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GENERALISED WITT ALGEBRAS AND IDEALIZERS

S. J. SIERRA AND Š. ŠPENKO

Abstract. Let \( k \) be an algebraically closed field of characteristic zero, and let \( \Gamma \) be a finitely generated additive subgroup of \( k \). Results of Kaplansky-Santharoubane and Su classify intermediate series representations of the generalised Witt algebra \( W_\Gamma \) in terms of three families, one parameterised by \( k^2 \) and two by \( \mathbb{F}^3 \). In this note, we use the first family to construct a homomorphism \( \Phi \) from the enveloping algebra \( U(W_\Gamma) \) to a skew extension \( k[A^2] \times \Gamma \) of the coordinate ring of \( k^2 \). We show that the image of \( \Phi \) is contained in a (double) idealizer subring of this skew extension and that the representation theory of idealizers explains the three families. We further show that the image of \( U(W_\Gamma) \) under \( \Phi \) is not left or right noetherian, giving a new proof that \( U(W_\Gamma) \) is not noetherian.

We construct \( \Phi \) as an application of a general technique to create ring homomorphisms from shift-invariant families of modules. Let \( G \) be an arbitrary group and let \( A \) be a \( G \)-graded ring. A graded \( A \)-module \( M \) is an intermediate series module if \( M_g \) is one-dimensional for all \( g \in G \). Given a shift-invariant family of intermediate series \( A \)-modules parametrised by a scheme \( X \), we construct a homomorphism \( \Phi \) from \( A \) to a skew extension of \( k[X] \). The kernel of \( \Phi \) consists of those elements which annihilate all modules in \( X \).

1. Introduction

Fix an algebraically closed ground field \( k \) of characteristic zero, and let \( \Gamma \) be a finitely generated additive subgroup of \( k \). The generalised Witt algebra \( W_\Gamma \) is the Lie algebra generated by elements \( e_\gamma : \gamma \in \Gamma \), with \([e_\gamma, e_\delta] = (\delta - \gamma)e_{\delta + \gamma}\). Recall that an intermediate series representation of \( W_\Gamma \) is an indecomposable representation all of whose \( e_0 \)-eigenspaces are 1-dimensional. It is a theorem of Kaplansky and Santharoubane [KS85] (if \( \Gamma = \mathbb{Z} \)) and of Su [Su94] (for general \( \Gamma \)) that intermediate series representations of \( W_\Gamma \) come in three families (with two modules represented twice): one family parameterised by \( k^2 \) and two parameterised by \( \mathbb{F}^3 \). In this note we use the first family to construct a homomorphism \( \Phi \) from \( U(W_\Gamma) \) to \( T = k[A^2] \times \Gamma \), and show that the existence of the other two families is a consequence of the fact that the image of \( U(W_\Gamma) \) is a sub-idealizer in \( T \). We further use the homomorphism \( \Phi \) to give a new proof that the enveloping algebra of \( U(W_\Gamma) \) is not noetherian, a fact originally proved in [SW14].

Since our main method is to construct and then analyze a homomorphism from \( U(W_\Gamma) \) to an idealizer in \( T \), we recall some facts about idealizers. We first define \( T \) as a vector space where we write \( T = \bigoplus_{\gamma \in \Gamma} k[e(a, b)]t^\gamma \), with \( t^\alpha t^\beta = t^{\alpha + \beta} \) and \( t^\gamma f(a, b) = f(a + \gamma, b)t^\gamma =: f_t^{\gamma} \). Note that \( T \) is a bimodule over \( k[e(a, b)] \).

An intermediate series module \( M \) over a \( \Gamma \)-graded ring is an indecomposable \( \Gamma \)-graded module with each \( M_g \) one-dimensional. It is a generalisation of a point module over an \( \mathbb{N} \)-graded ring, which is a cyclic graded module with Hilbert series \( 1/(1-t) \).

For \( p = (a, \beta) \in k^2 \), let \( I(p) \) be the ideal \( (\alpha - a, b - \beta) \) of \( k[e(a, b)] \). Let \( V(p) = T/I(p)T \). It is easy to see that the \( V(p) \) are all of the intermediate series right \( T \)-modules; more precisely, the right ideals \( J \) of \( T \) such that \( T/J \) is an intermediate series module are precisely the \( I(p)T \). Likewise, the intermediate series left \( T \)-modules are the \( T/IT(p) \). These families are preserved under degree shifting.

We now consider a subring of \( T \). Fix \( p_0 \in k^2 \), and let \( S = S(p_0) = k \oplus I(p_0)T \). The ring \( S \) is an idealizer in \( T \): the largest subalgebra of \( T \) such that the right ideal \( I(p_0)T \) becomes a two-sided ideal in \( S \). It is known [Rog84] that the representation theory of idealizers involves blowing up. Here for \( p \neq p_0 \) we have that \( V(p) \cong S/(S \cap I(p)T) \) is an intermediate series right \( S \)-module. On the other hand, to define an intermediate series right \( S \)-module at \( p_0 \), we need to consider a point \( q \) infinitely near to \( p_0 \); that is, an ideal \( I(q) \) with \( I(p_0)^2 \subseteq I(q) \subseteq I(p_0) \) of \( k[e(a, b)] \) such that \( I(p_0)/I(q) \) is one-dimensional. Such ideals are parameterised by the exceptional \( \mathbb{F}^3 \) in the blowup \( Bl_{p_0}(k^2) \); more specifically, we can write

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\[ I(q) = (y(a - \alpha) - x(b - \beta), (a - \alpha)^2, (a - \alpha)(b - \beta), (b - \beta)^2) \] for some \( [x : y] \in \mathbb{P}^1 \). For such \( I(q) \) we have that \( I(p_0) + I(q)T \) is a right ideal of \( S \). Let

\[ P(q) = S/(I(p_0) + I(q)T). \]

Then \( P(q) \) is an intermediate series right \( S \)-module. In fact, we have constructed all right ideals \( J \) of \( S \) such that \( S/J \) is an intermediate series \( S \)-module; they are parameterised by \( \text{Bl}_{p_0}(\mathbb{A}^2) \) but it is sometimes more convenient to consider them as parameterised by \( \mathbb{A}^2 \setminus \{ p_0 \} \) together with \( \mathbb{P}^1 \).

