the Quasi-shear wave compresses the fractures. The magnitude of SWS also decreases with the increasing patchiness of two immiscible fluids. In the low frequency limit, the uniform saturation case shows an abrupt drop of SWS as it approaches full water saturation. This may no longer be observed in the patchy saturation case since the variation curves could become smooth. At very high frequencies (e.g. $100\omega'_0$), the results are reversed as the abrupt change occurs for the patchy saturation case near full supercritical CO₂ saturation. At intermediate frequencies (e.g. ω'_0), we show observable non-monotonic behaviour of SWS with respect to changes in water saturation. While SWS generally decreases with water saturation, it may start to increase with water saturation due to the impact of relative permeability and patch effects on lowering the characteristic frequency.

Figure 3. The variation of SWS with water saturation at various values of q and frequencies.

Figure 4 shows the variation of SWS with frequencies relative to ω'_0 at different water saturations. It is clear that SWS decreases with frequency. For the uniform saturation case ($q = 1$), increasing water saturation generally results in a decreasing SWS. For the patchy saturation case ($q = 10q_0$), there can be certain points where SWS starts to increase with water saturation (e.g. at ω'_0 frequency). The presence of patchiness could also lead to a lower characteristic frequency where the dispersion occurs. This effect is more clearly shown in Figure 5 where the characteristic frequency depends on both water saturation and the patch parameter q . At intermediate water saturations, the characteristic frequency is more sensitive to q than it is at high or very low water saturations. As q goes lower, it is more likely that the characteristic frequency ω_f could sit below its value ω'_0 at full water saturation.

(a) $q = 1$

(b) $q = 1$

(c) $q = 10q_0$

(d) $q = 10q_0$

Figure 4. The variation of SWS and velocities with frequency for various water/supercritical- CO_2 saturations. (a) SWS varying with frequency for the uniform saturation case; (b) S-wave velocities varying with frequency for the uniform saturation case; (c) SWS varying with frequency for the patchy saturation case; (d) S-wave velocities varying with frequency for the patchy saturation case. The dashed line represents the fracture related reference frequency ω'_0 , and the dotted line represents the microcrack related reference frequency ω_0 .

Figure 5. The variation of the grain-scale characteristic frequency ω_m (red curves) and the fracture-scale characteristic frequency ω_f (blue curves) with water saturation at various values of the patch parameter q .

Modelling experimental observations

Amalokwu et al. (2014) investigated the effects of water saturation on SWS through experimental measurements in a synthetic silica-cemented sandstone containing fractures aligned at 45 degree to the fracture normal. The measured results show SWS being fairly constant between $S_w = 0 - 0.7$, and steadily decreasing as it approaches full water saturation. Amalokwu et al. (2014) combined the partial saturation model of White (1975) and the single-fluid squirt model of Chapman (2003) to explain this saturation dependence, and discussed the importance of wave-induced fluid flow mechanisms due to the presence of partial liquid/gas saturation and fractures.

Here we attempt to model their experimental results by using our unified multi-fluid squirt-patchy theory. According to equation (1), the elastic constants consist of the isotropic

$C^0(\lambda, \mu)$ of the matrix and additional corrections associated with C^1 , C^2 , C^3 from pores, microcracks and fractures. The original form of our model is limited to low porosity. To overcome this restriction, we follow Chapman et al. (2003) to estimate the Lamé parameters λ^0 and μ^0 assuming that we know V_P^0 , V_S^0 , ρ^0 at some frequency f_0 from the isotropic reference rock. The practical version of our model can then be written as

$$C_{IJ}(\omega) = C_{IJ}^0(\Lambda, Y, \omega) - \phi_0^\ominus C_{IJ}^1(\lambda^0, \mu^0, \omega) - \varepsilon_0^\ominus C_{IJ}^2(\lambda^0, \mu^0, \omega) - \varepsilon_0^\dagger C_{IJ}^3(\lambda^0, \mu^0, \omega) \quad (16)$$

where the reference constants Λ and Y are given by

$$\Lambda = \lambda^0 + \Phi_{c,p}(\lambda^0, \mu^0, f_0); \quad Y = \mu^0 + \Phi_{c,p}(\lambda^0, \mu^0, f_0) \quad (17)$$

with $\Phi_{c,p}$ representing perturbations from microcracks and pores.

Table 2, Common parameters for modelling experimental data used by Amalokwu et al. (2014) and this work. The use of White's mixing law is replaced with our consistent squirt-patchy model in this work.

V_P^0 (m/s)	3250	Porosity (unfractured)	30.43%
V_S^0 (m/s)	2200	Crack density	0
ρ^0 (Kg/m ³)	1800	Aspect ratio	0.088
Measurement frequency (kHz)	650	Solid density (Kg/m ³)	2590
Fracture density	0.0298	Fracture length (m)	0.005
Water viscosity (Pa.s)	1×10^{-3}	Air viscosity (Pa.s)	2×10^{-5}
Water density (Kg/m ³)	1000	Air density (Kg/m ³)	1.2
Water modulus (Pa)	2.25×10^9	Air modulus (Pa)	1×10^5

