THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Boundary behaviour for a singular perturbation problem

Citation for published version:

Karakhanyan, A & Shahgholian, H 2016, 'Boundary behaviour for a singular perturbation problem', *Nonlinear Analysis: Theory, Methods and Applications*, vol. 138, pp. 176-188.
<https://doi.org/10.1016/j.na.2015.12.024>

Digital Object Identifier (DOI):

[10.1016/j.na.2015.12.024](https://doi.org/10.1016/j.na.2015.12.024)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Peer reviewed version

Published In:

Nonlinear Analysis: Theory, Methods and Applications

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/281275934>

Boundary behaviour for a singular perturbation problem

Article in *Nonlinear Analysis* · December 2016

DOI: 10.1016/j.na.2015.12.024

CITATIONS

0

READS

178

2 authors:

[Aram Karakhanyan](#)

The University of Edinburgh

38 PUBLICATIONS 103 CITATIONS

[SEE PROFILE](#)

[Henrik Shahgholian](#)

KTH Royal Institute of Technology

121 PUBLICATIONS 1,037 CITATIONS

[SEE PROFILE](#)

All content following this page was uploaded by [Henrik Shahgholian](#) on 26 August 2015.

The user has requested enhancement of the downloaded file. All in-text references [underlined in blue](#) are added to the original document and are linked to publications on ResearchGate, letting you access and read them immediately.

BOUNDARY BEHAVIOR FOR A SINGULAR PERTURBATION PROBLEM

A. L. KARAKHANYAN AND H. SHAHGOLIAN

ABSTRACT. In this paper we study the boundary behaviour of the family of solutions $\{u^\varepsilon\}$ to singular perturbation problem $\Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon)$, $|u^\varepsilon| \leq 1$ in $B_1^+ = \{x_n > 0\} \cap \{|x| < 1\}$, where a smooth boundary data f is prescribed on the flat portion of ∂B_1^+ . Here $\beta_\varepsilon(\cdot) = \frac{1}{\varepsilon} \beta(\frac{\cdot}{\varepsilon})$, $\beta \in C_0^\infty(0, 1)$, $\beta \geq 0$, $\int_0^1 \beta(t) dt = M > 0$ is an approximation of identity. If $\nabla f(z) = 0$ whenever $f(z) = 0$ then the level sets $\partial\{u^\varepsilon > 0\}$ approach the fixed boundary in tangential fashion with uniform speed. The methods we employ here uses delicate analysis of local solutions, along with elaborated version of the so-called monotonicity formulas and classification of global profiles.

To Juan-Luis Vazquez on the occasion of his 70th Birthdate.

1. INTRODUCTION

In this paper we study the boundary behaviour of the family of solutions $\{u^\varepsilon\}$ to singular perturbation problem

$$(1.1) \quad \begin{cases} \Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon), & \text{in } B_1^+, \\ u^\varepsilon = f, & \text{on } B_1', \\ |u^\varepsilon| \leq 1, & \text{in } B_1^+, \end{cases}$$

in the half unit ball $B_1^+ = \{x_n > 0\} \cap \{|x| < 1\}$, where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $B_1' = B_1 \cap \{x_n = 0\}$. The perturbed right hand side β_ε , satisfies certain conditions that are specified below. Also, the boundary data f is a smooth function satisfying the following condition (specially on the flat portion of

2000 *Mathematics Subject Classification.* Primary 35R35. *Key words and phrases.* Free boundary problem, regularity, contact points.

H. Shahgholian was partially supported by the Swedish Research Council.

the boundary)

$$(1.2) \quad \nabla f(z) = 0 \text{ whenever } f(z) = 0.$$

Under these conditions we show that close to a "touching" point between the free and the fixed boundary, the free boundary touches the fixed one in a uniformly tangential fashion. Here free boundary refers to the zero level surface of our solution, $\partial\{u^\varepsilon > 0\}$.

Our analysis is based on utilization of the monotonicity formula and classification of global/blow-up solutions. The analogous problem for minimisers of the functional

$$J(u) = \int_{B_1^+} |\nabla u|^2 + \lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u\leq 0\}}$$

is studied in [7], where $\lambda_+^2 - \lambda_-^2 > 0$.

Problem (1.1) appears in the mathematical theory of combustion as a model with high activation energy, which is of order $\frac{1}{\varepsilon}$, in an ε -strip approximation of the flame, see [9] Chapter 4.3. The family $\{\beta_\varepsilon(\cdot)\}$ renders such approximation (see (1.3) below). Also, for more recent mathematical treatment see [2, 3, 4] and references therein.

Problem set-up and Standing Assumption:

To fix the ideas we suppose that

$$(1.3) \quad \beta_\varepsilon(\cdot) = \frac{1}{\varepsilon} \beta\left(\frac{\cdot}{\varepsilon}\right), \quad \beta \in C_0^\infty(0, 1), \quad \beta \geq 0, \quad \int_0^1 \beta(t) dt = M > 0.$$

Observe that by definition of $\beta_\varepsilon(t)$ we have

$$\int_0^\varepsilon \beta_\varepsilon(t) dt = \int_0^1 \beta(t) dt = M > 0.$$

The limit function, obtained as $\varepsilon \rightarrow 0$ solves locally the following free boundary problem

$$(1.4) \quad \begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cup \{u < 0\}, \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 2M & \text{on } \partial\{u > 0\}. \end{cases}$$

in a very weak sense, see [2], [3, 4].

Let f be a smooth function on $\{x_n = 0\} \cap B_1$ such that (1.2) is satisfied. It is known that under (1.2) the family $\{u^\varepsilon\}$ is uniformly bounded in Lipschitz

norm [6]

$$(1.5) \quad \sup_{x \in B_{1/2}^+} |\nabla u^\varepsilon(x)| \leq L,$$

with a positive constant $L > 0$, which is independent of ε for any solution of (1.1).

Assumptions (1.3) are standard (see [2], [6]), however one can relax the assumption $\beta \in C_0^\infty(0, 1)$ to $\beta \in C_0^{0,1}(0, 1)$ in the proof of the Lipschitz norm estimate (1.5).

Non-degeneracy: Throughout the paper we shall assume a linear non-degeneracy at the origin, standard for such problems, which is

$$(1.6) \quad \int_{B_r^+} u \geq C_0 r^{n+1},$$

for a universal C_0 .