Left intermediate series \( S \)-modules are also parameterised by \( \text{Bl}_{p_0}(\mathbb{A}^2) \). For \( p \in \mathbb{A}^2 \setminus \{ p_0 \} \), the left intermediate series module \( T/TT(p) \) is isomorphic to \( \left( I(p_0) \oplus \bigoplus_{0 \neq \nu \in \Gamma} \mathbb{k}[a, b]t^\nu \right) / \left( (I(p_0) \cap I(p)) \oplus \bigoplus_{0 \neq \nu \in \Gamma} t^\nu I(p) \right) \).

We can extend this construction to a family of modules parameterised by \( \text{Bl}_{p_0}(\mathbb{A}^2) \) by adding the \( \mathbb{P}^1 \) of points \( q \) infinitely near to \( p_0 \):

\[ Q(q) = \frac{I(p_0) \oplus \bigoplus_{0 \neq \nu \in \Gamma} \mathbb{k}[a, b]t^\nu}{I(q) \oplus \bigoplus_{0 \neq \nu \in \Gamma} t^\nu I(p_0)}. \]

Consider now right intermediate series modules over the double idealiser

\[ R = \mathbb{k}[a, b] + (I(p_0)T \cap TI(p_1)) \]

and assume for simplicity that \( p_0, p_1 \in \mathbb{A}^2 \) have distinct \( \Gamma \)-orbits. These correspond to points of the double blowup \( \text{Bl}_{p_0, p_1}(\mathbb{A}^2) \). More precisely, the \( V(p) \) are intermediate series modules for \( p \in \mathbb{A}^2 \setminus \{ p_0, p_1 \} \). From the inclusion \( R \subseteq \mathbb{k} \oplus I(p_0)T \) we obtain a family \( P(q) \) parameterised by the \( \mathbb{P}^1 \) of points infinitely near to \( p_0 \). Finally, from the inclusion \( R \subseteq \mathbb{k} \oplus TI(p_1) \) we obtain a family \( Q(q) \) of right modules parameterised by the \( \mathbb{P}^1 \) of points infinitely near to \( p_1 \) and constructed similarly to the construction of the left modules \( Q(q) \) over \( S \).

Let \( \Gamma \) now be an arbitrary group (more generally, a monoid) and let \( A \) be a \( \Gamma \)-graded ring. We give a general result in Theorem 2.2 (respectively, Theorem 2.5) which constructs a ring homomorphism (respectively, an anti-homomorphism) \( \Phi : A \to \mathbb{k}[X] \times \Gamma \), where \( X \) is a shift-invariant family of right (respectively, left) intermediate series \( A \)-modules; this generalises constructions in [ATV91, RZ08, V96].

When we apply this technique to \( U(W_T) \), we show that the image of \( \Phi \) is contained in a double idealizer \( R \) inside the ring \( T \) defined in the second paragraph, and we show in Propositions 3.3, 3.6 that the right intermediate series \( R \)-modules constructed above restrict to precisely the intermediate series representations of \( W_T \). This gives a unified geometric description of what have until now been seen as three distinct families of representations.

We further show in Proposition 4.3 that the image of \( U(W_T) \) under \( \Phi \) is neither right nor left noetherian. For \( \Gamma = \mathbb{Z} \) this was proved in [SW15] as the main step in proving the non-noetherianity of \( U(W) \). It follows that \( U(W_T) \) is neither right nor left noetherian; other proofs are given in [SW14, SW15].

The general behaviour of idealizers leads one to expect that at idealizers in \( T \) at ideals of points in \( \mathbb{k}[a, b] \) will not be noetherian since no points have dense \( \Gamma \)-orbits; see [Sie11] for a precise statement of a related result for \( \mathbb{N} \)-graded rings. However, infinite orbits are dense in \( \mathbb{A}^2 \). Thus one expects that the factors \( \Phi(U(W_T))[b = \beta] \), which live on the \( \Gamma \)-invariant line \( (b = \beta) \) in \( \mathbb{A}^2 \), are noetherian for all \( \beta \in \mathbb{k} \), and we also show in Proposition 4.6 that this is indeed the case.

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2. Intermediate series modules and ring homomorphisms

It is well-known that ring homomorphisms can be constructed from shift-invariant families of modules. Let \( A \) be a (connected \( \mathbb{N} \)-) graded ring, generated in degree 1. A point module over \( A \) is a cyclic graded \( A \)-module with Hilbert series \( 1/(1-t) \). Suppose that (right) \( A \)-point modules are parameterised by a projective scheme \( X \). Let the point module corresponding to \( x \in X \) be \( M_x \). Then the shift functor \( \Psi : M \mapsto M[1]_{\geq 0} \) induces an automorphism \( \sigma \) of \( X \) so that \( \Psi(M^{(x)}) \cong M^{\sigma(x)} \).