Figure 6 shows the fitting model for SWS at 45 degrees to fracture normal. The modelling parameters are displayed in Table 2, which are inferred from Amalokwu et al. (2014) and Amalokwu et al. (2016). By matching SWS at dry condition, we use the reference parameters $V_P^0 = 3250$ m/s, $V_S^0 = 2200$ m/s and $\rho^0 = 1800$ Kg/m³ to derive the Lamé parameters as $\lambda^0 = 1.59$ GPa and $\mu^0 = 8.71$ GPa. In the case of high porosity, the effect of grain-scale microcrack on frequency-dependent anisotropy can be neglected, as has been demonstrated by Chapman et al. (2003). We therefore set crack density $\varepsilon_0^\ominus$ to zero. We then scan through the combinations of the characteristic frequency ω'_0 at full water saturation and the non-dimensional parameter q to calculate the misfits between the data and the calculated SWS. From Figure 6a, the minimum misfit is achieved by taking ω/ω'_0 and q as 0.45 and 0.004, respectively. The corresponding relaxation time is 1.7×10^{-8} s, which is close to the value given by Amalokwu et al. (2014). The intermediate value of q suggests contribution from patch effects. Figure 6b shows the best fitting curve. The decreasing SWS trend observed by Amalokwu et al. (2014) can therefore be well modelled with our theory by considering the coupling of squirt and patchy mechanisms.

Figure 6. Model fit to the SWS trend measured by Amalokwu et al. (2014). (a) Misfits between data and the calculated SWS at various combinations of ω'_0 and q . The location of

the minimum misfit is labelled with white circle; (b) Best fitting curve to the measured SWS.

Discussion

We have studied frequency-dependent anisotropy in a fractured rock saturated by two fluids through the derivation of frequency-dependent elastic constants. Our model shows that S-wave velocity, attenuation and SWS are sensitive to fluid through coupled effects from squirt and patch mechanisms. There are various approaches to modelling patchy saturation. White (1975) described fluid heterogeneity by introducing gas-filled spherical pockets in a water saturated porous rock. Later authors have discussed the ‘bubble’ effect on velocity dispersion and attenuation (e.g. Dutta and Odé 1979, Carcione et al. 2003, Quintal et al. 2008). Johnson (2001) proposed a Biot-based theory by considering two geometrical parameters that would potentially allow us to deduce the effective sizes and shapes of the patches. Toms et al. (2007) proposed a model that considers randomly distributed fluid patches in 3D space, and discussed how the distribution of shapes and sizes of the fluid patches would affect dispersion and attenuation.

Alternatively, in this paper we model patch effects using a non-dimensional parameter q rather than considering any geometrical distribution. This is a simple approach that allows us to describe the multi-fluid saturated rock through a unified model in which the squirt-flow mechanism, the uneven fluid pressures, and the relative permeability effects are incorporated. Previous idea of modelling velocity dispersion in partially saturated rocks uses the effective fluid approach combined with the single-fluid squirt theory (Amalokwu et al. 2014). In this study, our modelling approach explicitly describes the two-phase fluid effects through the calculation of elastic constants. We emphasize the importance of the relative permeability effect as it affects the effective fluid mobility and the characteristic frequency, leading to non-monotonic variations of shear modulus and SWS with changing water saturation. In

place of defining an explicit relative permeability model we could alternately define a class of effective fluid models where fluid mobility is simply averaged (through an effective viscosity, in a generalization of the Amalokwu et al. (2014) approach) according to:

$$M_f = \frac{1}{\bar{q}} S_w M_w + \frac{q}{\bar{q}} (1 - S_w) M_g \quad (18)$$

Figure 7 compares the results from such an approach to those from our relative permeability model. In the case where the relative permeability effect is ignored, increasing water saturation would mainly lead to a stiffening effect that results in a monotonic increase in shear modulus and a decrease in SWS. Considering relative permeability, the effective fluid mobility would be lowered at intermediate saturations, leading to a lower characteristic frequency that gives rise to stronger dispersion on the elastic moduli. This leads to non-monotonic variations with respect to fluid saturation, and implies that inversion problem for monitoring fluid saturation from frequency-dependent anisotropy could be non-unique when relative permeability effects are important.

Figure 7. (a) The variation of shear modulus with water saturation for a range of q values at the reference frequency ω_0 . (b) The variation of SWS with water saturation for a range of q

values at the reference frequency ω'_0 . The red curves represent the case with relative permeability effects; The blue curves represent the case given by equation (18) without relative permeability effects.

In our study, we used the same relative permeability model for the grain-scale pores and cracks and meso-scale fractures, which results in the ratio of two characteristic frequencies ω_m/ω_f being the normalized fracture size. It has been suggested that the presence of fractures in a water-gas saturated rock could significantly affect the relative permeability behaviours (Lu et al. 2016). It would therefore be ambiguous to relate fracture size with the ω_m/ω_f ratio. Such effect was neglected in our study as we only choose a single relative permeability model for all inclusions. Future work will address the potential impact of such fracture related relative permeability variations within our model by considering alternative relative permeability models such as the Brooks-Corey model.

We use our theory to match experimental measurements of saturation effects on SWS by Amalokwu et al. (2014). Amalokwu et al. (2016) attempted to fit the data by using White's model to estimate an effective fluid bulk modulus that accounts for the patch-related dispersion. This effective modulus was then used as an input to the model of Chapman (2003) to include squirt-flow effects. In our modelling approach, a good fit can only be achieved at an intermediate q value with the frequency falling in the dispersive range. This confirmed the conclusion of Amalokwu et al. (2016) that both patch and squirt mechanisms are required to fit laboratory observation of the SWS behaviour.