Remark 1.1. *If large enough negative and positive phases are present then one can prove that u^+ is non-degenerate. Namely, let $x_0 \in \partial\{u > 0\}$, if there is a unit vector e , such that*

$$(1.7) \quad \begin{aligned} \liminf_{r \rightarrow 0} \frac{|\{u > 0\} \cap \{(x - x_0) \cdot e > 0\} \cap B_r(x_0)|}{|B_r|} &= \alpha_1 \\ \liminf_{r \rightarrow 0} \frac{|\{u < 0\} \cap \{(x - x_0) \cdot e < 0\} \cap B_r(x_0)|}{|B_r|} &= \alpha_2 \end{aligned}$$

with $\alpha_1 + \alpha_2 > \frac{1}{2}$ then there exists a tame constant $C > 0$ such that $\sup_{B_r(x_0)} u^\varepsilon \geq Cr$ [4] Theorem 6.3.

Our main result is the following theorem, stating tangential touch between the free and fixed boundary.

Theorem 1. *Let u^ε be a solution to our problem (1.1), satisfying non-degeneracy (1.6), and suppose f, β , satisfy the assumptions above. Then, there are $\varepsilon_0 > 0$, a radius $r_0 > 0$, and a modulus of continuity $\sigma(x)$ depending on f, n, M and L such that*

$$\partial\{u^\varepsilon > 0\} \cap B_{r_0}^+ \subset \{x_n < \sigma(|x|)|x|\} \quad \forall \varepsilon \in (0, \varepsilon_0).$$

It seems plausible that when $f \equiv 0$, one should obtain stronger result, such as the free boundary is locally a $C^{1,\alpha}$ -graph, close to touching points. Such a result needs more careful analysis of the local problem, and techniques will depend strongly on the choice of β .

2. TECHNICALITIES

In this section we gather a few standard results that are needed for our analysis of problem (1.1).

Proposition 2.1. *Let $v_j = \frac{u^{\varepsilon_j}(r_j x)}{r_j}$, with u^{ε_j} being a solution of (1.1). Then, after passing to a subsequence, there exists v so that*

- (i) $v_j \rightarrow v$ uniformly on compact subsets of $\mathbb{R}_+^n$ and in $C^{0,\alpha}(\overline{B_R^+})$, $0 < \alpha < 1$, for each $R > 0$,
- (ii) for each R , $v_j \rightharpoonup v$ weakly in $W^{1,2}(B_R^+)$,
- (iii) $\nabla v_j(x) \rightarrow \nabla v(x)$ for a.e. x .

Next we introduce the Alt-Caffarelli-Friedman monotonicity formulae and state its properties [1].

Lemma 2.2. *Let h_1, h_2 be two non-negative continuous sub-solutions of $\Delta u = 0$ in $B(x^0, R)$ ($R > 0$). Assume further that $h_1 h_2 = 0$ and that $h_1(x^0) = h_2(x^0) = 0$, and set (for $0 < r < R$)*

$$\varphi(r) = \varphi(r, h_1, h_2, x^0) = \frac{1}{r^4} \left(\int_{B(x^0, r)} \frac{|\nabla h_1|^2 dx}{|x - x^0|^{n-2}} \right) \left(\int_{B(x^0, r)} \frac{|\nabla h_2|^2 dx}{|x - x^0|^{n-2}} \right).$$

Then

$$(2.1) \quad \frac{d}{dr} \varphi(r) \geq \frac{2\varphi(r)}{r} A_r,$$

where $A_r > 0$ is given by (see [1] Lemmas 2.2-2.3)

$$(2.2) \quad \sqrt{A_r} = \frac{C_n}{r^{n-1}} \text{Area}(\partial B_r \setminus (\text{supp } h_1 \cup \text{supp } h_2)).$$

Lemma 2.3. *Let w be a solution of $\Delta w = \beta(w)$ in $\mathbb{R}_+^n$ such that $|\nabla w(x)| \leq L$, in $\mathbb{R}_+^n$, and $w = 0$ on $\{x_n = 0\}$. Then either $w \equiv 0$ or $w > 0$.*

Proof. Let $h_1 = \max(w, 0)$ and $h_2 = -\min(w, 0)$, then h_1, h_2 are nonnegative subharmonic functions. Then applying Lemma 2.2 we see that for $\sqrt{A_r}$ (2.2) we have the lower bound

$$\sqrt{A_r} \geq \frac{C_n}{r^{n-1}} \frac{n}{2} |B_1| r^{n-1} \geq C_n > 0.$$

Thus $\varphi(r) \geq \varphi(r_0) \left(\frac{r}{r_0}\right)^{C_n} \rightarrow \infty$ if $r \rightarrow \infty$ for fixed $r_0 > 0$. On the other hand from $|\nabla v(x)| \leq L$, in $\mathbb{R}_+^n$, it follows that $\varphi(r) \leq (n|B_1|)^2 L^4$. Thus either $h_2 \equiv 0$ or $h_1 \equiv 0$. The latter cannot be true unless $v \equiv 0$ since v is subharmonic in $\mathbb{R}_+^n$. Thus $h_2 \equiv 0$ yielding $v \geq 0$. To finish the proof we have to show that if v is not a trivial solution then $v > 0$. Because $\beta \geq 0$ and $\beta \in C_0^\infty(0,1)$ it follows that there is $K > 0$ large such that $Ks - \beta(s) \geq 0$ for any $s \geq 0$. Indeed, one can take the concave envelope of the graph of β and choose $K > 0$ to be the largest slope of the supporting lines. Thus we have

$$\Delta v - Kv = \beta(v) - Kv \leq 0$$

and by applying the strong maximum principle ([5] Theorem 3.5) it follows that $v > 0$ in $\mathbb{R}_+^n$. $\square$

An important tool to be used in the classification of global solutions is Weiss' monotonicity formula, which is based on a Pohozaev type identity.

