THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Cylinders in rational surfaces

Citation for published version:

Cheltsov, I 2016 'Cylinders in rational surfaces' ArXiv.

Link:

[Link to publication record in Edinburgh Research Explorer](#)

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

CYLINDERS IN RATIONAL SURFACES

IVAN CHELTSOV

ABSTRACT. We answer a question of Ciro Ciliberto about cylinders in rational surfaces which are obtained by blowing up the plane at points in general position.

Let S be a smooth rational surface. A *cylinder* in S is an open subset $U \subset S$ such that

$$U \cong \mathbb{C}^1 \times Z$$

for an affine curve Z . Here, the curve Z is just $\mathbb{P}^1$ with finitely many missing points.

The surface S contains many cylinders, and it seems hopeless to describe all of them. Instead of doing this, one can consider a similar problem for *polarized* rational surfaces. This problem has a significant application in affine geometry (see [8, 9, 10]).

Fix an ample $\mathbb{Q}$ -divisor A on the surface S .

Definition 1. An A -polar cylinder in S is a Zariski open subset U in S such that

- (C) $U \cong \mathbb{C}^1 \times Z$ for some affine curve Z , i.e., U is a cylinder in S ,
- (P) there is an effective $\mathbb{Q}$ -divisor D on S such that $D \sim_{\mathbb{Q}} A$ and $U = S \setminus \text{Supp}(D)$.

One can always choose an ample divisor A such that S contains an A -polar cylinder. This follows from [8, Proposition 3.13]. On the other hand, we have

Theorem 2 ([10, 2]). *Suppose that S is a smooth del Pezzo surface and $A \in \mathbb{Q}_{>0}[-K_S]$. Then S contains an A -polar cylinder $\iff K_S^2 \geq 4$.*

Theorem 3 ([11, 4]). *Suppose that S is a smooth del Pezzo surface and $A \notin \mathbb{Q}_{>0}[-K_S]$. If $K_S^2 \geq 3$, then S contains an A -polar cylinder.*

The paper [4] also contains a generalization of Theorem 2. To describe it, we put

$$\mu_A = \inf \left\{ \lambda \in \mathbb{Q}_{>0} \mid \text{the } \mathbb{Q}\text{-divisor } K_S + \lambda A \text{ is pseudo-effective} \right\}.$$

It is well-known that $\mu_A \in \mathbb{Q}$ (see, for example, [12, Theorem 2.1] or [13, Theorem 1]). The number μ_A is known as the Fujita invariant of the divisor A (see [7, Definition 3.1]). Let Δ_A be the smallest extremal face of the Mori cone $\overline{\text{NE}}(S)$ that contains $K_S + \mu_A A$. Denote by r_A the dimension of the face Δ_A . Then $r_A + K_S^2 \leq 9$, since S is rational. The number r_A is known as the Fujita rank of the divisor A (see [4, Definition 2.1.1]).

Theorem 4 ([4, Theorem 2.2.3]). *Suppose that S is a smooth del Pezzo surface and*

$$r_A + K_S^2 \leq 3.$$

Then S does not contain A -polar cylinders.

All varieties are assumed to be algebraic, projective and defined over $\mathbb{C}$.

During the conference *Complex affine geometry, hyperbolicity and complex analysis*, which was held in Grenoble in October 2016, Ciro Ciliberto asked

Question 5. *Suppose that S is a blow up of $\mathbb{P}^2$ at points in general position and*

$$r_A + K_S^2 \leq 3.$$

Is it true that S does not contain A -polar cylinders?

Ciliberto also suggested to consider Question 5 modulo [5, Conjecture 2.3].

In this paper, we show that the answer to Question 5 is Yes. To be precise, we prove

Theorem 6. *Suppose that S satisfies the following generality condition:*

(★) *the self-intersection of every smooth rational curve in S is at least -1 .*

If $r_A + K_S^2 \leq 3$, then S does not contain A -polar cylinders.

By [6, Proposition 2.4], if S is obtained by blowing up $\mathbb{P}^2$ at points in general position, then the condition (★) in Theorem 6 is satisfied. Thus, the answer to Question 5 is Yes.

Remark 7. If S is a del Pezzo surface, then the condition (★) in Theorem 6 is satisfied. In fact, if $K_S^2 \geq 1$, then this condition is equivalent to the ampleness of the divisor $-K_S$. This shows that Theorem 6 is a generalization of Theorem 4.

By [9, Corollary 3.2], Theorem 13 implies

Corollary 8. *Suppose that S satisfies the condition (★), and A is a $\mathbb{Z}$ -divisor. Put*

$$V = \operatorname{Spec} \left(\bigoplus_{n \geq 0} H^0(S, \mathcal{O}_S(nA)) \right).$$

If $r_A + K_S^2 \leq 3$, then V does not admit an effective action of the group $\mathbb{C}_+$.

The inequality $r_A + K_S^2 \leq 3$ in Theorem 6 is sharp:

Example 9. Suppose that S satisfies the condition (★) in Theorem 6, and $K_S^2 \leq 3$. Then there exists a blow up $f: S \rightarrow \mathbb{P}^2$ of $9 - K_S^2$ different points. Put $k = 4 - K_S^2 \geq 1$. Denote the $9 - K_S^2$ exceptional curves of the blow up f by $E_1, E_2, E_3, E_4, E_5, G_1, \dots, G_k$. Let $\mathcal{C}$ be the unique conic in $\mathbb{P}^2$ that passes through $f(E_1), f(E_2), f(E_3), f(E_4), f(E_5)$. Let L be a general line in $\mathbb{P}^2$ tangent to $\mathcal{C}$, and let $\mathcal{P}$ be the pencil generated by $\mathcal{C}$ and $2L$. Let C_i be the conic in $\mathcal{P}$ that contains $f(G_i)$. Then

$$\mathbb{P}^2 \setminus \left(\mathcal{C} \cup L \cup C_1 \cup \dots \cup C_k \right)$$

is a cylinder. Denote the proper transforms of $\mathcal{C}$ and L on S by $\tilde{\mathcal{C}}$ and $\tilde{L}$, respectively. Similarly, denote by $\tilde{C}_i$ the proper transform of the conic C_i on the surface S . Then

$$S \setminus \left(\tilde{\mathcal{C}} \cup \tilde{L} \cup E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup \tilde{C}_1 \cup \dots \cup \tilde{C}_k \cup G_1 \cup \dots \cup G_k \right) \cong \mathbb{P}^2 \setminus \left(\mathcal{C} \cup L \cup C_1 \cup \dots \cup C_k \right).$$