The following result goes back to [ATV90] (see also [V96]), although in this form it is due to Rogalski and Zhang.
Theorem 2.1. \cite{AV90} There is an invertible sheaf $\mathcal{L}$ on $X$ so that there is a homomorphism $\phi : A \to B(X, \mathcal{L}, \sigma)$ of graded rings, where $B(X, \mathcal{L}, \sigma)$ is the twisted homogeneous coordinate ring defined in \cite{AV90}. If $A$ is noetherian then $\phi$ is surjective in large degree.

The kernel of $\phi$ is equal in large degree to

$$J = \bigcap \{ \text{Ann}_A(M) \mid M \text{ is a } C\text{-point module for some commutative } k\text{-algebra } C \}.$$

The purpose of this section is to give a version of this theorem for a (not necessarily connected graded) algebra graded by an arbitrary monoid $\Gamma$.

We first need some notation. Let $\Gamma$ be a monoid and let $A$ be a $\Gamma$-graded ring. If $M$ is a $\Gamma$-graded right $A$-module and $\gamma \in \Gamma$, we define the shift $M(\gamma)$ of $M$ by $\gamma$ as:

$$M(\gamma) = \bigoplus_{\delta \in \Gamma} M(\gamma)\delta,$$

where $M(\gamma)\delta = M_{\gamma\delta}$. We note that

$$(M(\gamma))((\delta)) = M(\gamma)e = M_{\gamma\delta} = M(\gamma)e,$$

so $M(\gamma)$ is again a $\Gamma$-graded right $A$-module. Note that

$$(M(\gamma))(\delta) = M(\gamma)e = M_{\gamma\delta} = M(\gamma)e,$$

and so $(M(\gamma))(\delta)$ is canonically isomorphic to $M(\gamma)$.

If $M$ is a left module we define $M(\gamma)\delta = M_{\delta\gamma}$. Then (2.1) becomes:

$$A, M(\gamma)\delta = A, M_{\delta\gamma} \subseteq M_{\delta\gamma} = M(\gamma)e\delta,$$

as needed. We have

$$(M(\gamma))(\delta) = M(\gamma)e\delta = M_{\delta\gamma} = M(\delta)e\gamma,$$

so $(M(\gamma))(\delta)$ is canonically isomorphic to $M(\delta\gamma)$.

If $A$ is a $\Gamma$-graded ring, an intermediate series module over $A$ is a $\Gamma$-graded left or right $A$-module $M$ so that $\dim M_\gamma = 1$ for all $\gamma \in \Gamma$. We will use a shift-invariant family of intermediate series modules to construct a ring homomorphism from $A$ to a $\Gamma$-graded ring, giving a version of Theorem 2.1 in this setting.

Our notation for smash products is that if $\Gamma$ acts on $A$ then $A \rtimes \Gamma = \bigoplus_{\gamma \in \Gamma} A^\gamma$, where $t^\gamma t^\delta = t^{\gamma\delta}$ and $t^\gamma r = r^\gamma t^\gamma$ for all $r \in A$, $\gamma \in \Gamma$.

Theorem 2.2. Let $\Gamma$ be a monoid with identity $e$ and let $A$ be a $\Gamma$-graded ring. Let $X$ be a reduced affine scheme that parameterises a set of intermediate series right $A$-modules, in the sense that for $x \in X$ there is a module $M^x$ with basis $\{ v^x_\gamma \mid \gamma \in \Gamma \}$, and that there is a $k$-linear function $\phi : A \to k[X]$ so that

$$v^x_\gamma r = \phi(r)(x)v^x_\gamma$$

for all $\gamma \in \Gamma$, $r \in A_\gamma$. Further suppose that shifting defines a group antiisomorphism $\sigma : \Gamma \to \text{Aut}_k(A), \gamma \mapsto \sigma^\gamma$ so that $M^x(\sigma^\gamma) \cong M^{\sigma^\gamma}(x)$. Here we require that the isomorphism maps $v^x_\gamma \mapsto v^{\sigma^\gamma}(x)$.

In this setting the map

$$\Phi : A \to k[X] \rtimes \Gamma, \quad r \in A_\gamma \mapsto \phi(r)t^\gamma$$

is a graded homomorphism of algebras. Further,

$$\ker \Phi = \bigcap_{x \in X} \text{Ann}_A M^x.$$

Proof. Let $\Gamma$ act on $k[X]$ by $f^\gamma = (\sigma^\gamma)^*(f)$, so $\sigma$ defines a homomorphism from $\Gamma \to \text{Aut}_k(k[X])$.

Let $r \in A_\gamma, s \in A_\delta$, and let $\alpha : V^x(\gamma) \to V^{\sigma^\gamma}(x)$ be the given isomorphism. Then:

$$\alpha(v^x_\gamma s) = v^{\sigma^\gamma}(x)s = \phi(s)(\sigma^\gamma)(x)v^{\sigma^\gamma}(x) = \alpha(\phi(s)(\sigma^\gamma(x)))v^{\sigma^\gamma}(x).$$

So

$$v^x_\gamma s = \phi(s)^x(x)v^{\sigma^\gamma}(x).$$

Now, using (2.2), we obtain:

$$\phi(rs)(x)v^{\sigma^\gamma}(x) = v^x_\gamma rs = \phi(r)(x)v^x_\gamma s = \phi(r)(x)v^{\gamma^\sigma}(x)v^{\sigma^\gamma}(x).$$
and so
\[(2.3)\]
\[\phi(rs) = \phi(r)\phi(s)^\gamma.\]
Then by (2.3) we have
\[\Phi(rs) = \phi(rs)t^\gamma = \phi(r)\phi(s)^\gamma t^\gamma = \phi(r)\phi(s)^\gamma t^\gamma = \Phi(r)\Phi(s).\]

