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REGULARITY FOR ENERGY-MINIMIZING AREA-PRESERVING DEFORMATIONS

ARAM L. KARAKHANYAN

ABSTRACT. In this paper we establish the square integrability of the nonnegative hydrostatic pressure p , that emerges in the minimization problem

$$\inf_{\mathcal{K}} \int_{\Omega} |\nabla \mathbf{v}|^2, \quad \Omega \subset \mathbb{R}^2$$

as the Lagrange multiplier corresponding to the incompressibility constraint $\det \nabla \mathbf{v} = 1$ a.e. in Ω . Our method employs the Euler-Lagrange equation for the mollified Cauchy stress $\mathbf{C}$ satisfied in the image domain $\Omega^* = \mathbf{u}(\Omega)$. This allows to construct a convex function ψ , defined in the image domain, such that the measure of the normal mapping of ψ controls the L^2 norm of the pressure. As a by-product we conclude that $\mathbf{u} \in C_{\text{loc}}^{\frac{1}{2}}(\Omega)$ if the dual pressure (introduced in [6]) is nonnegative.

1. INTRODUCTION

Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and $\mathcal{K} = \{\mathbf{v} \in W^{1,2}(\Omega, \mathbb{R}^2), \det \nabla \mathbf{v} = 1 \text{ a.e. in } \Omega\}$. For $\mathbf{v} \in \mathcal{K}$ we define the stored energy as

$$(1.1) \quad E[\mathbf{v}] = \int_{\Omega} |\nabla \mathbf{v}|^2, \quad \mathbf{v} \in \mathcal{K}.$$

Let us recall the definition of local minimizers [1], [2], [6].

Definition 1.1. We say that an area-preserving deformation $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^2)$ is a *local minimizer* if for all area preserving (or incompressible) deformations $\mathbf{w} \in W^{1,2}(\Omega, \mathbb{R}^2)$ with $\text{supp}(\mathbf{w} - \mathbf{u}) \subset \Omega$ the following holds

$$(1.2) \quad \int_{\Omega} |\nabla \mathbf{u}|^2 \leq \int_{\Omega} |\nabla \mathbf{w}|^2.$$

Our primary interest is to analyze the properties of the local minimizers of $E[\cdot]$ and the integrability of the hydrostatic pressure p sought as the Lagrange multiplier corresponding to the incompressibility constraint $\det \nabla \mathbf{v} = 1$. The sufficiently regular local minimizers solve the system

$$(1.3) \quad \begin{cases} \operatorname{div} \mathbf{T} = 0 & \text{in } \Omega, \\ \det \nabla \mathbf{u} = 1 & \text{a.e. in } \Omega, \end{cases}$$

where $\mathbf{T} = \nabla \mathbf{u} + p(\nabla \mathbf{u})^{-t}$ is the first Piola-Kirchhoff tensor and $(\nabla \mathbf{u})^{-t}$ is the transpose of the inverse matrix, see [7], pages 371 and 379. Since $\det \nabla \mathbf{u} = 1$ we have

$$(1.4) \quad (\nabla \mathbf{u})^{-1} = \begin{pmatrix} u_2^2 & -u_2^1 \\ -u_1^2 & u_1^1 \end{pmatrix}, \quad (\nabla \mathbf{u})^{-t} = \begin{pmatrix} u_2^2 & -u_1^2 \\ -u_2^1 & u_1^1 \end{pmatrix}.$$

From (1.4) we deduce that (1.3) is equivalent to the system

$$(1.5) \quad \begin{cases} \operatorname{div}[\nabla u^1 - p \mathcal{J} \nabla u^2] = 0, \\ \operatorname{div}[\nabla u^2 + p \mathcal{J} \nabla u^1] = 0, \\ \det \nabla \mathbf{u} = 1. \end{cases}$$

Here $\mathcal{J}$ is the 90° counterclockwise rotation

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$$(1.6) \quad \mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $\mathbf{u} \in W^{1,2}(\Omega)$ the equations (1.3) or (1.5) cannot be justified. In fact the term $p(\nabla \mathbf{u})^{-t}$ is not well-defined unless $\nabla \mathbf{u}$ is better than L^2 integrable, see [2]. The lack of higher integrability of $\nabla \mathbf{u}$ produces a number of technical difficulties, see [6]. To circumvent them author and N. Chaudhuri succeeded to compute the first variation of the energy (1.6) in the image domain $\Omega^* = \mathbf{u}(\Omega)$ under very weak assumptions (note that $\mathbf{u}$ is open map [10]). For $\mathbf{u} \in W^{s,l}(\Omega)$ with $s > \frac{2}{l} + 1$ this was done in [8], Theorem 5.1. Below we formulate one of the main results from [2] relevant to the present work.

