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DISCRETE BILINEAR OPERATORS AND COMMUTATORS

ÁRPÁD BÉNYI AND TADAHIRO OH

ABSTRACT. We discuss boundedness properties of certain classes of discrete bilinear operators that are similar to those of the continuous bilinear pseudodifferential operators with symbols in the Hörmander classes $BS_{1,0}^\omega$. In particular, we investigate their relation to discrete analogues of the bilinear Calderón-Zygmund singular integral operators and show compactness of their commutators.

1. INTRODUCTION

The study of bilinear operators within Fourier analysis goes back to the seminal work of Coifman and Meyer in the 1970's on the Calderón commutators [10, 11, 12]. A nowadays classical result, known as the Coifman-Meyer multiplier theorem, concerns the boundedness of bilinear pseudodifferential operators with symbols in the class $BS_{1,0}^0(\mathbb{R}^d)$ from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ for $1 < p, q \leq \infty$, $\frac{1}{2} < r < \infty$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. This further led to the general study of the bilinear Hörmander classes $BS_{\rho,\delta}^\omega$ including, when appropriate, their symbolic calculus, boundedness properties, and applications to PDEs; see, for example, [1, 2, 18, 22] and the references therein for a succinct introduction to the subject. More precisely, the bilinear pseudodifferential operators considered in these works are a priori defined on appropriate test functions and are given by

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta, \quad (1.1)$$

where the symbol σ satisfies estimates of the form:

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \leq C_{\alpha,\beta,\gamma} (1 + |\xi| + |\eta|)^{\omega + \delta|\alpha| - \rho(|\beta| + |\gamma|)}, \quad (1.2)$$

for any $x, \xi, \eta \in \mathbb{R}^d$, any multi-indices α, β, γ , and some positive constants $C_{\alpha,\beta,\gamma} > 0$. The class of symbols satisfying (1.2) is denoted by $BS_{\rho,\delta}^\omega(\mathbb{R}^d)$, or simply $BS_{\rho,\delta}^\omega$ when it is clear from the context to which space the variables x, ξ, η belong. For example, any bilinear partial differential operator with variable coefficients that have bounded derivatives

$$L(f, g) = \sum_{|\alpha| + |\beta| \leq \omega} a_{\alpha,\beta}(x) \frac{\partial^\alpha f}{\partial x^\alpha} \frac{\partial^\beta g}{\partial x^\beta} \quad (1.3)$$

can be realized as an operator of order ω with $L = T_\sigma$ as in (1.1), where $\sigma \in BS_{1,0}^\omega$.

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Despite Hörmander's initial undertakings [17] “the use of Fourier transformations has been emphasized; as a result no singular integral operators are apparent...”, the later works made clear that, for symbols of order zero such as $BS_{1,0}^0$, the associated operators are bilinear Calderón-Zygmund singular integral operators, and therefore appropriate tools from the theory of singular integrals provide an alternative argument for boundedness results without the use of Littlewood-Paley theory as in [11]; see also [4]. More precisely, if we formally invert the Fourier transforms in (1.1), T_σ has an integral representation on the physical side as

$$T_\sigma(f, g)(x) = \langle K_\sigma(x, y, z), (f \otimes g)(y, z) \rangle,$$

where $(f \otimes g)(y, z) = f(y)g(z)$, $\langle \cdot, \cdot \rangle$ denotes the usual distribution-test function pairing (in y and z), and the kernel $K_\sigma(x, y, z)$ is an appropriate distribution that is singular along some variety. It is known [15, 7] that if the symbol $\sigma \in BS_{1,0}^0$, then K_σ is a bilinear Calderón-Zygmund kernel, that is, a function on \mathbb{R}^{3d} such that, away from the diagonal $\mathbb{D} = \{(x, y, z) \in \mathbb{R}^{3d} : x = y = z\}$, we have

$$\begin{aligned} |K_\sigma(x, y, z)| &\lesssim (|x - y| + |y - z| + |z - x|)^{-2d}, \\ |\nabla K_\sigma(x, y, z)| &\lesssim (|x - y| + |y - z| + |z - x|)^{-2d-1}, \end{aligned}$$

and

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_\sigma(x, y, z) f(y) g(z) dy dz \quad (1.4)$$

for $x \notin \text{supp}(f) \cap \text{supp}(g)$. Moreover, [2, Theorem 5.1] shows that, given multi-indices $\alpha, \beta, \gamma \in (\mathbb{N} \cup \{0\})^d$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$,

$$\sup_{(x,y,z) \in \mathbb{R}^{3d} \setminus \mathbb{D}} |\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K_\sigma(x, y, z)| (|x - y| + |y - z| + |z - x|)^N < \infty. \quad (1.5)$$

The goal of this note is to put forth the conceivability of an appropriate theory of *discrete* bilinear operators of the same flavor as the continuous ones discussed above. As such, it is inspired by the works [9] and [13]. The essential insight of [9] was to characterize the space of infinite matrices that model (linear) pseudodifferential operators and to define an appropriate notion of order that is reminiscent, if not the same, of that appearing in the definition of the linear Hörmander class $S_{1,0}^\omega$. It is within this framework that several applications to numerical approximations in PDEs were then taken up in [13]. Our own modest intention instead stems purely from exploring some appropriate boundedness results for the discrete analogues of classes of operators defined via infinite Calderón-Zygmund-like tensors, as well as showing the compactness of such operators and multiplication by bounded sequences with compact support.

2. DEFINITIONS

A discrete weight function is a non-negative function $w : \mathbb{Z}^d \rightarrow [0, \infty)$. Given $1 \leq p \leq \infty$, the *weighted space of p -summable sequences* $\ell_w^p(\mathbb{Z}^d)$ (or simply ℓ_w^p when the indexing set of the sequences is understood contextually) consists of all functions

$f : \mathbb{Z}^d \rightarrow \mathbb{C}$, that is sequences $f = \{f_k\}_{k \in \mathbb{Z}^d}$, such that the norm

$$\|f\|_{\ell_w^p} = \left(\sum_{k \in \mathbb{Z}^d} w(k)^p |f_k|^p \right)^{\frac{1}{p}}$$

is finite, with appropriate modifications when $p = \infty$. One of the standard classes of discrete weights is that of “power” type; given $s \in \mathbb{R}$, we define $w_s : \mathbb{Z}^d \rightarrow (0, \infty)$ by

$$w_s(k) = \langle k \rangle^s \quad (2.1)$$

for $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, where $\langle k \rangle = (1 + |k|^2)^{\frac{1}{2}}$ with $|k|^2 = |k_1|^2 + \dots + |k_d|^2$. For this particular discrete weight function w_s in (2.1), we simply write $\ell_s^p(\mathbb{Z}^d)$ for $\ell_{w_s}^p(\mathbb{Z}^d)$. When $p = 2$, we have $f \in \ell_s^2$ if and only if $\mathcal{F}^{-1}(f) \in H^s(\mathbb{T}^d)$, where the latter space denotes the usual L^2 -based Sobolev space¹ and \mathcal{F}^{-1} denotes the inverse Fourier transform. In [13], the space ℓ_s^2 is referred to as “discrete Sobolev space” (and denoted by h^s in [13]). For general $1 \leq p \leq \infty$, we have the following relation; $f \in \ell_s^p$ if and only if $\mathcal{F}^{-1}(f)$ belongs to the so-called Fourier-Lebesgue space $\mathcal{FL}^{s,p}(\mathbb{T}^d)$ which plays an important role in the study of nonlinear PDEs; see, for example, [19, 20]. It is easy to see that the dual of ℓ_s^p is $\ell_{-s}^{p'}$ where p' is the Hölder conjugate of p .