Since \(\Phi\) is graded, \(\ker \Phi\) is a graded ideal of \(A\). If \(r \in A\) is homogeneous then
\[\Phi(r) = 0 \iff \phi(r) = 0 \iff v^r_\gamma r = 0 \text{ for all } x \in X.\]
Let \(\gamma \in \Gamma\). Then
\[v^r_\gamma r = 0 \text{ for all } x \in X \iff v^r_\gamma (x)r = 0 \text{ for all } x \in X \iff v^r_\gamma r = 0 \text{ for all } x \in X,
using the isomorphism between \(M^r(\gamma)\) and \(M^{\sigma^r_\gamma(x)}\). So
\[\Phi(r) = 0 \iff v^r_\gamma r = 0 \text{ for all } x \in X, \gamma \in \Gamma \iff r \in \bigcap_{x \in X} \text{Ann}_A M^x.\]

(\text{The reason we require } X \text{ in the theorem statement to be reduced is that we are constructing } \Phi \text{ from the closed points of } X, \text{ and so effectively from the reduced induced structure on } X.)

Remark 2.3. We need the map \(\sigma\) in Theorem 2.2 to be an antihomomorphism because of the equations:
\[M^\sigma^r(\gamma) \cong M^r(\gamma \delta) = (M^r(\gamma))(\delta) \cong M^{\sigma^r_\gamma(x)}(\delta) = M^{\sigma^r_\gamma(x)}(\delta).

Remark 2.4. There is a universal module \(M\) for the family \(\{M^x \mid x \in X\}\), which is isomorphic as a \(\kk[X]\)-module to \(\bigoplus_{\gamma \in \Gamma} \kk[X] v^r_\gamma\). The module structure is given by
\[(2.4)\]
\[v^r_\gamma s = \phi(s)^\gamma v^r_\delta\]
for \(s \in A_\delta\). If we consider the natural right action of \(A\) on \(M = \kk[X] \rtimes \Gamma\) then we have \(t^\gamma \cdot s = t^\gamma \Phi(s) = t^\gamma \phi(s) t^\gamma = \phi(s)^\gamma t^\gamma\delta\) for \(s \in A_\delta\). This agrees with (2.4) if we set \(v^r_\gamma = t^\gamma\), and so \(M \cong \kk[X] \rtimes \Gamma\).

The theorem for left modules is:

Theorem 2.5. Let \(\Gamma\) be a monoid with identity \(e\) and let \(A\) be a \(\Gamma\)-graded ring. Let \(X\) be a reduced affine scheme that parameterises a set of intermediate series left \(A\)-modules, in the sense that the left module \(N^x\) has a basis \(\{v^r_\gamma \mid \gamma \in \Gamma\}\) and that there is a \(\kk\)-linear function \(\phi : A \to \kk[X]\) so that
\[rv^r_\gamma = \phi(r)(x)v^r_\gamma\]
for all \(\gamma \in \Gamma, r \in A_\gamma\). Further suppose that shifting defines a group homomorphism \(\sigma : \Gamma \to \text{Aut}_{\kk}[X], \gamma \mapsto \sigma^\gamma\)
so that \(N^x(\gamma) \cong N^{\sigma^\gamma(x)}\). Here we require that the isomorphism maps \(v^r_\gamma \mapsto \sigma^\gamma(v^r_\delta)\).

In this setting the map
\[\Phi : A \to \kk[X] \rtimes \Gamma^{\text{op}}\]
\[r \in A_\gamma \mapsto \phi(r)t^\gamma\]
is a graded antihomomorphism of algebras. Further,
\[\ker \Phi = \bigcap_{x \in X} \text{Ann}_A N^x.\]

Proof. We repeat the proof above to ensure that the change of notation from right to left is handled correctly. Again, let \(f^\gamma = (\sigma^\gamma)^* f\), so \(\sigma\) defines a homomorphism from \(\Gamma^{\text{op}} \to \text{Aut}_{\kk}[X]\). Let \(r \in A_\gamma, s \in A_\delta\), and let \(\alpha : V^r(\delta) \to V^{\sigma^r_\gamma(x)}(x)\) be the given isomorphism. Then:
\[\alpha(rv^r_\delta) = rv^{\sigma^r_\gamma(x)} = \phi(r)(\sigma^r_\gamma(x))v^{\sigma^r_\gamma(x)} = \alpha(\phi(r)(\sigma^r_\gamma(x))v^{\sigma^r_\gamma(x)}).\]
So
\[(2.5)\]
\[rv^r_\delta = \phi(r)(\sigma^r_\delta(x))v^{\sigma^r_\gamma(x)}\]
Now, using (2.5), we obtain:
\[\phi(rs)(x)v^{\sigma^r_\gamma(x)} = rs v^r_\delta = \phi(s)(x)rv^r_\delta = \phi(s)(x)\phi(r)(\sigma^r_\delta(x))v^{\sigma^r_\gamma(x)}\]
and so

(2.6) \[ \phi(rs) = \phi(s)\phi(r)^\delta. \]

Then by (2.6) we have

\[ \Phi(rs) = \phi(s)\phi(r)^\delta r^\delta = \phi(s)\phi(r)^\delta t^{\delta_{\sigma\gamma}} = \phi(s)t^\delta \phi(r)t^\gamma = \Phi(s)\Phi(r). \]

The proof of the last statement is identical to the proof in Theorem 2.2. \(\square\)

**Remark 2.6.** We need the map \(\sigma\) in Theorem 2.5 to be a homomorphism because:

\[ N^{\sigma_{\gamma}(x)} = N^x(\gamma\delta) = (N^x(\delta))(\gamma) = N^{\sigma_{\gamma}(\sigma_{\gamma}(x))}. \]

Note also that a graded anti-homomorphism from a \(\Gamma\)-graded algebra should map to a \(\Gamma^{op}\)-graded algebra, as we indeed have.

