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GLOBAL WELL-POSEDNESS FOR THE MASSLESS CUBIC DIRAC EQUATION

NIKOLAOS BOURNAVEAS AND TIMOTHY CANDY

ABSTRACT. We show that the cubic Dirac equation with zero mass is globally well-posed for small data in the scale invariant space $\dot{H}^{\frac{n-1}{2}}(\mathbb{R}^n)$ for $n = 2, 3$. The proof proceeds by using the Fierz identities to rewrite the equation in a form where the null structure of the system is readily apparent. This null structure is then exploited via bilinear estimates in spaces based on the null frame spaces of Tataru. We hope that the spaces and estimates used here can be applied to other nonlinear Dirac equations in the scale invariant setting. Our work complements recent results of Bejenaru-Herr who proved a similar result for $n = 3$ in the massive case.

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1. INTRODUCTION

Given a mass $m \geq 0$ we consider the nonlinear Dirac equation

$$\begin{aligned} -i\gamma^\mu \partial_\mu \psi + m\psi &= F(\psi) \\ \psi(0) &= \psi_0 \end{aligned} \tag{1}$$

for a spinor $\psi(t, x) : \mathbb{R}^{1+n} \rightarrow \mathbb{C}^N$ where $N = 2^{\lfloor \frac{n+1}{2} \rfloor}$ and $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$. The Gamma matrices γ^μ are constant $N \times N$ matrices satisfying the anti-commutativity properties

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2I g^{\mu\nu}$$

where $g^{\mu\nu}$ is the Minkowski metric $g = \text{diag}(1, -1, \dots, -1)$, and repeated upper and lower indices are summed over $\mu = 0, \dots, n$. We are interested in the special case of (1) where the nonlinearity F is cubic

and has some additional structure. More precisely, we consider the Lorentz invariant cubic nonlinearities

$$F(\psi) = \begin{cases} (\bar{\psi}\psi)\psi \\ (\bar{\psi}\gamma^\mu\psi)\gamma_\mu\psi \end{cases} \quad (2)$$

which are known as the Soler model [41], and the Thirring model [49] respectively. Here $\bar{\psi} = \psi^\dagger\gamma^0$ is the Dirac adjoint, and ψ^\dagger is the complex conjugate transpose of the vector ψ . The nonlinear Dirac equation is an important equation in relativistic quantum mechanics, and models the self-interaction of Dirac fermions, we refer the reader to [17, 48] for more on the physical background of the Dirac equation.

The nonlinear Dirac equation (1) with cubic nonlinearity (2) and mass $m = 0$ is invariant under the scaling $\psi(t, x) \mapsto \lambda^{\frac{1}{2}}\psi(\lambda t, \lambda x)$. Thus the scale invariant regularity is $s_c = \frac{n-1}{2}$ and it is expected that we have some form of ill-posedness for data $\psi_0 \in H^s(\mathbb{R}^n)$ with $s < \frac{n-1}{2}$. In terms of the well-posedness of the Cauchy problem, in the $n = 3$ case, work of Tzvetkov [50] via the method of commuting vector fields, shows that we have global existence in time for small smooth data in the case $|F(\psi)| \lesssim |\psi|^p$, $p > 2$. This extends earlier results of Reed [39], Dias-Figueira [14], and Escobedo-Vega [15]. In the low regularity setting, Machihara-Nakanishi-Ozawa [31] obtained global existence for small data in $H^s(\mathbb{R}^3)$ in the almost critical case $s > 1$ for positive mass $m > 0$, and cubic nonlinearities (2). This was improved to radial data (or data with some additional angular regularity) in $H^1(\mathbb{R}^3)$ by Machihara-Nakamura-Nakanishi-Ozawa [30]. Very recently, Bejenaru-Herr [2] proved that provided $m > 0$ and $F(\psi) = (\bar{\psi}\psi)\psi$, we have global well-posedness and scattering for small data in the critical space $H^1(\mathbb{R}^3)$.

On the other hand, in the $n = 2$ case, it was shown by Pecher [36, 37] that we have local well-posedness from data in $H^s(\mathbb{R}^2)$ in the almost critical case $s > \frac{1}{2}$. In the $n = 1$ case, global well-posedness for the Thirring model with large data in $H^s(\mathbb{R})$ with $s \geq 1$ is due to Delgado [13], this was improved to $s > \frac{1}{2}$ by Selberg-Tesfahun [40]. The critical case $s = 0$ was considered by the second author in [7] where it was shown that the Thirring model is globally well-posed for large data in $L^2(\mathbb{R})$. The question of scattering for the massive case $m > 0$ is still open. As well as the above mentioned results, if $n = 1$ it is known that the Thirring model $F(\psi) = (\bar{\psi}\gamma^\mu\psi)\gamma_\mu\psi$ is completely integrable [20, 51], and the stability of stationary solutions has been studied [10, 38]. The existence of stationary solutions in $n = 3$ is also known [9, 32, 45].

In the current article we are interested in the global well-posedness of small data in the critical space $\dot{H}^{\frac{n-1}{2}}(\mathbb{R}^n)$ for $n = 2, 3$. Our main result is the following.

Theorem 1.1. *Let $n = 2, 3$, $m = 0$, and $s \geq \frac{n-1}{2}$. Assume F is as in (2). There exists $\epsilon > 0$ such that if $\psi_0 \in \dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n)$ with*

$$\|\psi_0\|_{\dot{H}^{\frac{n-1}{2}}(\mathbb{R}^n)} < \epsilon$$

then we have a global solution $\psi \in C(\mathbb{R}, \dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n))$ to (1) with

$$\sup_{t \in \mathbb{R}} \|\psi(t)\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n)} \lesssim \|\psi_0\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n)}.$$

Moreover, the solution ψ depends continuously on the initial data and is the unique limit of smooth solutions. Finally, there exists $\psi_{\pm\infty} \in C(\mathbb{R}, \dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n))$ with $\gamma^\mu\partial_\mu\psi_{\pm\infty} = 0$ such that

$$\lim_{t \rightarrow \pm\infty} \|\psi(t) - \psi_{\pm\infty}(t)\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n)} = 0.$$

Remark 1.2. If $m > 0$ then for small data in $\psi_0 \in \dot{H}^{\frac{n-1}{2}}(\mathbb{R}^n)$ we have existence of a solution up to time $T \ll m^{-1}$, see Remark 7.2. This is essentially due to the fact that for times $T \ll m^{-1}$ the solution to the wave equation and Klein-Gordon equation is more or less the same. Of course if $m > 0$ and $n = 3$, then we already have global existence due to the work of Bejenaru-Herr [2]. On the other hand, local existence for $m > 0$ with data in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ is new in the case $n = 2$. Similarly, it is possible to use finite speed of propagation to deduce local in time existence for large data in $\dot{H}^{\frac{n-1}{2}}(\mathbb{R}^n)$ (this is true for any $m \geq 0$).

Remark 1.3. If $n = 3$ and we let $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, then Theorem 1.1 also holds in the cases where $F(\psi)$ is given by

$$(\bar{\psi}\gamma^5\psi)\gamma^5\psi, \quad (\bar{\psi}\psi)\gamma^5\psi, \quad (\bar{\psi}\gamma^5\psi)\psi.$$

In other words, we can more or less handle any nonlinearity built up using the bilinear Dirac null forms $\bar{\psi}\psi$ and $\bar{\psi}\gamma^5\psi$.

Remark 1.4. The nonlinear Dirac equation (1) together with the nonlinearity (2), satisfies *conservation of charge* $\|\psi(t)\|_{L_x^2} = \|\psi(0)\|_{L_x^2}$. Thus in Theorem 1.1 we may replace the homogeneous Sobolev spaces \dot{H}^s with the inhomogeneous spaces H^s . We should point out that the Dirac equation has other conserved quantities. However, they are not strictly positive, and thus do not appear to be immediately useful in the large data theory.

The first step in the proof of Theorem 1.1 is to rewrite the equation so that the null structure of the system is easy to exploit. The standard way to do this is to use projections to reduce (1) to studying the scalar half wave equations $(\partial_t \pm |\nabla|)$. In particular this method was used in the recent work of Pecher [36] and Bejenaru-Herr [2]. In the current article, we instead work with a *vector valued* formulation. Working in the vector valued setting has two key advantages. The first is that it only makes use of the derivatives¹ ∂_μ (as opposed to the Fourier multipliers $|\nabla|$), and thus behaves well under changes of coordinates. The second advantage is that, after an application of a *Fierz type identity* [16], the null structure hidden in the nonlinearities (2) manifests itself in products of vector valued waves traveling in opposite directions. On the other hand, the cost of avoiding the $\partial_t \pm |\nabla|$ formulation of (1) and using a vector valued formulation, is that the function spaces we construct need to retain this vectorial information in order to be able to prove the bilinear estimates that are needed to close an iteration argument.

The second, and more difficult, step in the proof of Theorem 1.1 is to construct appropriate function spaces and prove a number of bilinear null form estimates. The spaces used are a combination of vector valued version of the null frame spaces of Tataru [47], together with $X^{s,b}$ and energy type components. In slightly more detail, we define a norm that is schematically of the form $\|\psi\|_F = \|\psi\|_{L_t^\infty \dot{H}_x^{\frac{n-1}{2}}} + \|\gamma^\mu \partial_\mu \psi\|_Y$ and take $Y = L_t^1 \dot{H}_x^{\frac{n-1}{2}} + X^{-\frac{1}{2},1} + NF$ where $X^{-\frac{1}{2},1}$ is an $X^{s,b}$ type space with ℓ^1 sum over distances to the cone, and NF is based on the null frame spaces of Tataru [47]. The construction of the required spaces and the study of their basic properties is rather involved, and takes up a significant portion of the current paper. However, we believe that these spaces should be applicable to other endpoint well-posedness results for related systems such as the Dirac-Klein-Gordon, Maxwell-Dirac, Chern-Simons-Dirac etc. We

¹This can be thought of as a higher dimensional analogue of the two $n = 1$ formulations of the Dirac equation $(\partial_t \pm \partial_x)$ and $(\partial_t \pm |\partial_x|)$. The first formulation has the benefit that it is easy to write in the null coordinates $(t \pm x)$, and this is more or less the key property that led to the $L^2(\mathbb{R})$ critical result in [7].

plan to return to this problem in the future.

To explain the key difficulties in the proof of Theorem 1.1, note that the well-posedness theory for (1) would follow easily via standard energy estimates provided we had the $L_t^2 L_x^\infty$ estimate

$$\|\psi\|_{L_t^2 L_x^\infty(\mathbb{R}^{1+n})} \lesssim \|\psi(0)\|_{\dot{H}_x^{\frac{n-1}{2}}(\mathbb{R}^n)} \quad (3)$$

for solutions to (1) with $F = 0$. Unfortunately, it is well-known that this estimate *just* fails in the $n = 3$ case, and is far from true in the $n = 2$ case. Thus the low regularity well-posedness theory for (1) has more or less proceeded by trying to find suitable substitutes for the missing $L_t^2 L_x^\infty$ Strichartz estimate (3). One approach, used by Pecher [36, 37], is to move to the *bilinear* setting, and exploit the additional structure of the nonlinearity (2) via bilinear estimates in $X^{s,b}$ type spaces. While this works in the subcritical setting, it does not appear sufficient to handle the critical case $s = \frac{n-1}{2}$, see Remark 3.3. An alternative approach used in the work of Machihara-Nakamura-Nakanishi-Ozawa [30], is to exploit the fact that (3) is in fact true for *radial* data in $n = 3$. This is not quite enough on its own, as the Dirac equation does not commute with rotations, and thus radial data does not lead to radial solutions. Instead, Machihara-Nakamura-Nakanishi-Ozawa proved a version of (3) with additional regularity in the angular variable.

The approach in the current article relies on the following crucial observation of Tataru [47]. Although the estimate (3) fails for $n = 2, 3$ (and $m = 0$), the solution *can* be placed into spaces of the form $L^2 L^\infty$ provided we work in rotated null frames (t_ω, x_ω) where $\sqrt{2}t_\omega = (t, x) \cdot (1, \omega)$, $x_\omega = (t, x) - \frac{1}{\sqrt{2}}t_\omega(1, \omega)$ where $\omega \in \mathbb{S}^{n-1}$ is a direction on the sphere. These spaces exploit the fact that if $\text{supp } \hat{f} \subset \{|\xi| \approx \lambda, |\frac{\xi}{|\xi|} - \omega| \leq (T\lambda)^{-\frac{1}{2}}\}$, then for times $|t| \leq T$ we expect $(e^{it|\nabla|}f)(x) \approx f(x + t\omega)$. A computation then shows that $\|\mathbb{1}_{|t| < T}(t)e^{it|\nabla|}f\|_{L_{t_\omega}^2 L_{x_\omega}^\infty} \lesssim (\frac{\lambda}{T})^{\frac{n-1}{2}} \|f\|_{L^2}$. Of course to exploit this concentration property, requires localising the Fourier support to small sets. Thus to control a general function, we need to use many frames simultaneously.

In the $n = 1$ case, the gain formed by working in null frames is particularly easy to observe as the solution can only propagate in 2 directions $x \pm t$. More precisely, note that in the case $n = 1$, we can write the solution to (1) as $\psi(t, x) = f(x - t) + g(x + t)$. Clearly $\psi \notin L_t^2 L_x^\infty$, however, we *do* have

$$\|f(x - t)\|_{L_{x-t}^2 L_{x+t}^\infty(\mathbb{R}^{1+1})} = \|f\|_{L_x^2(\mathbb{R})}.$$

Thus despite the fact that (3) fails, we can place our solution in spaces of the form $L_{t \pm x}^2 L_{t \mp x}^\infty$. This simple observation played a key role in the $n = 1$ proof of critical well-posedness [7]. In higher dimensions, the solution can now travel in many directions $\omega \in \mathbb{S}^{n-1}$, thus instead of a fixed frame $L_{t_\omega}^2 L_{x_\omega}^\infty$, following the work of Tataru [47] we are forced to work in atomic Banach spaces made up of ℓ^1 sums of $L_{t_\omega}^2 L_{x_\omega}^\infty$ functions for various directions ω .