Let ϵ_1, ϵ_2 and x be rational numbers such that $\frac{1}{2} > \epsilon_1 > \frac{\epsilon_2}{2} > 0$ and $1 > x > 1 - \frac{1-2\epsilon_1}{2k}$. Put $A = -K_S + x(G_1 + \cdots + G_k)$. Then A is ample and $r_A = k$, since

$$A \sim_{\mathbb{Q}} \left(1 + \epsilon_1 - \frac{\epsilon_2}{2}\right) \tilde{\mathcal{C}} + \epsilon_2 \tilde{L} + \left(\epsilon_1 - \frac{\epsilon_2}{2}\right) \sum_{i=1}^5 E_i + \frac{1-2\epsilon_1}{2k} \sum_{i=1}^k \tilde{C}_i + \left(x + \frac{1-2\epsilon_1}{2k} - 1\right) \sum_{i=1}^k G_i.$$

Thus, the surface S contains an A -polar cylinder, and $r_A + K_S^2 = 4$.

The inequality $r_A + K_S^2 \geq 4$ does not always imply the existence of A -polar cylinders:

Example 10. Let $f: S \rightarrow \mathbb{P}^2$ be a blow up of 9 points such that $|-K_S|$ is base point free. Then $|-K_S|$ is a pencil. Suppose that all curves in the pencil $|-K_S|$ are irreducible. This easily implies that the surface S satisfies the generality condition (★) in Theorem 6. Suppose, in addition, that all singular curves in the pencil $|-K_S|$ do not have cusps. Let E_1, E_2, E_3 and E_4 be any four f -exceptional curves. Fix $x \in \mathbb{Q}$ such that $0 < x < 1$. Put $A = -K_S + x(E_1 + E_2 + E_3 + E_4)$. Then A is ample. Moreover, we have $r_A = 4$. Furthermore, if $x > \frac{7}{8}$, then it follows from Example 9 that S contains an A -polar cylinder. But S does not contain A -polar cylinders for $x \leq \frac{1}{4}$ by Lemma 12 and Remark 13.

The condition (★) in Theorem 6 cannot be omitted:

Example 11 (cf. [3, Example 4.3]). Let L_1 and L_2 be two distinct lines in $\mathbb{P}^2$. Then

$$\mathbb{P}^2 \setminus (L_1 \cup L_2) \cong \mathbb{C}^1 \times \mathbb{C}^*.$$

Let P_1 be a point in $L_1 \setminus L_2$. Let P_2, P_3, P_4, P_5, P_6 and P_7 be general points in $L_2 \setminus L_1$. Let $f: \hat{S} \rightarrow \mathbb{P}^2$ be the blow up of these seven points $P_1, P_2, P_3, P_4, P_5, P_6$ and P_7 . Denote by $F_1, F_2, F_3, F_4, F_5, F_6$ and F_7 the f -exceptional curves such that $f(F_i) = P_i$. Let $g: \tilde{S} \rightarrow \hat{S}$ be the blow up of the point in F_1 contained in the proper transform of L_1 . Denote by G the g -exceptional curve. Let $\tilde{F}_1$ be the proper transform on $\tilde{S}$ of the curve F_1 . Let $h: \bar{S} \rightarrow \tilde{S}$ be the blow up of the point $\tilde{F}_1 \cap G$. Denote by H the h -exceptional curve. Let $e: \mathcal{S} \rightarrow \bar{S}$ be the blow up of a general point in H . Denote by $\mathcal{E}$ the e -exceptional curve. Denote the proper transforms of the curves $H, G, F_1, F_2, F_3, F_4, F_5, F_6, F_7, L_1$ and L_2 on the surface $\mathcal{S}$ by the symbols $\mathcal{H}, \mathcal{G}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7, \mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Fix a positive rational number ϵ such that $\epsilon < \frac{1}{3}$. Then

$$-K_{\mathcal{S}} \sim_{\mathbb{Q}} (2 - \epsilon)\mathcal{L}_1 + (1 + \epsilon)\mathcal{L}_2 + (1 - \epsilon)\mathcal{F}_1 + \epsilon \sum_{i=2}^7 \mathcal{F}_i + (2 - 2\epsilon)\mathcal{G} + (2 - 3\epsilon)\mathcal{H} + (1 - 3\epsilon)\mathcal{E}.$$

We also have $\mathcal{S} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{G} \cup \mathcal{H} \cup \mathcal{E}) \cong \mathbb{P}^2 \setminus (L_1 \cup L_2)$. Let $\pi: \mathcal{S} \rightarrow S$ be the contraction of the curves $\mathcal{L}_1, \mathcal{G}$ and $\mathcal{H}$. Then S is a smooth surface. We have $K_S^2 = 2$, the divisor $-K_S$ is nef and big, and $\pi(\mathcal{E}) \cdot \pi(\mathcal{E}) = \pi(\mathcal{L}_2) \cdot \pi(\mathcal{L}_2) = -2$. In particular, the surface S does not satisfy the generality condition (★) in Theorem 6. Let L_{12} be the line in $\mathbb{P}^2$ that contains P_1 and P_2 , and let $\mathcal{L}_{12}$ be its proper transform on $\mathcal{S}$.

Fix a positive rational number x such that $\epsilon > x > 3\epsilon - 1$. Then

$$\begin{aligned} -K_S + x\mathcal{L}_{12} \sim_{\mathbb{Q}} (2 - \epsilon)\mathcal{L}_1 + (1 + \epsilon)\mathcal{L}_2 + (1 - \epsilon)\mathcal{F}_1 + (\epsilon - x)\mathcal{F}_2 + \\ + \epsilon(\mathcal{F}_3 + \mathcal{F}_4 + \mathcal{F}_5 + \mathcal{F}_6 + \mathcal{F}_7) + (2 + x - 2\epsilon)\mathcal{G} + (2 + x - 3\epsilon)\mathcal{H} + (1 + x - 3\epsilon)\mathcal{E}. \end{aligned}$$

Put $A = -K_S + x\pi(\mathcal{L}_{12})$. Then the divisor A is ample and $r_A = 1$, so that $r_A + K_S^2 = 3$. On the other hand, the surface S contains an A -polar cylinder, since

$$A \sim_{\mathbb{Q}} (1 + \epsilon)\pi(\mathcal{L}_2) + (1 - \epsilon)\pi(\mathcal{F}_1) + (\epsilon - x)\pi(\mathcal{F}_2) + \epsilon \sum_{i=3}^7 \pi(\mathcal{F}_i) + (1 + x - 3\epsilon)\pi(\mathcal{E}),$$

and $S \setminus (\pi(\mathcal{L}_2) \cup \pi(\mathcal{F}_1) \cup \pi(\mathcal{F}_2) \cup \pi(\mathcal{F}_3) \cup \pi(\mathcal{F}_4) \cup \pi(\mathcal{F}_5) \cup \pi(\mathcal{F}_6) \cup \pi(\mathcal{F}_7) \cup \pi(\mathcal{E})) \cong \mathbb{C}^1 \times \mathbb{C}^*$.