Consider now an infinite tensor $\Theta : \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{C}$; Θ will be identified with the collection of its elements $\Theta = \{\Theta(j, k, \ell)\}_{(j,k,\ell) \in \mathbb{Z}^{3d}}$. For appropriate sequence spaces X, Y, Z , a tensor Θ induces an, a priori formally defined, bilinear operator $\mathcal{T}_\Theta : X \times Y \rightarrow Z$ acting on pairs of sequences $(f, g) \in X \times Y$ via the formula

$$(\mathcal{T}_\Theta(f, g))_j = \sum_{k \in \mathbb{Z}^d} \sum_{\ell \in \mathbb{Z}^d} \Theta(j, k, \ell) f_k g_\ell, \quad j \in \mathbb{Z}^d. \quad (2.2)$$

The operator in (2.2) bears a strong resemblance to the continuous version of the bilinear Calderón-Zygmund singular integral operator in (1.4), which, under an appropriate condition, can be identified with the bilinear pseudodifferential operator in (1.1). Naturally, one is interested in conditions on the tensor Θ that make the bilinear operator \mathcal{T}_Θ continuous on appropriate function spaces X, Y, Z . For example, assuming that $\|\Theta\|_{\ell_j^1 \ell_k^2 \ell_\ell^2} < \infty$, we have that $\mathcal{T}_\Theta : \ell^2(\mathbb{Z}^d) \times \ell^2(\mathbb{Z}^d) \rightarrow \ell^1(\mathbb{Z}^d)$ is a bounded bilinear operator. Indeed, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\mathcal{T}_\Theta(f, g)\|_{\ell^1} &= \sum_{j \in \mathbb{Z}^d} |(\mathcal{T}_\Theta(f, g))_j| \leq \sum_{j \in \mathbb{Z}^d} \sum_{k, \ell \in \mathbb{Z}^d} |\Theta(j, k, \ell)| |f_k| |g_\ell| \\ &\leq \|\Theta\|_{\ell_j^1 \ell_k^2 \ell_\ell^2} \|f\|_{\ell^2} \|g\|_{\ell^2}. \end{aligned}$$

One can easily extend this argument to more general Hölder triples of exponents $(p, q, r) \in [1, \infty]^3$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$; for example, assuming that the mixed Lebesgue norm² $\|\Theta(j, k, \ell)\|_{\ell_j^r \ell_k^{p'} \ell_\ell^{q'}} < \infty$, we can prove that $\mathcal{T}_\Theta : \ell^p(\mathbb{Z}^d) \times \ell^q(\mathbb{Z}^d) \rightarrow \ell^r(\mathbb{Z}^d)$ is a bounded bilinear operator.

¹Recall that $H^s(\mathbb{T}^d) = L_s^2(\mathbb{T}^d)$, where $L_s^2(\mathbb{T}^d)$ is the Bessel potential space; see also [5].

²By using duality, we can take the mixed-Lebesgue $\ell_j^r \ell_k^{p'} \ell_\ell^{q'}$ -norm in any order and thus it suffices to assume that the minimum mixed-Lebesgue norm is finite to guarantee the boundedness of the operator $\mathcal{T}_\Theta : \ell^p(\mathbb{Z}^d) \times \ell^q(\mathbb{Z}^d) \rightarrow \ell^r(\mathbb{Z}^d)$.

Remark 2.1. Given an infinite matrix $\sigma : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{C}$, we can (formally) define a discrete linear operator \mathcal{L}_σ by

$$(\mathcal{L}_\sigma(f))_j = \sum_{k \in \mathbb{Z}^d} \sigma(j, k) f_k, \quad j \in \mathbb{Z}^d, \quad (2.3)$$

for $f = \{f_k\}_{k \in \mathbb{Z}^d}$. Let F be a function on the torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ such that its Fourier coefficient $\widehat{F}(k)$ agrees with f_k for any $k \in \mathbb{Z}^d$. Then, we can associate the operator \mathcal{L}_σ , acting on sequences, with the following operator, acting on functions on \mathbb{T}^d :

$$L_\sigma(F)(x) = \int_{\mathbb{T}^d} K_\sigma(x, y) F(y) dy, \quad (2.4)$$

where the kernel K_σ is given by

$$K_\sigma(x, y) = \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \sigma(j, k) e^{2\pi i j \cdot x} e^{-2\pi i k \cdot y}.$$

Hence, prior knowledge on continuous linear operators on \mathbb{T}^d of the form (2.4) provides insights on discrete linear operators on \mathbb{Z}^d of the form (2.3).

Given $f = \{f_k\}_{k \in \mathbb{Z}^d}$ and $g = \{g_k\}_{k \in \mathbb{Z}^d}$, let F and G be functions on \mathbb{T}^d such that $\widehat{F}(k) = f_k$ and $\widehat{G}(k) = g_k$ for any $k \in \mathbb{Z}^d$. Then, we can express a discrete bilinear operator \mathcal{T}_Θ in (2.2) as a continuous bilinear operator T_Θ on \mathbb{T}^d , formally given by

$$T_\Theta(f, g)(x) = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} K_\Theta(x, y, z) f(y) g(z) dy dz,$$

where the kernel K_Θ is given by

$$K_\Theta(x, y, z) = \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \sum_{\ell \in \mathbb{Z}^d} \Theta(j, k, \ell) e^{2\pi i j \cdot x} e^{-2\pi i k \cdot y} e^{-2\pi i \ell \cdot z}.$$

Thus, boundedness of \mathcal{T}_Θ from $\ell_{s_1}^p(\mathbb{Z}^d) \times \ell_{s_2}^q(\mathbb{Z}^d)$ to $\ell_{s_3}^r(\mathbb{Z}^d)$ is equivalent to boundedness of T_Θ on the Fourier-Lebesgue spaces: $\mathcal{FL}^{s_1, p}(\mathbb{T}^d) \times \mathcal{FL}^{s_2, q}(\mathbb{T}^d) \rightarrow \mathcal{FL}^{s_3, r}(\mathbb{T}^d)$.