It is worth comparing the results presented here with the work of Bejenaru-Herr [2] on the positive mass case $m > 0$. There it was observed that if $m > 0$ and $n = 3$, then the estimate (3) is true, provided we localise to frequencies $\lesssim 1$, or small angular caps. Unfortunately, while the additional dispersion given by the positive mass is helpful for small frequencies, the loss of scaling and the additional curvature of the characteristic surface complicates the analysis for high frequencies. In particular, to control the high

frequency components of the evolution, the work of Bejenaru-Herr required the use of null frames adapted to the hyperboloid $\tau = \pm\sqrt{|\xi|^2 + m}$.

The outline of the paper is as follows. In Subsection 1.1 we rewrite the equation (1) in a more accessible form, and use this formulation to provide a simple proof of a bilinear null form estimate in $L_{t,x}^2$. The main notation used is introduced in Section 2. In Section 3, we define the function spaces used to prove Theorem 1.1. The main linear estimates we require are stated in Section 4. In Sections 5 and 6 we prove our key bilinear and trilinear estimates. The proof of Theorem 1.1 is then given in Section 7. In Sections 8 and 9 we prove the linear estimates stated in Section 4. Finally, in Section 10, we prove a version of the energy inequality needed in the proof of Theorem 1.1.

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1.1. Structure of Dirac equation. We take the standard representations of the Gamma matrices in the terms of the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, for $n = 2$ we take

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^2, \quad \gamma^2 = -i\sigma^1$$

and if $n = 3$ we let

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \quad j = 1, 2, 3, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

To proceed further, we note that we have the special case of a Fierz type identity²

$$(\bar{\psi}\gamma^\mu\psi)\gamma_\mu\psi = \begin{cases} (\bar{\psi}\psi)\psi & n = 2 \\ (\bar{\psi}\psi)\psi - (\bar{\psi}\gamma^5\psi)\gamma^5\psi & n = 3. \end{cases} \quad (4)$$

This somewhat magical identity is the key to showing that the Thirring model nonlinearity is also a null form, and also shows that in $n = 2$ the Thirring and Soler models are identical. Define

$$\sigma \cdot \nabla = \sigma^j \partial_j = \begin{cases} \sigma^1 \partial_1 + \sigma^2 \partial_2 & n = 2 \\ \sigma^1 \partial_1 + \sigma^2 \partial_2 + \sigma^3 \partial_3 & n = 3 \end{cases} \quad \text{and} \quad \beta = \begin{cases} \sigma^3 & n = 2 \\ 0 & n = 3. \end{cases}$$

We claim that (1) with $m = 0$ is a special case of the system

$$\begin{aligned} (\partial_t + \sigma \cdot \nabla)u &= B_1(u, v)v + B_2(u, v)\beta u \\ (\partial_t - \sigma \cdot \nabla)v &= B_3(u, v)u + B_4(u, v)\beta v \end{aligned} \quad (5)$$

²This is essentially a special case of a *Fierz Identity* [16] which states that, in the $n = 3$ case, given $z_j \in \mathbb{C}^4$ we have

$$(\bar{z}_1\gamma^\mu z_2)(\bar{z}_3\gamma_\mu z_4) = (\bar{z}_1 z_4)(\bar{z}_3 z_2) - \frac{1}{2}(\bar{z}_1\gamma^\mu z_4)(\bar{z}_3\gamma_\mu z_2) - \frac{1}{2}(\bar{z}_1\gamma^\mu\gamma^5 z_4)(\bar{z}_3\gamma_\mu\gamma^5 z_2) - (\bar{z}_1\gamma^5 z_4)(\bar{z}_3\gamma^5 z_2),$$

see also [33]. The appendix to [34] contains the identities in general dimensions. Rearranging the Fierz identity easily gives the identity (4). Alternatively one can show (4) by first noting that for vectors $w_j \in \mathbb{C}^2$ we have $\sum_{j=1}^3 (w_1^\dagger \sigma^j w_2) \sigma^j w_3 = 2(w_1^\dagger w_3)w_2 - (w_1^\dagger w_2)w_3$ and then computing the identity by hand.

where the $B_j(u, v)$ are a linear combination of the bilinear forms

$$u^\dagger v, \quad v^\dagger u, \quad v^\dagger \beta v, \quad u^\dagger \beta u \quad (6)$$

and $u, v : \mathbb{R}^{1+n} \rightarrow \mathbb{C}^2$. To prove the claim, if $n = 3$ we decompose $\psi = \begin{pmatrix} u+v \\ u-v \end{pmatrix}$ into left and right spinors³, then a short computation using the Fierz identity (4) shows that the pair (u, v) is a solution to (5) with $B_1(u, v) = B_3(u, v) = 2i(u^\dagger v + v^\dagger u)$ in the Soler model case, and $B_1(u, v) = 4i(v^\dagger u)$, $B_3(u, v) = 4i(u^\dagger v)$ in the Thirring model case (note that $\beta = 0$ when $n = 3$ so the B_2, B_4 terms vanish). On the other hand, in the $n = 2$ case, if we multiply both sides of (1) by $\beta = \gamma^0 = \sigma^3$ and use the Fierz type identity (4) together with $\gamma^0 \gamma^1 = \sigma^1$, $\gamma^0 \gamma^2 = \sigma^2$ we get (5) with $B_1 = B_3 = B_4 = 0$ and $B_2(u, v) = (u^\dagger \beta u)$. To summarise, Theorem 1.1 follows from the following.

Theorem 1.5. *Let $n = 2, 3$ and $s \geq \frac{n-1}{2}$. There exists $\epsilon > 0$ such that if $(u(0), v(0)) \in \dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n)$ with*

$$\|u(0)\|_{\dot{H}^{\frac{n-1}{2}}(\mathbb{R}^n)} + \|v(0)\|_{\dot{H}^{\frac{n-1}{2}}(\mathbb{R}^n)} < \epsilon$$

then we have a global solution $(u, v) \in C(\mathbb{R}, \dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n))$ to (5) such that

$$\sup_{t \in \mathbb{R}} \|(u, v)(t)\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n)} \lesssim \|(u, v)(0)\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n)}.$$

Moreover, the solution (u, v) depends continuously on the initial data and is the unique limit of smooth solutions. Finally, there exists $u_{\pm\infty}, v_{\pm\infty} \in C(\mathbb{R}, \dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n))$ with $(\partial_t + \sigma \cdot \nabla)u_{\pm\infty} = (\partial_t - \sigma \cdot \nabla)v_{\pm\infty} = 0$ such that

$$\lim_{t \rightarrow \pm\infty} \left(\|u(t) - u_{\pm\infty}(t)\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n)} + \|v(t) - v_{\pm\infty}(t)\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s(\mathbb{R}^n)} \right) = 0.$$

To prove Theorem 1.5, we need to study the linear operator $(\partial_t \pm \sigma \cdot \nabla)$. To start with, note that in the $n = 2$ case, we have

$$(\partial_t \pm \sigma \cdot \nabla)\beta = \beta(\partial_t \mp \sigma \cdot \nabla). \quad (7)$$

In particular, if $(\partial_t + \sigma \cdot \nabla)u = 0$, then $(\partial_t - \sigma \cdot \nabla)\beta u = 0$. This has the important, and very useful, consequence that to study the nonlinearity in (5), it suffices to study products $u^\dagger v$ where u and v are solutions to

$$\begin{aligned} (\partial_t + \sigma \cdot \nabla)u &= 0 \\ (\partial_t - \sigma \cdot \nabla)v &= 0 \end{aligned} \quad (8)$$

since, clearly, products like $u^\dagger \beta u$ can be reduced to products of the form $u^\dagger v$ after an application of (7). We now claim that the product $u^\dagger v$ is a *null form*, in other words it satisfies improved bilinear estimates when compared to a product like $|u|^2$. This is intuitively clear as since u and v should resemble waves traveling in opposite directions, we expect that their product should decay faster than a corresponding product like $|u|^2$. An estimate that makes this idea more explicit, is the following.

Lemma 1.6. *Let $n = 1, 2, 3$. Assume (u, v) solve (8) with $u(0) = f$ and $v(0) = g$. Then*

$$\|u^\dagger v\|_{L^2_{t,x}(\mathbb{R}^{n+1})} \lesssim \|f\|_{L^2_x(\mathbb{R}^n)} \|g\|_{\dot{H}^{\frac{n-1}{2}}(\mathbb{R}^n)}. \quad (9)$$

³Essentially we are decomposing into standard left and right spinors $\psi = \psi_L + \psi_R$ where $\psi_R = \frac{1}{2}(I - \gamma^5)\psi$ and $\psi_L = \frac{1}{2}(I + \gamma^5)\psi$ and then writing $\psi_L = \begin{pmatrix} u \\ u \end{pmatrix}$ and $\psi_R = \begin{pmatrix} v \\ -v \end{pmatrix}$.

Note that this estimate is certainly *not* true for a product like $|u|^2$, if $n = 1, 2$ this is easy to see as $u \notin L_{t,x}^4(\mathbb{R}^{1+2})$ for solutions to (8). It is also worth noting that (9) is closely related to the missing $L_t^2 L_x^\infty$ Strichartz estimate. More precisely, if we had an $L_t^2 L_x^\infty$ control over v , then the bilinear estimate (9) would follow from a simple application of Hölder's inequality. Thus, in some cases, Lemma 1.6 can form a suitable substitute to the missing endpoint Strichartz estimate.

One way to prove Lemma 1.6 (at least in the case $n = 2, 3$) is to introduce potentials ϕ and φ such that

$$(\partial_t - \sigma \cdot \nabla)\phi = u, \quad (\partial_t + \sigma \cdot \nabla)\varphi = v.$$

Then a short computation shows that $\square\phi = \square\varphi = 0$ and furthermore, that $u^\dagger v$ is made up of a linear combination of the classical null forms

$$\partial_t \phi \partial_t \varphi - \nabla \phi \cdot \nabla \varphi, \quad \partial_\mu \phi \partial_\nu \varphi - \partial_\nu \phi \partial_\mu \varphi.$$

Lemma 1.6 then follows by applying the sharp bilinear null form estimates of Foschi-Klainerman [18]. Alternatively, and more in the spirit of the current article, we present a softer argument that just relies on a decomposition into traveling waves, followed by Hölder's inequality and a change of variables. This is similar to the approach used by Tataru [47] and Klainerman-Rodnianski [23].

Proof of Lemma 1.6. In the $n = 1$ case, we can reduce the estimate (9) to a product of the form $\|f(x-t)g(x+t)\|_{L_{t,x}^2(\mathbb{R}^{1+1})}$ and so lemma follows by a simple change of variables. On the other hand, if $n = 2, 3$ we begin by decomposing v into an average of traveling waves. More precisely, define $\Pi_\omega = \frac{1}{2}(I + \sigma \cdot \omega)$ and $\widehat{\Pi_\pm f} = \Pi_{\pm \frac{x}{|x|}} \widehat{f}$, note that $\Pi_\omega^\dagger = \Pi_\omega$ and $\Pi_\omega^2 = \Pi_\omega$. Then writing the solution v in Polar coordinates gives

$$\begin{aligned} v(t, x) &= e^{it|\nabla|} \Pi_+ g + e^{-it|\nabla|} \Pi_- g \\ &= \int_{\mathbb{S}^{n-1}} \Pi_\omega \int_0^\infty e^{ir(t+x \cdot \omega)} \widehat{g}(r\omega) r^{n-1} dr d\mathbb{S}(\omega) + \int_{\mathbb{S}^{n-1}} \Pi_{-\omega} \int_0^\infty e^{-ir(t-x \cdot \omega)} \widehat{g}(r\omega) r^{n-1} dr d\mathbb{S}(\omega) \\ &= \int_{\mathbb{S}^{n-1}} \Pi_\omega g_\omega(t + x \cdot \omega) d\mathbb{S}(\omega) \end{aligned} \tag{10}$$

where $g_\omega(a) = \int_0^\infty [e^{ira} \widehat{g}(r\omega) + e^{-ira} \widehat{g}(-r\omega)] r^{n-1} dr$. Consequently, by the self-adjointness of the projections Π_ω , we have the bound

$$\begin{aligned} \|u^\dagger v\|_{L_{t,x}^2} &\leq \int_{\mathbb{S}^{n-1}} \|u^\dagger(t, x) \Pi_\omega g_\omega(t + x \cdot \omega)\|_{L_{t,x}^2} d\mathbb{S}(\omega) \\ &= \int_{\mathbb{S}^{n-1}} \|(\Pi_\omega u)^\dagger(t - \omega \cdot x, x) g_\omega(t)\|_{L_{t,x}^2} d\mathbb{S}(\omega) \\ &\leq \sup_{\omega \in \mathbb{S}^{n-1}} \|(\Pi_\omega u)^\dagger(t - \omega \cdot x, x)\|_{L_t^\infty L_x^2} \int_{\mathbb{S}^{n-1}} \|g_\omega\|_{L_t^2} d\mathbb{S}(\omega). \end{aligned}$$

It is easy enough to check that by undoing the Polar coordinates, and using an application of Holder in the ω variables we obtain

$$\int_{\mathbb{S}^{n-1}} \|g_\omega\|_{L_t^2} d\mathbb{S}(\omega) \lesssim \|g\|_{\dot{H}^{\frac{n-1}{2}}}.$$

Thus we reduce (9) to proving

$$\sup_{\omega \in \mathbb{S}^{n-1}} \|(\Pi_\omega u)^\dagger(t - \omega \cdot x, x)\|_{L_t^\infty L_x^2} \lesssim \|f\|_{L_x^2}.$$

Note that $(-x \cdot \omega, x)$ is a parameterisation of the null plane $NP(\omega)$ orthogonal to the null vector $(1, \omega)$,

$$NP(\omega) = \{(t, x) \in \mathbb{R}^{1+n} \mid (t, x) \cdot (1, \omega) = 0\}.$$