Recall that S is a smooth rational surface. Let $C_1, \dots, C_n$ be irreducible curves on it. Fix non-negative rational numbers $\lambda_1, \dots, \lambda_n$. Put $D = \lambda_1 C_1 + \dots + \lambda_n C_n$.

In the rest of the paper, we prove Theorem 6. Before doing this, let us prove

Lemma 12. *In the assumptions of Example 10, suppose also that $D \sim_{\mathbb{Q}} A$ and $x \leq \frac{1}{4}$. Then the log pair (S, D) is log canonical.*

Proof. Suppose that (S, D) is not log canonical at some point $P \in S$. Then $\text{mult}_P(D) > 1$. Let $\mathcal{C}$ be the curve in the pencil $| -K_S |$ that contains P . We may assume that $\mathcal{C} = C_1$. Put $\Delta = \lambda_2 C_2 + \dots + \lambda_n C_n$. Then $\lambda_1 < 1$, because

$$C_1 + x(E_1 + E_2 + E_3 + E_4) \sim_{\mathbb{Q}} \lambda_1 C_1 + \Delta.$$

and the log pair $(S, C_1 + x(E_1 + E_2 + E_3 + E_4))$ is log canonical (by assumption).

We have $\lambda_1 > 0$. Indeed, if $\lambda_1 = 0$, then $1 \geq 4x = C_1 \cdot \Delta \geq \text{mult}_P(D) > 1$.

If C_1 is smooth at P , then the inversion of adjunction (see [1, Theorem 7]) gives

$$1 \geq 4x = C_1 \cdot \Delta \geq \left(C_1 \cdot \Delta \right)_P > 1.$$

Thus, the curve C_1 has a simple node at P . Then $P \notin E_1 \cup E_2 \cup E_3 \cup E_4$.

We may assume that one of the curves E_1, E_2, E_3, E_4 is not contained in $\text{Supp}(\Delta)$. Indeed, if this is not the case, then we can swap D with the divisor

$$(1 + \mu)D - \mu \left(C_1 + x(E_1 + E_2 + E_3 + E_4) \right),$$

where μ is the largest positive rational number such that the latter divisor is effective. Without loss of generality, we may assume that $E_4 \not\subset \text{Supp}(\Delta)$. Then

$$1 - x = E_4 \cdot (\lambda_1 C_1 + \Delta) = \lambda_1 + E_4 \cdot \Delta \geq \lambda_1.$$

Put $m = \text{mult}_P(\Delta)$. Then $4x = C_1 \cdot \Delta \geq 2m$, so that $m \leq 2x$.

Let $f: \tilde{S} \rightarrow S$ be the blow up of the point P . Denote by F the f -exceptional curve. Denote by $\tilde{C}_1$ and $\tilde{\Delta}$ the proper transforms on $\tilde{S}$ of the divisors C_1 and Δ , respectively. Then $(\tilde{S}, \lambda_1 \tilde{C}_1 + \tilde{\Delta} + (2\lambda_1 + m - 1)F)$ is not log canonical at some point $Q \in F$, since

$$K_{\tilde{S}} + \lambda_1 \tilde{C}_1 + \tilde{\Delta} + (2\lambda_1 + m - 1)F \sim_{\mathbb{Q}} f^*(K_S + D).$$

Moreover, we have $2\lambda_1 + m - 1 \leq 1$, since we already proved that $\lambda_1 \leq 1 - x$ and $m \leq 2x$.

If we have $Q \notin \tilde{C}_1$, then $(\tilde{S}, \tilde{\Delta} + F)$ is not log canonical at Q , so that

$$\frac{1}{2} \geq 2x \geq m = F \cdot \tilde{\Delta} \geq (F \cdot \tilde{\Delta})_P > 1$$

by the inversion of adjunction. This shows that $Q \in \tilde{C}_1$.

The curve $\tilde{C}_1$ is smooth and intersects F transversally at Q . We know that $m \leq 2x \leq 1$. Thus, we can apply [1, Theorem 13] to $(\tilde{S}, \lambda_1 \tilde{C}_1 + \tilde{\Delta} + (2\lambda_1 + m - 1)F)$. This gives

$$4x - 2m = \tilde{\Delta} \cdot \tilde{C}_1 \geq (\tilde{\Delta} \cdot \tilde{C}_1)_Q > 2(1 - (2\lambda_1 + m - 1))$$

or

$$m = \tilde{\Delta} \cdot F \geq (\tilde{\Delta} \cdot F)_Q > 2(1 - \lambda_1).$$

In both cases we obtain a contradiction, since $m \leq 2x$ and $\lambda_1 \leq 1 - x$. $\square$

Denote by U the complement in the surface S to the union of the curves $C_1, \dots, C_n$. To prove Theorem 6, we need the following remark originated from [8, Lemma 4.11].

Remark 13 ([4]). Suppose that $U \cong \mathbb{C}^1 \times Z$ for an affine curve Z . Then the exact sequence

$$\bigoplus_{i=1}^n \mathbb{Z}[C_i] \longrightarrow \text{Pic}(S) \longrightarrow \text{Pic}(U)$$

implies that $n \geq \text{rank Pic}(S) = 10 - K_S^2$, because we have $\text{Pic}(U) = 0$, since $\text{Pic}(Z) = 0$. Moreover, the embeddings $Z \hookrightarrow \mathbb{P}^1$ and $\mathbb{C}^1 \hookrightarrow \mathbb{P}^1$ induce a commutative diagram

$$\begin{array}{ccccccc} \mathbb{P}^1 \times \mathbb{P}^1 & \longleftarrow & \mathbb{C}^1 \times \mathbb{P}^1 & \longleftarrow & \mathbb{C}^1 \times Z \cong U & \hookrightarrow & S \\ \downarrow \bar{p}_2 & & \downarrow p_2 & & \downarrow p_Z & & \downarrow \psi \\ \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 \end{array}$$