Lemma 2.4. *Let v be a solution of $\Delta v = \beta_\varepsilon(v)$. Then for any $\psi \in C_0^{0,1}(\mathbb{R}^n, \mathbb{R}^n)$, $\psi(x) = (\psi^1(x), \dots, \psi^n(x))$ there holds*

$$\begin{aligned} \int_{\mathbb{R}_+^n} v_i v_j \psi_j^i &= \int_{\mathbb{R}_+^n} \operatorname{div} \psi \left[\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right] \\ &\quad + \int_{\{x_n=0\}} (\nabla v \cdot \nu) \nabla v \cdot \psi - \int_{\{x_n=0\}} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) \psi \cdot \nu \end{aligned}$$

where $\mathcal{B}_\varepsilon(t) = \int_0^t \beta_\varepsilon(s) ds$ and $v_i(x) = \partial_{x_j} v(x)$, $\psi_i^j(x) = \partial_{x_i} \psi_j^j$.

Proof. Let us fix a Lipschitz continuous $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\operatorname{supp} \psi \subset B_r$ for some $r > 0$, then we have from the divergence theorem

$$\begin{aligned} 2 \int_{B_r^+} v_i v_j \psi_j^i &= 2 \int_{\partial B_r^+} (\nabla v \cdot \nu) \nabla v \cdot \psi - \int_{B_r^+} (\nabla |\nabla v|^2 + 2 \nabla v \Delta v) \cdot \psi \\ &= 2 \int_{\partial B_r^+} (\nabla v \cdot \nu) \nabla v \cdot \psi - 2 \int_{B_r^+} \left(\nabla \frac{|\nabla v|^2}{2} + \nabla v \beta_\varepsilon(v) \right) \cdot \psi \\ &= 2 \int_{\partial B_r^+} (\nabla v \cdot \nu) \nabla v \cdot \psi - 2 \int_{B_r^+} \nabla \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) \cdot \psi. \end{aligned}$$

Now applying the divergence theorem again we obtain

$$\begin{aligned} 2 \int_{B_r^+} v_i v_j \psi_j^i &= 2 \int_{\partial B_r^+ \cap \{x_n=0\}} (\nabla v \cdot \nu) \nabla u \cdot \psi - 2 \int_{\partial B_r^+ \cap \{x_n=0\}} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) \psi \cdot \nu \\ &\quad + 2 \int_{B_r^+} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) \operatorname{div} \psi. \end{aligned}$$

Hence the proof is completed. $\square$

The integral identity in Lemma 2.4 allows to construct a functional defined on the upper half balls $B_r^+(x_0) = B_r(x_0) \cap \{x_n > 0\}$, $x_0 \in \{x_n = 0\}$ which is monotone function of r for any fixed $\varepsilon > 0$. This is done by choosing ψ appropriately.

Lemma 2.5. *Let v be a solution $\Delta v = \beta_\varepsilon(v)$ in $B_{r_0}(x_0)$ and $v = g_\varepsilon$ on $\{x_n = 0\} \cap B_{r_0}(x_0)$, $x_0 \in \{x_n = 0\}$ such that $g_\varepsilon \in C^{1,\alpha}$, and $g_\varepsilon(x) = |x|\ell(x)$ with $\ell(x) \leq C|x|^\alpha$, $\alpha > 0$. Introduce the functional $W_\varepsilon(r)$, $r < r_0$*

(2.3)

$$W_\varepsilon(r) := \frac{1}{r^n} \int_{B_r^+} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) - \frac{1}{2r^{n+1}} \int_{\partial B_r^+ \cap \{x_n > 0\}} v^2 + C \|g_\varepsilon\|_{C^{1,\alpha}(B_{r_0} \cap \{x_n=0\})} r^\alpha,$$

for some positive large constant $C > 0$. Then W_ε is monotone non-decreasing function of r , for any fixed ε . Moreover, for any $0 < S < R < r_0$ the following formula holds

$$(2.3) \quad W_\varepsilon(R) - W_\varepsilon(S) = \int_S^R \frac{d\rho}{\rho^n} \int_{\partial B_\rho \cap \{x_n > 0\}} \left(\nabla v \cdot \nu - \frac{v}{\rho} \right)^2.$$

Proof. Let us take $\psi(x) = x\eta_\delta(x)$ where

$$\eta_\delta(x) := \begin{cases} 1 & \text{if } x \in B_\rho, \\ \frac{\rho+\delta-|x|}{\delta} & \text{if } x \in B_{\rho+\delta} \setminus B_\rho, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_{\rho+\delta}, \end{cases}$$

and $\delta > 0$ is a small parameter. Then, by direct computation

$$\partial_{x_j} \psi^i(x) := \begin{cases} \delta_{ij} & \text{if } x \in B_\rho, \\ \delta_{ij} \eta - \frac{1}{\delta} \frac{x_1 x_j}{|x|} & \text{if } x \in B_{\rho+\delta} \setminus B_\rho, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_{\rho+\delta}, \end{cases}$$

and

$$\operatorname{div} \psi(x) := \begin{cases} N & \text{if } x \in B_\rho, \\ N\eta - \frac{|x|}{\delta} & \text{if } x \in B_{\rho+\delta} \setminus B_\rho, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus B_{\rho+\delta}. \end{cases}$$

Hence, plugging the last two formulas into the identity of Lemma 2.4 and noting that

$$\int_{\partial B_\rho^+ \cap \{x_n=0\}} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) \eta_\delta(x \cdot \nu) = 0,$$

because $x \cdot \nu = 0$ if $x \in \{x_n = 0\}$, and

$$\int_{\partial B_r^+ \cap \{x_n=0\}} (\nabla v \cdot \nu)(\nabla v \cdot x) \eta_\delta = \int_{\partial B_\rho^+ \cap \{x_n=0\}} (\nabla v \cdot \nu) \left(\sum_{i=1}^{n-1} \partial_{x_i} f_j(x) x_i \right) \eta_\delta \sim \rho^{n+\alpha}$$

because $\nabla v \cdot x$ is the tangential derivative on $\{x_n = 0\}$. We finally obtain

$$\begin{aligned} & \int_{B_\rho^+} |\nabla v|^2 + \int_{B_{\rho+\delta}^+ \setminus B_\rho^+} \nabla u \cdot \left(\nabla v \eta_\delta - \frac{1}{\delta} \nabla v \frac{x \otimes x}{|x|} \right) \\ &= n \int_{B_\rho^+} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) + \int_{B_{\rho+\delta}^+ \setminus B_\rho^+} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) \left(n\eta_\delta - \frac{|x|}{\delta} \right). \end{aligned}$$