Proposition 1.2. *Let $\mathbf{u} \in \mathcal{K}$ be a local minimizer of (1.1). Consider the matrix*

$$(1.7) \quad \sigma_{ij}(y) = \sum_m u_m^i(\mathbf{u}^{-1}(y)) u_m^j(\mathbf{u}^{-1}(y))$$

where $y \in \mathbf{u}(\Omega) = \Omega^*$ and $\mathbf{u}^{-1}$ is the inverse of $\mathbf{u}$ ($\mathbf{u}^{-1}$ is well-defined see Remark 3.3 [10]). If ρ_ε is a mollification kernel and $\sigma^\varepsilon = \sigma * \rho_\varepsilon$ then there is a C^∞ function q^ε such that

$$(1.8) \quad \operatorname{div} \sigma^\varepsilon(y) + \nabla q^\varepsilon(y) = 0 \quad y \in \Omega^*,$$

The regularized equation (1.8) in the image domain plays the crucial role in the proof of Theorem A (see below), notably it links (1.3) to the Monge-Ampère equation and from there we infer that $\{q^\varepsilon\}$ is uniformly bounded in $L^2_{\text{loc}}(\Omega^*)$.

Theorem A. *Let $\mathbf{u} \in \mathcal{K}$ be a local minimizer of $E[\cdot]$. If there is a sequence of $q^{\varepsilon_j} \geq 0$ solving (1.8) such that q^{ε_j} converges to a nonnegative Radon measure in $B_1 \subset \Omega^*$, then there is a convex function ψ^ε defined in B_1 such that*

$$D^2 \psi^\varepsilon = \operatorname{adj} \sigma^\varepsilon + q^\varepsilon \mathbb{I}$$

where $\operatorname{adj} \sigma^\varepsilon = (\sigma^\varepsilon)^{-1} \det \sigma^\varepsilon$ and $\mathbb{I}$ is the identity matrix. Moreover,

- there is a subsequence $q^{\varepsilon_{j(m)}}$ and $q \in L^2_{\text{loc}}(\Omega^*)$ such that $q^{\varepsilon_{j(m)}} \rightarrow q$ strongly in $L^2_{\text{loc}}(\Omega^*)$,
- there is a convex function $\psi : B_1 \mapsto \mathbb{R}$ such that $\psi^{\varepsilon_{j(m)}} \rightarrow \psi$ uniformly on the compact subsets of B_1 .

In [2] the authors found a representation for q^ε given by a sum of Calderón-Zygmund type singular integrals of $\sigma_{ij}^\varepsilon(y)$. As a result q^ε inherits the "half" of the integrability of $\nabla \mathbf{u}$. In other words $\{q^\varepsilon\}$ is uniformly bounded in $L^{1+\frac{\delta}{2}}_{\text{loc}}(\Omega^*)$ if $\nabla u \in L^{2+\delta}(\Omega)$, $\delta > 0$ and in $L^1_{\text{loc}}(\Omega^*)$ if $|\nabla \mathbf{u}|^2 \in L \log(2+L)(\Omega)$. This observation gives rise to the following question: Does the higher integrability of the pressure q translate to $\nabla \mathbf{u}$?

Theorem A gives a partial answer to this question: if $B_1 \subset \Omega^*$, $q \in L^{2+\delta}(B_1)$, $\delta > 0$ and $\sigma \in L^2(B_1)$ then it follows from Lemma 7.1 ^{1°} that $D^2 \psi = \operatorname{adj} \sigma + q \mathbb{I}$ and $D^2 \psi \in L^2(B_{\frac{7}{8}})$. Since by (1.7) $\sigma(y) = [\nabla \mathbf{u}(\nabla \mathbf{u})^t] \circ \mathbf{u}^{-1}(y)$, $y \in \Omega^*$ we infer that $\det \operatorname{adj} \sigma = 1$, which is equivalent to the Monge-Ampère equation

$$\det [D^2 \psi - q \mathbb{I}] = 1$$

satisfied a.e. in B_1 . Hence from the regularity theory available for the Monge-Ampère equation we will conclude higher integrability for $D^2 \psi$ in $B_{\frac{1}{2}}$, which translates to $\nabla \mathbf{u}$ in Ω through the equation $D^2 \psi = \operatorname{adj} \sigma + q \mathbb{I}$ and the inverse mapping theorem.

As one can observe from (1.8), the pressure q^ε is defined modulo a constant. The assumption $q^{\varepsilon_j} \geq 0$ seems a natural one since from a purely physical point of view the pressure must be nonnegative. From Theorem A

we can conclude that the first equation in (1.3) is well defined in Ω . Moreover applying the duality argument from [6] we infer that there is a function $P : \Omega^* \mapsto \mathbb{R}$ such that the pair $(\mathbf{u}^{-1}, P)$ is a solution the corresponding Euler-Lagrange equations in Ω^* , see Theorem 2 [6]. Combining Theorem A with this observation we obtain

Theorem B. *Let $\mathbf{u} : \Omega \mapsto \mathbb{R}^n$ and $q \in L^2(\Omega^*)$ be as in Theorem A.*

- 1° *Then $p(x) = q(\mathbf{u}(x))$, $x \in \Omega$ is locally L^2 integrable in Ω , $p(x)(\nabla \mathbf{u})^{-t} \in L^2_{loc}(\Omega)$ and the pair $(\mathbf{u}, p)$ solves the equation*

$$\operatorname{div}[\nabla \mathbf{u} + p(\nabla \mathbf{u})^{-t}] = 0 \quad \text{in } \Omega$$

in the weak sense.