$\swarrow \pi$ $\searrow \phi$
 $\mathcal{S}$

where p_Z , p_2 and $\bar{p}_2$ are projections to the second factors, ψ is the map induced by p_Z , the map π is a birational morphism resolving the indeterminacy of ψ , and ϕ is a morphism. Let $\mathcal{E}_1, \dots, \mathcal{E}_m$ be the π -exceptional curves (if π is an isomorphism, we simply put $m = 0$). Let C be the section of the projection $\bar{p}_2$, which is the complement of $\mathbb{C}^1 \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Denote by $\mathcal{C}_1, \dots, \mathcal{C}_n$ the proper transforms on $\mathcal{S}$ of the curves $C_1, \dots, C_n$, respectively. Similarly, denote by the symbol $\mathcal{C}$ the proper transform of the curve C on the surface $\mathcal{S}$. By construction, a general fiber of the morphism ϕ is $\mathbb{P}^1$, and the curve $\mathcal{C}$ is its section. Then the curve $\mathcal{C}$ is either one of the curves $\mathcal{C}_1, \dots, \mathcal{C}_n$ or one of the curves $\mathcal{E}_1, \dots, \mathcal{E}_m$. All the other curves among $\mathcal{C}_1, \dots, \mathcal{C}_n$ and $\mathcal{E}_1, \dots, \mathcal{E}_m$ are mapped by ϕ to points in $\mathbb{P}^1$. Without loss of generality, we may assume that either $\mathcal{C} = \mathcal{C}_1$ or $\mathcal{C} = \mathcal{E}_m$. We have

$$K_S + \sum_{i=1}^n \lambda_i \mathcal{C}_i + \sum_{i=1}^m \mu_i \mathcal{E}_i = \pi^* \left(K_S + \sum_{i=1}^n \lambda_i C_i \right)$$

for some rational numbers $\mu_1, \dots, \mu_m$. Let $\mathcal{F}$ be a general fiber of ϕ . If $\mathcal{C} = \mathcal{C}_1$, then

$$-2 + \lambda_1 = \left(K_S + \sum_{i=1}^n \lambda_i \mathcal{C}_i + \sum_{i=1}^m \mu_i \mathcal{E}_i \right) \cdot \mathcal{F} = \pi^* \left(K_S + \sum_{i=1}^n \lambda_i C_i \right) \cdot \mathcal{F} = \left(K_S + \sum_{i=1}^n \lambda_i C_i \right) \cdot \pi(\mathcal{F}).$$

Similarly, if $\mathcal{C} = E_m$, then

$$-2 + \mu_m = \left(K_S + \sum_{i=1}^n \lambda_i \mathcal{C}_i + \sum_{i=1}^m \mu_i \mathcal{E}_i \right) \cdot \mathcal{F} = \pi^* \left(K_S + \sum_{i=1}^n \lambda_i C_i \right) \cdot \mathcal{F} = \left(K_S + \sum_{i=1}^n \lambda_i C_i \right) \cdot \pi(\mathcal{F}).$$

In particular, if the divisor $K_S + \sum_{i=1}^n \lambda_i C_i$ is pseudo-effective, then

- if $\mathcal{C} = \mathcal{C}_1$, then $\lambda_1 \geq 2$, so that (S, D) is not log canonical along $C_1 = \pi(\mathcal{C})$,
- if $\mathcal{C} = E_m$, then (S, D) is not log canonical at the point $\pi(\mathcal{C})$.

To prove Theorem 6, we also need

Theorem 14 ([2, Theorem 1.12]). *Suppose that S is a del Pezzo surface, $K_S^2 \leq 3$, and*

$$D \sim_{\mathbb{Q}} -K_S.$$

Let P be a point in S . Suppose that the log pair (S, D) is not log canonical at the point P . Then $|-K_S|$ contains a unique divisor T such that (S, T) is not log canonical at P . Moreover, the support of D contains all the irreducible components of $\text{Supp}(T)$.

Let $G_1, \dots, G_r$ be disjoint smooth rational curves on S such that $G_1^2 = \dots = G_r^2 = -1$. Let $\epsilon_1, \dots, \epsilon_r$ be arbitrary non-negative rational numbers. Suppose that

$$-K_S + \sum_{i=1}^r \epsilon_i G_i \sim_{\mathbb{Q}} D = \sum_{i=1}^n \lambda_i C_i.$$

Lemma 15. *If $C_i \neq G_j$ for all i and j , then (S, D) is log canonical along $G_1 \cup \dots \cup G_r$.*

Proof. Suppose that the log pair (S, D) is not log canonical at some point $P \in G_1 \cup \dots \cup G_r$. Then $\text{mult}_P(D) > 1$. If $P \in G_1$ and $G_1 \not\subset \text{Supp}(D)$, then

$$1 \geq 1 - \epsilon_1 = \left(-K_S + \sum_{i=1}^r \epsilon_i G_i \right) \cdot G_1 = D \cdot G_1 \geq \text{mult}_P(D) > 1.$$

Similarly, we see that $P \notin G_2 \cup \dots \cup G_r$ provided that $C_i \neq G_j$ for all i and j . $\square$

Let $g: S \rightarrow \bar{S}$ be the contraction of the curves $G_1, \dots, G_r$. Observe that $K_{\bar{S}}^2 = r + K_S^2$. Put $\bar{C}_1 = g(C_1), \dots, \bar{C}_n = g(C_n)$, and put $\bar{D} = \lambda_1 \bar{C}_1 + \dots + \lambda_n \bar{C}_n$. Then $\bar{D} \sim_{\mathbb{Q}} -K_{\bar{S}}$.

Lemma 16. *Suppose that $r + K_S^2 = 1$, and S satisfies the condition $(\star)$ in Theorem 6. Suppose also that $\lambda_i > 0$ for every i . Then U is not a cylinder.*

Proof. First we observe that the surface $\bar{S}$ also satisfies the condition $(\star)$ in Theorem 6. Then $-K_{\bar{S}}$ is ample by Remark 7. Thus, if $K_{\bar{S}}^2 = 1$, then U is not a cylinder by Theorem 2.

Let us prove the assertion by induction on $K_S^2 \leq 1$. We may assume that $K_S^2 \leq 0$.