**Remark 2.7.** We likewise obtain the universal left module for the \(N^x\) from \(\Phi\). Set \(N = k[X] \rtimes \Gamma^{op}\). The left action induced by \(\Phi\) is \(r \cdot \delta = \delta \Phi(r)\) because \(\Phi\) is an anti-homomorphism, so we get

\[ r \cdot t^\delta = t^\delta \Phi(r) = t^\delta \phi(r)t^\gamma = \phi(r)^{\delta_{\delta\gamma}} = \phi(r)^{\delta_{\gamma}} \]

for \(r \in A_{\gamma}\), which is the structure we expect.

**Remark 2.8.** Let Bir\((X)\) be the group of birational self-maps of \(X\). In the settings above, suppose that shifting defines elements of Bir\((X)\), in the sense that \(\sigma\) maps \(\Gamma\) to Bir\((X)\). We get a generalization of Theorems 2.2 and 2.3 by replacing \(k[X]\) and Aut\((k[X])\) with \(\delta(X)\) and Bir\((X)\), respectively.

3. Intermediate series modules over higher rank Witt algebras

Let \(\Gamma\) be a rank \(n\) \(\mathbb{Z}\)-submodule of \(\mathbb{A}\). The rank \(n\) Witt algebra \(W_{\Gamma}\) (or higher rank Witt algebra if \(n \geq 2\), sometimes called the centerless higher rank Virasoro algebra) is the Lie algebra with \(k\)-basis \(\{e_\nu \mid \nu \in \Gamma\}\) and bracket

\[ [e_\mu, e_\nu] = (\nu - \mu)e_{\nu + \mu} \]

for \(\nu, \mu \in \Gamma\). The rank one Witt algebra is the “usual” Witt algebra, which we denote by \(W\).

As \(U(W_{\Gamma})\) is \(\Gamma\)-graded one can consider intermediate series modules as in Section 2. They are the standard intermediate series modules of Lie algebras, called also Harish-Chandra modules over \((W_{\Gamma}, k\mathfrak{e}_0)\); i.e., modules of the form \(\bigoplus_{\gamma \in \Gamma} V_\gamma\), where \(V_\gamma\) is the \(\gamma\)-eigenspace for \(e_0\) and has dimension 1.

The intermediate series \(W_{\Gamma}\)-modules have been classified in [Su94, Theorem 2.1], generalizing the classification [KSS5] for the Witt algebra. There are three families of indecomposable intermediate series \(W_{\Gamma}\)-modules:

\[ V_{(\alpha, \beta)} = \bigoplus_{\nu \in \Gamma} k_{\nu}, \quad e_\mu v_\nu = (\alpha + \beta \mu + \nu)v_{\mu + \nu}, \]

\[ A_{(\alpha, \beta)} = \bigoplus_{\nu \in \Gamma} k_{\nu}, \quad e_\mu v_\nu = \begin{cases} \nu v_{\mu + \nu} & \nu \neq 0, \mu + \nu \neq 0, \\ (\alpha + \beta \mu)v_\mu & \nu = 0, \\ 0 & \mu + \nu = 0, \end{cases} \]

\[ B_{(\alpha, \beta)} = \bigoplus_{\nu \in \Gamma} k_{\nu}, \quad e_\mu v_\nu = \begin{cases} (\mu + \nu)v_{\mu + \nu} & \nu \neq 0, \mu + \nu \neq 0, \\ 0 & \nu = 0, \\ (\alpha + \beta \mu)v_0 & \mu + \nu = 0, \end{cases} \]

where \((\alpha, \beta) \in \mathbb{A}^2\). Note that \(A_{(\alpha, \beta)}\), \(B_{(\alpha, \beta)}\) are only defined where \((\alpha, \beta) \neq (0, 0)\) and depend up to isomorphism (rescaling of \(v_0\)) only on \([\alpha : \beta] \in \mathbb{P}^1\). We will therefore denote them by \(A_{[\alpha : \beta]}\), \(B_{[\alpha : \beta]}\). Note also that we have \(A_{[0:1]} \cong V_{(0,1)}\) (by \(v_0 \mapsto v_0\) and \(v_\nu \mapsto \nu v_\nu\) when \(\nu \neq 0\)) and \(B_{[1:0]} \cong V_{(0,0)}\) (by \(v_0 \mapsto \nu v_0\) and \(v_\nu \mapsto \nu v_\nu\) when \(\nu \neq 0\)).