In other words, we need to control the integral of $\Pi_\omega u$ over $L^2(NP(\omega))$. By a change of variables we deduce that

$$e^{\mp i(t-x \cdot \omega)|\nabla|} f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{\mp i(t-x \cdot \omega)|\xi|} e^{ix \cdot \xi} d\xi = \int_{\mathbb{R}^n} \left[\widehat{f}(\xi) e^{\mp i t |\xi|} J^{-1}(\xi) \right](y) e^{iy \cdot x} dy$$

where $J(\xi) = 1 \pm \frac{\xi}{|\xi|} \cdot \omega \approx \theta(\omega, \mp \xi)^2$ is the Jacobian of the change of variables $y = \xi \pm |\xi| \omega$, and $\theta(\xi, \xi')$ denotes the angle of the two vectors $\xi, \xi' \in \mathbb{R}^n$. Hence using the “null form” estimate $|\Pi_\omega \Pi_{\pm \frac{\xi}{|\xi|}}| \lesssim \theta(\omega, \mp \xi)$ (see (15) below) together with Plancheral, we have

$$\begin{aligned} \|e^{\mp i(t-x \cdot \omega)|\nabla|} \Pi_\omega \Pi_{\pm \frac{\xi}{|\xi|}} f(x)\|_{L_x^2} &= \left\| \left[\Pi_\omega \Pi_{\pm \frac{\xi}{|\xi|}} \widehat{f}(\xi) e^{\mp i t |\xi|} J^{-1}(\xi) \right](y) \right\|_{L_y^2} \\ &= \|J^{-\frac{1}{2}}(\xi) \Pi_\omega \Pi_{\pm \frac{\xi}{|\xi|}} \widehat{f}\|_{L_\xi^2} \\ &\lesssim \|\theta(\omega, \mp \xi)^{-1} \theta(\omega, \mp \xi) \widehat{f}\|_{L_\xi^2} = \|f\|_{L^2}. \end{aligned} \quad (11)$$

If we apply this inequality to $u = e^{it|\nabla|} \Pi_- f + e^{-it|\nabla|} \Pi_+ f$ we obtain (9). Thus lemma follows. \square

2. NOTATION

Throughout this article we take $n = 2, 3$. We use the notation $a \lesssim b$ to denote the inequality $a \leq Cb$ for some constant $C > 0$ which is independent of the variables under consideration. Similarly, we write $a \ll b$ if $a \leq Cb$ with a small constant $C < \frac{1}{4}$. For a complex valued $n \times m$ matrix A , we let A^\dagger denote the conjugate transpose. If $\Omega \subset \mathbb{R}^{1+n}$, we define $\mathbb{1}_\Omega(t, x)$ to be the corresponding indicator function.

Let $L_t^q L_x^r(\mathbb{R}^{n+1})$ denote the usual mixed-norm Lebesgue space with the associated norm

$$\|u\|_{L_t^q L_x^r(\mathbb{R}^{n+1})} = \left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}^n} |u(t, x)|^r dx \right]^{\frac{q}{r}} dt \right)^{\frac{1}{q}}.$$

Occasionally we omit the domain \mathbb{R}^{n+1} when we can do so without causing confusion. Most functions that occur in this paper are \mathbb{C}^2 valued, although occasionally we make use of scalar valued maps as well. The Schwartz class of smooth functions on \mathbb{R}^n with rapidly decreasing derivatives is denoted by $\mathcal{S}(\mathbb{R}^n)$, we let $\mathcal{S}'(\mathbb{R}^n)$ denote its dual, the collection of all tempered distributions. For a function $f \in \mathcal{S}(\mathbb{R}^n)$ we let

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx$$

denote the spatial Fourier transform. Similarly for $u(t, x) \in \mathcal{S}(\mathbb{R}^{n+1})$ we let $\tilde{u}(\tau, \xi)$ denote the space-time Fourier transform. The Fourier transform is extended to \mathcal{S}' by duality in the usual manner. For $s > -\frac{n}{2}$ we define the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ as the completion of \mathcal{S} using the norm

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \|\ |\xi|^s \widehat{f}(\xi) \|_{L_\xi^2(\mathbb{R}^n)}.$$

Fix $\Phi \in C_0^\infty(\mathbb{R})$ with $\text{supp } \Phi \subset \{2^{-1} \leq a \leq 2\}$ and for $a \neq 0$

$$\sum_{\lambda \in 2^{\mathbb{Z}}} \Phi(\lambda^{-1} a) = 1. \quad (12)$$

We define the (homogeneous) Besov-Lipschitz spaces $\dot{B}_{p,q}^s$ via the norm

$$\|f\|_{\dot{B}_{p,q}^s} = \left(\sum_{\lambda \in 2^{\mathbb{Z}}} \left(\lambda^s \|P_\lambda f\|_{L^p} \right)^q \right)^{\frac{1}{q}}$$

where $\widehat{P_\lambda f} = \Phi(\lambda^{-1}|\xi|)\widehat{f}(\xi)$ is the Fourier cutoff to the region $|\xi| \approx \lambda$. Given a Banach space X , we let $C(\mathbb{R}, X)$ denote the collection of all continuous maps $u : \mathbb{R} \rightarrow X$.

Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$ denote the standard unit sphere in \mathbb{R}^n . If $\xi, \xi' \in \mathbb{R}^n$, then we let $\theta(\xi, \xi')$ denote the positive, smallest, angle between the unit vectors $\frac{\xi}{|\xi|}, \frac{\xi'}{|\xi'|} \in \mathbb{S}^{n-1}$. We frequently use the estimate $\theta(\xi, \xi') \approx 1 - \frac{\xi}{|\xi|} \cdot \frac{\xi'}{|\xi'|}$ as well as the more explicit⁴

$$\frac{49}{50}\theta(\omega, \omega') \leq |\omega - \omega'| \leq \theta(\omega, \omega') \quad (13)$$

which holds for $\omega, \omega' \in \mathbb{S}^{n-1}$ provided $\theta(\omega, \omega') \leq \frac{1}{4}$. Given a subset $\kappa \subset \mathbb{S}^{n-1}$ and vector $\omega \in \mathbb{S}^{n-1}$, we let $\theta(\omega, \kappa) = \inf\{\theta(\omega, \omega') \mid \omega' \in \kappa\}$.

We often restrict the Fourier transform of a function to lie in a certain subsets of \mathbb{R}^{n+1} . To exploit this restriction, we make use of Bernstein's inequality which states that if $\text{supp } \widehat{f} \subset \Omega$ and $p \geq 2$, then for any $q \leq p$ we have

$$\|f\|_{L^p} \lesssim |\Omega|^{\frac{1}{q} - \frac{1}{p}} \|f\|_{L^q}.$$

Similarly, when considering products, if $\text{supp } \widehat{f} \subset \Omega$ and $\text{supp } \widehat{g} \subset \Omega'$, we observe that the product fg satisfies $\widehat{fg} \subset \Omega + \Omega'$.

2.1. Null Coordinates. As mentioned in the introduction, the standard (t, x) coordinate frame is not sufficient to give the bilinear estimates that we require in the present paper. Instead, to exploit the type of arguments leading used in the proof of Lemma 1.6, we need the flexibility to be able to work in adapted *null* coordinate frames which are chosen depending on the Fourier support of the function under consideration. The definitions are as follows.

Let $\omega \in \mathbb{S}^{n-1}$ and $\vartheta = \frac{1}{\sqrt{2}}(1, \omega) \in \mathbb{R}^{1+n}$. We define the *null coordinates* $(t_\omega, x_\omega) \in \mathbb{R} \times \mathbb{R}^n$ as

$$t_\omega = (t, x) \cdot \vartheta = \frac{1}{\sqrt{2}}(t + \omega \cdot x), \quad x_\omega = x - \frac{1}{\sqrt{2}}[(t, x) \cdot \vartheta]\omega.$$

Note that $t_\omega \vartheta$ is the projection of (t, x) onto the span of the null vector ϑ , while $(-\omega \cdot x_\omega, x_\omega)$ is a parameterisation of the associated null hyperplane $\{(t, x) \in \mathbb{R}^{1+n} \mid (t, x) \cdot \vartheta = 0\}$. Moreover we have the identity

$$(t, x) = t_\omega \vartheta + (-\omega \cdot x_\omega, x_\omega).$$

To facilitate the computations we use later, we also decompose $x_\omega = x_\omega^\perp - \frac{1}{\sqrt{2}}x_\omega^1 \omega$ where

$$x_\omega^\perp = \frac{1}{\sqrt{2}}(t - \omega \cdot x), \quad x_\omega^1 = x - (x \cdot \omega)\omega.$$

⁴This can be deduced by letting $\theta = \theta(\omega, \omega')$,

$$|\omega - \omega'|^2 = 2 - 2\omega \cdot \omega' = 2 - 2\cos(\theta) = 4\sin^2\left(\frac{\theta}{2}\right).$$

The estimate now follows by using the estimate $\frac{\sin(a)}{a}x \leq \sin(x) \leq x$ for $0 < x < a$ and the fact that $8\sin(\frac{1}{8}) \geq \frac{49}{50}$.

Thus x_ω^\perp denotes the component of the vector (t, x) on the null cone in the direction $(1, -\omega)$, while x_ω^\perp is the remaining component orthogonal to ω . We can translate from the (t_ω, x_ω) coordinate frame back into the standard (t, x) frame by using the identities

$$\begin{aligned} t &= \frac{1}{\sqrt{2}}t_\omega - \omega \cdot x_\omega & x &= x_\omega + \frac{1}{\sqrt{2}}t_\omega \omega \\ &= \frac{1}{\sqrt{2}}(t_\omega + x_\omega^\perp) & &= x_\omega^\perp + \frac{1}{\sqrt{2}}(t_\omega - x_\omega^\perp)\omega. \end{aligned}$$

We also make use of the dual or frequency variables in null frames. If (τ, ξ) denote the Fourier variables associated to (t, x) , then we define the corresponding null frame versions $(\tau_\omega, \xi_\omega)$ by letting $(\tau, \xi) \cdot (t, x) = (\tau_\omega, \xi_\omega) \cdot (t_\omega, x_\omega)$. In other words we let

$$\tau_\omega = \frac{1}{\sqrt{2}}(\tau + \xi \cdot \omega), \quad \xi_\omega = \xi - \tau\omega = \xi_\omega^\perp - \sqrt{2}\xi_\omega^\perp \omega$$

where as before ξ_ω^\perp denotes the component of ξ_ω orthogonal to ω , and $\xi_\omega^\perp = \frac{1}{\sqrt{2}}(\tau - \xi \cdot \omega)$. We can translate from $(\tau_\omega, \xi_\omega)$ to (τ, ξ) by using the identities

$$\begin{aligned} \tau &= \frac{1}{\sqrt{2}}\tau_\omega - \frac{1}{2}\omega \cdot \xi_\omega & \xi &= \xi_\omega + \left(\frac{1}{\sqrt{2}}\tau_\omega - \frac{1}{2}\omega \cdot \xi_\omega\right)\omega \\ &= \frac{1}{\sqrt{2}}(\tau_\omega + \xi_\omega^\perp) & &= \xi_\omega^\perp + \frac{1}{\sqrt{2}}(\tau_\omega - \xi_\omega^\perp)\omega. \end{aligned}$$

Finally we note the fundamental fact that the symbol of the wave operator $\square = \partial^\gamma \partial_\gamma$ satisfies the key inequality

$$\tau^2 - |\xi|^2 = 2\tau_\omega \xi_\omega^\perp - |\xi_\omega^\perp|^2.$$

This simple identity plays an important role in the arguments used in this paper.

If we have a function $\phi(t, x)$ on \mathbb{R}^{1+n} , by default we use (t, x) coordinates. If we want to specify that ϕ is in (t_ω, x_ω) coordinates we write ϕ^* , thus

$$\phi(t, x) = \phi^*(t_\omega, x_\omega).$$

This convention also applies to the Fourier transform, $\widehat{\phi}(t, \xi)$ denotes the Fourier transform with respect to x , while $\widehat{\phi}^*(t_\omega, \xi_\omega)$ is the Fourier transform with respect to x_ω . A similar comment applies to the spacetime Fourier transform $\widetilde{\phi}(\tau, \xi)$.

2.2. The Projections Π_ω and Π_\pm . Let $\omega \in \mathbb{S}^{n-1}$ and define the projections Π_ω by

$$\Pi_\omega = \frac{1}{2}(I + \sigma \cdot \omega)$$

where I denotes the 2×2 identity matrix. The properties of the matrices σ implies that we have the important identities

$$I = \Pi_\omega + \Pi_{-\omega}, \quad \sigma \cdot \omega = \Pi_\omega - \Pi_{-\omega}, \quad \Pi_\omega^\dagger = \Pi_\omega, \quad \Pi_\omega \Pi_{-\omega} = 0, \quad \Pi_\omega^2 = \Pi_\omega. \quad (14)$$

Moreover we have the crucial (and well known) *null structure* estimate $|\Pi_\omega \Pi_{\omega'}| \lesssim \theta(\omega, -\omega')$ which follows from the orthogonality of the projections $\Pi_{\pm\omega}$ by writing

$$|\Pi_\omega \Pi_{\omega'}| = |(\Pi_\omega - \Pi_{-\omega'})\Pi_{\omega'}| = \frac{1}{2}|(\omega + \omega') \cdot \sigma \Pi_{\omega'}| \lesssim |\omega + \omega'| \lesssim \theta(\omega, -\omega'). \quad (15)$$

This angle estimate plays a crucial role in eliminating a number of dangerous bilinear interactions.

Aside from using the projections Π_ω to exploit the null structure present in the Thirring model, they can also be used to decompose the Dirac equation into half wave operators $\partial_t \pm i|\nabla|$. More precisely, define the Fourier multipliers Π_\pm as

$$\widehat{\Pi_\pm f}(\xi) = \Pi_{\pm \frac{\xi}{|\xi|}} \widehat{f}(\xi).$$

Then using the identities (14) we see that the Dirac equation $(\partial_t \pm \sigma \cdot \nabla)u = F$ is equivalent to

$$\begin{aligned} (\partial_t \pm i|\nabla|)\Pi_+ u &= \Pi_+ F \\ (\partial_t \mp i|\nabla|)\Pi_- u &= \Pi_- F. \end{aligned}$$

This formulation for the Dirac equation has played a crucial role in the low regularity well-posedness theory developed over the last decade or so. See for instance the work of D'Ancona-Foschi-Selberg [11, 12], and Pecher [35, 36] and the second author [8], for the Dirac equation coupled to a scalar field, as well as the of the current authors [5] for related ideas for the Spacetime-Monopole equation.