Suppose that $C_1 = G_1$. Then there exists a commutative diagram

$$\begin{array}{ccc} & S & \\ f \swarrow & & \searrow g \\ \widehat{S} & \xrightarrow{h} & \overline{S} \end{array}$$

where $f: S \rightarrow \widehat{S}$ is a contraction of the curve $C_1 = G_1$, and h is a birational morphism. Denote by $\widehat{G}_2, \dots, \widehat{G}_r$ the proper transforms on $\widehat{S}$ of the curves $G_2, \dots, G_r$, respectively. Denote by $\widehat{C}_2, \dots, \widehat{C}_n$ the proper transforms on $\widehat{S}$ of the curves $C_2, \dots, C_n$, respectively. Then $K_{\widehat{S}}^2 = K_S^2 + 1$ and

$$-K_{\widehat{S}} + \sum_{i=2}^r \epsilon_i \widehat{G}_i \sim_{\mathbb{Q}} \sum_{i=2}^n \lambda_i \widehat{C}_i.$$

By induction $\widehat{S} \setminus (\widehat{C}_2 \cup \dots \cup \widehat{C}_n) \cong U$ is not a cylinder. Thus, we may assume that $C_1 \neq G_1$. Similarly, we may assume that $C_i \neq G_j$ for all possible i and j .

We have $\lambda_i \leq 1$ for each i . Moreover, if $\lambda_i = 1$, then $i = 1$ and $n = 1$, so that $D = C_1$. This follows from the equality $-K_{\overline{S}} \cdot \overline{D} = 1$, because the divisor $-K_{\overline{S}}$ is ample.

Suppose that U is a cylinder. Then $n \geq 10 - K_S^2$ by Remark 13. Then $\lambda_i < 1$ for every i .

Let us use notations of Remark 13. Then $\pi(\mathcal{C})$ is a point. Denote this point by P . By Remark 13 and Lemma 15, (S, D) is not log canonical at P and $P \notin G_1 \cup \dots \cup G_r$.

Put $\overline{P} = g(P)$. Then $(\overline{S}, \overline{D})$ is not log canonical at $\overline{P}$.

By Theorem 14, there is $\overline{T} \in |-K_{\overline{S}}|$ such that $(\overline{S}, \overline{T})$ is not log canonical at the point $\overline{P}$. Moreover, Theorem 14 also implies that the curve $\overline{T}$ is one of the curves $\overline{C}_1, \dots, \overline{C}_n$. Therefore, we may assume that $\overline{T} = \overline{C}_1$. Note that the curve $\overline{T}$ is singular at $\overline{P}$.

Denote by $m_1, \dots, m_r$ the multiplicities of the curve $\overline{T}$ at $g(G_1), \dots, g(G_r)$, respectively. Since the curve $\overline{T}$ is singular at the point $\overline{P}$ by construction, it is smooth away of this point. This implies that either $m_i = 0$ (and $g(G_i) \notin \overline{T}$) or $m_i = 1$ (and $g(G_i) \in \overline{T}$) for all i . Denote by T the proper transform of the curve $\overline{T}$ on the surface S . Then

$$T \sim g^*(\overline{T}) - \sum_{i=1}^r m_i G_i \sim -K_S + \sum_{i=1}^r (1 - m_i) G_i.$$

Let us replace D by a divisor $(1 + \mu)D - \mu T$ for an appropriate rational number $\mu > 0$. The goal is to obtain an effective divisor that do not contain $T = C_1$ in its support. Put

$$D' = \frac{1}{1 - \lambda_1} D - \frac{\lambda_1}{1 - \lambda_1} C_1 = \sum_{i=2}^n \frac{\lambda_i}{1 - \lambda_1} C_i.$$

Then D' is an effective divisor that does not contain $T = C_1$ in its support. But

$$D' \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^r \frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} G_i.$$

Thus, if $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} \geq 0$ for every i , then (S, D') is not log canonical at P by Remark 13. In this case, the singularities of the log pair

$$\left(\overline{S}, \sum_{i=2}^n \frac{\lambda_i}{1 - \lambda_1} \overline{C}_i \right)$$

are not log canonical at $\overline{P}$, since $P \notin G_1 \cup \dots \cup G_r$. The latter is impossible by Theorem 14. Therefore, at least one of the numbers

$$\frac{\epsilon_1 + (m_1 - 1)\lambda_1}{1 - \lambda_1}, \frac{\epsilon_2 + (m_2 - 1)\lambda_1}{1 - \lambda_1}, \dots, \frac{\epsilon_r + (m_r - 1)\lambda_1}{1 - \lambda_1}$$

must be negative. Thus, we may assume that there exists $k \leq r$ such that

$$\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} < 0$$

for every $i \leq k$, and $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} \geq 0$ for every $i > k$ (if any). Then $m_1 = \dots = m_k = 0$. We may also assume that $\epsilon_1 \leq \dots \leq \epsilon_k$. Put

$$D'' = \frac{1}{1 - \epsilon_1} D - \frac{\epsilon_1}{1 - \epsilon_1} C_1 = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} C_1 + \sum_{i=2}^n \frac{\lambda_i}{1 - \epsilon_1} C_i.$$

Then D'' is an effective divisor such that

$$D'' \sim_{\mathbb{Q}} -K_S + \sum_{i=2}^r \frac{\epsilon_i - \epsilon_1(1 - m_i)}{1 - \epsilon_1} G_i = \sum_{i=2}^k \frac{\epsilon_i - \epsilon_1}{1 - \epsilon_1} G_i + \sum_{i=k+1}^r \frac{\epsilon_i - \epsilon_1(1 - m_i)}{1 - \epsilon_1} G_i.$$

Let $e: \tilde{S} \rightarrow \overline{S}$ be the blow up of the point $g(G_1)$, and let $\tilde{G}_1$ be its exceptional curve. Denote by $\tilde{C}_1, \dots, \tilde{C}_n$ the proper transforms on $\tilde{S}$ of the curves $C_1, \dots, C_n$, respectively. Denote by $\tilde{D}''$ the proper transform of the divisor D'' on the surface $\tilde{S}$. Then

$$\tilde{D}'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} \tilde{C}_1 + \sum_{i=2}^n \frac{\lambda_i}{1 - \epsilon_1} \tilde{C}_i \sim_{\mathbb{Q}} -K_{\tilde{S}}.$$