**Remark 3.1.** Note that \(A_{[\alpha : \beta]}\) contains a simple submodule \(\bigoplus_{\alpha \neq \beta \in \Gamma} k_{v_\nu}\) with a 1-dimensional trivial quotient. On the other hand, \(B_{[\alpha : \beta]}\) has the 1-dimensional trivial submodule \(k_{v_\nu}\), and the quotient is a simple module. This is explained by the isomorphism \(B_{[\alpha : \beta]} \cong A_{[\alpha : \beta]}^t\), where \(^t\) denotes the adjoint. (If \(M = \bigoplus_{\gamma \in \Gamma} k_{v_\gamma}\) is a left \(\Gamma\)-graded \(W_{\Gamma}\)-module, the adjoint (or restricted dual) of \(M\) is the left \(\Gamma\)-graded \(W_{\Gamma}\)-module \(M'_\gamma = \text{Hom}_k(M_{-\gamma}, k)\), \(v'_\gamma = v^*_{-\gamma}\), and \(e_\mu v'_\nu = -v^*_{-\gamma}e_{\mu}\).)
Remark 3.2. We use a slightly different presentation of the families $A_{[\alpha;\beta]}$, $B_{[\alpha;\beta]}$ than in [Su94]. In loc.cit the last two families are replaced by $\hat{A}(a')$ defined by
\[ e_\mu v'_\nu = (\nu + \mu)v'_{\mu+\nu}, \quad \nu \neq 0, \quad e_\mu v_0 = \mu(1 + (\mu + 1)a')v'_\mu, \]
and by $\hat{B}(a')$ defined by
\[ e_\mu v'_\nu = \nu v'_{\mu+\nu}, \quad \nu \neq -\mu, \quad e_\mu v_{-\mu} = -\mu(1 + (\mu + 1)a')v'_0, \]
for $a' \in \mathbb{k} \cup \{\infty\}$. If $a' = \infty$ then $1 + (\mu + 1)a'$ in the above definition is regarded as $\mu + 1$. Note that $\hat{A}(a')$ (resp. $\hat{B}(a')$) is isomorphic to $A_{[1+a',a']}$ (resp. $B_{[1+a',a']}$) if $a' \neq \infty$ and to $A_{[1;1]}$ (resp. $B_{[1;1]}$) if $a' = \infty$, for $v_\nu = \nu v'_\nu$ (resp. $v_\nu = \frac{1}{\nu}v'_\nu$) if $\nu \neq 0$, and $v_0 = v'_0$.

For the Witt algebra the choice of the basis is the same in [KSS5], however there $a' \in \mathbb{k}$ and modules are classified up to inversion: replacing $v_\nu$ by $-v_{-\nu}$.

Let us show how to obtain the intermediate series modules using results of Section 2.

Proposition 3.3. Let $\Gamma$ act on $\mathbb{k}[a, b]$ as $t^\nu p(a, b) = p(a + \nu, b)t^\nu$, and let $T := \mathbb{k}[a, b] \rtimes \Gamma$. The map $\phi : W_T \to T$, $\phi(e_\mu) = (a + b\mu)t^\mu$, induces an anti-homomorphism $\Phi : U(W_T) \to T$. Consequently, $T$ is a left $U(W)$-module via $e_\mu p(a, b)t^\nu = (a + \nu + b\mu)p(a, b)t^{\mu+\nu}$.

Proof. Note that $\mathbb{k}^2$ parametrises a set of intermediate series modules $N^{(\alpha, \beta)} := V_{(\alpha, \beta)}$ and $e_\mu v_0^{(\alpha, \beta)} = (a + b\mu)((\alpha, \beta))v^{(\alpha, \beta)}_\mu$. Further, $N^{(\alpha, \beta)}(\nu) \cong N^{(\alpha + \nu, \beta)}$ and hence $\sigma^{\nu}((\alpha, \beta)) = (\alpha + \nu, \beta)$ (using the notation of Section 2). The proposition therefore follows by Theorem 2.5 and Remark 2.7. 

Remark 3.4. Let $\Gamma = \mathbb{Z}$ and $T = \mathbb{k}[a, b] \rtimes \mathbb{Z}$. We may compose the map $\Phi$ of Proposition 3.3 with the canonical anti-automorphism $e_n \mapsto -e_n$ of $U(W)$ to obtain a homomorphism $\Phi' : U(W) \to T, e_n \mapsto (-a - bn)t^n$.

Recall that in [SW15] a homomorphism $\hat{\phi}$ was constructed from $U(W_T)$ to $T' := \mathbb{k}[u, v, v^{-1}]/(uv - vu - v^2, uw - wu - wv, vw - vw)$, defined by $\hat{\phi}(e_n) = (u - (n - 1)w)v^{n-1}$. The reader may verify that $\alpha : T' \to T$ defined by
\[ u \mapsto (b - a)t, \quad v \mapsto t, \quad w \mapsto bt \]
is an isomorphism of graded rings and that $\alpha \hat{\phi} = \Phi'$. Thus Proposition 3.3 generalises the construction of $\hat{\phi}$.

We now discuss applications of $\Phi$ to the representation theory of $W_T$. For $p = (\alpha, \beta) \in \mathbb{A}^2$ we denote by $I(p)$ the ideal $(a - \alpha, b - \beta)$ in $\mathbb{k}[a, b]$. For $q$ infinitely near to $p$, corresponding to $[x : y] \in \mathbb{P}^1$, we denote by $I(q)$ the ideal $(y(a - \alpha) - x(b - \beta), (a - \alpha)^2, (a - \alpha)(b - \beta), (b - \beta)^2)$.

Let $B = \Phi(U(W_T))$, and note that $B$ is contained in the double idealizer $R = \mathbb{k}[a, b] + I(0, 0)T \cap TI(0, 1)$. From the discussion in the introduction, then, we expect three families of intermediate series $U(W_T)$-modules, one parameterised by $\mathbb{A}^2 \setminus \{(0, 0), (0, 1)\}$ and two parameterised by $\mathbb{P}^1$. Note that because $\Phi$ is an anti-homomorphism, right $B$-modules will correspond to left $U(W_T)$-modules.

By construction of $\Phi$ we have $V(\alpha, \beta) \cong T/I(p)T$, considered as a $B$-module. Removing $V(0, 0)$ and $V(0, 1)$ we obtain the two-dimensional family we expect. We next show that we also obtain the two $\mathbb{P}^1$-families $A_{[\alpha;\beta]}$ and $B_{[\alpha;\beta]}$.

Proposition 3.5. Let $[x : y] \in \mathbb{P}^1$ and let $I(q) = (ya - xb, a^2, ab, b^2)$ define a point infinitely near to $(0, 0)$. Let
\[ P(q) = \frac{\mathbb{k}[a, b] + I(0, 0)T}{I(0, 0) + I(q)T}. \]
Then $A_{[x:y]} \cong P(q)$.