- 2° *Let $\mathbf{v} = \mathbf{u}^{-1}$ and Q be the dual pressure in Ω corresponding to $\mathbf{v}$, $Q(\mathbf{v}(z)) = P(z)$. If $Q \geq 0$ then $\mathbf{u} \in C^{\frac{1}{2}}_{loc}(\Omega)$.*

The paper is organized as follows: Section 2 is devoted to the construction of the family of functions ψ^ε . Then we prove uniform estimates for this family using some geometric ideas and the Poincaré-Wirtinger's theorem for the functions of bounded variation (or BV -functions, see [4]). This is contained in Section 3. A lower estimate for the $\det \operatorname{adj} \sigma^\varepsilon$ is established in Section 4. Next, in order to prove Theorem A, we recall the notion of generalized solution of the Monge-Ampère equation and define the corresponding normal mapping in Section 5. The proof of Theorem A is given in Section 6. Section 7 contains a brief discussion of the properties of the convex function ψ and its Legendre-Fenchel transformation. Finally, Section 8 contains the proof of Theorem B.

2. THE EULER-LAGRANGE EQUATION IN IMAGE DOMAIN

In this section we construct a convex function ψ^ε such that the mollification of the Cauchy stress tensor $\mathbf{C}_{ij} = \sigma_{ij} + q\delta_{ij}$ is the Hessian of ψ^ε .

We start by recalling that if $\mathbf{w}$ is C^∞ divergence free vectorfield in 2D then there is a scalar C^∞ function φ such that $\mathbf{w} = \mathcal{J} D\varphi = (-D_2\varphi, D_1\varphi)$.

Suppose that $B_1 \subset \Omega^*$. From the mollified equation (1.8) it follows that the vectorfields $(\sigma_{11}^\varepsilon + q^\varepsilon, \sigma_{12}^\varepsilon)$ and $(\sigma_{21}^\varepsilon, \sigma_{22}^\varepsilon + q^\varepsilon)$ are divergence free in Ω^* . Hence there are two scalar functions $\varphi_1^\varepsilon, \varphi_2^\varepsilon$ such that $\varphi_i^\varepsilon \in C^\infty(B_1)$, $i = 1, 2$ and

$$(2.1) \quad \begin{aligned} (\sigma_{11}^\varepsilon + q^\varepsilon, \sigma_{12}^\varepsilon) &= \mathcal{J} D\varphi_1^\varepsilon = (-\partial_2\varphi_1^\varepsilon, \partial_1\varphi_1^\varepsilon), \\ (\sigma_{21}^\varepsilon, \sigma_{22}^\varepsilon + q^\varepsilon) &= \mathcal{J} D\varphi_2^\varepsilon = (-\partial_2\varphi_2^\varepsilon, \partial_1\varphi_2^\varepsilon). \end{aligned}$$

Since

$$(2.2) \quad [\sigma_{ij}(z)] = \begin{pmatrix} |\nabla u^1(\mathbf{u}^{-1}(z))|^2 & \nabla u^1(\mathbf{u}^{-1}(z)) \cdot \nabla u^2(\mathbf{u}^{-1}(z)) \\ \nabla u^1(\mathbf{u}^{-1}(z)) \cdot \nabla u^2(\mathbf{u}^{-1}(z)) & |\nabla u^2(\mathbf{u}^{-1}(z))|^2 \end{pmatrix}$$

and $\sigma_{ij}^\varepsilon = \sigma_{ij} * \rho_\varepsilon$, where ρ_ε is a mollifying kernel, we conclude that σ_{ij}^ε is symmetric. Moreover the gradient matrix of the mapping $\Phi^\varepsilon = (\varphi_1^\varepsilon, \varphi_2^\varepsilon)$ is

$$(2.3) \quad \nabla \Phi^\varepsilon = \begin{pmatrix} \partial_1\varphi_1^\varepsilon & \partial_2\varphi_1^\varepsilon \\ \partial_1\varphi_2^\varepsilon & \partial_2\varphi_2^\varepsilon \end{pmatrix} = \begin{pmatrix} \sigma_{12}^\varepsilon & -\sigma_{11}^\varepsilon - q^\varepsilon \\ \sigma_{22}^\varepsilon + q^\varepsilon & -\sigma_{21}^\varepsilon \end{pmatrix}.$$

Therefore the mapping $\Phi = (\varphi_1^\varepsilon, \varphi_2^\varepsilon)$ is divergence free, because

$$\operatorname{div} \Phi^\varepsilon = \partial_1\varphi_1^\varepsilon + \partial_2\varphi_2^\varepsilon = \sigma_{12}^\varepsilon - \sigma_{21}^\varepsilon = 0$$

and the matrix σ_{ij}^ε is symmetric.

Thus, there is a scalar function ψ^ε such that $\Phi^\varepsilon = \mathcal{J} \nabla \psi^\varepsilon$. In other words $\varphi_1^\varepsilon = -\partial_2 \psi^\varepsilon, \varphi_2^\varepsilon = \partial_1 \psi^\varepsilon$, which in view of (2.1) implies the following identity for the Hessian of ψ^ε

$$(2.4) \quad D^2 \psi^\varepsilon(y) = \begin{pmatrix} \sigma_{22}^\varepsilon(y) + q^\varepsilon(y) & -\sigma_{21}^\varepsilon(y) \\ -\sigma_{21}^\varepsilon(y) & \sigma_{11}^\varepsilon(y) + q^\varepsilon(y) \end{pmatrix}.$$

Furthermore, $\det D^2 \psi^\varepsilon = \det \operatorname{adj} \sigma^\varepsilon + (q^\varepsilon)^2 + q^\varepsilon \operatorname{Tr} \sigma^\varepsilon$ and $\det(D^2 \psi - q^\varepsilon \mathbb{I}) = \det \operatorname{adj} \sigma^\varepsilon$, where $\mathbb{I} = \delta_{ij}$ is the identity matrix.