Previous work has shown that meso-scale fractures could lead to dispersion and attenuation in the seismic frequency band. This is still true for our work, but the situation becomes

complicated due to effects of the relative permeability and uneven fluid pressures. Although the frequency-dependent SWS is primarily affected by fracture size a_f , the SWS – a_f relationship is complicated since the characteristic frequency is further influenced by fluid saturation and the patch parameter q .

Conclusions

We have shown how seismic anisotropy of rocks saturated with two immiscible fluids can be influenced by coupled patch and squirt effects. Our modelling approach explicitly describes the multi-phase fluid effects on frequency-dependent anisotropy through the calculation of elastic constants in which the squirt-flow mechanism, the uneven fluid pressures, and the relative permeability effects are incorporated. We show that patchy saturation tends to increase S-wave velocity in the isotropic case and reduce SWS in the anisotropic case. The effects of relative permeability could reduce the fluid mobility, leading to a characteristic frequency lower in partially saturated rocks compared to the fully saturated case. This results in non-monotonic variations of moduli with respect to changes in water saturation, which may complicate the inversion of fluid saturation from seismic data. While the squirt and patch effects on shear properties are weaker than that on bulk properties, the impact on SWS in fractured rocks is potentially significant. The theory conveniently models previously published data, emphasising the requirement to include coupled squirt and patch mechanisms for the analysis of saturation effects on seismic anisotropy.

Acknowledgements

Zhaoyu Jin was supported by the Principal's Career Development PhD Scholarships and Edinburgh Global Research Scholarship from The University of Edinburgh. Mark Chapman was partially supported by the Edinburgh Anisotropy Project (EAP), British Geological Survey. Giorgos Papageorgiou acknowledges financial support from the program PETROMAKS2 of the Research Council of Norway (RCN grant number: 267765). We thank

Dr. Xu Liu and an anonymous reviewer for constructive comments that helped improve the manuscript.

References

- Al-Harrasi, O.H., Kendall, J.M. and Chapman, M., 2011. Fracture characterization using frequency-dependent shear wave anisotropy analysis of microseismic data. *Geophysical Journal International*, 185(2), pp.1059-1070.
- Amalokwu, K., Best, A.I. and Chapman, M., 2016. Effects of aligned fractures on the response of velocity and attenuation ratios to water saturation variation: A laboratory study using synthetic sandstones. *Geophysical Prospecting*, 64(4), pp.942-957.
- Amalokwu, K., Chapman, M., Best, A.I., Sothcott, J., Minshull, T.A. and Li, X.Y., 2014. Experimental observation of water saturation effects on S-wave splitting in synthetic rock with fractures aligned at oblique angles. *Geophysical Journal International*, 200(1), 17-24.
- Benson, S., Pini, R., Reynolds, C. and Krevor, S., 2013. Relative permeability analyses to describe multi-phase flow in CO₂ storage reservoirs. *Global CCS Institute*.
- Carcione, J.M., Gurevich, B., Santos, J.E. and Picotti, S., 2013. Angular and frequency-dependent wave velocity and attenuation in fractured porous media. *Pure and Applied Geophysics*, 170(11), pp.1673-1683.
- Carcione J. M., Helle H. B. and Pham N. H. 2003. White's model for wave propagation in partially saturated rocks: Comparison with poroelastic numerical experiments. *Geophysics* 68(4), 1389-1398.
- Chapman, M., 2003. Frequency-dependent anisotropy due to meso-scale fractures in the presence of equant porosity. *Geophysical Prospecting*, 51(5), pp.369-379.

- Chapman, M., Maultzsch, S., Liu, E. and Li, X.Y., 2003. The effect of fluid saturation in an anisotropic multi-scale equant porosity model. *Journal of Applied Geophysics*, 54(3), 191-202.
- Chesnokov, E.M., Queen, J.H., Vichorev, A.A., Lynn, H.B., Hooper, J.M., Bayuk, I.O., Castagna, J.A. and Roy, B., 2001. Frequency dependent anisotropy. In *SEG Annual Meeting*. Society of Exploration Geophysicists.
- Dutta N. C. and Odé H. 1979. Attenuation and dispersion of compressional waves in fluid-filled porous rocks with partial gas saturation (White model)-Part I: Biot theory. *Geophysics* 44(11), 1777-1788.
- Dvorkin, J., Mavko, G. and Nur, A., 1995. Squirt flow in fully saturated rocks. *Geophysics*, 60(1), pp.97-107.
- Galvin, R.J. and Gurevich, B., 2015. Frequency-dependent anisotropy of porous rocks with aligned fractures. *Geophysical Prospecting*, 63(1), pp.141-150.
- Johnson, D.L., 2001. Theory of frequency dependent acoustics in patchy-saturated porous media. *The Journal of the Acoustical Society of America*, 110(2), pp.682-694.
- Lu, H., Lu, X., Fan, J., Zhao, M., Wei, H., Zhang, B. and Lu, Y., 2016. Controlling effect of fractures on gas accumulation and production within the tight sandstone: A case study on the Jurassic Dibei gas reservoir in the eastern part of the Kuqa foreland basin, China. *Journal of Natural Gas Geoscience*, 1(1), pp.61-71.
- Marson-Pidgeon, K. and Savage, M.K., 1997. Frequency-dependent anisotropy in Wellington, New Zealand. *Geophysical Research Letters*, 24(24), pp.3297-3300.
- Maultzsch, S., Chapman, M., Liu, E. and Li, X.Y., 2003. Modelling frequency - dependent seismic anisotropy in fluid - saturated rock with aligned fractures: implication of fracture size estimation from anisotropic measurements. *Geophysical Prospecting*, 51(5), pp.381-392.