Proof. If $w \in \mathbb{k}[a, b] + I(0, 0)T$ let $\overline{w}$ be the image of $w$ in $P(q)$. If $x \neq 0$ we choose a basis
\[ v_\nu = \begin{cases} \overline{aw} & \nu \neq 0, \\ 1 & \nu = 0 \end{cases} \]
for $P(q)$. 

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Using the anti-homomorphism, we compute for \( \nu \neq 0 \)
\[
e_{\mu}v_{\nu} = a^\nu (a + b\mu) t^{\nu} = a(a + b\mu + \nu)t^{\nu + \nu} = \nu a t^{\nu + \nu} = \begin{cases} 
\nu v_{\nu + \mu} & \nu + \mu \neq 0, \\
0 & \nu + \mu = 0.
\end{cases}
\]
and
\[
e_{\mu}v_{0} = (a + b\mu)t^{\nu} = \frac{(a + \frac{y}{x}a\mu)}{t^{\nu}} = \left(1 + \frac{y}{x}\mu\right)v_{\mu},
\]
so \( P(q) \cong A_{[x:y]} \) as claimed.

If \( y \neq 0 \) we pick a basis
\[
v_{\nu} = \begin{cases} 
b(t^\nu) & \nu \neq 0, \\
1 & \nu = 0,
\end{cases}
\]
and obtain \( e_{\mu}v_{\nu} = \nu v_{\nu + \mu}, e_{\mu}v_{0} = (\frac{x}{y} + \mu)v_{\mu}, e_{\mu}v_{-\mu} = 0. \) Thus \( P(q) \cong A_{[x:y]} \) again. \( \Box \)

In the next result, note the change of sides from the left modules \( Q(q) \) defined in the introduction.

**Proposition 3.6.** Let \( [x : y] \in \mathbb{P}^1 \) and let \( I(q) = (ya - x(b - 1), a^2, a(b - 1), (b - 1)^2) \) define a point infinitely near to \((0, 1)\). Let
\[
Q(q) = \frac{I(0, 1) \oplus \bigoplus_{0 \neq \nu \in \Gamma} k[a, b] t^\nu}{I(q) \oplus \bigoplus_{0 \neq \nu \in \Gamma} I(0, 1) t^\nu}.
\]
Then \( B_{[x:y]} \cong Q(q) \).

**Proof.** If \( x \neq 0 \) we choose a basis
\[
v_{\nu} = \begin{cases} 
t^\nu & \nu \neq 0, \\
1 & \nu = 0
\end{cases}
\]
for \( Q(q) \). We compute for \( \nu + \mu \neq 0, \nu \neq 0 \)
\[
e_{\mu}v_{\nu} = (a + b\mu + \nu)t^{\nu + \nu} = (\mu + \nu)t^{\nu + \nu} = (\mu + \nu)v_{\nu + \mu}
\]
and
\[
e_{\mu}v_{0} = a(a + b\mu)t^{\nu} = 0, \quad e_{\mu}v_{-\mu} = a + b\mu - \mu = \left(1 + \frac{y}{x}\mu\right)v_{0}.
\]
If \( y \neq 0 \) we pick a basis
\[
v_{\nu} = \begin{cases} 
t^\nu & \nu \neq 0, \\
\frac{1}{b} & \nu = 0
\end{cases}
\]
We get \( e_{\mu}v_{\nu} = \nu v_{\nu + \mu}, e_{\mu}v_{0} = 0, e_{\mu}v_{-\mu} = \left(\frac{x}{y} + \mu\right)v_{0}. \) \( \Box \)

4. FACTORS OF \( U(W_{\Gamma}) \)

In this section we generalise techniques from [SW15] to show that \( B = \Phi(U(W_{\Gamma})) \) is not left or right noetherian. This in particular implies that \( U(W_{\Gamma}) \) is not left or right noetherian, which was proved earlier in [SW13, SW15].

For \( 0 \neq \mu \in \Gamma \), let
\[
p_{\mu} = e_{\mu}e_{3\mu} - e_{2\mu}^2 - \mu e_{4\mu}.
\]

**Lemma 4.1.** We have \( \Phi(p_{\mu}) = \mu^2 b(1 - b) t^{4\mu} \).

**Proof.** Let us compute
\[
\Phi(e_{\mu}e_{3\mu} - e_{2\mu}^2 - \mu e_{4\mu}) = ((a + 3\mu b)(a + \mu b + 3\mu) - (a + 2\mu b)(a + 2\mu b + 2\mu) - \mu(a + 4\mu b)) t^{4\mu} = \mu^2 b(1 - b) t^{4\mu}.
\]
\( \Box \)

Fix \( 0 \neq \mu \in \Gamma \) and let \( I = B \Phi(p_{\mu})B \).

**Lemma 4.2.** For all \( \nu \in \Gamma \) we have \( b(1 - b) t^{\nu} \in I \). In particular, \( I \) does not depend on the choice of \( \mu \).

Consequently, \( I = b(1 - b) k[a, b] \rtimes \Gamma \).

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Proof. We have
\[
\Phi(e_{\nu-4\mu})b(1-b)t^{4\mu} - b(1-b)t^{4\mu}\Phi(e_{\nu-4\mu}) = (\Phi(e_{\nu-4\mu}) - \Phi(e_{\nu-4\mu}) - 4\mu)b(1-b)t^\nu = -4\mu b(1-b)t^\nu.
\]
Thus the first claim follows by Lemma 4.1. Note that \(I \subseteq b(1-b)k_{[a, b] \times \Gamma}\), and as \(b(1-b) \in I\) and \(a \in B\), we have \(b(1-b)k_{[a, b] \times \Gamma} \subseteq I\). Since also \((a + b_\mu)t^\mu \in B\), we easily obtain by induction on \(n\) that \(b(1-b)k^n_{[a, b] \times \Gamma} \subseteq I\) for all \(n \geq 0\), and thus the last claim. \(\Box\)

Proposition 4.3. The ideal \(I\) is not finitely generated as a left or right ideal of \(B\).