Lemma 2.1. *If $q^\varepsilon \geq C$ for some $C \in \mathbb{R}$, independent of ε , then $\psi^\varepsilon(y) - \frac{C}{2}|y|^2$ are convex for any $\varepsilon > 0$.*

Proof: Let $e = (a, b) \in \mathbb{R}^2$ and $\partial_e = a\partial_1 + b\partial_2$. Then using (2.2) and (2.4) we conclude

$$\begin{aligned} \partial_{ee} \psi^\varepsilon(z) &= a^2 \partial_{11} \psi^\varepsilon + 2ab \partial_{12} \psi^\varepsilon + b^2 \partial_{22} \psi^\varepsilon \\ &= a^2 \sigma_{22}^\varepsilon + 2ab \sigma_{12}^\varepsilon + b^2 \sigma_{11}^\varepsilon + q^\varepsilon(z)(a^2 + b^2) \\ &= |a \nabla_x u^2(\mathbf{u}^{-1}(z)) + b \nabla_x u^1(\mathbf{u}^{-1}(z))|^2 + q^\varepsilon(z)(a^2 + b^2) \\ &\geq C(a^2 + b^2). \end{aligned}$$

Therefore $\psi(z) - \frac{C}{2}|z|^2$ is convex. $\square$

Remark 2.2. *The pressure $q^\varepsilon(z)$ is defined modulo a constant as it is seen from the equation (1.8). In particular, ψ^ε is determined modulo a quadratic polynomial. Thus if $q_0^\varepsilon(z) = q^\varepsilon(z) - C$ then $\psi_0^\varepsilon(z) = \psi^\varepsilon(z) - \frac{C}{2}|z|^2$ solves $\det(D^2 \psi_0^\varepsilon - q_0^\varepsilon(z) \mathbb{I}) = \det \operatorname{adj} \sigma^\varepsilon$ and (2.4) holds with ψ^ε and q^ε replaced by ψ_0^ε and q_0^ε respectively.*

3. UNIFORM ESTIMATES FOR ψ^ε

Lemma 3.1. *Suppose that the sequence q^ε converges to a nonnegative Radon measure q . Then there is a positive constant C such that $\sup_{\partial B_1} |\psi^\varepsilon| \leq C$.*

Proof: By Helmholtz-Weyl decomposition [3], $\Phi^\varepsilon = Dh^\varepsilon + \mathcal{J} D\eta^\varepsilon$ where h^ε solves the Neumann problem

$$(3.1) \quad \begin{cases} \Delta h^\varepsilon = 0 & \text{in } B_1, \\ Dh^\varepsilon \cdot \nu = \Phi^\varepsilon \cdot \nu & \text{on } \partial B_1. \end{cases}$$

Moreover $-\Delta \eta^\varepsilon = \operatorname{curl} \Phi^\varepsilon = \sigma_{11}^\varepsilon + \sigma_{22}^\varepsilon + 2q^\varepsilon$ and $\eta^\varepsilon = 0$ on ∂B_1 .

By Poincaré-Wirtinger's theorem $\tilde{\Phi}^\varepsilon = \Phi^\varepsilon - \int_{B_1} \Phi^\varepsilon \in BV(B_1, \mathbb{R}^2)$, i.e. $\varphi_i^\varepsilon - \int_{B_1} \varphi_i^\varepsilon \in BV(B_1), i = 1, 2$. Since Φ^ε is defined modulo a constant (see (2.3)), in what follows, we take $\tilde{\Phi}^\varepsilon = \Phi^\varepsilon - \int_{B_1} \Phi^\varepsilon$. Thus the estimate

$$(3.2) \quad \|\tilde{\Phi}^\varepsilon\|_{L^1(B_1)} = \left\| \Phi^\varepsilon - \int_{B_1} \Phi^\varepsilon \right\|_{L^1(B_1)} \leq C \sup \left\{ \left| \int_{B_1} \Phi^\varepsilon \operatorname{div} \xi \right|, \forall \xi \in C_0^1(B_1, \mathbb{R}^2), |\xi| \leq 1 \right\}$$

is true, with $C > 0$ independent from ε .