- Mavko, G. and Mukerji, T., 1998. Bounds on low-frequency seismic velocities in partially saturated rocks. *Geophysics*, 63(3), pp.918-924.
- Mavko, G., Mukerji, T. and Dvorkin, J., 2009. The rock physics handbook: Tools for seismic analysis of porous media. Cambridge university press.
- Papageorgiou, G. and Chapman, M., 2017. Wave-propagation in rocks saturated by two immiscible fluids. *Geophysical Journal International*, 209(3), pp.1761-1767.
- Papageorgiou, G., Falcon-Suarez, I., Chapman, M. and Best, A., 2018. Pressure-varying CO₂ distribution affects the ultrasonic velocities of synthetic sandstones. *International Journal of Greenhouse Gas Control*, 74, pp.1-8.
- Qian, Z., Chapman, M., Li, X.Y., Dai, H.C., Liu, E., Zhang, Y. and Wang, Y., 2007. Use of multicomponent seismic data for oil-water discrimination in fractured reservoirs. *The Leading Edge*, 26(9), pp.1176-1184.
- Quintal B., Schmalholz S. M. and Podladchikov Y. Y. 2008. Low-frequency reflections from a thin layer with high attenuation caused by interlayer flow. *Geophysics* 74(1), N15-N23.
- Tillotson, P., Chapman, M., Best, A.I., Sothcott, J., McCann, C., Shangxu, W. and Li, X.Y., 2011. Observations of fluid-dependent shear-wave splitting in synthetic porous rocks with aligned penny-shaped fractures. *Geophysical Prospecting*, 59(1), pp.111-119.
- Toms, J., Müller, T.M. and Gurevich, B., 2007. Seismic attenuation in porous rocks with random patchy saturation. *Geophysical Prospecting*, 55(5), pp.671-678.
- White J. E. 1975. Computed seismic speeds and attenuation in rocks with partial gas saturation. *Geophysics* 40(2), 224-232.
- Zatsepin, S.V. and Crampin, S., 1997. Modelling the compliance of crustal rock—I. Response of shear-wave splitting to differential stress. *Geophysical Journal International*, 129(3), pp.477-494.

Appendix

In this appendix, we give detailed calculation of the elastic constants of the fractured rock saturated by two immiscible fluids. Our model consists of an isotropic collection of grain-scale randomly oriented ellipsoidal microcracks and spherical pores with a set of meso-scale perfectly aligned fractures. Following Chapman (2003), we assume that the fracture has the same aspect ratio as the crack. The radii of the fractures are larger than the grain size but smaller than the seismic wavelength. The cracks and pores are connected to c_1 other elements and the fractures are connected to c_2 elements. Cracks and pores are allowed to be connected with any kind of inclusion while fractures can only be connected with pores or cracks. It is also assumed that each crack or pore is connected to at most one fracture. We use superscripts $\ominus$, $\ominus$ and $\dagger$ to represent pore, crack and fracture, and subscripts g and w to denote fluid types.

The fluid mass content in each inclusion can be represented by the following equations

$$\begin{aligned} m_w^{\ominus} &= S_w \rho_w^{\ominus} \phi^{\ominus}; \quad m_g^{\ominus} = (1 - S_w) \rho_g^{\ominus} \phi^{\ominus}; \\ m_w^{\ominus} &= S_w \rho_w^{\ominus} \phi^{\ominus}; \quad m_g^{\ominus} = (1 - S_w) \rho_g^{\ominus} \phi^{\ominus}; \\ m_w^{\dagger} &= S_w \rho_w^{\dagger} \phi^{\dagger}; \quad m_g^{\dagger} = (1 - S_w) \rho_g^{\dagger} \phi^{\dagger} \end{aligned} \quad (\text{A1})$$

where S_w is the saturation that is assumed equal in each inclusion. ρ_f is the fluid density which can be further estimated from its undisturbed density ρ_f^0 , the fluid pressure P_f , and fluid bulk modulus K_f through

$$\rho_f \simeq \rho_f^0 \left(1 + \frac{P_f}{K_f}\right) \quad (\text{A2})$$

Assuming a unit volume of the rock, we can use the void fraction (porosity) ϕ to represent the volume of the inclusion under the applied stress σ_{ii} , i.e.

$$\begin{aligned}
\phi^{\odot} &= \phi_0^{\odot} \left(1 - \frac{3(1-\nu)\sigma_{ii}}{4\mu(1+\nu)} + \frac{3P^{\odot}}{4\mu}\right); \\
\phi^{\ominus} &= \phi_0^{\ominus} \left(1 - \frac{\sigma_i}{\sigma_c} + \frac{P^{\ominus}}{\sigma_c}\right); \\
\phi^{\dagger} &= \phi_0^{\dagger} \left(1 - \frac{\sigma_f}{\sigma_c} + \frac{P^{\dagger}}{\sigma_c}\right)
\end{aligned} \tag{A3}$$

where ϕ_0 is the unstressed inclusion volume, μ is the mineral shear modulus, ν is the Poisson's ratio, σ_i is the normal stress acting on the crack face, σ_f is the normal stress acting on the fracture face, and σ_c is related to the aspect ratio r of the crack through equation (10).