Notice that from Lebesgue's theorem on the absolute continuity of integrals we have that

$$\int_{B_{\rho+\delta}^+ \setminus B_\rho^+} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) n\eta_\delta \rightarrow 0, \quad \int_{B_{\rho+\delta}^+ \setminus B_\rho^+} |\nabla v|^2 \eta_\delta \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Therefore, sending $\delta \rightarrow 0$, we end up with the identity

$$\begin{aligned} & \int_{B_\rho^+} |\nabla v|^2 - \rho \int_{\partial B_\rho \cap \{x_n > 0\}} (\partial_\nu v)^2 \\ &= n \int_{B_\rho^+} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) - \rho \int_{\partial B_\rho \cap \{x_n > 0\}} \left(\frac{|\nabla v|^2}{2} + \mathcal{B}_\varepsilon(v) \right) + O(\rho^{n+\alpha}). \end{aligned}$$

On the other hand we have that

$$\int_{B_\rho} |\nabla v|^2 = \int_{\partial B_\rho^+} v \partial_\nu v - \int_{B_\rho^+} v \beta_\varepsilon(v) = \int_{\partial B_\rho^+ \cap \{x_n > 0\}} v \partial_\nu v - O(\rho^{n+\alpha}).$$

Thus multiplying both sides of the last identity by ρ^{-n-1} the proof follows. $\square$

Next lemma takes care of the limit of W_ε when $\varepsilon \rightarrow 0$.

Lemma 2.6. *Let v_ε be as in Lemma 2.5 and $v_\varepsilon \rightarrow v$ for a suitable subsequence $\varepsilon = \varepsilon_j$, see Proposition 2.1, then $W_\varepsilon(r) \rightarrow W(r)$ where*

$$W(r) = \frac{1}{r^n} \int_{B_r^+} \left(\frac{|\nabla v|^2}{2} + \gamma(x) \right) - \frac{1}{2r^{n+1}} \int_{\partial B_r^+ \cap \{x_n > 0\}} v^2 + C \|g_0\|_{C^{1,\alpha}(B_{r_0} \cap \{x_n = 0\})} r^\alpha$$

$g_0 = \lim_{\varepsilon \rightarrow 0} g_\varepsilon$, W_ε is given by (2.3) and $\gamma(x) \in L^\infty$ is the weak-star limit of $\mathcal{B}_\varepsilon(v_\varepsilon)$. Furthermore, for any $0 < S < R < \frac{1}{r_j}$ the following holds

$$(2.4) \quad W(R) - W(S) = \int_S^R \frac{d\rho}{\rho^n} \int_{\partial B_\rho \cap \{x_n > 0\}} \left(\nabla v \cdot \nu - \frac{v}{\rho} \right)^2.$$

Proof. Since $\mathcal{B}_\varepsilon \in L^\infty$ then it follows that there is $\gamma(x) \in L^\infty$ such that for any $R > 0$ and $\eta \in L^1(B_R^+)$ there holds

$$\int_{B_R^+} \mathcal{B}_\varepsilon(v_\varepsilon(x)) \eta(x) dx \rightarrow \int_{B_R^+} \gamma(x) \eta(x) dx,$$

in other words γ is the weak-star limit. Note that $\gamma(x) \geq M \chi_{\{v > 0\}}$. Indeed, if $v(x) > 0$ then by uniform convergence there is $\delta > 0$ such that $v_\varepsilon(y) \geq \frac{v(x)}{2}$ if $y \in B_\delta(x)$. Thus $\mathcal{B}_\varepsilon(x) = M$ for sufficiently small ε . To fix the ideas we set $v_j := v_{\varepsilon_j}$ for some $\varepsilon_j \rightarrow 0$ such that $v_j \rightarrow v$. Thus by almost everywhere convergence of $\nabla v_j(x) \rightarrow \nabla v(x)$ (recall Proposition 2.1), (1.5) and Lebesgue's dominated convergence theorem we infer that

$$\int_{B_R^+} \left(\frac{|\nabla v_j|^2}{2} + \mathcal{B}_{\varepsilon_j}(v) \right) \rightarrow \int_{B_R^+} \left(\frac{|\nabla v|^2}{2} + \gamma(x) \right).$$

On the other hand by Fubini's theorem

$$\int_S^R \frac{d\rho}{\rho^n} \int_{\partial B_\rho \cap \{x_n > 0\}} \left(\nabla v_j \cdot \nu - \frac{v_j}{\rho} \right)^2 = \int_{(B_R \setminus B_S) \cap \{x_n > 0\}} \left(\nabla v_j(x) \cdot \frac{x}{|x|} - \frac{v_j(x)}{|x|} \right)^2 \frac{dx}{|x|^n}$$

and again using almost everywhere convergence of $\nabla v_j(x) \rightarrow \nabla v(x)$, (1.5) and Lebesgue's dominated convergence theorem we infer that

$$\begin{aligned} & \int_{(B_R \setminus B_S) \cap \{x_n > 0\}} \left(\nabla v_j(x) \cdot \frac{x}{|x|} - \frac{v_j(x)}{|x|} \right)^2 \frac{dx}{|x|^n} \longrightarrow \\ & \longrightarrow \int_{(B_R \setminus B_S) \cap \{x_n > 0\}} \left(\nabla v(x) \cdot \frac{x}{|x|} - \frac{v(x)}{|x|} \right)^2 \frac{dx}{|x|^n} = \\ & = \int_S^R \frac{d\rho}{\rho^n} \int_{\partial B_\rho \cap \{x_n > 0\}} \left(\nabla v \cdot \nu - \frac{v}{\rho} \right)^2. \end{aligned}$$

Consequently, we have $W_\varepsilon \rightarrow W$ and in view of (2.5) W is nondecreasing function of r . $\square$

Remark 2.7. *Actually one can say more about $\gamma(x)$ in Lemma 2.6. Namely, there is a function $M(x)$ such that*

$$(2.5) \quad \gamma(x) = M\chi_{\{v>0\}} + M(x)\lfloor\partial\{v>0\}, \quad 0 \leq M(x) \leq M$$

see [3] (3.7) page 726. Clearly we can take

$$(2.6) \quad \gamma(x) = M\chi_{\{v>0\}} \quad \text{in } W \quad \text{if } \mathcal{L}^n(\partial\{v>0\}) = 0,$$

where $\mathcal{L}^n$ is the n -dimensional Lebesgue measure. This is certainly true for linear function $v = Cx_n$, $C > 0$.