On the other hand after integration by parts we get

$$(3.3) \quad \int_{B_1} \tilde{\Phi}^\varepsilon \operatorname{div} \xi = \int_{B_1} \Phi^\varepsilon \operatorname{div} \xi = - \int_{B_1} \xi \nabla \Phi^\varepsilon$$

for any $\xi \in C_0^1(B_1, \mathbb{R}^2)$ which in conjunction with (2.3) gives

$$\begin{aligned}
(3.4) \quad \left| \int_{B_1} \varphi_1^\varepsilon \operatorname{div} \xi \right| &= \left| - \int_{B_1} \xi D \varphi_1^\varepsilon \right| \\
&= \left| \int_{B_1} \xi^1 \sigma_{12}^\varepsilon - \xi^2 (\sigma_{11}^\varepsilon + q^\varepsilon) \right| \\
&\leq \int_{B_1} [|\sigma_{11}^\varepsilon| + |\sigma_{12}^\varepsilon| + q^\varepsilon].
\end{aligned}$$

Similarly, one can check that $\left| \int_{B_1} \varphi_2^\varepsilon \operatorname{div} \xi \right| \leq \int_{B_1} [|\sigma_{12}^\varepsilon| + |\sigma_{22}^\varepsilon| + q^\varepsilon]$. Because $\sigma_{ij} \in L^1$ and q^ε converges to a nonnegative Radon measure it follows that

$$\|\tilde{\Phi}^\varepsilon\|_{BV(B_1)} \leq C (\|\sigma_{ij}\|_{L^1(B_1)} + \|q\|_{\mathcal{M}(B_1)}),$$

where $\mathcal{M}(B_1)$ is the space of measures in B_1 .

Using Theorems 2.10 and 2.11 from [4] we conclude that the trace $\Phi_0^\varepsilon \in L^1(\partial B_1)$ of $\tilde{\Phi}^\varepsilon$ is well-defined and satisfies the following uniform estimate

$$(3.5) \quad \|\tilde{\Phi}_0^\varepsilon\|_{L^1(\partial B_1)} \leq C \|\tilde{\Phi}^\varepsilon\|_{BV(B_1)} \leq C (\|\sigma_{ij}\|_{L^1(B_1)} + \|q\|_{\mathcal{M}(B_1)}).$$

In particular (3.5) implies that the Neumann problem (3.1) for h^ε is well-defined.

Next we have that $\Phi^\varepsilon = \mathcal{J} \nabla \psi^\varepsilon = \nabla h^\varepsilon + \mathcal{J} \nabla \eta^\varepsilon$ or equivalently

$$\nabla \psi^\varepsilon - \nabla \eta^\varepsilon = -\mathcal{J} \nabla h^\varepsilon.$$

In particular $\psi^\varepsilon - \eta^\varepsilon$ is harmonic in B_1 . We want to estimate the tangential component of $\nabla \psi^\varepsilon$ on the boundary ∂B_1 . Let τ be a unit tangent vector to ∂B_1 , then

$$\nabla \psi^\varepsilon \cdot \tau = \nabla \eta^\varepsilon \cdot \tau - \mathcal{J} \nabla h^\varepsilon \cdot \tau = \nabla h^\varepsilon \cdot \nu,$$

where $\nu = \mathcal{J} \tau$ is a unit vector normal to ∂B_1 . Using polar coordinates (r, θ) , $\theta \in (0, 2\pi)$, we obtain that

$$(3.6) \quad \psi^\varepsilon(\theta) = \psi^\varepsilon(0) + \int_0^\theta \nabla h \cdot \nu d\theta = \psi^\varepsilon(0) + \int_0^\theta \Phi_0^\varepsilon \cdot \nu d\theta.$$

Without loss of generality we assume that $\psi^\varepsilon(0) = 0$ (see Remark 2.2). Thus

$$|\psi^\varepsilon(\theta)| \leq C \|\Phi_0^\varepsilon\|_{L^1(\partial B_1)}, \quad \forall \theta \in (0, 2\pi).$$

The desired result now follows from (3.5). □

Lemma 3.2. *Retain the assumptions of previous lemma. Then there is a constant C , such that $\inf_{B_1} \psi^\varepsilon \geq C$ uniformly in ε .*

Proof: It suffices to prove that $\nabla \psi^\varepsilon \in L^1(\partial B_1)$ uniformly in ε . Indeed, ψ^ε is convex hence if ψ^ε tends to $-\infty$ then the $\nabla \psi^\varepsilon$ becomes uniformly large on ∂B_1 .

From Lemma 3.5 we have that

$$\nabla \psi^\varepsilon = \nabla \eta^\varepsilon - \mathcal{J} \nabla h^\varepsilon = \mathcal{J} (-\mathcal{J} \nabla \eta^\varepsilon - \nabla h^\varepsilon) = -\mathcal{J} \tilde{\Phi}^\varepsilon$$

implying the estimate

$$\|\nabla \psi^\varepsilon\|_{L^1(\partial B_1)} \leq \|\tilde{\Phi}_0^\varepsilon\|_{L^1(\partial B_1)}.$$

The proof now follows if we recall (3.5). □

4. LOWER ESTIMATE FOR $\det(\operatorname{adj} \sigma^\varepsilon)$

Lemma 4.1. *Let $\sigma^\varepsilon = \sigma * \rho_\varepsilon$, where $\sigma(z) = [\nabla \mathbf{u}(\nabla \mathbf{u})^t] \circ \mathbf{u}^{-1}(z)$, $z \in \Omega^\star$ then for any $\varepsilon > 0$*

$$\det(\operatorname{adj} \sigma^\varepsilon(z)) \geq 1 \quad z \in \Omega^\star.$$