$P^{\odot}$, $P^{\ominus}$ and $P^{\dagger}$ refer to the average pressures of the pore, crack and fracture. In our model, each inclusion is saturated by two fluids, the pressures of which are allowed to be different via the patch parameter q (i.e. $P_g = qP_w$). We therefore intend to write the inclusion pressure as a weighted average of these two fluid pressures. Following Papageorgiou and Chapman (2017), we take water saturation as the weighting factor and give the following expressions

$$P^{\odot} = \tilde{q}P_w^{\odot}; \quad P^{\ominus} = \tilde{q}P_w^{\ominus}; \quad P^{\dagger} = \tilde{q}P_w^{\dagger} \tag{A4}$$

where $\tilde{q} = S_w + q(1 - S_w)$.

By expanding equations (A1), neglecting high order terms and taking the derivatives, we can write the fluid mass flow in each inclusion in terms of the applied stress and fluid pressure as

$$\begin{aligned}
\partial_t m_w^{\odot} &= \frac{3}{4\mu} S_w \rho_w^0 \phi_0^{\odot} \partial_t \left[\left(1 + \frac{4\mu}{3\tilde{q}K_w}\right) P^{\odot} - \frac{(1-\nu)\sigma_{ii}}{(1+\nu)} \right]; \\
\partial_t m_g^{\odot} &= \frac{3}{4\mu} (1 - S_w) \rho_g^0 \phi_0^{\odot} \partial_t \left[\left(1 + \frac{4\mu q}{3\tilde{q}K_g}\right) P^{\odot} - \frac{(1-\nu)\sigma_{ii}}{(1+\nu)} \right]; \\
\partial_t m_w^{\ominus} &= \frac{1}{\sigma_c} S_w \rho_w^0 \phi_0^{\ominus} \partial_t \left[\left(1 + \frac{\sigma_c}{\tilde{q}K_w}\right) P^{\ominus} - \sigma_i \right]; \\
\partial_t m_g^{\ominus} &= \frac{1}{\sigma_c} (1 - S_w) \rho_g^0 \phi_0^{\ominus} \partial_t \left[\left(1 + \frac{q\sigma_c}{\tilde{q}K_g}\right) P^{\ominus} - \sigma_i \right];
\end{aligned}$$

$$\begin{aligned}\partial_t m_w^\dagger &= \frac{1}{\sigma_c} S_w \rho_w^0 \phi_0^\dagger \partial_t \left[\left(1 + \frac{\sigma_c}{\tilde{q} K_w} \right) P^\dagger - \sigma_f \right]; \\ \partial_t m_g^\dagger &= \frac{1}{\sigma_c} (1 - S_w) \rho_g^0 \phi_0^\dagger \partial_t \left[\left(1 + \frac{q \sigma_c}{\tilde{q} K_g} \right) P^\dagger - \sigma_f \right]\end{aligned}\quad (\text{A5})$$

Calculation of the frequency-dependent anisotropic elastic constants

In the single-fluid model of Chapman (2003), the fluid mass exchange between inclusions was described by equations

$$\partial_t (m^\ominus - m^\odot) = \frac{c_1 \rho^0}{g} (P^\odot - P^\ominus); \quad (\text{A6})$$

$$\partial_t m^\dagger = \frac{c_2 \rho^0}{g} (E_p - P^\dagger) \quad (\text{A7})$$

where E_p is the expected pressure of the pore-crack system defined by Chapman (2003). The single-fluid coefficient g is given as

$$\frac{1}{g} = l M_f \quad (\text{A8})$$

where l is a characteristic length scale, and M_f is the fluid mobility defined as the ratio of rock permeability κ to fluid viscosity η , i.e.

$$M_f = \frac{\kappa}{\eta} \quad (\text{A9})$$

Equations (A6) and (A7) are essentially based on Darcy's law approximated by Chapman et al. (2002) in which the fluid mass variation is proportionally related to the pressure gradients between the inclusions via the fluid mobility. In the multi-fluid case, for each fluid we assume that the mass variation is proportional to the corresponding fluid pressure gradient between the inclusions through a fluid mobility that is scaled by the relative permeability, i.e. the Darcy constants g_w and g_g for water and gas are written as

$$\frac{1}{g_w} = l\kappa_w M_w; \quad \frac{1}{g_g} = l\kappa_g M_g \quad (\text{A10})$$

where κ_w and κ_g are relative permeabilities of water and gas. M_w and M_g are the mobilities of water and gas given by

$$M_w = \frac{\kappa}{\eta_w}; \quad M_g = \frac{\kappa}{\eta_g} \quad (\text{A11})$$

where η_w and η_g are viscosities of water and gas.