3. TOWARDS A CLASS OF DISCRETE BILINEAR SYMBOLS

The discussion at the end of the previous section makes it clear that a sufficiently fast decay in each one of the entries of the infinite tensor Θ is sufficient to guarantee boundedness of the associated bilinear operator \mathcal{T}_Θ on products of discrete Lebesgue spaces. A natural question then is whether one can impose an appropriate condition on the “discrete kernel” Θ reminiscent of those for the continuous bilinear Hörmander classes defined in (1.2) (and for their distributional kernels (1.5)) that would guarantee boundedness of the discrete bilinear operator \mathcal{T}_Θ defined in (2.2).

One of the first observations towards answering this question is the following; if a tensor Θ is almost diagonal, namely, if Θ decays rapidly away from the main diagonal $\{(j, k, \ell) \in \mathbb{Z}^{3d} : j = k = \ell\}$, then \mathcal{T}_Θ is bounded from $\ell^p(\mathbb{Z}^d) \times \ell^q(\mathbb{Z}^d)$ to $\ell^r(\mathbb{Z}^d)$. One of the goals of this section is to make this remark precise. It is worth noting that the sufficiency of almost diagonal conditions is rather natural in view of the estimates (1.5) on distributional kernels corresponding to the Hörmander classes $BS_{1, \delta}^0$, $0 \leq \delta < 1$, as well as their appearance elsewhere in multilinear harmonic analysis; see, for example, [8,

14] for almost diagonal estimates stemming from wavelet discretizations of multilinear operators.

Let $\omega \in \mathbb{R}$ and $N \in \mathbb{N}$. Given a tensor $\Theta : \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{C}$, define the following norm:

$$\|\Theta\|_{\omega, N} := \sup_{j, k, \ell \in \mathbb{Z}^d} \frac{|\Theta(j, k, \ell)| \langle |j - k| + |j - \ell| \rangle^{2N}}{\langle |j| + |k| \rangle^\omega \langle |j| + |\ell| \rangle^\omega}. \quad (3.1)$$

Then, we have the following boundedness of the bilinear operator \mathcal{T}_Θ .

Proposition 3.1. *Let $s_1, s_2, \omega \in \mathbb{R}$ and set*

$$N_0 = N_0(d, \omega, s_1, s_2) := d + \omega_+ + \frac{1}{2} \{ |s_1 + \omega| + |s_2 + \omega| \}, \quad (3.2)$$

where $\omega_+ := \max(\omega, 0)$. Suppose that $\|\Theta\|_{\omega, N} < \infty$ for some $N > N_0$. Then, \mathcal{T}_Θ defined in (2.2) is a bounded bilinear operator from $\ell_{s_1+\omega}^p(\mathbb{Z}^d) \times \ell_{s_2+\omega}^q(\mathbb{Z}^d)$ to $\ell_{s_1+s_2}^r(\mathbb{Z}^d)$ for any $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Remark 3.2. Proposition 3.1 is of particular interest when $s_1 = s_2 = \omega =: \frac{s}{2}$. In this case, for $N > d + \frac{1}{2}s_+ + |s|$, Proposition 3.1 implies boundedness of \mathcal{T}_Θ from $\ell_s^p(\mathbb{Z}^d) \times \ell_s^q(\mathbb{Z}^d)$ to $\ell_s^r(\mathbb{Z}^d)$. In particular, when $s_1 = s_2 = \omega = 0$, the hypothesis of Proposition 3.1 reduces to

$$\|\Theta\|_{0, N} = \sup_{j, k, \ell \in \mathbb{Z}^d} |\Theta(j, k, \ell)| \langle |j - k| + |j - \ell| \rangle^{2N} < \infty \quad (3.3)$$

for some $N > d$, which resembles the bound (1.5) in the continuous case (with $\alpha = \beta = \gamma = 0$). See also Corollary 3.3 below.

Proof of Proposition 3.1. Given $f = \{f_k\}_{k \in \mathbb{Z}^d} \in \ell_{s_1+\omega}^p$ and $g = \{g_\ell\}_{\ell \in \mathbb{Z}^d} \in \ell_{s_2+\omega}^q$, set $F = \{F_k\}_{k \in \mathbb{Z}^d}$ and $G = \{G_\ell\}_{\ell \in \mathbb{Z}^d}$ by $F_k = \langle k \rangle^{s_1+\omega} |f_k|$ and $G_\ell = \langle \ell \rangle^{s_2+\omega} |g_\ell|$ such that $F \in \ell^p$ and $G \in \ell^q$ with $\|F\|_{\ell^p} = \|f\|_{\ell_{s_1+\omega}^p}$ and $\|G\|_{\ell^q} = \|g\|_{\ell_{s_2+\omega}^q}$.

From the definition (3.1), we have

$$\begin{aligned} \|\mathcal{T}_\Theta(f, g)\|_{\ell_{s_1+s_2}^r}^r &= \sum_{j \in \mathbb{Z}^d} \left| \langle j \rangle^{s_1+s_2} \sum_{k, \ell \in \mathbb{Z}^d} \Theta(j, k, \ell) f_k g_\ell \right|^r \\ &\leq \sum_{j \in \mathbb{Z}^d} \left(\sum_{k, \ell \in \mathbb{Z}^d} |\Theta(j, k, \ell)| \langle j \rangle^{s_1+s_2} \langle k \rangle^{-s_1-\omega} \langle \ell \rangle^{-s_2-\omega} F_k G_\ell \right)^r \\ &\lesssim \|\Theta\|_{\omega, N}^r \sum_{j \in \mathbb{Z}^d} \left(\sum_{k, \ell \in \mathbb{Z}^d} \frac{\langle |j| + |k| \rangle^\omega \langle |j| + |\ell| \rangle^\omega \langle j \rangle^{s_1+s_2}}{\langle |j - k| + |j - \ell| \rangle^{2N} \langle k \rangle^{s_1+\omega} \langle \ell \rangle^{s_2+\omega}} F_k G_\ell \right)^r. \end{aligned} \quad (3.4)$$

By the triangle inequality, we have $\langle |j| + |k| \rangle \lesssim \langle j \rangle \langle j - k \rangle$. Combining this with a trivial bound $\langle |j| + |k| \rangle^{-1} \lesssim \langle j \rangle^{-1}$, we obtain

$$\langle |j| + |k| \rangle^\omega \lesssim \langle j \rangle^\omega \langle j - k \rangle^{\omega_+}, \quad (3.5)$$

where $\omega_+ = \max(\omega, 0)$. Similarly, we have

$$\langle |j| + |\ell| \rangle^\omega \lesssim \langle j \rangle^\omega \langle j - \ell \rangle^{\omega_+}. \quad (3.6)$$

By a version of Peetre's inequality [23], we have

$$\langle j \rangle^{s_i+\omega} \lesssim \min \left(\langle k \rangle^{s_i+\omega} \langle j - k \rangle^{|s_i+\omega|}, \langle \ell \rangle^{s_i+\omega} \langle j - \ell \rangle^{|s_i+\omega|} \right), \quad i = 1, 2. \quad (3.7)$$