2.3. Solution Operators. Define the unitary operator $\mathcal{U}_\pm(t)$ on $L_x^2(\mathbb{R}^n)$ by the formula

$$\mathcal{U}_\pm(t)[f] = e^{\mp it|\nabla|}\Pi_+ f + e^{\pm it|\nabla|}\Pi_- f.$$

If we note that $\sigma \cdot \nabla = i|\nabla|(\Pi_+ + \Pi_-)$ then a short computation shows that

$$(\partial_t \pm \sigma \cdot \nabla)\mathcal{U}_\pm(t)[f] = 0$$

and $\mathcal{U}_\pm(0)f = f$. Thus $\mathcal{U}_\pm(t)f$ gives the homogeneous solution to $(\partial_t \pm \sigma \cdot \nabla)u = 0$ with data $u(0) = f$.

2.4. Sets and Multipliers. The global well-posedness result in Theorem 1.1 depends on a number of sharp bilinear estimates. The proof of these bilinear estimates relies on being able to localise to certain frequency regions. The key tool to do this is the standard technique of dyadic decomposition.

Take $\Phi \in C_0^\infty(\mathbb{R})$ as in (12) and let $\Phi_0(\xi) = \sum_{\lambda \leq 2^{-1}} \Phi(\frac{\xi}{\lambda})$ with $\Phi_0(0) = 1$. Define the Fourier multipliers P_λ , C_d , and C_d^\pm via

$$\widehat{P_\lambda f}(\xi) = \Phi\left(\frac{|\xi|}{\lambda}\right)\widehat{f}(\xi), \quad \widetilde{C_d F}(\tau, \xi) = \Phi\left(\frac{||\tau| - |\xi||}{d}\right)\widetilde{F}(\tau, \xi), \quad \widetilde{C_d^\pm F}(\tau, \xi) = \Phi\left(\frac{|\tau \pm |\xi||}{d}\right)\widetilde{F}(\tau, \xi).$$

Note that P_λ restricts the Fourier support to the set $\{2^{-1}\lambda \leq |\xi| \leq 2\lambda\}$, C_d restricts the Fourier support to be at distance $\approx d$ from the cone, and C_d^\pm restricts the Fourier support onto the forward and backward components of the cone. Similarly we define multipliers $C_{\leq d}$, $C_{\leq d}^\pm$ as

$$\widetilde{C_d F}(\tau, \xi) = \Phi_0\left(\frac{||\tau| - |\xi||}{d}\right)\widetilde{F}(\tau, \xi), \quad \widetilde{C_d^\pm F}(\tau, \xi) = \Phi_0\left(\frac{|\tau \pm |\xi||}{d}\right)\widetilde{F}(\tau, \xi),$$

thus $C_{\leq d}^\pm$ and $C_{\leq d}$ are the (smooth) restriction of the Fourier support to the sets $\{||\tau| - |\xi|| \leq d\}$ and $\{|\tau \pm |\xi|| \leq d\}$. Note that if $F \in L^2(\mathbb{R}^{1+n})$ we can decompose

$$F = \sum_{d \in 2^{\mathbb{Z}}} C_d^\pm F \tag{16}$$

where the sum converges in $L^2(\mathbb{R}^{1+n})$. This is not true for $F \in L_t^\infty L_x^2$ for instance, as the righthand side of (16) vanishes for functions with Fourier transforms supported on the lightcone, i.e. solutions to the wave equation. Thus some care has to be taken when decomposing functions into dyadic distances

from the cone, as in general, (16) only holds modulo solutions to the wave equation.

The number of \pm signs that will be floating around in various formula throughout this article can be daunting. To alleviate this somewhat, we define

$$\mathfrak{C}_d^\pm = \Pi_+ C_d^\pm + \Pi_- C_d^\mp.$$

Thus \mathfrak{C}_d^\pm is the vector valued analogue of the C_d^\pm multipliers. Note that \mathfrak{C}_d^\pm roughly corresponds to localising spacetime frequencies to distance $\sim d$ from the characteristic surface of the equation $(\partial_t \pm \sigma \cdot \nabla)u = 0$. In a similar vein, we define

$$S_{\lambda,d}^\pm u = \mathfrak{C}_d^\pm P_\lambda u.$$

The multipliers $\mathfrak{C}_{\leq d}^\pm$ and $S_{\lambda,\leq d}^\pm$ are defined in the obvious manner.

As well as the above multipliers, we also need to be able to decompose into angular regions. Let $\alpha \ll 1$ and define \mathcal{C}_α to be a finitely overlapping cover of \mathbb{S}^{n-1} where every cap $\kappa \in \mathcal{C}_\alpha$ has radius α . We use $\omega(\kappa)$ to denote the centre of the cap $\kappa \in \mathcal{C}_\alpha$ and so $\kappa = \{\omega \in \mathbb{S}^{n-1} \mid \theta(\omega, \omega(\kappa)) \leq \alpha\}$. For constants $C > 1$ and $\kappa \in \mathcal{C}_\alpha$, we also define $C\kappa = \{\omega \in \mathbb{S}^{n-1} \mid \theta(\omega, \omega(\kappa)) \leq C\alpha\}$.

Given a subset $\kappa \in \mathcal{C}_\alpha$ we define the sets

$$A_\lambda(\kappa) = \{(\tau, \xi) \in \mathbb{R}^{1+n} \mid \lambda 2^{-1} \leq |\xi| \leq 2\lambda, \operatorname{sgn}(\tau) \frac{\xi}{|\xi|} \in \kappa\}, \quad A_\lambda^\pm(\kappa) = \{\xi \in \mathbb{R}^n \mid \lambda 2^{-1} \leq |\xi| \leq 2\lambda, \mp \frac{\xi}{|\xi|} \in \kappa\}$$

note that $A_\lambda(\kappa) \subset \mathbb{R}^{1+n}$ while $A_\lambda^\pm(\kappa) \subset \mathbb{R}^n$. These sets decompose the annulus $\{|\xi| \approx \lambda\}$ into radially directed, rectangularly shaped sets of size $\lambda \times (\alpha\lambda)^{n-1}$. Similarly we let

$$A_{\alpha,\lambda}(\kappa) = \{\lambda 2^{-1} \leq |\xi| \leq 2\lambda, \|\tau - |\xi|\| \leq c\alpha^2\lambda, \operatorname{sgn}(\tau) \frac{\xi}{|\xi|} \in \kappa\},$$

and

$$A_{\alpha,\lambda}^\pm(\kappa) = \{\lambda 2^{-1} \leq |\xi| \leq 2\lambda, |\tau \pm |\xi|| \leq c\alpha^2\lambda, \mp \frac{\xi}{|\xi|} \in \kappa\},$$

where $c \ll 1$ is some small constant. Clearly we have $\mathbb{R} \times A_\lambda^\pm(\kappa) \subset A_\lambda(\kappa)$ and $A_{\alpha,\lambda}^\pm \subset A_{\alpha,\lambda}(\kappa)$.

For each of the angular sets defined above, we need the corresponding Fourier cutoffs. Fix $\alpha \ll 1$ and let Φ_κ be a smooth partition of unity on \mathbb{S}^{n-1} subordinate to the caps $\kappa \in \mathcal{C}_\alpha$. Note that we may ensure that, after a rotation to centre the cap κ on the ξ_1 axis, we have for $\xi \neq 0$ the derivative bounds

$$|\partial_{\xi_1}^N [\Phi_\kappa(\frac{\xi}{|\xi|})]| \lesssim |\xi|^{-N}, \quad |\partial_{\xi_j}^N [\Phi_\kappa(\frac{\xi}{|\xi|})]| \lesssim (\alpha|\xi|)^{-N} \quad j \neq 1. \quad (17)$$

We now define the corresponding Fourier multiplier

$$\widehat{R_\kappa^\pm f}(\xi) = \Phi_\kappa(\mp \frac{\xi}{|\xi|}) \widehat{f}(\xi)$$

and take

$$R_{\kappa,d}^\pm = C_{\ll d}^\pm R_\kappa^\pm, \quad P_{\lambda,\kappa}^\pm = P_\lambda R_\kappa^\pm, \quad P_{\lambda,\kappa}^{\pm,\alpha} = C_{\ll \alpha^2 \lambda}^\pm P_\lambda R_\kappa^\pm.$$

The multipliers and corresponding sets are summarised in Table 1.

We would like to pretend that the operators introduced above are idempotent, i.e. satisfy $P^2 = P$. Unfortunately, this clearly fails (although it is *almost* the case, in the sense that P^2 is a cutoff to the same region of frequency space). Thus, to work around this difficulty, we introduce cutoffs to slight

Sets	Fourier multipliers
$\{ \xi \approx \lambda\}$	P_λ
$\{ \tau - \xi \approx d\}$	C_d
$\{ \tau \pm \xi \approx d\}$	C_d^\pm
$\{\mp \frac{\xi}{ \xi } \in \kappa\}$	R_κ^\pm
$\{ \tau \pm \xi \ll d, \mp \frac{\xi}{ \xi } \in \kappa\}$	$R_{\kappa,d}^\pm$
$A_\lambda^\pm(\kappa)$	$P_{\lambda,\kappa}^\pm$
$A_{\alpha,\lambda}^\pm(\kappa)$	$P_{\lambda,\kappa}^{\pm,\alpha}$

TABLE 1. Sets and corresponding Fourier multipliers

enlargements of the sets used above. More precisely, if A is one of the sets defined above, then we let $\natural A$ be the set which is $\frac{101}{100}$ times larger, thus $A \subset \natural A$. For example, we let

$$\natural A_\lambda^\pm(\kappa) = \{\lambda 2^{-1} \frac{100}{101} \leq |\xi| \leq 2\lambda \frac{101}{100}, \frac{\xi}{|\xi|} \in \frac{101}{100}\kappa\}.$$

The sets $\natural A_\lambda(\kappa)$, $\natural A_{\lambda,\alpha}^\pm(\kappa)$, and $\natural A_{\lambda,\alpha}(\kappa)$ are defined similarly. Moreover, if A is one of the previous sets, we let $\natural P$ denote a corresponding multiplier that is 1 on A , and has support inside the corresponding set $\natural A$. For instance $\natural P_{\lambda,\kappa}^\pm$ restricts the Fourier transform to the set $\natural A_\lambda^\pm(\kappa)$. Note that we always have identities of the form $\natural P_{\lambda,\kappa}^\pm P_{\lambda,\kappa}^\pm = P_{\lambda,\kappa}^\pm$ and furthermore we may assume that the new multipliers $\natural R_\kappa^\pm$ still satisfy the derivative bounds (17).

2.5. Estimate on coordinates in $A_{\alpha,\lambda}(\kappa)$. For later use, we record here the following useful estimate on the dual coordinates $(\tau_\omega, \xi_\omega)$. We start by noting that

$$|\xi_\omega^1| = \frac{1}{\sqrt{2}}|\tau| - |\xi| + |\xi| - \text{sgn}(\tau)\xi \cdot \omega, \quad |\xi_\omega^\perp|^2 = \left||\xi| + \xi \cdot \omega\right| \times \left||\xi| - \xi \cdot \omega\right|, \quad |\tau_\omega| \leq |\tau| + |\xi|. \quad (18)$$

In particular, if $\alpha \ll 1$, $\kappa \in \mathcal{C}_\alpha$, and $(\tau, \xi) \in A_{\lambda,\alpha}(\kappa)$ then have

$$|\xi_\omega^1| \lesssim (\max\{\alpha, \theta(\omega, \kappa)\})^2 \lambda, \quad |\xi_\omega^\perp| \lesssim \theta(\omega, \kappa) \lambda, \quad |\tau_\omega| \lesssim \lambda. \quad (19)$$

Clearly the same bounds also hold for $(\tau, \xi) \in \natural A_{\lambda,\alpha}(\kappa)$.

A slightly sharper estimate is available if $\omega \notin 2\kappa$. More precisely, the additional assumption on ω implies that $\alpha \leq \theta(\omega, \kappa)$ and so $||\tau| - |\xi|| \ll \theta^2(\omega, \kappa)\lambda$. Consequently

$$|\xi_\omega^1| \approx \theta^2(\omega, \kappa)\lambda, \quad |\xi_\omega^\perp| \lesssim \theta(\omega, \kappa)\lambda, \quad |\tau_\omega| \lesssim \lambda. \quad (20)$$

3. FUNCTION SPACES

A standard method used to handle the critical wave equation, is to take a Banach space Y , and then define a norm at scale λ via

$$\|u\|_{F_\lambda^\pm} = \|P_\lambda u\|_{L_t^\infty L_x^2} + \|(\partial_t \pm \nabla \cdot \sigma)P_\lambda u\|_Y$$

A good first choice for Y , (one that has worked well for the critical wave equation in high dimensions), is to take

$$Y = L_t^1 L_x^2 + X^{-\frac{1}{2},1}$$

where $X_\lambda^{b,1}$ is the relevant $X^{s,b}$ type space with an ℓ^1 sum in the distance to the cone, at scale $|\xi| \approx \lambda$. The idea is that away from the light cone, we use the $X^{s,b}$ type spaces, while close to the light cone, where the symbol blows up, we use the $L_t^1 L_x^2$ type norm.