We can then extend the single-fluid mass exchange equations to the two-fluid case, and describe each fluid flow between inclusions by the following equations:

$$\begin{aligned} \partial_t(m_w^\ominus - m_w^\odot) &= \frac{c_1 \rho_w^0}{g_w} (P_w^\odot - P_w^\ominus) = \frac{c_1 \rho_w^0}{\tilde{q} g_w} (P^\odot - P^\ominus); \\ \partial_t(m_g^\ominus - m_g^\odot) &= \frac{c_1 \rho_g^0}{g_g} (P_g^\odot - P_g^\ominus) = \frac{q c_1 \rho_g^0}{\tilde{q} g_g} (P^\odot - P^\ominus); \\ \partial_t m_w^\dagger &= \frac{c_2 \rho_w^0}{g_w} (E_{P_w} - P_w^\dagger) = \frac{c_2 \rho_w^0}{\tilde{q} g_w} (E_P - P^\dagger); \\ \partial_t m_g^\dagger &= \frac{c_2 \rho_g^0}{g_g} (E_{P_g} - P_g^\dagger) = \frac{q c_2 \rho_g^0}{\tilde{q} g_g} (E_P - P^\dagger) \end{aligned} \quad (\text{A12})$$

Substituting equations (A5) into the above equations yields

$$\begin{aligned} \frac{\phi_0^\ominus}{\sigma_c} \partial_t [(1 + K_c) P^\ominus - \sigma_i] - \frac{3\phi_0^\odot}{4\mu} \partial_t \left[(1 + K_p) P^\odot - \frac{(1-\nu)\sigma_{ii}}{(1+\nu)} \right] &= \frac{c_1}{g} (P^\odot - P^\ominus); \\ \frac{\phi_0^\dagger}{\sigma_c} \partial_t [(1 + K_c) P^\dagger - \sigma_f] &= \frac{c_2}{g} (E_P - P^\dagger) \end{aligned} \quad (\text{A13})$$

where the Darcy constant $\frac{1}{g}$ given in equation (A8) now depends on the effective mobility of the two-fluid mixture that is calculated as a weighted average of water mobility M_w and gas mobility M_g influenced by the relative permeability and the patch parameter q through the following equation

$$M_f = \frac{\kappa_w}{\tilde{q}} M_w + \frac{q\kappa_g}{\tilde{q}} M_g \quad (\text{A14})$$

K_c and K_p are related to the effective fluid bulk modulus K_f via

$$K_c = \frac{\sigma_c}{K_f}; \quad K_p = \frac{4\mu}{3K_f} \quad (\text{A15})$$

where K_f is a Reuss average of the two fluid bulk moduli weighted by q , i.e.

$$\frac{1}{K_f} = \frac{S_w}{\tilde{q}K_w} + \frac{(1-S_w)q}{\tilde{q}K_g} \quad (\text{A16})$$

Taking the Fourier transform of equations (A13) and considering the conservation of mass throughout all elements, we have the solutions of pressures in each element:

$$\begin{aligned} P^\ominus(\omega) &= D_1(\omega)\sigma_{ii} + D_2(\omega)\sigma_f; \\ P^\ominus(\omega) &= G_1(\omega)\sigma_i + G_2(\omega)\sigma_{ii} + G_3(\omega)\sigma_f; \\ P^\dagger(\omega) &= F_1(\omega)\sigma_{ii} + F_2(\omega)\sigma_f \end{aligned} \quad (\text{A17})$$

where $D_1(\omega)$, $D_2(\omega)$, $G_1(\omega)$, $G_2(\omega)$, $G_3(\omega)$, $F_1(\omega)$ and $F_2(\omega)$ are frequency-dependent parameters derived as

$$\begin{aligned} D_1 &= \left[(1-\iota)\gamma + \frac{(1-\iota)\beta}{1+i\omega/\omega_f} + \frac{1+i\omega\gamma/\omega_m}{1+i\omega/\omega_m} \left(\iota + \frac{\iota\beta}{1+i\omega/\omega_f} \right) \right]^{-1} \\ &\times \left[\frac{\iota}{3(1+K_c)} + (1-\iota)\gamma' \right. \\ &\quad \left. - \frac{i\omega/\omega_m}{1+i\omega/\omega_m} \left(\frac{1}{3(1+K_c)} - \gamma' \right) \left(\iota + \frac{\iota\beta}{1+i\omega/\omega_f} \right) \right] \end{aligned} \quad (\text{A18})$$

$$D_2 = \left[(1 - \iota)\gamma + \frac{(1 - \iota)\beta}{1 + i\omega/\omega_f} + \frac{1 + i\omega\gamma/\omega_m}{1 + i\omega/\omega_m} \left(\iota + \frac{\iota\beta}{1 + i\omega/\omega_f} \right) \right]^{-1} \times \frac{\beta}{(1 + K_c)(1 + i\omega/\omega_f)} \quad (\text{A19})$$

$$G_1 = \frac{i\omega/\omega_m}{(1 + K_c)(1 + i\omega/\omega_m)} \quad (\text{A20})$$

$$G_2 = \frac{1 + i\omega\gamma/\omega_m}{1 + i\omega/\omega_m} D_1 - \frac{i\omega\gamma'/\omega_m}{1 + i\omega/\omega_m} \quad (\text{A21})$$

$$G_3 = \frac{1 + i\omega\gamma/\omega_m}{1 + i\omega/\omega_m} D_2 \quad (\text{A22})$$

$$F_1 = \frac{1}{1 + i\omega/\omega_f} \left[\frac{1 + i\omega\gamma/\omega_m}{1 + i\omega/\omega_m} \iota D_1 + (1 - \iota) D_1 + \frac{i\omega\iota/\omega_m}{1 + i\omega/\omega_m} \left(\frac{1}{3(1 + K_c)} - \gamma' \right) \right] \quad (\text{A23})$$