We close this section by proving a simple convergence result for the function $\gamma(x)$ defined in (2.5).

Lemma 2.8. *Let v be a limit of singular perturbation problem. Consider the scaled functions $v_j(x) = \frac{v(r_j x)}{r_j}$, $r_j \rightarrow 0$ and v_0 be a blow-up limit corresponding to $\{r_j\}$. Then the functions*

$$\gamma_j(x) = M\chi_{\{v_j>0\}} + M(r_j x)\lfloor\partial\{v_j>0\} \in L^\infty(\mathbb{R}_+^n)$$

weak-star converge to γ_0 in $L^1(B_R^+)$ for any fixed $R > 0$, where

$$(2.7) \quad \gamma_0(x) = M\chi_{\{v_0>0\}} + M_0(x)\lfloor\partial\{v_0>0\}, \quad 0 \leq M_0(x) \leq M.$$

Proof. Let $x \in \mathbb{R}_+^n$ such that $v_0(x) > 0$ then $v_j(y) \geq \frac{v_0(x)}{2}$ for all $y \in B_\delta(x)$ provided that δ is small. Therefore $\gamma_j = M$ in $B_\delta(x)$ for j large. Analogously, if $v_0(x) < 0$ then $\gamma_j = 0$ in a neighbourhood of x if j is large enough. These imply that

$$\gamma_j(x) \rightarrow M\chi_{\{v_0>0\}} \quad \text{weak-star in } L^\infty(B_R^+ \cap \{v_0 \neq 0\}).$$

The fact that $0 \leq \gamma_j \leq M$ (see (2.5) and the discussion preceding it) implies that there exists $0 \leq \gamma_0 \leq M$ such that

$$\gamma_j(x) \rightarrow \gamma_0(x) \quad \text{weak-star in } L^\infty(B_R^+).$$

Then necessarily $\gamma_0(x) = M\chi_{\{v_0>0\}}$ in $B_R^+ \cap \{v_0 \neq 0\}$, and consequently there is $0 \leq M_0(x) \leq M$ such that (2.7) holds. $\square$

Remark 2.9. *The conclusion of Lemma (2.8) remains true if we consider the blow down limit instead, i.e. if we let $r_j \rightarrow \infty$.*

3. PROOF OF THEOREM 1

Proof. It suffices to show that for any $\delta > 0$ there is $r_0, \varepsilon_0 > 0$ depending on f, n, M and L such that

$$(3.1) \quad \partial\{u^\varepsilon > 0\} \cap B_r^+ \subset \{x_n < \delta|x|\} \cap B_r^+, \quad \forall r < r_0 \text{ and } \varepsilon < \varepsilon_0.$$

Suppose, towards a contradiction, that (3.1) fails, then for some fixed δ_0 there is $\{r_j\}_{j=1}^\infty, \{\varepsilon_j\}_{j=1}^\infty$, and $x_j \in \partial\{u^{\varepsilon_j} > 0\} \cap \partial B_{r_j}^+$ such that

$$(3.2) \quad x_j \in \{x_n > \delta_0|x|\} \cap \partial B_{r_j}^+ \quad \text{and} \quad u^{\varepsilon_j}(x_j) = 0$$

introduce the scaled function

$$(3.3) \quad v_j(x) = \frac{u^{\varepsilon_j}(r_j x)}{r_j}, \quad x \in B_{\frac{1}{r_j}}^+$$

then we have from (1.5) that

$$(3.4) \quad |\nabla v_j(x)| \leq L, \quad x \in B_{\frac{1}{r_j}}^+$$

and

$$(3.5) \quad \Delta v_j(x) = \frac{r_j}{\varepsilon_j} \beta \left(\frac{v_j(x)}{\varepsilon_j/r_j} \right) \quad x \in B_{\frac{1}{r_j}}^+.$$

Furthermore, in view of (1.2) it follows that the corresponding scaled boundary data is

$$(3.6) \quad f_j(x) := \frac{f(r_j x)}{r_j} = o(r_j) \quad \text{and} \quad \|f_j(x)\|_{C^{1,\alpha}(\overline{B_M^+})} \rightarrow 0$$

for any fixed $M > 0$. Observe that (3.2) translates to the limit configuration such that

$$(3.7) \quad \exists x_0 \in \{x_n > \delta_0|x|\} \cap \partial B_1^+ \quad \text{and} \quad v(x_0) = 0.$$

There are three possible scenarios.

Case 1: There is a subsequence, still denoted j , such that $\frac{r_j}{\varepsilon_j} \rightarrow 0$. It follows from (3.5) that $0 \leq \Delta v_j \leq \sup \beta \frac{r_j}{\varepsilon_j} \rightarrow 0$. Thus owing to (3.4), (3.6) and Proposition 2.1 it follows that $v_j \rightarrow v$ such that v solves the following problem

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}_+^n, \\ |\nabla v(x)| \leq L & x \in \mathbb{R}_+^n, \\ v = 0 & \text{on } \{x_n = 0\}. \end{cases}$$

From Liouville's theorem it follows that v is linear, which contradicts (3.7).

Case 2: There is a subsequence, still denoted j , such that $\frac{r_j}{\varepsilon_j} \rightarrow a > 0$. Without loss of generality we assume that $a = 1$. From (3.5) and Proposition 2.1 it follows that we can extract a subsequence $j(k)$ such that $v_{j(k)} \rightarrow v$ and $\Delta v = \beta(v)$. Thus applying Lemma 2.3 we see that $v > 0$ which is again in contradiction with (3.7).