If now apply our Y type norm to our well-posedness problem, the essentially point would be to control the term

$$\|(\partial_t + \sigma \cdot \nabla)^{-1}[(u^\dagger v)v]\|_Y.$$

Let $F = u^\dagger v$. Since the product $u^\dagger v$ is a null form, we should be able to put $F \in L_{t,x}^2$ (i.e. as in Lemma 1.6). If we also let $F_\lambda = P_\lambda F$, and $v_\mu = P_\mu v$, we need to prove estimates of the form

$$\|F_\lambda v_\mu\|_Y \lesssim \mu \|F_\lambda\|_{L_{t,x}^2} \|v_\mu\|_{F^-}. \quad (21)$$

This estimate is essentially true for $Y = L_t^1 L_x^2 + X_\lambda^{-\frac{1}{2},1}$ except for one particularly bad case where the output $F_\lambda v_\mu$ is concentrated near the null cone (so we are forced to use the $L_t^1 L_x^2$ space), v_μ is also close to null cone (so is essentially a homogeneous solution), but F is far from the null cone. Our only option is put $F_\lambda v_\mu$ in $L_t^1 L_x^2$, but then since $F_\lambda \in L^2$, we need $v_\mu \in L_t^2 L_x^\infty$ which fails (since v_μ is essentially a homogeneous solution). Note that this interaction is not a null interaction (as F_λ is far from the cone) so null structure doesn't help.

The key observation, due to Tataru, is that we do have a $L^2 L^\infty$ type estimate, provided we look at null coordinates (t_ω, x_ω) instead. This means that we can control the product in $L_{t_\omega}^1 L_{x_\omega}^2$ but not $L_t^1 L_x^2$. Thus, for certain interactions, we need to replace the $L_t^1 L_x^2$ component of the Y norm, with a $L_{t_\omega}^1 L_{x_\omega}^2$ type norm instead. This is possible but the construction of the required function spaces is a little involved.

In the rest of this section, we construct an appropriate replacement for the space Y . Essentially we will take Y to be roughly $L_t^1 L_x^2 + X^{s,b} + NF$ with an added term to deal with the regions far from the cone. Here NF are the null frame spaces originally appearing in the work of Tataru [47], and developed further by Tao [46]. See also the results in [28, 29, 44, 43].

3.1. $X^{s,b}$ type norms. We define the Dirac version of the Bourgain-Klainerman-Machedon spaces by using the norm⁵

$$\|u\|_{\dot{X}_\pm^{b,q}} = \left(\sum_{d \in 2\mathbb{Z}} d^{qb} \|\mathfrak{C}_d^\pm u\|_{L_{t,x}^2}^q \right)^{\frac{1}{q}}.$$

The norm $\dot{X}_\pm^{b,q}$ is related to the more standard norms $\dot{X}_\pm^{b,q}$ adapted to the cone $\{\tau \pm |\xi| = 0\}$ by the formula

$$\|u\|_{\dot{X}_\pm^{b,q}} \approx \|\Pi_+ u\|_{\dot{X}_\pm^{b,q}} + \|\Pi_- u\|_{\dot{X}_\mp^{b,q}} \quad (22)$$

where

$$\|u\|_{\dot{X}_\pm^{b,q}} = \left(\sum_{d \in 2\mathbb{Z}} d^{qb} \|C_d^\pm u\|_{L_{t,x}^2}^q \right)^{\frac{1}{q}}$$

and we recall that $\mathfrak{C}_d^\pm = C_d^\pm \Pi_+ + C_d^\mp \Pi_-$. Note that this implies that $\|\Pi_+ u\|_{\dot{X}_\pm^{b,q}} \approx \|\Pi_+ u\|_{\dot{X}_\pm^{b,q}}$ and a similar equality in the Π_- case. The equivalence of the two norms (22) follows by simply using the self-adjointness of the projections Π_\pm to obtain $\int (\Pi_\pm u)^\dagger \Pi_\mp v dx = \int u^\dagger \Pi_\pm \Pi_\mp v dx = 0$.

⁵Note that $\|\cdot\|_{\dot{X}_\pm^{b,q}}$ is not technically a norm, as it vanishes for distributions with Fourier support on the cone, thus it is only a *semi-norm*. However we make the (fairly) standard abuse of notation and refer to all semi-norms as norms.

The $\dot{\mathcal{X}}_{\pm}^{b,q}$ norm is designed to exploit the fact that, at least for small times or small data, we expect that the Π_+ component of the solution to

$$(\partial_t \pm \sigma \cdot \nabla)u = F$$

to concentrate close to the cone $\{\tau \pm |\xi| = 0\}$. The $X^{s,b}$ type norms have been a standard tool in the low regularity theory of nonlinear dispersive PDE since the work of Bourgain [4], Kenig-Ponce-Vega [21], and Klainerman-Machedon [22]. See also the earlier work of Beals [1] who used similar spaces in the study of singularity formation for the nonlinear wave equation.

We make use of the following basic results.

Lemma 3.1. *Let $u \in L_{t,x}^2$ with $\text{supp } \widetilde{\Pi_+ u} \subset \{|\tau \pm |\xi|| \approx d\}$ and $\text{supp } \widetilde{\Pi_- u} \subset \{|\tau \mp |\xi|| \approx d\}$. Then we can write*

$$u(t, x) = \frac{1}{2\pi} \int_{|\tau| \approx d} e^{it\tau} \mathcal{U}_{\pm}(t)[f_{\tau}] d\tau \quad (23)$$

where $f_{\tau} \in L_x^2$ has the same ξ support as \hat{u} , and $\|f_{\tau}\|_{L_x^2} \lesssim \|\tilde{u}(\tau, \xi)\|_{L_{\xi}^2}$.

Proof. We simply let

$$f_{\tau}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} [\widetilde{\Pi_+ u}(\tau \mp |\xi|, \xi) + \widetilde{\Pi_- u}(\tau \pm |\xi|, \xi)] e^{ix \cdot \xi} d\xi.$$

□

The identity (23) easily implies the well-known *transference principle*. Namely, if for every $\tau \in \mathbb{R}$ we have the bound $\|e^{it\tau} \mathcal{U}_{\pm}(t)f\|_X \lesssim \|f\|_{L_x^2}$, then $\|u\|_X \lesssim \|u\|_{\dot{\mathcal{X}}_{\pm}^{\frac{1}{2},1}}$ (for $u \in L_{t,x}^2$ say). In other words, any homogeneous estimate for the Dirac equation, immediately implies the same estimate holds for elements of $\dot{\mathcal{X}}_{\pm}^{\frac{1}{2},1}$. See for instance [47, Proposition 5.1] or [24, Proposition 3.7]. In particular, $\dot{\mathcal{X}}_{\pm}^{\frac{1}{2},1}$ controls the Strichartz norms $L_t^q L_x^r$. More precisely, if say $u \in L_{t,x}^2$ with $\text{supp } \hat{u} \subset \{|\xi| \approx \lambda\}$, then for any $\frac{1}{q} + \frac{n-1}{2r} \leq \frac{n-1}{4}$ with $(q, r) \neq (2, \infty)$ we have⁶

$$\|u\|_{L_t^q L_x^r} \lesssim \lambda^{n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}} \|u\|_{\dot{\mathcal{X}}_{\pm}^{\frac{1}{2},1}} \quad (24)$$

The transference principle can save a significant amount of work when working with $\dot{\mathcal{X}}_{\pm}^{\frac{1}{2},1}$ type norms. Finally, we recall the well-known fact that after truncating in time, homogenous solutions belong to $\dot{\mathcal{X}}_{\pm}^{\frac{1}{2},1}$.

Lemma 3.2 (Homogeneous solutions belong to $\dot{\mathcal{X}}_{\pm}^{\frac{1}{2},1}$). *Let $\rho \in C_0^{\infty}(\mathbb{R})$ and $T > 0$. Then*

$$\|\rho(\frac{\cdot}{T}) \mathcal{U}_{\pm}(t)f\|_{\dot{\mathcal{X}}_{\pm}^{\frac{1}{2},1}} \lesssim \|f\|_{L_x^2}$$

where the constant is independent of T .

⁶After an application of the triangle inequality and scaling, it is enough to consider the case $u = \mathcal{E}_1^{\pm} u$. We now apply (23) followed by the homogeneous Strichartz estimate to deduce

$$\|u\|_{L_t^q L_x^r} \leq \int_{|\tau| \approx 1} \|\mathcal{U}_{\pm}(t)f_{\tau}\|_{L_t^q L_x^r} d\tau \lesssim \lambda^{n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}} \int_{|\tau| \approx 1} \|f_{\tau}\|_{L_x^2} d\tau \lesssim \lambda^{n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}} \|u\|_{L_{t,x}^2}.$$

Proof. Let $\rho_T(t) = \rho(\frac{t}{T})$. If we observe that $[\rho_T(\widehat{\mathcal{U}}_{\pm}(t)f)](\tau, \xi) = \widehat{\rho}(\tau \pm |\xi|)\widehat{\Pi}_+ f(\xi) + \widehat{\rho}(\tau \mp |\xi|)\widehat{\Pi}_- f(\xi)$ then

$$\|\rho(\frac{t}{T})\mathcal{U}_{\pm}(t)f\|_{\dot{\mathcal{X}}_{\pm}^{\frac{1}{2},1}} \leq 2\|\rho_T\|_{\dot{B}_{2,1}^{\frac{1}{2}}} \|f\|_{L_x^2}.$$

Hence result follows by recalling that $\|\rho_T\|_{\dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R})} = \|\rho\|_{\dot{B}_{2,1}^{\frac{1}{2}}(\mathbb{R})} < \infty$. \square

Remark 3.3. An obvious question immediately arises, namely, can we simply prove Theorem 1.5 by iterating the equation in the norm $\dot{\mathcal{X}}_{\pm}^{\frac{1}{2},1}$? In other words, we are asking if we can bound the cubic term in $\dot{\mathcal{X}}_{\pm}^{-\frac{1}{2},1}$, which in view of Lemma 1.6 and the transference principle, would more or less require the estimate

$$\|F_{\lambda}v_{\mu}\|_{\dot{\mathcal{X}}_{+}^{-\frac{1}{2},1}} \lesssim \mu\|F_{\lambda}\|_{L_{t,x}^2}\|v_{\mu}\|_{\dot{\mathcal{X}}_{+}^{\frac{1}{2},1}} \quad (25)$$

for $\mu \ll \lambda$. Unfortunately, (25) fails. This can be seen by making the choice $\tilde{F} = \chi_{\Omega_1}$, $\tilde{v} = \chi_{\Omega_2}$ where

$$\Omega_1 = \{\lambda - 4 \leq |\tau| \leq \lambda + 4, \lambda - 4 \leq |\xi| \leq \lambda + 4\}, \quad \Omega_2 = \{|\tau| \leq 1, 2 \leq |\xi| \leq 3\}.$$

Note that if $\frac{1}{2}d \leq |\tau \pm |\xi|| \leq 2d$ and $\lambda \leq |\xi| \leq \lambda + 1$, (for $d \ll 1$ say), and $(\tau', \xi') \in \Omega_2$, then $(\tau - \tau', \xi - \xi') \in \Omega_1$ since

$$|\tau - \tau'| \leq |\tau \pm |\xi|| + |\xi| + |\tau'| \leq 2d + \lambda + 1 + 1 \leq \lambda + 4$$

and similarly

$$|\tau - \tau'| \geq |\xi| - |\tau \pm |\xi|| - |\tau'| \geq \lambda - 2d - 1 \geq \lambda - 4.$$

The argument for the $\xi - \xi'$ variable is similar. Therefore,

$$\|C_d(Fv)\|_{L_{t,x}^2} = \left\| \int_{\Omega_2} \chi_{\Omega_1}(\tau - \tau', \xi - \xi') d\tau' d\xi' \right\|_{L_{\tau,\xi}^2(|\tau \pm |\xi|| \approx d)} \gtrsim |\Omega_1| |\{|\tau \pm |\xi|| \approx d, \lambda \leq |\xi| \leq \lambda + 1\}|^{\frac{1}{2}} \approx d^{\frac{1}{2}}$$

and consequently

$$\|Fv\|_{X^{-\frac{1}{2},1}} \geq \sum_{d \ll 1} d^{-\frac{1}{2}} d^{\frac{1}{2}} = \infty.$$

On the other hand it is easy to check that the righthand side of (25) is finite, thus (25) fails. We make the remark that this counterexample does not include interactions close to the cone, thus null structure would not help. To summarise, endpoint $X^{s,b}$ type spaces together with bilinear estimates, do not appear to be enough to obtain critical well-posedness results.

3.2. Atomic Banach Spaces. The remaining function spaces used in this article have a complicated structure as they need to capture certain space-time integrability properties of our solution in arbitrary null frames. The method to define these spaces, going back to the work of Tataru [47], is via an *atomic* construction. The standard set up is as follows. We start with a subset $E \subset \mathcal{S}'$ such that for every $\phi \in \mathcal{S}$ we have

$$\sup_{f \in E} |f(\phi)| < \infty. \quad (26)$$

The set E consists of our *atoms*. We then define the atomic Banach space $\mathcal{A}(E)$ as

$$\mathcal{A}(E) = \left\{ \sum_{j \in \mathbb{N}} c_j f_j \mid (c_j)_{j \in \mathbb{N}} \in \ell^1(\mathbb{N}), f_j \in E \right\} \quad (27)$$

with the norm

$$\|f\|_{\mathcal{A}(E)} = \inf \left\{ \sum_{j \in \mathbb{N}} |c_j| \mid f = \sum_j c_j f_j, (c_j)_{j \in \mathbb{N}} \in \ell^1(\mathbb{N}), f_j \in E \right\} \quad (28)$$

It is easy to check that provided $c_j \in \ell^1(\mathbb{N})$ and $f_j \in E$, the condition (26) implies that the sum $\sum_j c_j f_j$ converges in \mathcal{S}' and thus $\mathcal{A}(E)$ is a well-defined subset of \mathcal{S}' . Moreover a standard computation shows that $\|\cdot\|_{\mathcal{A}(E)}$ is indeed a norm on $\mathcal{A}(E)$ (which is stronger than the standard Schwartz topology on \mathcal{S}'), and the pair $(\mathcal{A}(E), \|\cdot\|_{\mathcal{A}(E)})$ form a Banach space.