$$F_2 = \frac{1}{1 + i\omega/\omega_f} \left[\frac{i\omega/\omega_f}{1 + K_c} + \iota \frac{1 + i\omega\gamma/\omega_m}{1 + i\omega/\omega_m} D_2 + (1 - \iota) D_2 \right] \quad (\text{A24})$$

where

$$\gamma = \frac{3\phi_0^{\ominus}\sigma_c(1+K_p)}{4\mu\phi_0^{\ominus}(1+K_c)}; \quad \gamma' = \frac{(1-\nu)\gamma}{(1+\nu)(1+K_p)} \quad (\text{A25})$$

$$\iota = \frac{\phi_0^{\ominus}}{\phi_0^{\ominus} + r\phi_0^{\ominus}}; \quad \beta = \frac{\iota\phi_0^{\dagger}}{\phi_0^{\ominus}} \quad (\text{A26})$$

ω_m and ω_f are the micro-scale and fracture-scale characteristic frequencies derived as

$$\omega_m = \frac{\omega_0 M_f}{M_w}; \quad \omega_f = \frac{\omega'_0 M_f}{M_w} \quad (\text{A27})$$

where ω_0 and ω'_0 are the values of ω_m and ω_f at full water saturation, the explicit expressions of which have been given by Chapman (2003), and they are related via

$$\omega'_0 = \frac{\varsigma}{a_f} \omega_0 \quad (\text{A28})$$

where ς is the grain size that is assumed to be identified with the radii a of the pore and cracks, and a_f is the radius of the fracture.

We further introduce notations

$$L_2 = \lambda^2 + \frac{4}{3}\lambda\mu + \frac{4}{5}\mu^2 \quad (\text{A29})$$

$$L_4 = \lambda^2 + \frac{4}{3}\lambda\mu + \frac{4}{15}\mu^2 \quad (\text{A30})$$

$$k = \lambda + \frac{2}{3}\mu \quad (\text{A31})$$

With the relationship between the pressure in each inclusion and an imposed stress field being built, we can then follow Chapman (2003) to apply the equivalent medium theory for the calculation of the effective stiffness tensor. We use Cartesian coordinates in which the x_3 -axis is aligned with the fracture normal direction. The resulting medium is therefore

transversely isotropic with vertical axis of symmetry (VTI). With the adoption of the Voigt notation, the five independent elastic constants can be explicitly calculated as

$$\begin{aligned}
 C_{11} = & (\lambda + 2\mu) \\
 & - \phi_0^\ominus \left[\frac{L_2}{\sigma_c} + \frac{32}{15} \frac{(1-\nu)\mu}{(2-\nu)\pi r} - \left(\frac{L_2}{\sigma_c} + k \right) G_1 \right. \\
 & \left. - \left(\frac{3k^2}{\sigma_c} + 3k \right) G_2 - \left(\frac{\lambda k}{\sigma_c} + \lambda \right) G_3 \right] \\
 & - \phi_0^\odot \left[\frac{3}{4\mu} \frac{1-\nu}{1+\nu} \left(3\lambda^2 + 4\lambda\mu + \frac{36+20\nu}{7-5\nu} \mu^2 \right) \right. \\
 & \left. - \left(1 + \frac{3k}{4\mu} \right) (3kD_1 + \lambda D_2) \right] \\
 & - \phi_0^\dagger \left[\frac{\lambda^2}{\sigma_c} - 3k \left(\frac{\lambda}{\sigma_c} + 1 \right) F_1 - \lambda \left(\frac{\lambda}{\sigma_c} + 1 \right) F_2 \right]
 \end{aligned} \tag{A32}$$

$$\begin{aligned}
 C_{33} = & (\lambda + 2\mu) \\
 & - \phi_0^\ominus \left[\frac{L_2}{\sigma_c} + \frac{32}{15} \frac{(1-\nu)\mu}{(2-\nu)\pi r} - \left(\frac{L_2}{\sigma_c} + k \right) G_1 \right. \\
 & \left. - \left(\frac{3k^2}{\sigma_c} + 3k \right) G_2 - \left(\frac{\lambda + 2\mu}{\sigma_c} k + \lambda + 2\mu \right) G_3 \right] \\
 & - \phi_0^\odot \left[\frac{3}{4\mu} \frac{1-\nu}{1+\nu} \left(3\lambda^2 + 4\lambda\mu + \frac{36+20\nu}{7-5\nu} \mu^2 \right) \right. \\
 & \left. - \left(1 + \frac{3k}{4\mu} \right) (3kD_1 + (\lambda + 2\mu)D_2) \right] \\
 & - \phi_0^\dagger \left[\frac{(\lambda + 2\mu)^2}{\sigma_c} - 3k \left(\frac{\lambda + 2\mu}{\sigma_c} + 1 \right) F_1 \right. \\
 & \left. - (\lambda + 2\mu) \left(\frac{\lambda + 2\mu}{\sigma_c} + 1 \right) F_2 \right]
 \end{aligned} \tag{A33}$$

$$C_{44} = \mu - \phi_0^\ominus \left[\frac{4}{15} \frac{\mu^2}{\sigma_c} (1 - G_1) + \frac{8}{5} \frac{(1 - \nu)\mu}{(2 - \nu)\pi r} \right] - 15\mu\phi_0^\ominus \frac{1 - \nu}{7 - 5\nu} - 4\mu\phi_0^\dagger \frac{1 - \nu}{(2 - \nu)\pi r} \quad (\text{A34})$$