In view of the condition $N > N_0$, write $2N = N_1 + N_2$, where

$$N_i > d + \omega_+ + |s_i + \omega|, \quad i = 1, 2. \quad (3.8)$$

Then, from (3.4), (3.5), (3.6), and (3.7) with a trivial bound $\langle j - k \rangle^{N_1} \langle j - \ell \rangle^{N_2} \leq \langle |j - k| + |j - \ell| \rangle^{2N}$, we have

$$\|\mathcal{T}_\Theta(f, g)\|_{\ell^r_{s_1+s_2}} \lesssim \|\Theta\|_{\omega, N} \left(\sum_{j \in \mathbb{Z}^d} (a_j b_j)^r \right)^{\frac{1}{r}}, \quad (3.9)$$

where

$$a_j = \sum_{k \in \mathbb{Z}^d} \langle j - k \rangle^{\omega_+ + |s_1 + \omega| - N_1} F_k,$$

$$b_j = \sum_{\ell \in \mathbb{Z}^d} \langle j - \ell \rangle^{\omega_+ + |s_2 + \omega| - N_2} G_\ell.$$

Let $a = \{a_j\}_{j \in \mathbb{Z}^d}$ and $b = \{b_j\}_{j \in \mathbb{Z}^d}$. Then, it follows from (3.9), Hölder's inequality, Young's inequality (for a discrete convolution), and (3.8) that

$$\begin{aligned} \|\mathcal{T}_\Theta(f, g)\|_{\ell^r_{s_1+s_2}} &\lesssim \|\Theta\|_{\omega, N} \|a\|_{\ell^p} \|b\|_{\ell^q} \\ &= \|\Theta\|_{\omega, N} \|\langle \cdot \rangle^{\omega_+ + |s_1 + \omega| - N_1} * F\|_{\ell^p} \|\langle \cdot \rangle^{\omega_+ + |s_2 + \omega| - N_2} * G\|_{\ell^q} \\ &\lesssim \|\Theta\|_{\omega, N} \|f\|_{\ell^p_{s_1+\omega}} \|g\|_{\ell^q_{s_2+\omega}}. \end{aligned}$$

This proves Proposition 3.1. \square

When $\omega = 0$, we have the following corollary.

Corollary 3.3. *Let $s_1, s_2 \in \mathbb{R}$. Suppose that Θ satisfies (3.3) for some $N > d + \frac{1}{2}\{|s_1| + |s_2|\}$. Then, \mathcal{T}_Θ is a bounded bilinear operator from $\ell^p_{s_1}(\mathbb{Z}^d) \times \ell^q_{s_2}(\mathbb{Z}^d)$ to $\ell^r_{s_1+s_2}(\mathbb{Z}^d)$ for any $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. In particular, if Θ satisfies (3.3) for some $N > d$, then \mathcal{T}_Θ is bounded from $\ell^p(\mathbb{Z}^d) \times \ell^q(\mathbb{Z}^d)$ to $\ell^r(\mathbb{Z}^d)$.*

Remark 3.4. (i) We can slightly change the norm in (3.1) on the infinite tensor Θ and impose a more general condition $\|\Theta\|_{\omega_1, \omega_2, N} < \infty$ for $\omega_1, \omega_2 \in \mathbb{R}$ and some appropriately large N , where now

$$\|\Theta\|_{\omega_1, \omega_2, N} := \sup_{j, k, \ell \in \mathbb{Z}^d} \frac{|\Theta(j, k, \ell)| \langle |j - k| + |j - \ell| \rangle^{2N}}{\langle |j| + |k| \rangle^{\omega_1} \langle |j| + |\ell| \rangle^{\omega_2}}.$$

Essentially the same argument as in the proof of Proposition 3.1 yields boundedness of the corresponding bilinear operator $\mathcal{T}_\Theta : \ell^p_{s_1+\omega_1}(\mathbb{Z}^d) \times \ell^q_{s_2+\omega_2}(\mathbb{Z}^d) \rightarrow \ell^r_{s_1+s_2}(\mathbb{Z}^d)$.

(ii) From the perspective of [13, Definition 2.1], it is also natural to consider the following norm on the infinite tensor Θ :

$$\|\Theta\|_{0,0,\omega,N} := \sup_{j,k,\ell \in \mathbb{Z}^d} \frac{|\Theta(j,k,\ell)| \langle |j-k| + |j-\ell| \rangle^{2N}}{\langle |j| + |k| + |\ell| \rangle^\omega}. \quad (3.10)$$

We will comment further on (3.10) in Section 5.

Remark 3.5. We wish to end this section by discussing the appropriateness of the condition $\|\Theta\|_{\omega, N} < \infty$ in Proposition 3.1. As mentioned in the introduction, the insight in [9] had to do with finding an appropriate notion of order ω for an infinite matrix $\sigma : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{C}$. This notion requires further defining the finite difference operator Δ^α , $\alpha \in \mathbb{Z}^d$; see Section 5. However, for the purposes of boundedness of the associated (linear) operator $\mathcal{L}_\sigma : \ell_s^2(\mathbb{Z}^d) \rightarrow \ell_{s-\omega}^2(\mathbb{Z}^d)$, defined in (2.3), we only need the condition

$$|\sigma(j, k)| \lesssim \langle |j| + |k| \rangle^\omega \langle j - k \rangle^{-M} \quad (3.11)$$

for all $j, k \in \mathbb{Z}^d$ and for some M sufficiently large; see [13, Lemma 2.2]. A natural way to bilinearize such a linear operator is to consider the tensor operator $\mathcal{T}_\Theta = \mathcal{L}_{\sigma_1} \otimes \mathcal{L}_{\sigma_2}$ with σ_1, σ_2 satisfying the condition (3.11); that is, for all $j \in \mathbb{Z}^d$, we set

$$\begin{aligned} (\mathcal{T}_\Theta(f, g))_j &= (L_{\sigma_1}(f))_j (L_{\sigma_2}(g))_j \\ &= \sum_{k \in \mathbb{Z}^d} \sigma_1(j, k) f_k \sum_{\ell \in \mathbb{Z}^d} \sigma_2(j, \ell) g_\ell \\ &= \sum_{k, \ell \in \mathbb{Z}^d} \underbrace{\sigma_1(j, k) \sigma_2(j, \ell)}_{=: \Theta(j, k, \ell)} f_k g_\ell. \end{aligned}$$

It is easy to see that, if σ_1 and σ_2 satisfy (3.11), then Θ in the last summation above satisfies $\|\Theta\|_{\omega, N} < \infty$ in (3.1) with $N = \frac{M}{2}$.