Given a linear operator T , and a Banach space $X \subset \mathcal{S}'$, we often need to prove inequalities of the form

$$\|Tf\|_X \lesssim \|f\|_{\mathcal{A}(E)}. \quad (29)$$

In general, this can be broken down into two steps. The first step is to show that if $f = \sum_{j \in \mathbb{N}} c_j f_j$ is a decomposition of f into atoms $f_j \in E$, then

$$Tf = \sum_{j \in \mathbb{N}} c_j T f_j \quad (30)$$

with convergence in \mathcal{S}' say. The second is to obtain (29) in the special case where $f \in E$ is an atom. In other words show that we have the bound

$$\sup_{f \in E} \|Tf\|_X \lesssim 1. \quad (31)$$

It is a simple exercise to show that (30) and (31), together with the uniqueness of limits in \mathcal{S}' , implies the bound (29). Note that in general, it is *not* true that boundedness on atoms (31) directly implies the bound (29), see for instance [6] for an example related to the Hardy space. Thus some care has to be taken to first check the identity (30) as well as the boundedness on atoms. However, in the arguments used in the current paper, the identity (30) is almost immediately, and thus we often leave the proof of (30) to the reader. The reduction of (29) to (31) is used frequently in the arguments to follow.

As a special case of (29), note that if $X \subset \mathcal{S}'$ is a Banach space with $E \subset \{\|f\|_X \lesssim 1\}$ (thus the set of atoms is contained inside the unit ball of X), then we immediately deduce the continuous embedding $\mathcal{A}(E) \subset X$. Conversely, if the unit ball of X is contained in the set of atoms E , then we have $X \subset \mathcal{A}(E)$. Of course this condition can be weakened considerably, for instance if E contains a *dense* subset of $\{\|f\|_X \leq 1\}$, then we still have $X \subset \mathcal{A}(E)$. See [3] for a more general result of this nature.

3.3. Null Frame spaces - $NF^\pm(\kappa)$, $PW^\pm(\kappa)$, and $[NF^\pm]^*(\kappa)$. As mentioned previously, the wave equation satisfies improved regularity properties in certain null frames (t_ω, x_ω) . However, we cannot pick a fixed frame (t_ω, x_ω) to work in, and instead have to work in certain *averages* over directions $\omega \in \kappa$. The fact that we have to control our solution in many coordinates frames simultaneously forces us to use the rather complicated atomic construction (27) and (28) to define the necessary spaces. The construction below is heavily based on the original work of Tataru on the wave maps problem [47]. Accordingly we follow, as much as possible, the notation introduced in [47].

The first null frame space we introduce is based on $L_{t_\omega}^1 L_{x_\omega}^2$, and should be thought of as a suitable replacement for the $L_t^1 L_x^2$ norm. It is designed to capture the improved space-time estimates that we get in null coordinates, and will handle the case where we are very close to the cone, in which case the $\dot{\mathcal{X}}_\pm^{-\frac{1}{2}, 1}$ norm is not so effective. The definition is as follows.

Let $\kappa \in \mathcal{C}_\alpha$ be a cap on the sphere. We say that F is a $NF^\pm(\kappa)$ atom if there exists $\omega \notin 2\kappa$ such that

$$\|\Pi_{\pm\omega}F\|_{L_{t,\omega}^1 L_{x,\omega}^2} + \theta(\omega, \kappa)^{-1} \|\Pi_{\mp\omega}F\|_{L_{t,\omega}^1 L_{x,\omega}^2} \leq 1.$$

We then define the atomic Banach space $NF^\pm(\kappa)$ via⁷ (27) where we take E to be the set of all $NF^\pm(\kappa)$ atoms, thus

$$NF^\pm(\kappa) = \left\{ \sum_j c_j F_j \mid (c_j) \in \ell^1, F_j \text{ is a } NF^\pm(\kappa) \text{ atom} \right\}$$

with the obvious norm defined as in (28). We frequently make use of the immediate inequality

$$\|F\|_{NF^\pm(\kappa)} \leq \inf_{\omega \notin 2\kappa} \left(\|\Pi_{\pm\omega}F\|_{L_{t,\omega}^1 L_{x,\omega}^2} + \theta(\omega, \kappa)^{-1} \|\Pi_{\mp\omega}F\|_{L_{t,\omega}^1 L_{x,\omega}^2} \right).$$

The second null frame space we define forms a replacement for the missing $L_t^2 L_x^\infty$ Strichartz estimate and is based on $L_{t,\omega}^2 L_{x,\omega}^\infty$ type norms. Similar to the $NF^\pm(\kappa)$ space we use an atomic definition.

Let $\kappa \in \mathcal{C}_\alpha$. We say ψ is a $PW^\pm(\kappa)$ atom, if there exists $\omega \in 2\kappa$ such that

$$\|\Pi_{\pm\omega}\psi\|_{L_{t,\omega}^2 L_{x,\omega}^\infty} + \alpha^{-1} \|\Pi_{\mp\omega}\psi\|_{L_{t,\omega}^2 L_{x,\omega}^\infty} \leq 1.$$

The atomic Banach space $PW^\pm(\kappa)$ is then defined to be made up of sums of $PW^\pm(\kappa)$ atoms as in (27) with the induced norm (28). Provided we have two sufficiently separated caps κ and $\bar{\kappa}$, the null frame space $NF^\pm(\bar{\kappa})$ and the plane wave type space $PW^\pm(\kappa)$ have a simple relation via what is essentially an application of Holder's inequality.

Lemma 3.4. *Let $0 < \alpha, \beta \ll 1$. Assume $\kappa \in \mathcal{C}_\alpha$ and $\bar{\kappa} \in \mathcal{C}_\beta$ with $\theta(\kappa, \bar{\kappa}) \geq 5 \max\{\alpha, \beta\}$. Let $\psi : \mathbb{R}^{n+1} \rightarrow \mathbb{C}^2$, and F a scalar valued function. Then*

$$\|F\psi\|_{NF^\pm(\bar{\kappa})} \lesssim \|F\|_{L_{t,x}^2} \|\psi\|_{PW^\pm(\kappa)}.$$

More generally, for a fixed $\kappa \in \mathcal{C}_\alpha$, we have the orthogonality property

$$\left(\sum_{\substack{\bar{\kappa} \in \mathcal{C}_\beta \\ \theta(\kappa, \bar{\kappa}) \geq 5 \max\{\alpha, \beta\}}} \left\| P_{\lambda, \bar{\kappa}}^{\pm, \beta} \Pi_{\pm}(Fv) \right\|_{NF^\pm(\bar{\kappa})}^2 + \left\| P_{\lambda, \bar{\kappa}}^{\mp, \beta} \Pi_{\mp}(Fv) \right\|_{NF^\pm(\bar{\kappa})}^2 \right)^{\frac{1}{2}} \lesssim \|F\|_{L_{t,x}^2} \|v\|_{PW^\pm(\kappa)}. \quad (32)$$

Proof. We start by assuming ψ is a $PW^\pm(\kappa)$ atom ψ , thus there exists $\omega \in 2\kappa$ such that

$$\|\Pi_{\pm\omega}\psi\|_{L_{t,\omega}^2 L_{x,\omega}^\infty} + \alpha^{-1} \|\Pi_{\mp\omega}\psi\|_{L_{t,\omega}^2 L_{x,\omega}^\infty} \leq 1.$$

The assumption $\theta(\kappa, \bar{\kappa}) \geq 5 \max\{\alpha, \beta\}$ implies that $\theta(\omega, \bar{\kappa}) \geq \theta(\kappa, \bar{\kappa}) - \theta(\omega, \kappa) \geq 3 \max\{\alpha, \beta\}$. In particular, $\omega \notin 2\bar{\kappa}$ and $\theta(\omega, \bar{\kappa})^{-1} \lesssim \alpha^{-1}$. Hence via Holder's inequality, we obtain

$$\begin{aligned} \|F\psi\|_{NF^\pm(\bar{\kappa})} &\leq \|F \Pi_{\pm\omega}\psi\|_{L_{t,\omega}^1 L_{x,\omega}^2} + \theta(\omega, \bar{\kappa})^{-1} \|F \Pi_{\mp\omega}\psi\|_{L_{t,\omega}^1 L_{x,\omega}^2} \\ &\lesssim \|F\|_{L_{t,x}^2} \left(\|\Pi_{\pm\omega}\psi\|_{L_{t,\omega}^2 L_{x,\omega}^\infty} + \alpha^{-1} \|\Pi_{\mp\omega}\psi\|_{L_{t,\omega}^2 L_{x,\omega}^\infty} \right) \\ &\leq \|F\|_{L_{t,x}^2}. \end{aligned}$$

⁷Note that if F is a $NF^\pm(\kappa)$ atom then for every $\phi \in \mathcal{S}(\mathbb{R}^{1+n})$,

$$|F(\phi)| \leq \left(\|\Pi_{\pm\omega}F\|_{L_{t,\omega}^1 L_{x,\omega}^2} + \theta(\omega, \kappa)^{-1} \|\Pi_{\mp\omega}F\|_{L_{t,\omega}^1 L_{x,\omega}^2} \right) \|\phi\|_{L_{t,\omega}^\infty L_{x,\omega}^2} \lesssim (1 + |t| + |x|)^{n+1} \|\phi\|_{L_{t,x}^\infty}$$

and so (26) holds.

The argument for a general $\psi \in PW^\pm(\kappa)$ follows by decomposing $\psi = \sum_j c_j \psi_j$ where ψ_j are atoms, and noting that $F\psi = \sum_j c_j F\psi_j$ in \mathcal{S}' .

The proof of (32) is similar, but requires the additional complication of the orthogonality estimate

$$\left(\sum_{\substack{\bar{\kappa} \in \mathcal{C}_\beta \\ \theta(\kappa, \bar{\kappa}) \geq 5 \max\{\alpha, \beta\}}} \left\| P_{\lambda, \bar{\kappa}}^{\pm, \beta} \Pi_\pm G \right\|_{NF^\pm(\bar{\kappa})}^2 + \left\| P_{\lambda, \bar{\kappa}}^{\mp, \beta} \Pi_\mp G \right\|_{NF^\pm(\bar{\kappa})}^2 \right)^{\frac{1}{2}} \lesssim \|\Pi_{\pm\omega} G\|_{L_{t\omega}^1 L_{x\omega}^2} + \alpha^{-1} \|\Pi_{\mp\omega} G\|_{L_{t\omega}^1 L_{x\omega}^2}$$

which can be found in (iii) Corollary 8.2 below. \square

The final null frame space we require is a version of the energy type norm $L_t^\infty L_x^2$ in null frames. Given a cap $\kappa \subset \mathcal{C}_\alpha$, we define the norm $\|\cdot\|_{[NF^\pm]^*(\kappa)}$ as

$$\|u\|_{[NF^\pm]^*(\kappa)} = \sup_{\omega \notin 2\kappa} \left(\|\Pi_{\pm\omega} u\|_{L_{t\omega}^\infty L_{x\omega}^2} + \theta(\omega, \kappa) \|\Pi_{\mp\omega} u\|_{L_{t\omega}^\infty L_{x\omega}^2} \right).$$

It is easy enough to check that we have the duality relation

$$\left| \int u^\dagger v dx dt \right| \leq \|u\|_{NF^\pm(\kappa)} \|v\|_{[NF^\pm]^*(\kappa)} \quad (33)$$

and consequently, by a duality argument, we have the following counterpart to Lemma 3.4.

Lemma 3.5. *Let $0 < \alpha, \beta \ll 1$. Assume $\kappa \in \mathcal{C}_\alpha$ and $\bar{\kappa} \in \mathcal{C}_\beta$ with $\theta(\kappa, \bar{\kappa}) \geq 5 \max\{\alpha, \beta\}$. Let u, v take values in \mathbb{C}^2 . Then*

$$\|u^\dagger v\|_{L_{t,x}^2} \lesssim \|u\|_{[NF^\pm]^*(\bar{\kappa})} \|v\|_{PW^\pm(\kappa)}.$$

Remark 3.6. In the original work of Tataru [47], the null frame spaces were defined similarly but without the added complications of the projections $\Pi_{\pm\omega}$. The addition of the projections $\Pi_{\pm\omega}$ is needed to exploit the vector valued nature of the Dirac equation, and is motivated by the fact that if $\text{supp } \hat{f} \subset A_\lambda(\kappa)$, then from (10) we can write the homogeneous solution $\mathcal{U}_-(t)f$ in the form

$$\mathcal{U}_-(t)f = \int_\kappa \Pi_\omega f_\omega(\sqrt{2}t_\omega) d\mathbb{S}(\omega).$$

Thus the projections Π_ω appear naturally when we write the solution as an average of traveling waves. Furthermore, morally speaking, as $\Pi_\omega f_\omega(\sqrt{2}t_\omega)$ is a multiple of a $PW^\pm(\kappa)$ atom, we should have the bound

$$\|\rho(t)\mathcal{U}_-(t)f\|_{PW^+(\kappa)} \leq \int_\kappa \|f_\omega\|_{L^2(\mathbb{R})} d\mathbb{S}(\omega), \quad \|\rho(t)\mathcal{U}_-(t)f\|_{PW^-(\kappa)} \leq \alpha^{-1} \int_\kappa \|f_\omega\|_{L^2(\mathbb{R})} d\mathbb{S}(\omega)$$

where $\rho \in C_0^\infty(\mathbb{R})$ is a cutoff in time⁸ (see Corollary 8.8 below). In particular, as $\alpha \ll 1$, $\mathcal{U}_-(t)f$ obeys much better bounds in $PW^+(\kappa)$ than $PW^-(\kappa)$. Without the projections $\Pi_{\pm\omega}$ built into the spaces $PW^\pm(\kappa)$, this observation would be much harder to exploit. Finally, we note that the additional regularity given by placing $\mathcal{U}_-(t)f \in PW^+(\kappa)$, is a manifestation of the *null structure* of the Dirac equation, and plays a crucial role in the proof of Theorem 1.5.

⁸It is unclear to the authors if this is true without the cutoff ρ .

3.4. **The space \mathcal{N}_λ^\pm .** The space defined to hold the nonlinearity at scale λ , is made up of three components, a $L_t^1 L_x^2$ component, an $X^{s,b}$ component, and a null frame $L_{t_\omega}^1 L_{x_\omega}^2$ component. As previously, the definition is an atomic one, however, unlike the definition of $NF^\pm(\kappa)$ and $PW^\pm(\kappa)$, we require 3 different types of atoms.