Case 3: There is a subsequence, still denoted j , such that $\frac{r_j}{\varepsilon_j} \rightarrow \infty$. Introduce $\varepsilon'_j = \frac{\varepsilon_j}{r_j} \rightarrow 0$ then from (3.5) we get

$$(3.8) \quad \Delta v_j(x) = \beta_{\varepsilon'_j}(v_j(x)) \quad x \in B_{\frac{1}{r_j}}^+.$$

Observe that the boundary data for v_j is determined by $g_{\varepsilon'_j}(x) = f_j(x)$ and

$$\|g_{\varepsilon'_j}\|_{C^{1,\alpha}(B_{r_0} \cap \{x_n=0\})} \rightarrow 0$$

thanks to (3.6). Therefore by Lemma 2.6 we have for the limit function $v = \lim_{j \rightarrow \infty} v_j$

$$(3.9) \quad W(r) = \frac{1}{r^n} \int_{B_r^+} \left(\frac{|\nabla v|^2}{2} + \gamma(x) \right) - \frac{1}{2r^{n+1}} \int_{\partial B_r^+ \cap \{x_n > 0\}} v^2$$

is a monotone function of r , see (2.4). Let us now denote by W^r the functional in (3.9), where γ has been replaced by $\gamma(rx)$, i.e.

$$\begin{aligned} (3.10) \quad W(s\rho) &= \frac{1}{(s\rho)^n} \int_{B_{s\rho}^+} \left(\frac{|\nabla v|^2}{2} + \gamma(x) \right) - \frac{1}{2(s\rho)^{n+1}} \int_{\partial B_{s\rho}^+ \cap \{x_n > 0\}} v^2 \\ &= \frac{1}{s^n} \int_{B_s^+} \left(\frac{|\nabla v_\rho|^2}{2} + \gamma(\rho x) \right) - \frac{1}{2s^{n+1}} \int_{\partial B_s^+ \cap \{x_n > 0\}} v_\rho^2 \\ &=: W^\rho(s, 0, v_\rho) \end{aligned}$$

and W^0 the corresponding functional with γ_0 in Lemma 2.8. Observe that by (3.10) one has $W^\rho(s, 0, v_\rho) \geq W^\rho(\tau, 0, v_\rho)$ for $s \geq \tau$.

Then scaling the functional, and using monotonicity, we have

$$\begin{aligned} W(0^+, 0, v) &= \lim_{r \rightarrow 0} W^r(rs, 0, v) = \lim_{r \rightarrow 0} W^r(s, 0, v_r) = W^0(s, 0, v_0) = \\ &= \frac{1}{s^n} \int_{B_s^+} \left(\frac{|\nabla v_0|^2}{2} + \gamma_0(x) \right) - \frac{1}{2s^{n+1}} \int_{\partial B_s^+ \cap \{x_n > 0\}} v_0^2 \end{aligned}$$

which implies that the blow-up is homogeneous of degree one. In a similar way we conclude that the blow-down v_∞ is homogeneous of degree one, since

$$\begin{aligned}
0 = W(0^+, 0, v) - W(0^+, 0, v) &\longleftarrow W(rs, 0, v) - W(r\tau, 0, v) = \\
&= \int_{r\tau}^{rs} \frac{d\rho}{\rho^n} \int_{\partial B_\rho \cap \{x_n > 0\}} \left(\nabla v \cdot \nu - \frac{v}{\rho} \right)^2 \\
&= \int_\tau^s \frac{d\rho}{\rho^n} \int_{\partial B_\rho \cap \{x_n > 0\}} \left(\nabla v_r \cdot \nu - \frac{v_r}{\rho} \right)^2 \\
&\longrightarrow \int_\tau^s \frac{d\rho}{\rho^n} \int_{\partial B_\rho \cap \{x_n > 0\}} \left(\nabla v_0 \cdot \nu - \frac{v_0}{\rho} \right)^2,
\end{aligned}$$

for $s \geq \tau$.

Applying Lemma 2.2 to the limit and its even reflection across $x_n = 0$ we see that $v \geq 0$ and hence v_0 (the blow-up) and v_∞ (the blow down) must be linear functions. Thus

$$(3.11) \quad v_0(x) = C_0 x_n \text{ and } v_\infty(x) = C_\infty x_n.$$

Now it remains to check that $C_0 = C_\infty$. It suffices to show that v is homogeneous function of degree one. From upper-semi continuity (see Claim 4.2 in Appendix) it follows that the energy at the origin is larger than or equal to the energy at all other free boundary points in a small neighborhood. In particular if we take a sequence of regular free boundary points in $\mathbb{R}_+^n$, where $|\nabla u(z)| = \sqrt{2M}$, the energy is a fixed constant A , i.e.

$$\lim_{z \rightarrow 0} W(0^+, z, v) =: A,$$

at any regular free boundary point z (with $z_1 > 0$) where one necessarily has $|\nabla u(z)| = \sqrt{2M}$, which is simply the free boundary condition satisfied at regular points in classical sense.

Thus we obtain for $z \in \partial\{v > 0\}$ with $z_n > 0$

$$\begin{aligned}
A &= W(0^+, z, v) = \lim_{r_j \downarrow 0} W(r_j, z, v) \\
&= \lim_{r_j \downarrow 0} \frac{1}{r_j^n} \int_{B_{r_j}(z) \cap \{x_n > 0\}} \left(\frac{|\nabla v|^2}{2} + \gamma(x) \right) - \frac{1}{2r_j^{n+1}} \int_{\partial B_{r_j}^+(z) \cap \{x_n > 0\}} v^2 \\
&= \lim_{j \rightarrow \infty} \int_{B_1} \left(\frac{|\nabla v_j(y)|^2}{2} + \gamma(z + r_j y) \right) - \frac{1}{2} \int_{\partial B_1} v_j^2
\end{aligned}$$

where we set $v_j(y) = \frac{v(z+r_j y)}{r_j}$, and in the second equality we have used the fact that in $B_{r_1}(z)$, for some small r_1 , there is no fixed boundary presented, and hence the situation is like an interior case for the monotonicity function. Setting $\gamma_j(y) = \gamma(z + r_j y)$ and recalling Lemma 2.8 we see

$$\begin{aligned}
(3.12) \quad A &= \int_{B_1} \left(\frac{|\nabla v_0|^2}{2} + \gamma_0 \right) - \frac{1}{2} \int_{\partial B_1} v_0^2 \\
&= \frac{1}{2} \int_{\partial B_1} v_0 \nabla v_0 \cdot \nu + \int_{B_1} \gamma_0 - \frac{1}{2} \int_{\partial B_1} v_0^2
\end{aligned}$$

where to get the last line we used the divergence theorem, i.e. $\operatorname{div}(v_0 \nabla v_0) = |\nabla v_0|^2 + v_0 \Delta v_0 = |\nabla v_0|^2$ because $v_0 = c_0(x \cdot e)^+$ for some fixed unit direction e (recall that v_0 is a blow-up of v at a regular free boundary point z). Consequently, v_0 is homogeneous function of degree one i.e. $v_0(x) = \nabla v_0(x) \cdot x$ which gives that $\frac{1}{2} \int_{\partial B_1} v_0 \nabla v_0 \cdot \nu - \frac{1}{2} \int_{\partial B_1} v_0^2 = 0$. Returning to (3.12) we infer from (2.7) with $v_0 = c_0(x \cdot e)^+$ and (2.6)

$$\begin{aligned}
(3.13) \quad A &= \int_{B_1} \gamma_0 \\
&= M \chi_{\{v_0 > 0\}}.
\end{aligned}$$