4. SMOOTHING OF DISCRETE BILINEAR COMMUTATORS

Bilinear commutators are natural objects to consider within harmonic analysis. Given a bilinear operator T and an appropriate function b , one can consider the following *bilinear commutators*:

$$\begin{aligned} [T, b]_1(f, g) &= T(bf, g) - bT(f, g), \\ [T, b]_2(f, g) &= T(f, bg) - bT(f, g). \end{aligned} \quad (4.1)$$

Just as their linear counterparts, these commutators have a smoothing effect. More precisely, it was shown in [6] that if T is a bilinear Calderón-Zygmund operator and $b \in CMO(\mathbb{R}^d)$, then $[T, b]_i$, $i = 1, 2$, are compact bilinear operators from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \rightarrow L^r(\mathbb{R}^d)$ for all $1 < p, q < \infty$ and $1 \leq r < \infty$, satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. See [3] for a related discussion. The definition of a bilinear compact operator is most natural; given three normed spaces X, Y , and Z , we say that a bilinear operator $T : X \times Y \rightarrow Z$ is *compact* if the set $\{T(f, g) : \|f\|_X, \|g\|_Y \leq 1\}$ is pre-compact in Z . Clearly, compactness of T implies the continuity of T . The proof of the compactness statement mentioned above and other subsequent compactness results in the literature make use of the so-called Fréchet-Kolmogorov-Riesz theorem which provides a characterization of pre-compactness in the L^p -spaces; see, for example, Yosida's book [24]. The version of this theorem for the ℓ^p spaces [16, Theorem 4] reads as follows.

Lemma 4.1. *A subset $\mathcal{F} \subset \ell^p(\mathbb{Z}^d)$, $1 \leq p < \infty$, is totally bounded if and only if the following two conditions are satisfied:*

- (i) \mathcal{F} is pointwise bounded,

(ii) Given any $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that

$$\left(\sum_{|j| > j_0} |f_j|^p \right)^{\frac{1}{p}} < \varepsilon$$

for any $f = \{f_j\}_{j \in \mathbb{Z}^d} \in \mathcal{F}$.

We now define $\ell_c^\infty(\mathbb{Z}^d)$ and $c_0(\mathbb{Z}^d)$ by

$$\begin{aligned} \ell_c^\infty(\mathbb{Z}^d) &= \{ \{b_k\}_{k \in \mathbb{Z}^d} : b_k = 0 \text{ for all but a finite number of } k \}, \\ c_0(\mathbb{Z}^d) &= \{ \{b_k\}_{k \in \mathbb{Z}^d} : b_k \rightarrow 0 \text{ as } |k| \rightarrow \infty \}. \end{aligned}$$

Recall from [21, Example III.1.3] that $c_0(\mathbb{Z}^d)$ is the completion of $\ell_c^\infty(\mathbb{Z}^d)$ with respect to the ℓ^∞ -norm. In the following, we work with $c_0(\mathbb{Z}^d)$. In the context of discrete bilinear operators, this is a natural space to study commutators with a sequence b if we recall that in the continuous case, the space $CMO(\mathbb{R}^d)$ is the closure of $C_c^\infty(\mathbb{R}^d)$ with respect to the BMO (bounded mean oscillations)-norm. Our main result is the following.

Theorem 4.2. *Suppose that Θ satisfies (3.3) for some $N > d$ and $b \in c_0(\mathbb{Z}^d)$. Let \mathcal{T}_Θ be as in (2.2). Then, the bilinear commutators $[\mathcal{T}_\Theta, b]_i$, $i = 1, 2$, defined as in (4.1), are compact bilinear operators from $\ell^p(\mathbb{Z}^d) \times \ell^q(\mathbb{Z}^d)$ to $\ell^r(\mathbb{Z}^d)$ for any $1 \leq p, q \leq \infty$ and $1 \leq r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.*

Proof. We will only consider the first commutator $[\mathcal{T}_\Theta, b]_1$. The calculations for $[\mathcal{T}_\Theta, b]_2$ are similar. Given any $b \in \ell^\infty(\mathbb{Z}^d)$, $f \in \ell^p(\mathbb{Z}^d)$, and $g \in \ell^q(\mathbb{Z}^d)$, it follows from Corollary 3.3 (with $N > d$) and Hölder's inequality that

$$\|[\mathcal{T}_\Theta, b]_1(f, g)\|_{\ell^r} \lesssim \|b\|_{\ell^\infty} \|f\|_{\ell^p} \|g\|_{\ell^q}. \quad (4.2)$$

That is, $[\mathcal{T}_\Theta, b]_1 : \ell^p(\mathbb{Z}^d) \times \ell^q(\mathbb{Z}^d) \rightarrow \ell^r(\mathbb{Z}^d)$ is bounded with the operator norm $\|[\mathcal{T}_\Theta, b]_1\| \lesssim \|b\|_{\ell^\infty}$. In particular, this proves condition (i) in Lemma 4.1. Hence, it remains to prove the condition (ii). Without loss of generality, assume $b \neq 0$. Let $f = \{f_k\}_{k \in \mathbb{Z}^d}$ and $g = \{g_k\}_{k \in \mathbb{Z}^d}$ such that $\|f\|_{\ell^p}, \|g\|_{\ell^q} \leq 1$. Then, our goal is to show that, given $\varepsilon > 0$, there exists $j_0 = j_0(\varepsilon) \in \mathbb{N}$ (independent of f and g) such that

$$\left(\sum_{|j| > j_0} |([\mathcal{T}_\Theta, b]_1(f, g))_j|^r \right)^{\frac{1}{r}} \lesssim \varepsilon. \quad (4.3)$$

In view of (4.2) and the density of $\ell_c^\infty(\mathbb{Z}^d)$ in $c_0(\mathbb{Z}^d)$ with respect to the ℓ^∞ -norm, it suffices to prove (4.3) for $b \in \ell_c^\infty(\mathbb{Z}^d)$. Thus, we assume that there exists $j_1 \in \mathbb{N}$ such that

$$b_j = 0 \quad (4.4)$$

for any $|j| > j_1$.

Formally, we have

$$([\mathcal{T}_\Theta, b]_1(f, g))_j = \sum_{k \in \mathbb{Z}^d} \sum_{\ell \in \mathbb{Z}^d} \Theta(j, k, \ell) (b_k - b_j) f_k g_\ell.$$