Proof. We first compute
\[
\begin{align*}
(4.1) \quad & (a + b\nu_1)t^\nu_1 \cdots (a + b\nu_l)t^\nu_l p(a, b)b(1-b)t^\lambda = \\
& (a + b\nu_1) \cdots (a + b\nu_l + \nu_1 + \cdots + \nu_{l-1})p(a + \nu_1 + \cdots + \nu_{l-1} + \nu_l, b)b(1-b)t^{\nu_1 + \cdots + \nu_l + \lambda}, \\
(4.2) \quad & p(a, b)b(1-b)t^\lambda (a + b\nu_1) \cdots (a + b\nu_l)t^\nu_l = \\
& p(a, b)b(1-b)(a + b\nu_1 + \lambda) \cdots (a + b\nu_l + \lambda + \nu_1 + \cdots + \nu_{l-1})t^{\nu_1 + \cdots + \nu_l}.
\end{align*}
\]

Let us assume that \(I\) is finitely generated as a right ideal in \(B\). Then there exist \(\mu_1, \ldots, \mu_k \in \Gamma\) such that \(I = B(I_{\mu_1} + \cdots + I_{\mu_k})\). Let us take \(\mu \neq \mu_i\), \(1 \leq i \leq k\). It follows from (4.1) that \((B(I_{\mu_1} + \cdots + I_{\mu_k}))_\mu\) is contained in \((a, b)b(1-b)t^\mu\), a contradiction to Lemma 4.2.

Let us assume now that \(I\) is finitely generated as a right ideal in \(B\). Then there exist \(\mu_1, \ldots, \mu_k \in \Gamma\) such that \(I = (I_{\mu_1} + \cdots + I_{\mu_k})B\). For \(\mu \neq \mu_i\), \(1 \leq i \leq k\), we obtain from (4.2) that \(((I_{\mu_1} + \cdots + I_{\mu_k})B)_\mu\) is contained in \((a + \mu, b - 1)b(1-b)t^\mu\), which again contradicts Lemma 4.2. \(\Box\)

Remark 4.4. Note that the same proof works if \(\Gamma\) is a submonoid of \(k\). Lemma 4.2 yields in this case \(b(1-b)t^{n\mu} \in I\), for \(n \geq 4\). The proof of Proposition 4.3 can then be adapted in an obvious way to apply to this a slightly more general situation. In particular, \(\Phi(U(W_+))\) is not noetherian, where \(W_+\) is the subalgebra of \(W\) generated by \(\{e_n : n \geq 1\}\). (This last statement is proved in SW15.)

We now show that the image \(B_\beta\) of the map \(\phi_\beta : U(W) \to B/(b - \beta)\) induced from \(\Phi\) is noetherian for every \(\beta \in k\). This is an analogue of SW15 Proposition 2.1].

Lemma 4.5. We have \(B_0 \cong k + a(k_{[a] \times \Gamma})\), \(B_1 \cong k + (k_{[a] \times \Gamma})a\), \(B_\beta \cong k_{[a] \times \Gamma}\) for \(\beta \neq 0, 1\).

Proof. The lemma is obvious for \(\beta = 0, 1\). Assume therefore that \(\beta \neq 0, 1\). Let us compute
\[
(a + \beta \mu_1)t^\mu(a + \beta \nu_1)t^\nu - a(a + \beta_1\mu + \nu_1)(a + \beta_2\mu_2 + \nu_2) = (\mu + \beta_1\mu_2 + \nu_2)t^{\mu_2 + \nu_2} \in B_\beta.
\]
Subtracting \(\mu(a + b(\mu + \nu))t^{\mu_2 + \nu_2}\), we thus have \(\beta \mu_2
\]

Proposition 4.6. \(B_\beta\) is noetherian for every \(\beta \in k\).

Proof. For \(\beta \neq 0, 1\) this follows by [MR01] Theorem 4.5] using Lemma 4.5. Let us note that \(B_0 \cong B_1\) by conjugation with \(a\). It thus suffices to prove that \(B_0\) is right noetherian and \(B_1\) is left noetherian. We show that \(B_0\) is right noetherian, and following the same argument one can show that \(B_1\) is left noetherian.

We first note that \(I = a(k_{[a] \times \Gamma})\) is a maximal right ideal in \(C = k_{[a] \times \Gamma}\). To see this, let \(J \neq I\) be a right ideal which contains \(I\). Take an element \(e = \sum \alpha_{i\mu}t^{i\mu} \neq 0\) in \(J\) with the minimal number of nonzero coefficients. Since \(ce = \sum \alpha_{i\mu}t^{i\mu}j \in J\) and hence \(\sum \alpha_{i\mu}t^{i\mu} \in J\), the minimality assumption implies that \(J = k_{[a] \times \Gamma}\).

The proposition now follows by Rob72 Theorem 2.2] using Lemma 4.5. \(\Box\)

Remark 4.7. We remark that for any \(\beta\) the modules \(V(\alpha, \beta)\) are all faithful over \(B_\beta\), and it follows easily that the \(B_\beta\) are primitive. In general, the primitive factors of \(U(W_\Gamma)\) are unknown, even for \(\Gamma = Z\).
References