Since we have $g(G_1) \notin \overline{T}$, the point $g(G_1)$ is not the base point of the pencil $| -K_{\overline{S}} |$. Thus, the pencil $| -K_{\overline{S}} |$ contains a unique irreducible curve $\overline{R}$ that passes through $g(G_1)$. Let $\tilde{R}$ and R be the proper transforms of this curve on the surfaces $\tilde{S}$ and S , respectively. If $\overline{R}$ is singular at $g(G_1)$, then R is a smooth rational curve such that

$$R^2 \leq \tilde{R}^2 = -3.$$

This is impossible by our assumption. Then the curve R is smooth at $g(G_1)$ and $\tilde{R} \sim -K_{\tilde{S}}$. Since $\tilde{R}^2 = 0$ and $\tilde{C}_1 \cdot \tilde{R} = 1$, the divisor $\tilde{R}$ is nef. Then

$$0 = K_{\tilde{S}}^2 = \tilde{D}'' \cdot \tilde{R} = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} \tilde{C}_1 \cdot \tilde{R} + \sum_{i=2}^n \frac{\lambda_i}{1 - \epsilon_1} \tilde{C}_i \cdot \tilde{R} \geq \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} \tilde{C}_1 \cdot \tilde{R} = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1},$$

so that $\epsilon_1 \geq \lambda_1$. This is a contradiction, since $\epsilon_1 < \lambda_1$. $\square$

Lemma 17. *Suppose that $r + K_S^2 = 2$, and S satisfies the condition $(\star)$ in Theorem 6. Suppose also that $\lambda_i > 0$ for every i . Then U is not a cylinder.*

Proof. By Remark 7, the surface $\bar{S}$ is a del Pezzo surface. We also have $K_{\bar{S}}^2 = K_S^2 + r = 2$. Therefore, if $K_S^2 = 2$, then $r = 0$ and $S \cong \bar{S}$, so that U is not a cylinder by Theorem 2. Now arguing as in the proof of Lemma 16, we may assume that $C_i \neq G_j$ for all i and j .

By [2, Lemma 3.1] applied to $(\bar{S}, \bar{D})$, we have $\lambda_i \leq 1$ for each i .

Suppose that U is a cylinder. Then $n \geq 10 - K_S^2 \geq 9$ by Remark 13.

Let us use notations of Remark 13. Then $\pi(\mathcal{C})$ is a point. Denote this point by P . Then (S, D) is not log canonical at P and $P \notin G_1 \cup \dots \cup G_r$ by Remark 13 and Lemma 2. Put $\bar{P} = g(P)$. Then $(\bar{S}, \bar{D})$ is not log canonical at $\bar{P}$.

By Theorem 14, there is a curve $\bar{T} \in |-K_{\bar{S}}|$ such that $(\bar{S}, \bar{T})$ is not log canonical at $\bar{P}$, and all irreducible components of the curve $\bar{T}$ are among the irreducible curves $\bar{C}_1, \dots, \bar{C}_n$. In particular, the curve $\bar{T}$ is singular at $\bar{P}$. It is uniquely determined by this property.

Note that $\bar{T}$ has at most two irreducible component. Thus, we may assume that

- either $\bar{T} = \bar{C}_1$,
- or $\bar{T} = \bar{C}_1 + \bar{C}_2$ and $\lambda_1 \leq \lambda_2$.

If $\bar{T} = \bar{C}_1$, then $\bar{T}$ has a cusp at $\bar{P}$. If $\bar{T} = \bar{C}_1 + \bar{C}_2$, then $\bar{T}$ has a tacknode at $\bar{P}$. In both cases, the point $\bar{P}$ is the unique singular point of the curve $\bar{T}$.

If $\bar{T} = \bar{C}_1$, then $\lambda_1 < 1$, because

$$2 = -K_{\bar{S}} \cdot \bar{D} = \sum_{i=1}^n \lambda_i (-K_{\bar{S}} \cdot \bar{C}_i) = 2\lambda_1 + \sum_{i=2}^n \lambda_i (-K_{\bar{S}} \cdot \bar{C}_i) \geq 2\lambda_1 + \sum_{i=2}^n \lambda_i > 2\lambda_1.$$

Similarly, if $\bar{T} = \bar{C}_1 + \bar{C}_2$, then $\lambda_1 < 1$, because

$$2 = -K_{\bar{S}} \cdot \bar{D} = \lambda_1 + \lambda_2 + \sum_{i=3}^n \lambda_i (-K_{\bar{S}} \cdot \bar{C}_i) > \lambda_1 + \lambda_2 \geq 2\lambda_1.$$

Denote by $m_1, \dots, m_r$ the multiplicities of the curve $\bar{T}$ at $g(G_1), \dots, g(G_r)$, respectively. Then either $m_i = 0$ or $m_i = 1$. Let T be the proper transform of the curve $\bar{T}$ on S . Then

$$T \sim g^*(\bar{T}) - \sum_{i=1}^r m_i G_i \sim -K_S + \sum_{i=1}^r (1 - m_i) G_i.$$

Put $D' = \frac{1}{1-\lambda_1} D - \frac{\lambda_1}{1-\lambda_1} T$ and $\bar{D}' = \frac{1}{1-\lambda_1} \bar{D} - \frac{\lambda_1}{1-\lambda_1} \bar{T}$. If $\bar{T} = \bar{C}_1$, then

$$D' \sim_{\mathbb{Q}} \sum_{i=2}^n \frac{\lambda_i}{1-\lambda_1} C_i.$$

Similarly, if $\bar{T} = \bar{C}_1 + \bar{C}_2$, then

$$D' \sim_{\mathbb{Q}} \frac{\lambda_2 - \lambda_1}{1 - \lambda_1} C_2 + \sum_{i=3}^n \frac{\lambda_i}{1 - \lambda_1} C_i.$$

In both cases D' is an effective divisor that does not contain the curve C_1 in its support. Similarly, the divisor $\overline{D}'$ is effective and does not contain $\overline{C}_1$ in its support. But

$$D' \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^r \frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} G_i.$$

Thus, if $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} \geq 0$ for every i , then (S, D') is not log canonical at P by Remark 13. Then $\overline{D}' \sim_{\mathbb{Q}} -K_{\overline{S}}$ and $(\overline{S}, \overline{D}')$ is not log canonical at $\overline{P}$, which contradicts Theorem 14. Hence, at least one of the numbers $\frac{\epsilon_1 + (m_1 - 1)\lambda_1}{1 - \lambda_1}, \dots, \frac{\epsilon_r + (m_r - 1)\lambda_1}{1 - \lambda_1}$ is negative.

We may assume that $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} < 0 \iff i \leq k$ for some $k \leq r$, and $\epsilon_1 \leq \dots \leq \epsilon_k$. Then $m_i = 0$ and $\epsilon_i < \lambda_1$ for every $i \leq k$.