- (i) We say F is a NF_λ^\pm atom if there exists (a dyadic) $0 < \alpha \ll 1$ and a decomposition $F = \sum_{\kappa \in \mathcal{C}_\alpha} F_\kappa$ such that each $F_\kappa \in NF^\pm(\kappa)$ with

$$\text{supp } \widehat{\Pi_+ F_\kappa} \subset A_{\alpha, \lambda}^\pm(\kappa), \quad \text{supp } \widehat{\Pi_- F_\kappa} \subset A_{\alpha, \lambda}^\mp(\kappa) \quad (34)$$

and we have the angular square function estimate

$$\left(\sum_{\kappa \in \mathcal{C}_\alpha} \|F_\kappa\|_{NF^\pm(\kappa)}^2 \right)^{\frac{1}{2}} \leq 1.$$

- (ii) We say that F is a $P_\lambda(L_t^1 L_x^2)$ atom, or *energy atom*, if $\text{supp } \widehat{F} \subset \{|\xi| \approx \lambda\}$ and

$$\|F\|_{L_t^1 L_x^2} \leq 1.$$

- (iii) We say that F is a $\dot{\mathcal{X}}_\pm^{-\frac{1}{2}, 1}$ atom if

$$\text{supp } \widetilde{\Pi_+ F} \subset \{|\xi| \approx \lambda, |\tau \pm |\xi|| \approx d\}, \quad \text{supp } \widetilde{\Pi_- F} \subset \{|\xi| \approx \lambda, |\tau \mp |\xi|| \approx d\}$$

and

$$\|F\|_{L_{t,x}^2} \leq d^{\frac{1}{2}}.$$

We now define

$$\mathcal{N}_\lambda^\pm = \left\{ \sum_j c_j F_j \mid (c_j) \in \ell^1(\mathbb{N}), F_j \text{ is either a } N_\lambda^\pm \text{ atom, an energy atom, or a } \dot{\mathcal{X}}_\pm^{-\frac{1}{2}, 1} \text{ atom} \right\}$$

with the obvious norm given by (28). It is not so difficult to check that the condition (26) is satisfied, thus the space \mathcal{N}_λ^\pm is a well-defined atomic Banach space.

In the proof of Theorem 1.1, our aim will be to place the nonlinearity in \mathcal{N}_λ^\pm . Thus we shall frequently be aiming to estimate terms of the form $\|P_\lambda F\|_{\mathcal{N}_\lambda^\pm}$. To this end, we note that if $P_\lambda F \in L_t^1 L_x^2$, then $P_\lambda F$ is multiple of an energy atom. Hence $P_\lambda F \in \mathcal{N}_\lambda^\pm$ and we have the immediate bound

$$\|P_\lambda F\|_{\mathcal{N}_\lambda^\pm} \leq \|P_\lambda F\|_{L_t^1 L_x^2}. \quad (35)$$

Similarly, if we can write $P_\lambda F = \sum_{d \in 2^{\mathbb{Z}}} P_\lambda \mathfrak{C}_d^\pm F$, then as each $P_\lambda \mathfrak{C}_d^\pm F$ is a multiple of a $\dot{\mathcal{X}}_\pm^{-\frac{1}{2}, 1}$ atom, we have $P_\lambda F \in \mathcal{N}_\lambda^\pm$ and

$$\|P_\lambda F\|_{\mathcal{N}_\lambda^\pm} \leq \|P_\lambda F\|_{\dot{\mathcal{X}}_\pm^{-\frac{1}{2}, 1}}. \quad (36)$$

The general strategy to put $F \in \mathcal{N}_\lambda^\pm$ will be to decompose F into certain frequency regions, and then make use of the previous bounds. Of course we will be unable to always place the nonlinearity in as nice a space as $L_t^1 L_x^2$ (or $\dot{\mathcal{X}}_\pm^{-\frac{1}{2}, 1}$) and in certain frequency regions (notable when everything is close to the cone) we have to use the additional flexibility given by the NF_λ^\pm type atoms.

Remark 3.7. Let F be a NF_λ^\pm atom, and let $F = \sum_{\kappa \in \mathcal{C}_\alpha} F_\kappa$ be the corresponding decomposition into atoms. When we come to prove estimates for the F_κ , to use the fact that $F_\kappa \in NF^\pm(\kappa)$, we will be forced to decompose $F_\kappa = \sum_j F_\kappa^{(j)}$ into $NF^\pm(\kappa)$ atoms $F_\kappa^{(j)}$. Unfortunately this means that we may lose the support properties (34), as there is no guarantee that the $F_\kappa^{(j)}$ retain the same Fourier support as F_κ . However, as we can write

$$\Pi_+ F_\kappa = \mathfrak{h} P_{\lambda, \kappa}^{\pm, \alpha} \Pi_+ F_\kappa = \sum_j c_j \mathfrak{h} P_{\lambda, \kappa}^{\pm, \alpha} \Pi_+ F_\kappa^{(j)}$$

then as we have the bound

$$\begin{aligned} \|\mathfrak{h} P_{\lambda, \kappa}^{\pm, \alpha} \Pi_{\pm \omega} \Pi_+ F_\kappa^{(j)}\|_{L_{t, \omega}^1 L_{x, \omega}^2} + \theta(\omega, \kappa)^{-1} \|\mathfrak{h} P_{\lambda, \kappa}^{\pm, \alpha} \Pi_{\mp \omega} \Pi_+ F_\kappa^{(j)}\|_{L_{t, \omega}^1 L_{x, \omega}^2} \\ \lesssim \|\Pi_{\pm \omega} F_\kappa^{(j)}\|_{L_{t, \omega}^1 L_{x, \omega}^2} + \theta(\omega, \kappa)^{-1} \|\Pi_{\mp \omega} F_\kappa^{(j)}\|_{L_{t, \omega}^1 L_{x, \omega}^2} \end{aligned}$$

(see Lemma 3.14 below) the function $\mathfrak{h} P_{\lambda, \kappa}^{\pm, \alpha} \Pi_+ F_\kappa^{(j)}$ is again a, perhaps slightly larger, $NF^\pm(\kappa)$ atom. Thus we may always assume that the functions $F_\kappa^{(j)}$ satisfy the slightly larger support properties

$$\text{supp } \widehat{\Pi_+ F_\kappa^{(j)}} \subset \mathfrak{h} A_{\alpha, \lambda}^\pm(\kappa), \quad \text{supp } \widehat{\Pi_- F_\kappa^{(j)}} \subset \mathfrak{h} A_{\alpha, \lambda}^\mp(\kappa).$$

This observation is frequently used without mention in the remainder of the article.

When we come to prove estimates using the \mathcal{N}_λ^+ spaces, we often have to estimate a NF_λ^\pm atom in $L_{t, x}^2$. The following lemma is very useful in this regard.

Lemma 3.8. *Let $0 < \alpha \ll 1$ and assume $F = \sum_{\kappa \in \mathcal{C}_\alpha} F_\kappa$ is a NF_λ^\pm atom. Then*

$$\|\mathfrak{C}_d^\pm F\|_{L_{t, x}^2} \lesssim (\min\{d, \alpha^2 \lambda\})^{\frac{1}{2}}.$$

Proof. We only prove the $\pm = +$ case, the $-$ case is similar. Let $F = \sum_{\kappa \in \mathcal{C}_\alpha} F_\kappa$. By orthogonality in $L_{t, x}^2$, and the observation that $\mathfrak{C}_d^+ F_\kappa = 0$ for $d > \alpha^2 \lambda$, it is enough to show that

$$\|\mathfrak{C}_d^+ F_\kappa\|_{L_{t, x}^2} \lesssim d^{\frac{1}{2}} \|F_\kappa\|_{NF^+(\kappa)}$$

for $d \lesssim \alpha^2 \lambda$. Furthermore, by decomposing F_κ into $NF^+(\kappa)$ atoms, we reduce to proving that for $\omega \notin 2\kappa$ we have

$$\|C_d^\pm \mathfrak{h} P_{\lambda, \kappa}^{\pm, \alpha} \Pi_\pm G\|_{L_{t, x}^2} \lesssim d^{\frac{1}{2}} \left(\|\Pi_\omega G\|_{L_{t, \omega}^1 L_{x, \omega}^2} + \theta(\omega, \kappa)^{-1} \|\Pi_{-\omega} G\|_{L_{t, \omega}^1 L_{x, \omega}^2} \right). \quad (37)$$

Note that if $(\tau, \xi) \in \mathfrak{h} A_{\lambda, \alpha}^\pm(\kappa)$ and $|\tau \pm |\xi|| \lesssim d$, then from (20), we have $|\xi_\omega^1| \approx \lambda \theta(\omega, \kappa)^2$ and hence

$$\left| \tau_\omega - \frac{|\xi_\omega^\perp|}{2\xi_\omega^1} \right| = \frac{|\tau|^2 - |\xi|^2}{2|\xi_\omega^1|} \lesssim \frac{d}{\theta(\omega, \kappa)^2}.$$

Thus, for fixed ξ_ω , τ_ω varies in a set of size $\frac{d}{\theta(\omega, \kappa)^2}$. Therefore, by an application of Bernstein together with the null form estimate $|\Pi_{\pm \frac{\xi}{|\xi|}} \Pi_\omega| \lesssim \theta(\omega, \mp \xi) \approx \theta(\omega, \kappa)$, we have

$$\begin{aligned} \|C_d^\pm \mathfrak{h} P_{\lambda, \kappa}^{\pm, \alpha} \Pi_\pm G\|_{L_{t, x}^2} &\lesssim \|C_d^\pm \mathfrak{h} P_{\lambda, \kappa}^{\pm, \alpha} \Pi_\pm \Pi_\omega G\|_{L_{t, x}^2} + \|C_d^\pm \mathfrak{h} P_{\lambda, \kappa}^{\pm, \alpha} \Pi_\pm \Pi_{-\omega} G\|_{L_{t, x}^2} \\ &\lesssim \theta(\omega, \kappa) \times \frac{d^{\frac{1}{2}}}{\theta(\omega, \kappa)} \|\widehat{\Pi_\omega G}\|_{L_{\xi_\omega}^2 L_{\tau_\omega}^\infty} + \frac{d^{\frac{1}{2}}}{\theta(\omega, \kappa)} \|\widehat{\Pi_{-\omega} G}\|_{L_{\xi_\omega}^2 L_{\tau_\omega}^\infty} \\ &\lesssim d^{\frac{1}{2}} \left(\|\Pi_\omega G\|_{L_{t, \omega}^1 L_{x, \omega}^2} + \theta(\omega, \kappa)^{-1} \|\Pi_{-\omega} G\|_{L_{t, \omega}^1 L_{x, \omega}^2} \right) \end{aligned}$$

as required. \square

3.5. Iteration Space. We now have the basic building blocks of the Banach space with which to prove Theorem 1.1. Define

$$F_\lambda^\pm = \{ u \in L_t^\infty L_x^2 \mid \text{supp } \widehat{u} \subset \{|\xi| \approx \lambda\}, (\partial_t \pm \sigma \cdot \nabla)u \in \mathcal{N}_\lambda^\pm \}$$

with the associated norm

$$\|u\|_{F_\lambda^\pm} = \|u\|_{L_t^\infty L_x^2} + \|(\partial_t \pm \sigma \cdot \nabla)u\|_{\mathcal{N}_\lambda^\pm}.$$

We now sum up over frequencies to define

$$\|u\|_{F^{s,\pm}} = \left(\sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{2s} \|P_\lambda u\|_{F_\lambda^\pm}^2 \right)^{\frac{1}{2}}$$

and let

$$F^{s,\pm} = \{ u \in L_t^\infty \dot{H}_x^s \mid P_\lambda u \in F_\lambda^\pm, \|u\|_{F^{s,\pm}} < \infty \}.$$

The space F_λ^\pm is essentially enough to prove the multi-linear estimates that we require, and via bilinear estimates of the form in Lemma 1.6, it is possible to complete the proof of Theorem 1.1 with small data in the slightly smaller space $\dot{B}_{2,1}^{\frac{n-1}{2}}$ (i.e. with an ℓ^1 sum over frequencies instead of a ℓ^2 sum). To get the more general $\dot{H}^{\frac{n-1}{2}}$ result, we need some additional gain away from the light cone. To this end, motivated by the recent work⁹ of Bejenaru-Herr [2], we define an additional semi-norm $\|\cdot\|_{\mathcal{Y}^\pm}$ as

$$\|u\|_{\mathcal{Y}^\pm} = \sup_d d \|\mathfrak{e}_d^\pm u\|_{L_t^{\frac{4n}{3n-1}} L_x^2}.$$

Then we take G_λ^\pm as

$$G_\lambda^\pm = \{ u \in F_\lambda^\pm \mid \|u\|_{\mathcal{Y}^\pm} < \infty \}$$

with the norm

$$\|u\|_{G_\lambda^\pm} = \|u\|_{F_\lambda^\pm} + \lambda^{-\frac{n+1}{4n}} \|u\|_{\mathcal{Y}^\pm}$$

where the $\lambda^{-\frac{n+1}{4n}}$ term is to ensure that both components of the G_λ^\pm norm scale the same way. We now define

$$G^{s,\pm} = \{ u \in L_t^\infty \dot{H}_x^s \mid P_\lambda u \in G_\lambda^\pm, \|u\|_{G^{s,\pm}} < \infty \}$$

where

$$\|u\|_{G^{s,\pm}} = \left(\sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{2s} \|P_\lambda u\|_{G_\lambda^\pm}^2 \right)^{\frac{1}{2}}.$$

Corresponding to the function spaces $F^{s,\pm}$ and $G^{s,\pm}$, we aim to put the nonlinearity in the summed up versions of the \mathcal{N}_λ^\pm and $L_t^{\frac{4n}{3n-1}} L_x^2$ spaces. Namely, we define

$$\|F\|_{\mathcal{N}^{s,\pm}} = \left(\sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{2s} \|P_\lambda F\|_{\mathcal{N}_\lambda^\pm}^2 \right)^{\frac{1}{2}}$$

and

$$\|F\|_{(\mathcal{N} \cap \mathcal{Y})^{s,\pm}} = \left(\sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{2s} \|P_\lambda F\|_{\mathcal{N}_\lambda^\pm}^2 + \lambda^{2(s-\frac{n+1}{4n})} \|P_\lambda F\|_{L_t^{\frac{4n}{3n-1}} L_x^2}^2 \right)^{\frac{1}{2}}.$$

These spaces satisfy the following important properties.