$$C_{12} = \lambda - \phi_0^\ominus \left[\frac{L_4}{\sigma_c} - \frac{16}{15} \frac{(1 - \nu)\mu}{(2 - \nu)\pi r} - \left(\frac{L_4}{\sigma_c} + k \right) G_1 - \left(\frac{3k^2}{\sigma_c} + 3k \right) G_2 - \left(\frac{\lambda k}{\sigma_c} + \lambda \right) G_3 \right] - \phi_0^\ominus \left[\frac{3}{4\mu} \frac{1 - \nu}{1 + \nu} \left(3\lambda^2 + 4\lambda\mu - \frac{4(1 + 5\nu)}{7 - 5\nu} \mu^2 \right) - \left(1 + \frac{3k}{4\mu} \right) (3kD_1 + \lambda D_2) \right] - \phi_0^\dagger \left[\frac{\lambda^2}{\sigma_c} - 3k \left(\frac{\lambda}{\sigma_c} + 1 \right) F_1 - \lambda \left(\frac{\lambda}{\sigma_c} + 1 \right) F_2 \right] \quad (\text{A35})$$

$$C_{13} = \lambda - \phi_0^\ominus \left[\frac{L_4}{\sigma_c} - \frac{16}{15} \frac{(1 - \nu)\mu}{(2 - \nu)\pi r} - \left(\frac{L_4}{\sigma_c} + k \right) G_1 - \left(\frac{3k^2}{\sigma_c} + 3k \right) G_2 - (\lambda + \mu) \left(\frac{k}{\sigma_c} + 1 \right) G_3 \right] - \phi_0^\ominus \left[\frac{3}{4\mu} \frac{1 - \nu}{1 + \nu} \left(3\lambda^2 + 4\lambda\mu - \frac{4(1 + 5\nu)}{7 - 5\nu} \mu^2 \right) - \left(1 + \frac{3k}{4\mu} \right) (3kD_1 + (\lambda + \mu)D_2) \right] - \phi_0^\dagger \left[\frac{\lambda(\lambda + 2\mu)}{\sigma_c} - 3k \left(\frac{\lambda + \mu}{\sigma_c} + 1 \right) F_1 - \left(\frac{\lambda(\lambda + 2\mu)}{\sigma_c} + \lambda + \mu \right) F_2 \right] \quad (\text{A36})$$

C_{66} depends on C_{11} and C_{12} via

$$\begin{aligned} C_{66} &= \frac{1}{2}(C_{11} - C_{12}) \\ &= \mu - \phi_0^\ominus \left[\frac{4}{15} \frac{\mu^2}{\sigma_c} (1 - G_1) + \frac{8}{5} \frac{(1 - \nu)\mu}{(2 - \nu)\pi r} \right] \\ &\quad - 15\mu\phi_0^\ominus \frac{1 - \nu}{7 - 5\nu} \end{aligned} \quad (\text{A37})$$

The porosities of cracks and fractures $\phi_0^\ominus$, $\phi_0^\dagger$ can be further defined as more commonly used crack density $\varepsilon_0^\ominus$ and fracture density $\varepsilon_0^\dagger$ via the following equations (Mavko et al. 2009):

$$\varepsilon_0^\ominus = \frac{N_c}{V} a^3 = \frac{3\phi_0^\ominus}{4\pi r}; \quad \varepsilon_0^\dagger = \frac{N_f}{V} a_f^3 = \frac{3\phi_0^\dagger}{4\pi r} \quad (\text{A38})$$

where $\frac{N_c}{V}$ and $\frac{N_f}{V}$ represent the number of cracks and fractures per unit volume.

The isotropic limit of the model

In the absence of meso-scale fractures, we would expect no contribution from terms associated with $\varepsilon_0^\dagger$ by setting the fracture density to zero. This reduces our model to the isotropic limit that can be specified by two independent constants. We can then characterize the medium by using the effective bulk and shear moduli. The bulk modulus $K_{eff}(\omega)$ is given by $C_{11} - \frac{4}{3}C_{44}$, which is calculated to be consistent with the one proposed by Papageorgiou and Chapman (2017):

$$K_{eff}(\omega) = K_{dry} + 3\phi_0^\ominus \left(1 + \frac{K_m}{\sigma_c} \right) K_m A(\omega) + 3\phi_0^\ominus \left(1 + \frac{3K_m}{4\mu} \right) K_m B(\omega) \quad (\text{A39})$$

where K_m is the mineral bulk modulus, K_{dry} is the dry frame modulus given by

$$K_{dry} = K_m - K_m^2 \left(\frac{9\phi_0^\odot}{4\mu} \frac{1-\nu}{1+\nu} + \frac{\phi_0^\ominus}{\sigma_c} \right) \quad (\text{A40})$$

$A(\omega)$ and $B(\omega)$ are frequency-dependent constants defined as

$$A(\omega) = \frac{\frac{1+i\omega\gamma/\omega_m}{3(1+K_c)} + \gamma'}{1+(1+i\omega/\omega_m)\gamma}; \quad B(\omega) = \frac{\frac{1}{3(1+K_c)} + (1+i\omega/\omega_m)\gamma'}{1+(1+i\omega/\omega_m)\gamma} \quad (\text{A41})$$

The effective shear modulus $\mu_{eff}(\omega)$ is given by C_{44} , the form of which is expressed by equation (9).