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Fast interior point solvers for H^1 -regularized PDE-constrained optimization problems

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We consider Newton systems arising from the interior point solution of PDE-constrained optimization problems. In particular, we examine problems where the control variable is regularized by an H^1 -norm within the cost functional. We present preconditioned iterative methods for the resulting matrix systems, and justify the potency of our approach through numerical experiments.

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1 Problem statement

We consider linear, time-independent PDE-constrained optimization problems with additional bound constraints, of the form:

$$\begin{aligned} \min_{y,u} \quad & \frac{1}{2} \|y - \hat{y}\|_{L_2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{H^1(\Omega)}^2 \\ \text{s.t.} \quad & \mathcal{D}y = u, \quad \text{in } \Omega, \\ & y = f, \quad \text{on } \partial\Omega, \\ & y_a \leq y \leq y_b, \quad \text{a.e. in } \Omega, \\ & u_a \leq u \leq u_b, \quad \text{a.e. in } \Omega, \end{aligned}$$

where y and u denote state and control variables which we wish to determine, $\hat{y}$ is a given desired state, $\beta > 0$ a regularization parameter, $\mathcal{D}$ some differential operator, and y_a, y_b, u_a, u_b prescribed bound constraints on the state and control variables. The problem is solved in a domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, with boundary $\partial\Omega$. We highlight that the regularization term corresponding to the control is $\|u\|_{H^1(\Omega)}^2 = \|u\|_{L_2(\Omega)}^2 + \|\nabla u\|_{L_2(\Omega)}^2$ – problems with an L_2 -norm regularization term for the control are considered in [2], as are time-dependent variants of such problems.

Applying an interior point method and a *discretize-then-optimize* approach, as in [2], leads to a discrete Lagrangian of the following form:

$$\begin{aligned} \mathcal{L}(\vec{y}, \vec{u}, \vec{p}) = & \frac{1}{2} \vec{y}^\top M \vec{y} - \vec{y}_d^\top \vec{y} + \frac{\beta}{2} \vec{u}^\top (M + K) \vec{u} + \vec{p}^\top (A \vec{y} - M \vec{u} - \vec{f}) \\ & - \mu \sum_j \log(y_j - y_{a,j}) - \mu \sum_j \log(y_{b,j} - y_j) - \mu \sum_j \log(u_j - u_{a,j}) - \mu \sum_j \log(u_{b,j} - u_j), \end{aligned}$$

of which we wish to find the stationary point. Here $\vec{p}$ and $\vec{y}_d$ correspond to the discretized adjoint variable and desired state, $y_j, y_{a,j}, y_{b,j}, u_j, u_{a,j}$ and $u_{b,j}$ represent the values of y, y_a, y_b, u, u_a and u_b at the j -th finite element node, K and M are finite element stiffness and mass matrices, A is the finite element matrix related to the PDE operator $\mathcal{D}$, and μ is the chosen barrier parameter within the interior point method.

Applying Newton iteration (with Newton steps $\vec{s}_y, \vec{s}_u, \vec{s}_p$, and previous iterates $\vec{y}^*, \vec{u}^*, \vec{p}^*$) to the resulting first-order optimality conditions leads to matrix systems of the form

$$\begin{bmatrix} M + D_y & 0 & A^\top \\ 0 & \beta(M + K) + D_u & -M \\ A & -M & 0 \end{bmatrix} \begin{bmatrix} \vec{s}_y \\ \vec{s}_u \\ \vec{s}_p \end{bmatrix} = \begin{bmatrix} \mu(Y - Y_a)^{-1} \vec{e} - \mu(Y_b - Y)^{-1} \vec{e} + \vec{y}_d - M \vec{y}^* - A^\top \vec{p}^* \\ \mu(U - U_a)^{-1} \vec{e} - \mu(U_b - U)^{-1} \vec{e} - \beta(M + K) \vec{u}^* + M \vec{p}^* \\ \vec{f} - A \vec{y}^* + M \vec{u}^* \end{bmatrix},$$

where

$$D_y = (Y - Y_a)^{-1} Z_{y,a} + (Y_b - Y)^{-1} Z_{y,b}, \quad D_u = (U - U_a)^{-1} Z_{u,a} + (U_b - U)^{-1} Z_{u,b}.$$

Here, Y, U, Y_a, Y_b, U_a, U_b are diagonal matrices containing the entries of y, u (from the previous Newton iteration), y_a, y_b, u_a, u_b at each finite element node, and $Z_{y,a}, Z_{y,b}, Z_{u,a}, Z_{u,b}$ contain entries of the form $\frac{\mu}{y - y_a}, \frac{\mu}{y_b - y}, \frac{\mu}{u - u_a}, \frac{\mu}{u_b - u}$, respectively.

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2 Preconditioning

We now note that the Newton system is of saddle point form (see [1] for a survey of such systems). Using the justification provided in [3, 4], we consider the block triangular preconditioners

$$\mathcal{P}_1 = \begin{bmatrix} M + D_y & 0 & 0 \\ 0 & \beta(M + K) + D_u & 0 \\ A & -M & -\widehat{S}_1 \end{bmatrix}, \quad \mathcal{P}_2 = \begin{bmatrix} -\widehat{S}_2 & 0 & A^\top \\ 0 & \beta(M + K) + D_u & -M \\ 0 & -M & 0 \end{bmatrix},$$

where $\widehat{S}_1$ and $\widehat{S}_2$ are derived using ‘matching strategies’, as follows:

$$\begin{aligned} \widehat{S}_1 &= (A + \widehat{M})(M + D_y)^{-1}(A + \widehat{M})^\top, \\ \widehat{S}_2 &= -(A + \widehat{M})^\top M^{-1}(\beta(M + K) + D_u)M^{-1}(A + \widehat{M}), \end{aligned}$$

with $\widehat{M} = M[\text{diag}(\beta(M + K) + D_u)]^{-1/2}[\text{diag}(M + D_y)]^{1/2}$. Eigenvalue analysis concerning similar preconditioners for interior point methods can be found in [2].