Put $D'' = \frac{1}{1 - \epsilon_1} D - \frac{\epsilon_1}{1 - \epsilon_1} T$. Then D'' is an effective divisor. Indeed, if $\overline{T} = \overline{C}_1$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} C_1 + \sum_{i=2}^n \frac{\lambda_i}{1 - \epsilon_1} C_i.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} C_1 + \frac{\lambda_2 - \epsilon_1}{1 - \epsilon_1} C_2 + \sum_{i=3}^n \frac{\lambda_i}{1 - \epsilon_1} C_i.$$

Note that $\text{Supp}(D'') = \text{Supp}(D)$. On the other hand, we have

$$D'' \sim_{\mathbb{Q}} -K_S + \sum_{i=2}^r \frac{\epsilon_i - \epsilon_1(1 - m_i)}{1 - \epsilon_1} G_i.$$

Applying Lemma 16 to D'' , we see that U is not a cylinder. This is a contradiction. $\square$

Lemma 18. *Suppose that $r + K_S^2 = 3$, and S satisfies the condition $(\star)$ in Theorem 6. Suppose also that $\lambda_i > 0$ for every i . Then U is not a cylinder.*

Proof. By Remark 7, the surface $\overline{S}$ is a cubic surface in $\mathbb{P}^3$, because $K_{\overline{S}}^2 = r + K_S^2 = 3$. Moreover, if $K_S^2 = 3$, then $r = 0$ and $S \cong \overline{S}$, so that U is not a cylinder by Theorem 2. Arguing as in the proof of Lemma 16, we may assume that $C_i \neq G_j$ for all possible i and j .

Applying [2, Lemma 4.1] to $(\overline{S}, \overline{D})$, we see that $\lambda_i \leq 1$ for each i .

Suppose that U is a cylinder. Then $n \geq 10 - K_S^2 \geq 8$ by Remark 13.

Let us use notations of Remark 13. Then $\pi(\mathcal{C})$ is a point. Denote this point by P . By Remark 13 and Lemma 2, (S, D) is not log canonical at P and $P \notin G_1 \cup \dots \cup G_r$.

By Theorem 14, there is a curve $\overline{T} \in |-K_{\overline{S}}|$ such that $(\overline{S}, \overline{T})$ is not log canonical at $\overline{P}$, and all irreducible components of the curve $\overline{T}$ are among the irreducible curves $\overline{C}_1, \dots, \overline{C}_n$. Note that $\overline{T}$ is cut out on $\overline{S} \subset \mathbb{P}^3$ by a hyperplane that is tangent to $\overline{S}$ at the point $\overline{P}$.

The curve $\overline{T}$ has at most three irreducible component. Thus, we may assume that

- either $\overline{T} = \overline{C}_1$,
- or $\overline{T} = \overline{C}_1 + \overline{C}_2$ and $\lambda_1 \leq \lambda_2$.
- or $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$ and $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

If $\overline{T} = \overline{C}_1$, then $\overline{T}$ has a cusp at $\overline{P}$. If $\overline{T} = \overline{C}_1 + \overline{C}_2$, then $\overline{T}$ has a tacknode at $\overline{P}$. Finally, if $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$, then the curves $\overline{C}_1$, $\overline{C}_2$ and $\overline{C}_3$ are lines passing through $\overline{P}$. In all possible cases, the point $\overline{P}$ is the unique singular point of the curve $\overline{T}$.

If $\overline{T} = \overline{C}_1$, then $\lambda_1 < 1$, because

$$3 = -K_{\overline{S}} \cdot \overline{D} = \sum_{i=1}^n \lambda_i \left(-K_{\overline{S}} \cdot \overline{C}_i \right) = 3\lambda_1 + \sum_{i=2}^n \lambda_i \left(-K_{\overline{S}} \cdot \overline{C}_i \right) \geq 3\lambda_1 + \sum_{i=2}^n \lambda_i > 3\lambda_1.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then $\lambda_1 < 1$, because

$$3 = \lambda_1 \deg(\overline{C}_1) + \lambda_2 \deg(\overline{C}_2) + \sum_{i=3}^n \lambda_i \left(-K_{\overline{S}} \cdot \overline{C}_i \right) > \lambda_1 \left(\deg(\overline{C}_1) + \deg(\overline{C}_2) \right) = 3\lambda_1.$$

Finally, if $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$, then we also have $\lambda_1 < 1$, because

$$3 = -K_{\overline{S}} \cdot \overline{D} = \lambda_1 + \lambda_2 + \lambda_3 + \sum_{i=4}^n \lambda_i \left(-K_{\overline{S}} \cdot \overline{C}_i \right) > \lambda_1 + \lambda_2 + \lambda_3 \geq 3\lambda_1.$$

Denote by $m_1, \dots, m_r$ the multiplicities of the curve $\overline{T}$ at $g(G_1), \dots, g(G_r)$, respectively. Then either $m_i = 0$ or $m_i = 1$. Let T be the proper transform of the curve $\overline{T}$ on S . Then

$$T \sim g^*(\overline{T}) - \sum_{i=1}^r m_i G_i \sim -K_S + \sum_{i=1}^r (1 - m_i) G_i.$$

Put $D' = \frac{1}{1-\lambda_1} D - \frac{\lambda_1}{1-\lambda_1} T$ and $\overline{D}' = \frac{1}{1-\lambda_1} \overline{D} - \frac{\lambda_1}{1-\lambda_1} \overline{T}$. If $\overline{T} = \overline{C}_1$, then

$$D' \sim_{\mathbb{Q}} \sum_{i=2}^n \frac{\lambda_i}{1-\lambda_1} C_i.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then

$$D' \sim_{\mathbb{Q}} \frac{\lambda_2 - \lambda_1}{1 - \lambda_1} C_2 + \sum_{i=3}^n \frac{\lambda_i}{1 - \lambda_1} C_i.$$

Finally, if $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$, then

$$D' \sim_{\mathbb{Q}} \frac{\lambda_2 - \lambda_1}{1 - \lambda_1} C_2 + \frac{\lambda_3 - \lambda_1}{1 - \lambda_1} C_3 + \sum_{i=4}^n \frac{\lambda_i}{1 - \lambda_1} C_i.$$

In all cases D' is an effective divisor that does not contain the curve C_1 in its support. Similarly, the divisor $\overline{D}'$ is effective and does not contain $\overline{C}_1$ in its support. But

$$D' \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^r \frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} G_i.$$

Thus, if $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} \geq 0$ for every i , then (S, D') is not log canonical at P by Remark 13. Then $\overline{D}' \sim_{\mathbb{Q}} -K_{\overline{S}}$ and $(\overline{S}, \overline{D}')$ is not log canonical at $\overline{P}$, which contradicts Theorem 14. Hence, at least one of the numbers $\frac{\epsilon_1 + (m_1 - 1)\lambda_1}{1 - \lambda_1}, \dots, \frac{\epsilon_r + (m_r - 1)\lambda_1}{1 - \lambda_1}$ is negative.