Proof: Using the definition of $\sigma^\varepsilon(z)$ and the Cauchy-Schwarz inequality we get

$$\begin{aligned} \det(\operatorname{adj} \sigma^\varepsilon) &= \sigma_{11}^\varepsilon \sigma_{22}^\varepsilon - \sigma_{12}^\varepsilon \sigma_{21}^\varepsilon \\ &= \int_{B_1} \sigma_{11} \rho_\varepsilon \int_{B_1} \sigma_{22} \rho_\varepsilon - \left(\int_{B_1} \sigma_{12} \rho_\varepsilon \right)^2 \\ &\geq \left(\int_{B_1} \sqrt{\sigma_{11} \sigma_{22}} \rho_\varepsilon \right)^2 - \left(\int_{B_1} \sigma_{12} \rho_\varepsilon \right)^2 \\ &= \int_{B_1} (\sqrt{\sigma_{11} \sigma_{22}} - \sigma_{12}) \rho_\varepsilon \int_{B_1} (\sqrt{\sigma_{11} \sigma_{22}} + \sigma_{12}) \rho_\varepsilon. \end{aligned}$$

By definition we have $\sigma_{11} = |\nabla u^1|^2$, $\sigma_{22} = |\nabla u^2|^2$ and $\sigma_{12} = \sigma_{21} = \nabla u^1 \cdot \nabla u^2$. Let α be the angle between ∇u^1 and ∇u^2 . Recall that $\det \nabla \mathbf{u} = |\nabla u^1| |\nabla u^2| \sin \alpha = 1$. Then

$$\sqrt{\sigma_{11} \sigma_{22}} - \sigma_{12} = |\nabla u^1| |\nabla u^2| (1 - \cos \alpha) = |\nabla u^1| |\nabla u^2| 2 \sin^2 \frac{\alpha}{2} = \tan \frac{\alpha}{2}$$

and similarly have that

$$\sqrt{\sigma_{11} \sigma_{22}} + \sigma_{12} = |\nabla u^1| |\nabla u^2| (1 + \cos \alpha) = |\nabla u^1| |\nabla u^2| 2 \cos^2 \frac{\alpha}{2} = \cot \frac{\alpha}{2}.$$

Applying the Cauchy-Schwarz inequality one more time we obtain

$$\det(\operatorname{adj} \sigma^\varepsilon) \geq 1.$$

□

5. NORMAL MAPPING OF THE CONVEX FUNCTION ψ^ε

In this section we will employ some basic concepts from the theory of generalized solutions of Monge-Ampère equation. Our notation follow that of the paper [11]. Let ψ be a convex function defined in $B_1 \subset \mathbb{R}^2$. For $x \in B_1$ we let

$$\chi_\psi(x) = \{\xi \in \mathbb{R}^2 : \psi(y) \geq \psi(x) + \xi \cdot (y - x) \quad \forall y \in B_1\}.$$

For a set $E \subset B_1$ we define the mapping

$$(5.1) \quad \chi_\psi(E) = \bigcup_{x \in E} \chi_\psi(x).$$

χ_ψ is called the normal mapping of ψ . For smooth convex ψ , χ_ψ coincides with the gradient mapping of ψ .

Let

$$\mathcal{C} = \{E \subset B_1 : \chi_\psi(E) \text{ is Lebesgue measurable}\}.$$

Then $\mathcal{C}$ is a σ -algebra containing the Borel subsets of B_1 , see [11]. For each $E \in \mathcal{C}$ we define the set function

$$\omega(E) = |\chi_\psi(E)|$$

i.e. the Lebesgue measure of the normal mapping of E . It is easy to verify that for $\psi \in C^2(B_1)$ we have

$$\omega(E) = \int_E \det D^2 \psi, \quad \text{for all Borel } E \in B_1.$$

It follows from Aleksandrov's theorem, see [11], that

$$|\{\xi \in \mathbb{R}^2 : \xi \in \chi_\psi(x) \cap \chi_\psi(y), \text{ for } x \neq y, x, y \in B_1\}| = 0.$$

As a consequence, we get that ω is countably additive Radon measure.

Moreover, we have weak convergence for measure ω . Indeed, let ψ_j be a sequence of convex functions and $\psi_j \rightarrow \psi$ uniformly on compact subsets of B_1 . Let ω_j and ω be the Radon measures associated with ψ_j and ψ respectively. Then ω_j converges weakly on B_1 to ω in the space of measures $\mathcal{M}(B_1)$ [11], i.e.

$$(5.2) \quad \limsup_{j \rightarrow \infty} \omega_j(K) \leq \omega(K)$$

for any compact set $K \subset B_1$, and

$$(5.3) \quad \liminf_{j \rightarrow \infty} \omega_j(U) \geq \omega(U)$$

for any open set $U \subset B_1$.

6. PROOF OF THEOREM A

Let ω_j be the Radon measure corresponding to ψ^{ε_j} , for some sequence $\{\varepsilon_j\}$. By Lemmas 3.1 and 3.2 the sequence of convex functions $\{\psi^{\varepsilon_j}\}$ is uniformly bounded in B_1 . Thus for a subsequence, again denoted by $\{\psi^{\varepsilon_j}\}$ we have $\psi^{\varepsilon_j} \rightarrow \psi$ uniformly on the compact subsets of B_1 . Clearly ψ is convex. Let ω be the Radon measure corresponding to ψ . By Lemma 4.1 we have that

$$(6.1) \quad \begin{aligned} \omega_j(B_r(x_0)) &= \int_{B_r(x_0)} \det D^2 \psi^{\varepsilon_j} \\ &= \int_{B_r(x_0)} \det(\text{adj } \sigma^{\varepsilon_j}(z)) + q^{\varepsilon_j}(z) [|\nabla \mathbf{u}(\mathbf{u}^{-1}(z))|^2 * \rho_{\varepsilon_j}] + (q^{\varepsilon_j}(z))^2 dz \\ &\geq |B_r(x_0)| + \int_{B_r(x_0)} (q^{\varepsilon_j}(z))^2 dz \end{aligned}$$

for any open ball $B_r(x_0) \subset B_1$.