In practice it is sensible to use multigrid methods to apply the inverse of the $(1, 1)$ -block, and the approximate Schur complement inverses $\widehat{S}_1^{-1}$ and $\widehat{S}_2^{-1}$. We apply both preconditioners $\mathcal{P}_1$ and $\mathcal{P}_2$ within the GMRES algorithm.

3 Numerical results

We now implement an interior point method, coupled with our GMRES solver (with preconditioners $\mathcal{P}_1$ and $\mathcal{P}_2$), for a particular test problem. In more detail, we set $\mathcal{D} = -\nabla^2$, $\widehat{y} = e^{-64((x_1-0.5)^2 + (x_2-0.5)^2)}$, where $\mathbf{x} = [x_1, x_2]^\top \in \Omega = [0, 1]^2$, with state and control constraints prescribed based on the physical properties of the problem. The iterative solvers are run to a tolerance of 10^{-8} , with the outer (interior point) solver set to a tolerance of 10^{-6} . We solve for a range of step-sizes h and values of β , using MATLAB R2015a, on a quad-core 3.2 GHz processor. We observe that the number of interior point iterations, as well as the GMRES iteration count, behave robustly for a wide range of parameters. We therefore conclude that our preconditioning strategies are highly effective for the problem considered.

Table 1: Number of interior point (Newton) iterations required to achieve convergence (blue, left), and average number of GMRES steps per interior point iteration before a relative preconditioned residual norm of 10^{-8} is achieved (black, right), for the test problem considered.

$\mathcal{P}_1$		$\beta = 1$ $0 \leq y \leq 1.5 \times 10^{-5}$ $0 \leq u \leq 3 \times 10^{-4}$	$\beta = 10^{-1}$ $0 \leq y \leq 1.5 \times 10^{-4}$ $0 \leq u \leq 3 \times 10^{-3}$	$\beta = 10^{-2}$ $0 \leq y \leq 1.5 \times 10^{-3}$ $0 \leq u \leq 0.03$	$\beta = 10^{-3}$ $0 \leq y \leq 0.015$ $0 \leq u \leq 0.3$	$\beta = 10^{-4}$ $0 \leq y \leq 0.08$ $0 \leq u \leq 1.5$	$\beta = 10^{-5}$ $0 \leq y \leq 0.15$ $0 \leq u \leq 5$
h	2^{-2}	5 7.0	7 6.1	9 5.5	11 8.7	12 7.1	13 11.5
	2^{-3}	5 10.7	7 9.8	10 8.2	11 9.3	13 8.5	14 19.9
	2^{-4}	6 11.0	8 10.0	10 9.2	12 9.6	14 9.5	15 18.4
	2^{-5}	7 10.8	10 9.8	11 9.3	13 9.6	15 9.6	16 16.8
	2^{-6}	8 12.4	10 9.7	12 9.5	14 9.7	16 9.4	19 21.1
	2^{-7}	9 12.2	12 10.1	13 10.6	15 9.4	17 9.9	20 26.1
$\mathcal{P}_2$		$\beta = 1$ $0 \leq y \leq 1.5 \times 10^{-5}$ $0 \leq u \leq 3 \times 10^{-4}$	$\beta = 10^{-1}$ $0 \leq y \leq 1.5 \times 10^{-4}$ $0 \leq u \leq 3 \times 10^{-3}$	$\beta = 10^{-2}$ $0 \leq y \leq 1.5 \times 10^{-3}$ $0 \leq u \leq 0.03$	$\beta = 10^{-3}$ $0 \leq y \leq 0.015$ $0 \leq u \leq 0.3$	$\beta = 10^{-4}$ $0 \leq y \leq 0.08$ $0 \leq u \leq 1.5$	$\beta = 10^{-5}$ $0 \leq y \leq 0.15$ $0 \leq u \leq 5$
h	2^{-2}	5 14.2	7 12.0	9 7.8	11 6.1	12 5.5	13 5.6
	2^{-3}	5 12.8	7 11.1	10 10.2	11 8.9	13 9.9	14 9.3
	2^{-4}	6 13.2	8 14.4	10 10.7	12 10.4	14 10.9	15 10.5
	2^{-5}	7 13.0	10 14.4	11 13.5	13 13.3	15 11.2	16 12.6
	2^{-6}	8 11.8	10 13.2	12 14.9	14 14.6	16 14.8	19 14.6
	2^{-7}	9 11.6	12 13.4	13 15.0	15 14.8	17 15.3	20 16.2

References

- [1] M. Benzi, G. H. Golub, and J. Liesen, Numerical solution of saddle point problems, *Acta Numerica* **14**, pp. 1–137 (2005).
- [2] J. W. Pearson and J. Gondzio, Fast interior point solution of quadratic programming problems arising from PDE-constrained optimization, submitted (2016).
- [3] J. W. Pearson and A. J. Wathen, Fast iterative solvers for convection-diffusion control problems, *Electronic Transactions on Numerical Analysis* **40**, pp. 294–310 (2013).
- [4] J. W. Pearson and A. J. Wathen, A new approximation of the Schur complement in preconditioners for PDE-constrained optimization, *Numerical Linear Algebra with Applications* **19**, pp. 816–829 (2012).