From upper semicontinuity (Claim 4.2 in Appendix) we see that

$$W(0^+, 0, v) \geq \lim_{z \rightarrow 0} W(0^+, z, v) = A.$$

This in conjunction with the monotonicity of W (2.4) and scaling property of W (3.10) yield

$$\begin{aligned}
A &\leq W(0^+, 0, v) \leq \\
&\leq W(\infty, 0, v) = \lim_{j \rightarrow \infty} W(r_j, 0, v) = \lim_{j \rightarrow \infty} W^{r_j}(1, 0, v_j) = \\
&= \lim_{j \rightarrow \infty} \left[\int_{B_1^+} \left(\frac{|\nabla v_j|^2}{2} + \gamma(r_j x) \right) - \int_{\partial B_1^+ \cap \{x_n > 0\}} v_j^2 \right] \\
&= \int_{B_1^+} \left(\frac{|\nabla v_\infty|^2}{2} + \gamma_\infty \right) - \frac{1}{2} \int_{\partial B_1^+} v_\infty^2
\end{aligned}$$

where $v_j(x) = \frac{v(r_j x)}{r_j}$ and γ_∞ is the limit of the functions $\gamma_j(r_j x)$ as $r_j \rightarrow \infty$.

By (3.11) $v_\infty = C_\infty x_n, x \in \mathbb{R}_+^n$. In view of (2.6) it follows that $\gamma_\infty = M\chi\{u_\infty > 0\}$. Thus using the divergence theorem we conclude

$$\begin{aligned}
(3.14) \quad A &\leq \frac{1}{2} \int_{\partial B_1^+} v_\infty \nabla v_\infty \cdot \nu - \frac{1}{2} \int_{\partial B_1^+} v_\infty^2 + \int_{B_1^+} M\chi\{u_\infty > 0\} \\
&= \int_{B_1^+} M\chi\{u_\infty > 0\} = A.
\end{aligned}$$

Since by monotonicity

$$A \leq W(0^+, 0, v) \leq W(r, 0, v) \leq W(\infty, 0, v) \leq A$$

we see that, in view of (2.4), v is homogeneous. Applying Lemma 2.2 with $h_1 = v$ and h_2 being the even reflection of v across $x_n = 0$, we see that v must be linear. But being linear it has to be cx_n and hence this is in contradiction with (3.7). $\square$

4. APPENDIX

We define the subset of free boundary points T , which are touching the hyperplane $\{x_n = 0\}$ as follows:

$$(4.1) \quad T = \{z : \exists z^j \in \partial\{v > 0\}, z_n^j > 0, z^j \rightarrow z\},$$

where $\partial\{v > 0\}$ denotes free boundary points.

In what follows we set

$$(4.2) \quad W(\rho, z, v) = \frac{1}{\rho^n} \int_{B_\rho(z) \cap \{x_n > 0\}} \frac{|\nabla v|^2}{2} + \gamma(x) - \frac{1}{2\rho^{n+1}} \int_{\partial B_\rho(z) \cap \{x_n > 0\}} v^2$$

where $\gamma(x)$ is defined by (2.5) and $z \in \overline{\mathbb{R}_+^n} \cap \partial\{v > 0\}$.

Claim 4.1. *For any given $0 < \tau < 10^{-5}$, $k > 10^5$ there exists $R_0 > 0$ such that if for some free boundary point z with $z_1 > 0$ and $\text{dist}(z, T) := r_0 < R_0$ (see (4.1) for the definition of the class of touching free boundary points) then*

$$W(r_0, z, v) \leq (1 + \tau)W(kr_0, z, v).$$

Proof. If the claim fails then there are $k > 10^5$, $0 < \tau < 10^{-5}$ and a sequence $r_j \rightarrow 0$, and $x^j \rightarrow x_0 \in T$, with $\text{dist}(x^j, T) = r_j$ and

$$(4.3) \quad W(r_j, x^j, v) \geq (1 + \tau)W(kr_j, x^j, v).$$

Without loss of generality we take $x_0 = 0$. Scale the solution and the monotonicity function with r_j , that is

$$u_j(x) := \frac{v(x_j + r_j x)}{r_j},$$

so (4.3) translates to

$$W(1, 0, v_j) \geq (1 + \tau)W(k, 0, v_j).$$

Letting $r_j \rightarrow 0$ and noting that

$$\frac{x_n^j}{r_j} \rightarrow 0 \quad (\text{because of tangential touch})$$

we have that $w = \lim v_j$ satisfies

$$W(1, 0, w) \geq (1 + \tau)W(k, 0, w)$$

where now w is a global solution over $\mathbb{R}_+^n$ because $x_n^j/r_j \rightarrow 0$. Since also $w(x', 0) = 0$, we arrive at a contradiction to the monotonicity formula in Lemma 2.6. This proves the claim. $\square$

Claim 4.2. *The function $\omega(z) := W(0^+, z, v)$ is upper-semi continuous.*

Proof. Choose z on the free boundary, with $z_n > 0$, then $\omega(z) \leq W(r_0, z, v)$ for $r_0 \leq \text{dist}(z, T)$, because there is no touching point in this radius, and the problem is like an interior problem.