Now utilizing the weak convergence of the measures $\omega_j \rightarrow \omega$ and (5.2) we obtain the following uniform

$$\int_K (q^{\varepsilon_j}(z))^2 dz \leq C + \omega(K)$$

for any compact set $K \subset B_1$. Then a customary compactness argument in L^2 finishes the proof. $\square$

7. PROPERTIES OF ψ

The convex function ψ enjoys a number of remarkable properties which are summarized in the following

Lemma 7.1. *Let ψ be as in Theorem A. Then*

- 1° ψ is strictly convex and $\psi \in W_{\text{loc}}^{2,1}(B_1)$,
- 2° $\psi^* \in C^{1,1}$ where ψ^* is the Legendre-Fenchel transformation of ψ in $B_{\frac{1}{2}}$.

Proof: 1° Recall that q^ε is defined modulo a constant summand, see Remark 2.2. Thus, without loss of generality, we assume that $q^\varepsilon \geq 1$. Let y_0 be an arbitrary point in B_1 , then by Lemma 4.1 $\det D^2 \psi^\varepsilon \geq (q^\varepsilon)^2 \geq 1$. Thus we conclude that

$$\omega_j(U) \geq |U|, \quad \forall \text{ open } U \subset B_1.$$

Since $\omega_j \rightharpoonup \omega$ weakly and in view of (5.3) the above inequality implies

$$\omega(U) \geq |U|.$$

Now the strict convexity of ψ follows from Aleksandrov's theorem, see [9], Chapter 2.3 Theorem 2.

The mollified matrices $\sigma_{km}^{\varepsilon_j} \rightarrow \sigma_{km}$ strongly in $L_{\text{loc}}^1(B_1)$ as $\varepsilon_j \downarrow 0$ and $q^{\varepsilon_j} \rightarrow q$ in L_{loc}^2 at least for a subsequence. Moreover $\{\psi^{\varepsilon_j}\}$ is uniformly bounded thanks to Lemmas 3.1 and 3.2, hence for a suitable subsequence ψ^{ε_j} will uniformly converge to a convex function ψ in any compact subset of B_1 . Let us show that $D^2\psi = \text{adj}\sigma + q\mathbb{I}$ a.e in B_1 .

Indeed, let $\eta \in C_0^\infty(B_1)$ and compute

$$\begin{aligned} \int \partial_k \psi \partial_i \eta &= \int \partial_k \psi^{\varepsilon_j} \partial_i \eta + o(1) \\ &= - \int \partial_{ik} \psi^{\varepsilon_j} \eta + o(1) \\ &= - \int [(\text{adj}\sigma^{\varepsilon_j})_{ik} + q^{\varepsilon_j} \delta_{ik}] \eta + o(1) \\ &\longrightarrow - \int [(\text{adj}\sigma)_{ik} + q \delta_{ik}] \eta. \end{aligned}$$

Hence ψ has generalized second order derivatives in $L_{\text{loc}}^1(B_1)$ and $D^2\psi = \text{adj}\sigma + q\mathbb{I}$ a.e in B_1 .

2° Recall that the Legendre-Fenchel transformation ψ^* of ψ in $B_{\frac{1}{2}}$ is given by

$$\psi^*(z) = \sup_{y \in B_{\frac{1}{2}}} (z \cdot y - \psi(y)), \quad z \in \chi_\psi(B_{\frac{1}{2}}).$$

Notice that by part **1°** ψ is strictly convex, hence it can be shown that ψ^* is C^1 in the domain of ψ^* , see Chapter D of [5].

Let us denote $B = B_{\frac{1}{2}}$ and $B^* = \chi_\psi(B)$ where χ_ψ is the normal mapping of ψ . Notice that B^* is bounded because $\psi \in C^{0,1}(\overline{B_{\frac{1}{2}}})$. Denote $(B^\varepsilon)^* = \chi_{\psi^\varepsilon}(B)$, then $(\psi^\varepsilon)^*(z), z \in (B^\varepsilon)^*$ is smooth because ψ^ε is C^∞ . Furthermore from (2.4) we obtain

$$D^2(\psi^\varepsilon)^* = [D^2\psi^\varepsilon]^{-1} = \frac{1}{\det D^2\psi^\varepsilon} (\sigma^\varepsilon + q\mathbb{I})$$

or equivalently

$$\begin{aligned} \partial_{ij}(\psi^\varepsilon)^* &= \frac{\sigma_{ij}^\varepsilon + q\delta_{ij}}{\det \text{adj}\sigma + q^\varepsilon \text{Tr}\sigma^\varepsilon + (q^\varepsilon)^2} \\ &\leq \frac{1}{q^\varepsilon} \frac{\sigma_{ij}^\varepsilon + q\delta_{ij}}{\frac{1}{q^\varepsilon} + \text{Tr}\sigma^\varepsilon + q^\varepsilon} \\ &\leq \frac{1}{q^\varepsilon} \leq 1, \quad i = j \end{aligned}$$

if we assume that $q^\varepsilon \geq 1$, see Remark 2.2.