It follows from (4.4), (3.3), and Young's and Hölder's inequalities that, for $|j| > j_0 \gg j_1$, we have

$$\begin{aligned} |([\mathcal{T}_\Theta, b]_1(x, y))_j| &\leq \sum_{k \in \mathbb{Z}^d} \sum_{\ell \in \mathbb{Z}^d} \mathbf{1}_{|k| \leq j_1} \cdot |\Theta(j, k, \ell)| |b_k| |f_k| |g_\ell| \\ &\lesssim \|b\|_{\ell^\infty} \sum_{|k| \leq j_1} \frac{|f_k|}{\langle j - k \rangle^N} \sum_{\ell \in \mathbb{Z}^d} \frac{|g_\ell|}{\langle j - \ell \rangle^N} \\ &\lesssim \|b\|_{\ell^\infty} \langle j \rangle^{-N} \left(\sum_{|k| \leq j_1} |f_k| \right) \|g\|_{\ell^\infty} \\ &\lesssim \|b\|_{\ell^\infty} \langle j \rangle^{-N} j_1^{\frac{d}{p'}} \|f\|_{\ell^p} \|g\|_{\ell^q}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \|\mathbf{1}_{|j| > j_0} \cdot ([\mathcal{T}_\Theta, b]_1(f, g))_j\|_{\ell^r} &\lesssim \|b\|_{\ell^\infty} j_1^{\frac{d}{p'}} \|\mathbf{1}_{|j| > j_0} \cdot \langle j \rangle^{-N}\|_{\ell^r} \\ &\lesssim \|b\|_{\ell^\infty} j_1^{\frac{d}{p'}} \langle j_0 \rangle^{-N + \frac{d}{r}} < \varepsilon \end{aligned}$$

by choosing $j_0 \gg j_1$; in the calculations above, $\mathbf{1}_X$ denotes the characteristic function of the set X . This proves (4.3) and therefore completes the proof of Theorem 4.2. \square

5. TENSORS OF ORDER ω

In Section 3, we studied the boundedness property of the discrete bilinear operator \mathcal{T}_Θ for an infinite tensor $\Theta : \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{C}$ with a finite $\|\cdot\|_{\omega, N}$ -norm defined in (3.1). In this section, we seek for an analogous definition for infinite tensors as the one given in [9] for infinite matrices, and possible examples of such tensors. Following [9], we first define partial finite difference operators on the set of infinite tensors $\{\Theta(j, k, \ell)\}_{(j, k, \ell) \in \mathbb{Z}^{3d}}$; see also [13, Definition 2.1]. For $m \in \{1, \dots, d\}$, let $e_m = \{\delta_{mn}\}_{n \in \mathbb{Z}^d}$, where δ_{mn} is the Kronecker symbol. Given a tensor Θ , we denote by $\Theta_2^{m,+}$ and $\Theta_2^{m,-}$ the shifted tensors defined by

$$\Theta_2^{m,+}(j, k, \ell) = \Theta(j + e_m, k + e_m, \ell) \quad \text{and} \quad \Theta_2^{m,-}(j, k, \ell) = \Theta(j - e_m, k - e_m, \ell).$$

Then, we define the partial finite difference operators $\Delta_2^{m,+}$ and $\Delta_2^{m,-}$ acting on Θ :

$$\Delta_2^{m,+}\Theta := \Theta_2^{m,+} - \Theta \quad \text{and} \quad \Delta_2^{m,-}\Theta := \Theta_2^{m,-} - \Theta.$$

Similarly, we define $\Theta_3^{m,\pm}$ and $\Delta_3^{m,\pm}\Theta$ by

$$\Theta_3^{m,\pm}(j, k, \ell) = \Theta(j \pm e_m, k, \ell \pm e_m) \quad \text{and} \quad \Delta_3^{m,\pm}\Theta := \Theta_3^{m,\pm} - \Theta.$$

Let $i = 2, 3$. Then, for $t \in \mathbb{Z}$, we set $\Delta_{i,m}^t = (\Delta_i^{m, \text{sign}(t)})^{|t|}$ and $\Delta_{i,m}^0 = \text{Id}$. Finally, for $\alpha = \{\alpha_m\}_{m=1}^d \in \mathbb{Z}^d$, we set

$$\Delta_i^\alpha = \Delta_{i,1}^{\alpha_1} \cdots \Delta_{i,d}^{\alpha_d}.$$

Definition 5.1. Let $\omega \in \mathbb{R}$ and $N \in \mathbb{N}$. We say that Θ belongs to the class $BT^{\omega, N}(\mathbb{Z}^d)$ if, for all $\alpha, \beta \in \mathbb{Z}^d$, there exists $C_{N, \alpha, \beta} > 0$ such that

$$|\Delta_2^\alpha \Delta_3^\beta \Theta(j, k, \ell)| \leq C_{N, \alpha, \beta} \langle |j| + |k| + |\ell| \rangle^{\omega - |\alpha| - |\beta|} \langle |j - k| + |j - \ell| \rangle^{-2N} \quad (5.1)$$

for any $j, k, \ell \in \mathbb{Z}^d$.

Clearly, given $N \in \mathbb{N}$, the bilinear tensor classes $BT^{\omega, N}(\mathbb{Z}^d)$ of order ω are nested, that is, if $\omega_1 \leq \omega_2$, then $BT^{\omega_1, N}(\mathbb{Z}^d) \subseteq BT^{\omega_2, N}(\mathbb{Z}^d)$. Note that $BT^{\omega, N}(\mathbb{Z}^d)$ is a Fréchet space under the family of semi-norms:

$$\|\Theta\|_{\alpha, \beta, \omega, N} := \sup_{j, k, \ell \in \mathbb{Z}^d} \frac{|\Delta_2^\alpha \Delta_3^\beta \Theta(j, k, \ell)| \langle |j - k| + |j - \ell| \rangle^{2N}}{\langle |j| + |k| + |\ell| \rangle^{\omega - |\alpha| - |\beta|}} \quad (5.2)$$

for $\alpha, \beta \in \mathbb{Z}^d$. When $\alpha = \beta = 0$ (as elements in \mathbb{Z}^d), (5.2) reduces to the expression in (3.10). Let us point out that, even when $\alpha = \beta = 0$, (3.10) and (3.1) are, in general, not comparable. However, we have the following estimates:

- If $\omega \geq 0$, then $\|\Theta\|_{0, 0, 2\omega, N} \lesssim \|\Theta\|_{\omega, N} \lesssim \|\Theta\|_{0, 0, \omega, N}$,
- If $\omega \leq 0$, then $\|\Theta\|_{0, 0, \omega, N} \lesssim \|\Theta\|_{\omega, N} \lesssim \|\Theta\|_{0, 0, 2\omega, N}$.

An immediate consequence of Proposition 3.1 is given below.

Corollary 5.2. *Given $s_1, s_2, \omega \in \mathbb{R}$ and $N \in \mathbb{N}$, suppose that*

- *if $\omega \geq 0$, then $\Theta \in BT^{\omega, N}(\mathbb{Z}^d)$ for some $N > N_0$, where N_0 is as in (3.2),*
- *if $\omega < 0$, then $\Theta \in BT^{2\omega, N}(\mathbb{Z}^d)$ for some $N > N_0$.*

Let \mathcal{T}_Θ be as in (2.2). Then, \mathcal{T}_Θ is a bounded bilinear operator from $\ell_{s_1+\omega}^p(\mathbb{Z}^d) \times \ell_{s_2+\omega}^q(\mathbb{Z}^d)$ to $\ell_{s_1+s_2}^r(\mathbb{Z}^d)$ for any $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. In particular, if we define the bilinear tensor class $BT^0(\mathbb{Z}^d)$ of order zero by

$$BT^0(\mathbb{Z}^d) := \bigcup_{N > d} BT^{0, N}(\mathbb{Z}^d),$$

then \mathcal{T}_Θ is a bounded bilinear operator from $\ell^p(\mathbb{Z}^d) \times \ell^q(\mathbb{Z}^d)$ to $\ell^r(\mathbb{Z}^d)$ for any $\Theta \in BT^0(\mathbb{Z}^d)$.