Next by Claim 4.1, we have

$$(4.4) \quad \begin{aligned} \omega(z) &\leq W(r_0, z, v) \leq (1 + \tau)W(kr_0, z, v) \leq \\ &\leq (1 + \tau) \left[\left(\frac{k+1}{k} \right)^n W((k+1)r_0, 0, v) + I_k(z) \right], \end{aligned}$$

where

$$I_k(z) = \frac{1}{2k^n(k+1)r_0^{n+1}} \int_{\partial B_{(k+1)r_0}^+(0) \cap \{x_n > 0\}} v^2 - \frac{1}{2(kr_0)^{n+1}} \int_{\partial B_{kr_0}^+(z) \cap \{x_n > 0\}} v^2.$$

We wish to estimate I_k as follows

$$(4.5) \quad \begin{aligned} 2(r_0 k)^{n+1} I_k &= \frac{k}{k+1} \int_{\partial B_{(k+1)r_0}^+(0) \cap \{x_n > 0\}} v^2 - \int_{\partial B_{kr_0}^+(z) \cap \{x_n > 0\}} v^2 \\ &= \frac{k}{k+1} \left(\frac{k+1}{k} \right)^{n-1} \int_{\partial B_{kr_0}^+(0) \cap \{x_n > 0\}} v^2 ((1 + 1/k)y) - \\ &\quad - \int_{\partial B_{kr_0}^+(z) \cap \{x_n > 0\}} v^2 \\ &= \left(\frac{k+1}{k} \right)^{n-2} \int_{\partial B_{kr_0}^+(0) \cap \{x_n > 0\}} v^2 ((1 + 1/k)y) - \\ &\quad - \int_{\partial B_{kr_0}^+(z) \cap \{x_n > 0\}} v^2 \\ &= \left(\frac{k+1}{k} \right)^{n-2} \int_{\partial B_{kr_0}^+(0) \cap \{x_n > 0\}} v^2 ((1 + 1/k)y) - \\ &\quad - \int_{\partial B_{kr_0}^+(z) \cap \{0 < x_n < z_n\}} v^2 - \int_{\partial B_{kr_0}^+(z) \cap \{x_n > z_n\}} v^2 \\ &= \left(\frac{k+1}{k} \right)^{n-2} \int_{\partial B_{kr_0}^+(0) \cap \{x_n > 0\}} v^2 ((1 + 1/k)y) \\ &\quad - \int_{\partial B_{kr_0}^+(z) \cap \{x_n > z_n\}} v^2 - O((kr_0)^{n-2}(z_n)^3) \\ &= \left(1 + \frac{1}{k} \right)^{n-2} \int_{\partial B_{kr_0}^+(0) \cap \{x_n > 0\}} v^2 ((1 + 1/k)y) - \end{aligned}$$

$$- \int_{\partial B_{kr_0}^+(z) \cap \{y_n > 0\}} v^2(z+y) - O((kr_0)^{n-2}(z_n)^3).$$

Notice that by tangential touch

$$(4.6) \quad O((kr_0)^{n-2}(z_n)^3) = (kr_0)^{n-2}o(r_0^3).$$

Furthermore

$$\begin{aligned} |v^2((1+1/k)y) - v^2(z+y)| &= |v((1+1/k)y) - v(z+y)| |v((1+1/k)y) + v(z+y)| \\ &\leq L^2 \left[\left(\frac{|y|}{k} + |z| \right) \left(\left(2 + \frac{1}{k} \right) |y| + |z| \right) \right] \\ &\leq L^2 \left[(r_0 + |z|) \left(\left(2 + \frac{1}{k} \right) kr_0 + r_0 \right) \right] \\ &\leq 2(2k+2)L^2 r_0^2 \\ &= 4(k+1)L^2 r_0^2 \end{aligned}$$

where L is the Lipschitz constant defined in (1.5).

Combining this with (4.5) and (4.6) we obtain

$$|I_k| \leq \frac{O(1)}{k} + \frac{2(k+1)L^2}{k^2} + \frac{o(r_0^3)}{(kr_0)^3}.$$

Returning to (4.4) and letting $z \rightarrow T$, i.e. $r_0 \rightarrow 0$ we arrive at

$$\lim_{z \rightarrow 0} \omega(z) \leq (1+\tau) \left(\frac{k+1}{k} \right)^n \omega(0) + \frac{C}{k}.$$

Since by construction ε is any number in the interval $(0, 10^{-5})$ and k is any number such that $k > 10^5$ (see the statement of Claim 4.1) we first send $k \rightarrow \infty$ and then $\tau \rightarrow 0$ to conclude the desired result. $\square$

REFERENCES

- [1] L.A. CAFFARELLI, L. KARP, H. SHAHGHOLIAN, *Regularity of a free boundary with application to the Pompeiu problem*, Ann. Math., vol 151 (2000), 269–292.
- [2] H. BERESTYCKI, L.A. CAFFARELLI, L. NIRENBERG, *Uniform estimates for regularization of free boundary problems*. Analysis and partial differential equations, 567–619, Lecture Notes in Pure and Appl. Math., 122, Dekker, New York, 1990.

- [3] [L.A. CAFFARELLI, C. LEDERMAN, N. WOLANSKI, Pointwise and viscosity solutions for the limit of a two phase parabolic singular perturbation problem. Indiana Univ. Math. J. 46 \(1997\), no. 3, 719–740](#)
- [4] [L.A. CAFFARELLI, N. LEDERMAN, N. WOLANSKI Uniform estimates and limits for a two phase parabolic singular perturbation problem. Indiana Univ. Math. J. 46 \(1997\), no. 2, 453–489](#)
- [5] [D. GILBARG, N.S. TRUDINGER, Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001](#)
- [6] [A. GUREVICH, Boundary regularity for free boundary problems. Comm. Pure Appl. Math. 52 \(1999\), no. 3, 363–403](#)
- [7] [A.L. KARAKHANYAN, C.E. KENIG, H. SHAHGHOLIAN, *The behavior of the free boundary near the fixed boundary for a minimization problem*, Calc. of Var. and PDEs, vol. 28, 1, \(2007\), 15–31.](#)
- [8] [G.S. WEISS; Partial regularity for weak solutions of an elliptic free boundary problem. Comm. Partial Differential Equations 23 \(1998\), no. 3-4, 439–455.](#)
- [9] [YA. B. ZEL'DOVICH, G. I. BARENBLATT, V. B. LIBROVICH, AND G. M. MAKHVILADZE, Combustion and explosions, Consultants Bureau, New York, 1985.](#)

SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, MAYFIELD ROAD, EH9 3JZ,
EDINBURGH, SCOTLAND

E-mail address: `aram6k@gmail.com`

DEPARTMENT OF MATHEMATICS, ROYAL INSTITUTE OF TECHNOLOGY, 100 44 STOCK-
HOLM, SWEDEN

E-mail address: `henriksh@math.kth.se`