As for $i \neq j$, we use Lemma 4.1 to conclude

$$|\sigma_{12}^\varepsilon| \leq \sqrt{\sigma_{11}^\varepsilon \sigma_{22}^\varepsilon - 1} \leq \sqrt{\sigma_{11}^\varepsilon \sigma_{22}^\varepsilon} + 1 \leq \frac{\sigma_{11}^\varepsilon + \sigma_{22}^\varepsilon}{2} + 1.$$

Thus $|D^2(\psi^\varepsilon)^*| \leq C$ uniformly in ε .

Next, we extend $(\psi^\varepsilon)^*$ to B_R by the formula $\sup_{z \in B_R} (y \cdot z - \psi^\varepsilon(y))$ with $z \in B_R$ and $R = \sup_{\varepsilon} \|\nabla \psi^\varepsilon\|_{L^\infty(B_{\frac{1}{2}})}$. Thus in B_R we have a sequence of convex functions $(\psi^\varepsilon)^*$ with uniformly bounded Hessian matrices. By a

customary compactness argument we can show that for at least a subsequence we have $(\psi^{\varepsilon_j})^* \rightarrow \bar{\psi}$ for some convex function $\bar{\psi}$. It remains to show that $\psi^* = \bar{\psi}$ in B^* .

From the definition of $(\psi^\varepsilon)^*$ we have that $(\psi^\varepsilon)^*(z) + \psi^\varepsilon(y) \geq y \cdot z$ and after passing to the limit we obtain $\bar{\psi}(z) + \psi(y) \geq y \cdot z$ implying that $\bar{\psi}(z) \geq \psi^*(z)$. To get the converse inequality we use the uniform convergence

$$\bar{\psi}(z) \leftarrow (\psi^\varepsilon)^*(z) = \sup_{y \in B} (y \cdot z - \psi^\varepsilon(y)) \leq \sup_{y \in B} (y \cdot z - \psi(y)) + \sup_{y \in B} |\psi(y) - \psi^\varepsilon(y)| \rightarrow \psi^*(z).$$

This completes the proof. $\square$

Remark 7.2. At each point $z \in \text{int}B^*$, $B^* = \chi_\psi(B_{\frac{1}{2}})$ we can define the lower Gauss curvature [9]

$$\underline{\omega}^*(z_0) = \liminf_{r \rightarrow 0} \frac{|\chi_{\psi^*}(B_r(z_0))|}{|B_r(z_0)|}.$$

If there is a constant $m > 0$ such that $\underline{\omega}^*(z_0) \geq m > 0$ for a.e. $z_0 \in B^*$ then σ and q are bounded in $B_{\frac{1}{2}}$. In particular this will imply that $\mathbf{u}$ is Lipschitz in $\mathbf{u}^{-1}(B_{\frac{1}{2}}) \subset \Omega$.

8. PROOF OF THEOREM B

The part 1° follows from change of variable formula [10] and Theorem A. To prove part 2° we employ the duality principle of $\mathbf{u}$ and its inverse $\mathbf{v} = \mathbf{u}^{-1}$ in [6], i.e. $\mathbf{v}$ is a local minimizer of the dual problem in the image domain $\Omega^* = \mathbf{u}(\Omega)$. Hence we can apply Theorem A to the pair $(\mathbf{v}, P)$ where $\mathbf{v} = \mathbf{u}^{-1}$. Thus, there is a convex function η^ε such that $D^2\eta^\varepsilon = \text{adj}\tilde{\sigma}^\varepsilon + Q^\varepsilon \mathbb{I}$ where

$$\tilde{\sigma}_{ij}(z) = \sum_m v_m^i(\mathbf{v}^{-1}(z)) v_m^j(\mathbf{v}^{-1}(z)), \quad z \in \Omega$$

and $\tilde{\sigma}^\varepsilon = \tilde{\sigma} * \rho_\varepsilon$ and Q^ε are the mollifications of $\tilde{\sigma}$ and Q respectively. Note that $Q(\mathbf{v}(z)) = P(z)$, $z \in \Omega$. In particular, for any $B_r(x_0) \subset B_1 \subset \Omega$ we have

$$\begin{aligned} \int_{B_r(x_0)} |\nabla \mathbf{u}(x)|^2 dx &= \int_{B_r(x_0)} \text{Tr} \tilde{\sigma}_{ij}(x) dx \\ &= \int_{B_r(x_0)} \Delta \eta^\varepsilon - 2Q^\varepsilon \\ &\leq \int_{B_r(x_0)} \Delta \eta^\varepsilon \\ &= \int_{\partial B_r(x_0)} \nabla \eta^\varepsilon \cdot \nu \\ &\leq Cr \end{aligned}$$

with some tame constant C depending on the Lipschitz norms of η^ε , which is bounded by Lemmas 3.2 and 3.1. Now the result follows from Morrey's estimate. $\square$

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