Let us provide simple examples of bilinear tensors of order zero. Consider the following tensor:

$$\Theta_1(j, k, \ell) = \theta_j \cdot \mathbf{1}_{j=k=\ell} \cdot \mathbf{1}_{|j|+|k|+|\ell| \leq M}$$

with $\{\theta_j\}_{j \in \mathbb{Z}^d} \in \ell^\infty(\mathbb{Z}^d)$ and some $M \in \mathbb{N}$. Then, it is easy to see that $\Theta_1 \in BT^0(\mathbb{Z}^d)$. Next, given a function $\Phi : \mathbb{Z}^d \times \mathbb{Z}^d \rightarrow \mathbb{C}$ with a bound $|\Phi(x, y)| \lesssim \langle |x| + |y| \rangle^{-N}$ for any $x, y \in \mathbb{Z}^d$ and for some $N > d$, consider

$$\Theta_2(j, k, \ell) = \Phi(j - k, j - \ell) \cdot \mathbf{1}_{j=k+\ell}.$$

Noting that $\Delta_2^\alpha \Delta_3^\beta \Theta_2(j, k, \ell) = 0$ unless $\alpha = \beta = 0$, we see that $\Theta_2 \in BT^0(\mathbb{Z}^d)$.

We also state a compactness result on the bilinear commutators of a discrete bilinear operator \mathcal{T}_Θ with $\Theta \in BT^0(\mathbb{Z}^d)$ and $b \in c_0(\mathbb{Z}^d)$, which follows from Theorem 4.2 and Corollary 5.2.

Corollary 5.3. *Suppose that $\Theta \in BT^0(\mathbb{Z}^d)$ and $b \in c_0(\mathbb{Z}^d)$. Let \mathcal{T}_Θ be as in (2.2). Then, the bilinear commutators $[\mathcal{T}_\Theta, b]_i$, $i = 1, 2$, defined as in (4.1), are compact bilinear operators from $\ell^p(\mathbb{Z}^d) \times \ell^q(\mathbb{Z}^d)$ to $\ell^r(\mathbb{Z}^d)$ for any $1 \leq p, q \leq \infty$ and $1 \leq r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.*

Remark 5.4 (transposes). The (formal) transposes of a (continuous) bilinear operator $T : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ are defined by the duality relations

$$\langle T(f, g), h \rangle = \langle T^{*1}(h, g), f \rangle = \langle T^{*2}(f, h), g \rangle.$$

If T has kernel K as in (1.4), then its formal transposes T^{*1} and T^{*2} have kernels given by $K^{*1}(x, y, z) = K(y, x, z)$ and $K^{*2}(x, y, z) = K(z, y, x)$, respectively. In the discrete setting, for $\mathcal{T}_\Theta : \ell^p(\mathbb{Z}^d) \times \ell^q(\mathbb{Z}^d) \rightarrow \ell^r(\mathbb{Z}^d)$ with a discrete symbol Θ , using the natural duality pairing

$$\langle \mathcal{T}(f, g), h \rangle = \sum_{j \in \mathbb{Z}^d} (\mathcal{T}(f, g))_j h_j = \sum_{j, k, \ell \in \mathbb{Z}^d} \Theta(j, k, \ell) f_k g_\ell h_j,$$

it is easy to see that, given $\Theta \in BT^{\omega, N}(\mathbb{Z}^d)$ for some $N \in \mathbb{N}$, we have $(\mathcal{T}_\Theta)^{*i} = \mathcal{T}_{\Theta^{*i}}$ with $\Theta^{*i} \in BT^{\omega, N}(\mathbb{Z}^d)$, $i = 1, 2$, where $\Theta^{*1}(j, k, \ell) = \Theta(k, j, \ell)$ and $\Theta^{*2}(j, k, \ell) = \Theta(\ell, k, j)$. This is not surprising considering that if K is a bilinear Calderón-Zygmund kernel, then so are K^{*i} , $i = 1, 2$; see also [2, Theorem 2.1] for the symbolic calculus of the Hörmander class $BS_{1,0}^\omega(\mathbb{R}^d)$.

We conclude this paper by presenting some other natural examples of infinite tensors belonging to the classes introduced in Definition 5.1.

Given multi-indices $a, b \in (\mathbb{N} \cup \{0\})^d$, consider the following bilinear partial differential operator on \mathbb{T}^d :

$$T_{a,b}(F, G)(x) = \frac{\partial^a F}{\partial x^a} \frac{\partial^b G}{\partial x^b}, \quad x \in \mathbb{T}^d. \quad (5.3)$$

By taking the Fourier transform, we have

$$\mathcal{F}(T_{a,b}(F, G))(j) = \sum_{k \in \mathbb{Z}^d} \sum_{\ell \in \mathbb{Z}^d} (2\pi i k)^a (2\pi i \ell)^b \cdot \mathbf{1}_{j=k+\ell} \cdot f_k g_\ell,$$

where $f_k = \widehat{F}(k)$ and $g_\ell = \widehat{G}(\ell)$. By setting

$$\Theta_{a,b}(j, k, \ell) = (2\pi i k)^a (2\pi i \ell)^b \cdot \mathbf{1}_{j=k+\ell}, \quad (5.4)$$

we then have $\mathcal{T}_{\Theta_{a,b}}(f, g)(j) = \mathcal{F}(T_{a,b}(F, G))(j)$, $j \in \mathbb{Z}^d$. More generally, we can consider a tensor Θ_Φ of the form

$$\Theta_\Phi(j, k, \ell) = \Phi(k, \ell) \cdot \mathbf{1}_{j=k+\ell}. \quad (5.5)$$

Lemma 5.5. *Let $\Phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C})$. Suppose that there exists $\omega \in \mathbb{R}$ such that*

$$|\partial_x^\alpha \partial_y^\beta \Phi(x, y)| \lesssim \langle |x| + |y| \rangle^{\omega - |\alpha| - |\beta|} \quad (5.6)$$

for any multi-indices $\alpha, \beta \in (\mathbb{N} \cup \{0\})^d$ and $x, y \in \mathbb{R}^d$. Then, the tensor Θ_Φ defined in (5.5) belongs to $BT^{\omega+2N, N}(\mathbb{Z}^d)$ for any $N \in \mathbb{N}$. In particular, if $\Theta_{a,b}$ is the tensor of the bilinear partial differential operator given in (5.4), then $\Theta_{a,b} \in BT^{|a|+|b|+2N, N}(\mathbb{Z}^d)$.