We may assume that $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} < 0 \iff i \leq k$ for some $k \leq r$, and $\epsilon_1 \leq \dots \leq \epsilon_k$. Then $m_i = 0$ and $\epsilon_i < \lambda_1$ for every $i = 1, \dots, k$.

Put $D'' = \frac{1}{1 - \epsilon_1}D - \frac{\epsilon_1}{1 - \epsilon_1}T$. Then D'' is an effective divisor. Indeed, if $\overline{T} = \overline{C}_1$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1}C_1 + \sum_{i=2}^n \frac{\lambda_i}{1 - \epsilon_1}C_i.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1}C_1 + \frac{\lambda_2 - \epsilon_1}{1 - \epsilon_1}C_2 + \sum_{i=3}^n \frac{\lambda_i}{1 - \epsilon_1}C_i.$$

Finally, if $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1}C_1 + \frac{\lambda_2 - \epsilon_1}{1 - \epsilon_1}C_2 + \frac{\lambda_3 - \epsilon_1}{1 - \epsilon_1}C_3 + \sum_{i=4}^n \frac{\lambda_i}{1 - \epsilon_1}C_i.$$

In all cases $\text{Supp}(D'') = \text{Supp}(D)$. On the other hand, we have

$$D'' \sim_{\mathbb{Q}} -K_S + \sum_{i=2}^r \frac{\epsilon_i - \epsilon_1(1 - m_i)}{1 - \epsilon_1}G_i.$$

Applying Lemma 17 to D'' , we see that U is not a cylinder. This is a contradiction. $\square$

Let us show that Theorem 6 follows from Lemmas 16, 17 and 18.

Proof of Theorem 6. Let A be an ample $\mathbb{Q}$ -divisor on S , let μ_A be its Fujita invariant, and let r_A be its the Fujita rank. Suppose that $r_A + K_S^2 \leq 3$. Then

$$\mu_A A \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{r_A} a_i E_i,$$

where $E_1, \dots, E_{r_A}$ are smooth rational curves on S , and $a_1, \dots, a_{r_A}$ are positive rational numbers. Moreover, there is a birational morphism $f: S \rightarrow \widehat{S}$ such that

- the curves $E_1, \dots, E_{r_A}$ are all f -exceptional curves,
- the surface $\widehat{S}$ is smooth.

This is well-known. See, for example, [12, Theorem 2.1] or [13, Theorem 1].

Suppose, in addition, that the surface S satisfies the condition $(\star)$ in Theorem 6. Then the curves $E_1, \dots, E_{r_A}$ are disjoint and $E_1^2 = E_2^2 = \dots = E_{r_A}^2 = -1$.

To prove Theorem 6, we have to show that S does not contain A -polar cylinders. If $r_A + K_S^2 \geq 1$, then S does not contain A -polar cylinders by Lemmas 16, 17 and 18. Therefore, we may assume that $r_A + K_S^2 \leq 0$. Put $k = 1 - K_S^2 - r_A$.

There is a birational morphism $h: \widehat{S} \rightarrow \widetilde{S}$ such that $\widetilde{S}$ is smooth and $K_{\widetilde{S}}^2 = 1$. Let $H_1, \dots, H_k$ be the proper transforms of the h -exceptional curves on the surface S .

Then the curves $H_1, \dots, H_k$ are smooth rational curves such that $H_1^2 = \dots = H_k^2 = -1$. Moreover, the curves $E_1, \dots, E_{r_A}, H_1, \dots, H_k$ are disjoint. On the other hand, we have

$$\mu_A A \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{r_A} a_i E_i = -K_S + \sum_{i=1}^{r_A} a_i E_i + 0 \times H_1 + \dots + 0 \times H_k.$$

Therefore, the surface S does not contain A -polar cylinders by Lemma 16. $\square$

Acknowledgements. The author is very grateful to Ciro Ciliberto for asking Question 5.

REFERENCES

- [1] I. Cheltsov, *Del Pezzo surfaces and local inequalities*, Springer Proceedings in Mathematics and Statistics **79** (2014), 83–101.
- [2] I. Cheltsov, J. Park, J. Won, *Affine cones over smooth cubic surfaces*, Journal of the European Mathematical Society **18** (2016), 1537–1564.
- [3] I. Cheltsov, J. Park, J. Won, *Cylinders in singular del Pezzo surfaces*, Compositio Mathematica **152** (2016), 1198–1224.
- [4] I. Cheltsov, J. Park, J. Won, *Cylinders in del Pezzo surfaces*, to appear in International Mathematics Research Notices.
- [5] C. Ciliberto, B. Harbourne, R. Miranda, J. Ro, *Variations on Nagata’s conjecture*, Clay Mathematics Proceedings **18** (2013), 185–203.
- [6] T. de Fernex, *Negative curves on very general blow-ups of $\mathbb{P}^2$* , Projective Varieties with Unexpected Properties, 199–207, de Gruyter, Berlin, 2005.
- [7] B. Lehmann, S. Tanimoto, Yu. Tschinkel, *Balanced line bundles on Fano varieties*, to appear in Journal fr die reine und angewandte Mathematik.
- [8] T. Kishimoto, Yu. Prokhorov, M. Zaidenberg, *Group actions on affine cones*, CRM Proceedings Lecture Notes **54** (2011), 123–163.
- [9] T. Kishimoto, Yu. Prokhorov, M. Zaidenberg, $\mathbb{G}_a$ -actions on affine cones, Transformation Groups **18** (2013), 1137–1153.
- [10] T. Kishimoto, Yu. Prokhorov, M. Zaidenberg, *Unipotent group actions on del Pezzo cones*, Algebraic Geometry **1** (2014), 46–56.
- [11] A. Perepechko, *Flexibility of affine cones over del Pezzo surfaces of degree 4 and 5*, Functional Analysis and its Applications **47** (2013), 284289.
- [12] M. Reid, *Surfaces of small degree*, Mathematische Annalen **275** (1986), 71–80.
- [13] F. Sakai, *On polarized normal surfaces*, Manuscripta Mathematica **59** (1987), 109–127.

Ivan Cheltsov

School of Mathematics, University of Edinburgh, Edinburgh EH9 3FD, Scotland

I.Cheltsov@ed.ac.uk