Proof. Under $j = k + \ell$, we have

$$\begin{aligned} \mathbf{1}_{j=k+\ell} \cdot \frac{\langle |j| + |k| + |\ell| \rangle^{\omega+2N-|\alpha|-|\beta|}}{\langle |j-k| + |j-\ell| \rangle^{2N}} &\sim \mathbf{1}_{j=k+\ell} \cdot \langle |j| + |k| + |j-k| \rangle^{\omega-|\alpha|-|\beta|} \\ &\sim \mathbf{1}_{j=k+\ell} \cdot \langle |k| + |j-k| \rangle^{\omega-|\alpha|-|\beta|} \end{aligned}$$

and thus it suffices to show that, for any $\alpha, \beta \in \mathbb{Z}^d$, there exists $C_{\alpha, \beta} > 0$ such that

$$|\Delta_2^\alpha \Delta_3^\beta \Theta_\Phi(j, k, \ell)| \leq C_{\alpha, \beta} \cdot \mathbf{1}_{j=k+\ell} \cdot \langle |k| + |j-k| \rangle^{\omega-|\alpha|-|\beta|} \quad (5.7)$$

for any $j, k, \ell \in \mathbb{Z}^d$. For simplicity of notation, we drop $\mathbf{1}_{j=k+\ell}$ but it is understood that $j = k + \ell$ in the following. Moreover, since the constant $C_{N, \alpha, \beta}$ in (5.1) can depend on α and β , we only need to prove the bound (5.1) for

$$|j|, |k|, |\ell| \gg |\alpha| + |\beta|. \quad (5.8)$$

When $\alpha = \beta = 0$, we have

$$|\Phi(k, \ell)| \lesssim \langle |k| + |\ell| \rangle^\omega,$$

yielding (5.7). Let $m = 1, \dots, d$. By the mean value theorem, we have

$$\Delta_3^{m,+} \Theta_\Phi(j, k, \ell) = \Phi(j - \ell, \ell + e_m) - \Phi(j - \ell, \ell) = \partial_{y_m} \Phi(j - \ell, \ell + \varepsilon e_m)$$

for some $\varepsilon \in [0, 1]$. By iteratively applying difference operators with the mean value theorem, we have

$$\begin{aligned} \Delta_3^\beta \Theta_\Phi(j, k, \ell) &= \partial_y^\beta \Phi(j - \ell, \ell + \varepsilon_\beta) \\ &= \partial_y^\beta \Phi(k, j - k + \varepsilon_\beta) \end{aligned}$$

for some $|\varepsilon_\beta| \leq |\beta|$. Now, by applying $\Delta_2^\alpha = \Delta_{2,1}^{\alpha_1} \cdots \Delta_{2,d}^{\alpha_d}$ in an iterative manner together with the mean value theorem, we have

$$\Delta_2^\alpha \Delta_3^\beta \Theta_\Phi(j, k, \ell) = \partial_x^\alpha \partial_y^\beta \Phi(k + \varepsilon_\alpha, j - k + \varepsilon_\beta)$$

for some $|\varepsilon_\alpha| \leq |\alpha|$. In view of (5.6) and (5.8), we then obtain

$$\begin{aligned} |\Delta_2^\alpha \Delta_3^\beta \Theta_\Phi(j, k, \ell)| &\lesssim \langle |k + \varepsilon_\alpha| + |j - k + \varepsilon_\beta| \rangle^{\omega-|\alpha|-|\beta|} \\ &\lesssim \langle |k| + |j - k| \rangle^{\omega-|\alpha|-|\beta|}, \end{aligned}$$

which yields (5.7). □

Example 5.6. Next, let us consider the multiplication operator M_V on the physical side given by

$$M_V(F, G)(x) = V(x)F(x)G(x).$$

By taking the Fourier transform, we have

$$\mathcal{F}(M_V(F, G)) = \sum_{k, \ell \in \mathbb{Z}^d} \widehat{V}(j - k - \ell) f_k g_\ell.$$

Given $V \in C^\infty(\mathbb{T}^d)$, consider the tensor

$$\Theta_V(j, k, \ell) := \widehat{V}(j - k - \ell).$$

Note that we have $\Delta_2^\alpha \Delta_3^\beta \Theta_V(j, k, \ell) = 0$ unless $\alpha = \beta = 0$. By the smoothness of V , we have

$$|\Theta_V(j, k, \ell)| = |\widehat{V}(j - k - \ell)| \lesssim \langle j - k - \ell \rangle^{-K}$$

for any $K > 0$. However, note that, for $j = 2k = 2\ell$, we have $|\widehat{V}(j - k - \ell)| \lesssim 1$, while

$$\langle |j - k| + |j - \ell| \rangle^{-2N} \sim \langle j \rangle^{-2N} \rightarrow 0$$

as $|j| \rightarrow \infty$. This shows that we have $\Theta_V \in BT^{\omega, N}(\mathbb{Z}^d)$ only for $\omega \geq 2N$. Compare this with the linear case studied in Lemma 2.7 (ii) in [13].

Example 5.7. Lastly, consider the tensor

$$\Theta_{V, \Phi}(j, k, \ell) = \widehat{V}(j - k - \ell) \Phi(k, \ell),$$

where $V \in C^\infty(\mathbb{T}^d)$ and $\Phi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C})$ satisfies (5.6). The tensor for the bilinear partial differential operator with variable coefficients in (1.3) is given by a linear combination of such tensors. By arguing as in the proof of Lemma 5.5, we have

$$\Delta_2^\alpha \Delta_3^\beta \Theta_{V, \Phi}(j, k, \ell) = \widehat{V}(j - k - \ell) \partial_x^\alpha \partial_y^\beta \Phi(k + \varepsilon_\alpha, j - k + \varepsilon_\beta)$$

for some $|\varepsilon_\alpha| \leq |\alpha|$ and $|\varepsilon_\beta| \leq |\beta|$. Without loss of generality, assume (5.8). Then, we have

$$|\Delta_2^\alpha \Delta_3^\beta \Theta_{V, \Phi}(j, k, \ell)| \lesssim \langle j - k - \ell \rangle^{-K} \langle |k| + |\ell| \rangle^{\omega - |\alpha| - |\beta|}.$$

for any $K > 0$. When $|j| \lesssim |k| + |\ell|$, we have

$$|\Delta_2^\alpha \Delta_3^\beta \Theta_{V, \Phi}(j, k, \ell)| \lesssim \langle |j| + |k| + |\ell| \rangle^{\omega - |\alpha| - |\beta|}.$$

On the other hand, when $|j| \gg |k| + |\ell|$, we have

$$|\Delta_2^\alpha \Delta_3^\beta \Theta_{V, \Phi}(j, k, \ell)| \lesssim \langle j \rangle^{-K + \omega} \sim \langle |j| + |k| + |\ell| \rangle^{-K + \omega}$$

for any $K > 0$. Hence, we conclude that $\Theta_{V, \Phi} \in BT^{\omega + 2N, N}(\mathbb{Z}^d)$ for any $N \in \mathbb{N}$.

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