⁹In the work of Bejenaru-Herr, they also needed some additional integrability in time of functions supported away from the light cone. To accomplish this, they made use of the norm (in the notation used in the current paper)

$$\sup_d d \|\mathfrak{e}_d^\pm u\|_{L_t^{\frac{4}{3}} L_x^2(\mathbb{R}^{1+3})}.$$

In the current paper, this norm is too strong, and we need to use the slightly weaker \mathcal{Y}_λ^\pm norm (note that $\frac{4n}{3n-1} = \frac{3}{2}$ if $n = 3$, thus we need *less* integrability in time).

Theorem 3.9.

(i) (Energy inequality.) *Let $s \geq 0$. Then $F^{s,\pm}$ is a Banach space, and moreover we have the energy inequality*

$$\|u\|_{F^{s,\pm}} \leq \|u(0)\|_{\dot{H}^s} + C\|(\partial_t \pm \sigma \cdot \nabla)u\|_{\mathcal{N}^{s,\pm}}.$$

Similarly we have

$$\|u\|_{G^{s,\pm}} \leq \|u(0)\|_{\dot{H}^s} + C\|(\partial_t \pm \sigma \cdot \nabla)u\|_{(\mathcal{N} \cap \mathcal{Y})^{s,\pm}}$$

(here C is some constant independent of u).

(ii) (Stability with respect to time cutoffs.) *Let $\rho \in C_0^\infty(\mathbb{R})$ and $T > 0$. Then*

$$\|\rho(\frac{t}{T})u\|_{F^{s,\pm}} \lesssim \|u\|_{F^{s,\pm}}, \quad \|\mathbb{1}_{(-T,T)}(t)F\|_{\mathcal{N}^{s,\pm}} \lesssim \|F\|_{\mathcal{N}^{s,\pm}}$$

where the implied constants are independent of T . Similarly, if $\lambda \in 2^{\mathbb{Z}}$ and $T \geq \lambda^{-1}$, we have the bound

$$\|\rho(\frac{t}{T})u\|_{G_\lambda^\pm} \lesssim \|u\|_{G_\lambda^\pm}$$

where the implied constant is again independent of T .

(iii) (Scattering.) *Let $\rho \in C_0^\infty(\mathbb{R})$ with $\rho(t) = 1$ on $[-1, 1]$ and assume $\sup_{T>0} \|\rho(\frac{t}{T})u\|_{F^{s,\pm}} < \infty$. Then there exists $f_{-\infty}, f_{+\infty} \in \dot{H}^s$ such that*

$$\lim_{t \rightarrow \infty} \left(\|u(t) - \mathcal{U}_\pm(t)f_{+\infty}\|_{\dot{H}^s} + \|u(-t) - \mathcal{U}_\pm(-t)f_{-\infty}\|_{\dot{H}^s} \right) = 0.$$

Proof. We leave the proof to Section 10. □

Remark 3.10. Note that the previous theorem implies that we have the bound

$$\|\phi\|_{F^{s,\pm}} \leq \|\phi(0)\|_{\dot{H}^s} + \|(\partial_t \pm \sigma \cdot \nabla)\phi\|_{L_t^1 \dot{H}_x^s}.$$

In particular, $\|\phi\|_{F^{s,\pm}} < \infty$ for every $\phi \in \mathcal{S}$. A similar comment applies in the $G^{s,\pm}$ case.

Remark 3.11. Our eventual aim will be to construct a solution in $G^{\frac{n-1}{2},\pm}$, although this will require a significant amount of work. To alleviate this somewhat, we note that if $u \in F_\lambda^\pm$, then letting $v(t, x) = u(t, -x)$, a computation shows that¹⁰ $v \in F_\lambda^\mp$. Similarly we can check that if $u \in G_\lambda^\pm$ then $v \in G_\lambda^\mp$. On the other hand, if we reflect in both t and x , i.e. we let $w(t, x) = u(-t, -x)$, then a similar calculation shows that $\|u\|_{G_\lambda^\pm} = \|w\|_{G_\lambda^\pm}$ and $\|u\|_{F_\lambda^\pm} = \|w\|_{F_\lambda^\pm}$ while $(\Pi_\pm u)(t, x) = (\Pi_\mp w)(-t, -x)$. Together these observations often allow us to reduce to considering just the $+$ case, rather than both $+$ and $-$ cases.

In a similar vein, we observe that in the $n = 2$ case a computation using (7) shows that we have $\|\beta u\|_{G^{s,\pm}} \approx \|u\|_{G^{s,\mp}}$. As in the homogeneous case, this will allow us to deduce estimates for $u^\dagger \beta u$ from estimates of the form $u^\dagger v$.

The norm F_λ^\pm is fairly complicated due to its atomic structure. However it can be compared to the more standard $\dot{\mathcal{X}}_\pm^{\frac{1}{2},q}$ spaces by the following useful estimate.

¹⁰Essentially this boils down to showing that reflecting a \mathcal{N}_λ^\pm atom in x , gives a \mathcal{N}_λ^\mp atom, which is not too difficult to show.

Lemma 3.12. *Let $u \in F_\lambda^\pm$. Then*

$$\|u\|_{\dot{\mathcal{X}}_\pm^{\frac{1}{2},\infty}} \lesssim \|u\|_{F_\lambda^\pm}. \quad (38)$$

Proof. By a reflection, we may assume that $\pm = +$. The estimate

$$\|u\|_{\dot{\mathcal{X}}_+^{\frac{1}{2},\infty}} \approx \|(\partial_t + \sigma \cdot \nabla)u\|_{\dot{\mathcal{X}}_+^{-\frac{1}{2},\infty}}$$

shows that is enough to prove that $\|F\|_{\dot{\mathcal{X}}_+^{-\frac{1}{2},\infty}} \lesssim \|F\|_{\mathcal{N}_\lambda^+}$. The atomic definition of \mathcal{N}_λ^+ , implies that we need to consider three cases, F is a $\dot{\mathcal{X}}_+^{-\frac{1}{2},\infty}$ atom, F is an energy atom, and F is a NF_λ^+ atom. The first case is obvious due to the embedding $\dot{\mathcal{X}}_+^{-\frac{1}{2},1} \subset \dot{\mathcal{X}}_+^{-\frac{1}{2},\infty}$. On the other hand, if $F \in L_t^1 L_x^2$, then as $\text{supp } \widetilde{\mathfrak{C}}_d^+ F \subset \{|\tau| - |\xi| \approx d\}$, we see that for each fixed d

$$d^{-\frac{1}{2}} \|\mathfrak{C}_d^+ F\|_{L_{t,x}^2} \lesssim \|\widetilde{F}\|_{L_\xi^2 L_\tau^\infty} \lesssim \|F\|_{L_t^1 L_x^2}.$$

Taking the sup over d then gives the $F \in L_t^1 L_x^2$ case. Finally, if F is a NF_λ^+ atom, then by Lemma 3.8 we obtain $\|\mathfrak{C}_d^+ F\|_{L_{t,x}^2} \lesssim d^{\frac{1}{2}}$ and so we clearly have $\|F\|_{\dot{\mathcal{X}}_+^{-\frac{1}{2},\infty}} \lesssim 1$ as required. \square

Remark 3.13. Note that, if $\text{supp } u \subset \{|\xi| \approx \lambda\}$ and $u \in L_{t,x}^2(\mathbb{R}^{1+n})$, then by decomposing $u = \sum_{d \in 2^{\mathbb{Z}}} \mathfrak{C}_d^\pm u$ (which is possible as $u \in L_{t,x}^2$), by definition of \mathcal{N}_λ^\pm together with (24) we have

$$\|u\|_{F_\lambda^\pm} \leq \|u\|_{L_t^\infty L_x^2} + \|(\partial_t \pm \sigma \cdot \nabla)u\|_{\mathcal{N}_\lambda^\pm} \lesssim \|u\|_{\dot{\mathcal{X}}_\pm^{\frac{1}{2},1}} + \sum_{d \in 2^{\mathbb{Z}}} d^{-\frac{1}{2}} \|(\partial_t \pm \sigma \cdot \nabla)\mathfrak{C}_d^\pm u\|_{L_{t,x}^2} \lesssim \|u\|_{\dot{\mathcal{X}}_\pm^{\frac{1}{2},1}}. \quad (39)$$

In particular, we have the bounds

$$\|u\|_{\dot{\mathcal{X}}_\pm^{\frac{1}{2},\infty}} \lesssim \|u\|_{F_\lambda^\pm} \lesssim \|u\|_{\dot{\mathcal{X}}_\pm^{\frac{1}{2},1}},$$

thus F_λ^\pm is within a log factor of an $X^{s,b}$ spaces.

As mentioned previously, if we had access to a $L_t^2 L_x^\infty$ Strichartz estimate, then the proof of GWP would follow by an application of Hölder's inequality. However, the $L_t^2 L_x^\infty$ Strichartz estimate barely fails in $n = 3$, and is far from true in the $n = 2$ case. Despite this, provided we are away from the cone, we *can* control the $L_t^2 L_x^\infty$ by a simple application of Bernstein together with the previous lemma. More precisely, if $u \in F_\lambda^\pm$, then by Lemma 3.12

$$\|\mathfrak{C}_{\gtrsim \delta}^\pm u\|_{L_{t,x}^2} \lesssim \sum_{d \gtrsim \delta} \|\mathfrak{C}_d^\pm u\|_{L_{t,x}^2} \lesssim \|u\|_{\dot{\mathcal{X}}_\pm^{\frac{1}{2},\infty}} \sum_{d \gtrsim \delta} d^{-\frac{1}{2}} \lesssim \delta^{-\frac{1}{2}} \|u\|_{F_\lambda^\pm} \quad (40)$$

and consequently

$$\|\mathfrak{C}_{\gtrsim \delta}^\pm u\|_{L_t^2 L_x^\infty} \lesssim \lambda^{\frac{n}{2}} \|\mathfrak{C}_{\gtrsim \delta}^\pm u\|_{L_{t,x}^2} \lesssim \lambda^{\frac{n}{2}} \delta^{-\frac{1}{2}} \|u\|_{F_\lambda^\pm}.$$

This estimate, as well as the important $L_{t,x}^2$ bound (40), is used frequently in the remainder of this article as it essentially allows us to deal with the the region away from the light cone¹¹. The remaining close cone interaction is much more complicated, and requires the the full strength of the norms defined above.

¹¹This is true in the bilinear case. In the proof of the trilinear estimates, Lemma 3.12 is not enough to deal with the far cone regions and we require the addition decay in time provided by the \mathcal{Y}^\pm norms.

3.6. Disposable Multipliers. We use some notation originally due to Tao [46]. We say a Fourier multiplier \mathcal{M} is *disposable* on a Banach space X , if we have

$$\|\mathcal{M}F\|_X \lesssim \|F\|_X.$$

Clearly any Fourier multiplier with bounded symbol is disposable on $L^2_{t,x}$ by Plancheral. More generally we have the following.

Lemma 3.14 (Multipliers are disposable). *Let $\alpha, \beta \ll 1$ and $\kappa \in \mathcal{C}_\alpha, \bar{\kappa} \in \mathcal{C}_\beta$.*

- (i) *Let \pm_1 and \pm_2 be independent choices of signs. Then $P_{\lambda,\kappa}^{\pm_1,\alpha}$ is given by a convolution with an $L^1_{t,x}(\mathbb{R}^{1+n})$ kernel. In particular, $P_{\lambda,\kappa}^{\pm_1,\alpha}$ is disposable on $NF^{\pm_2}(\bar{\kappa})$, $PW^{\pm_2}(\bar{\kappa})$, and $[NF^{\pm_2}]^*(\bar{\kappa})$.*
- (ii) *Assume $\alpha \lesssim \beta$ and $\kappa \cap \bar{\kappa} \neq \emptyset$. Then $P_{\lambda,\kappa}^{\pm,\alpha}\Pi_\pm$ is disposable on $NF^+(\bar{\kappa})$, $PW^+(\bar{\kappa})$, and $[NF^+]^*(\bar{\kappa})$. Similarly $P_{\lambda,\kappa}^{\mp,\alpha}\Pi_\pm$ is disposable on $NF^-(\bar{\kappa})$, $PW^-(\bar{\kappa})$, and $[NF^-]^*(\bar{\kappa})$.*
- (iii) *The multipliers $C_d, C_{\lesssim d}, C_d^\pm, C_{\lesssim d}^\pm$ are disposable on $L_t^q L_x^2$ for $1 \leq q \leq \infty$.*
- (iv) *Let $d \gtrsim \lambda$. Then $P_\lambda C_d, P_\lambda C_{\lesssim d}$, and $P_\lambda C_{\gtrsim d}$ are disposable on $L_t^q L_x^r$ for any $1 \leq q, r \leq \infty$.*

Proof. (i) and (ii): We only show that $P_{\lambda,\kappa}^{\pm,\alpha}\Pi_\pm$ is disposable as the remaining case is similar (but easier). So assume that $\alpha \lesssim \beta$ and $\kappa \cap \bar{\kappa} \neq \emptyset$. The general idea is to show that the kernel of $P_{\lambda,\kappa}^{\pm,\alpha}\Pi_\pm$ belongs to $L^1_{t,x}$, and then apply Holder. There is a slight complication however, as the definition of the null frame spaces use the projections $\Pi_{\pm\omega}$ which do not commute with the Π_\pm . Thus showing that the kernel is in $L^1_{t,x}$ would not suffice and we need to prove a stronger estimate exploiting the null form estimate (15).

Let $\tilde{\rho}(\tau, \xi) = \Phi(\frac{|\xi|}{\lambda})\Phi_\kappa(\mp \frac{\xi}{|\xi|})\Phi_0(\frac{|\tau \pm |\xi||}{c\alpha^2\lambda})$ where c is the small constant used in the definition of $A_{\lambda,\alpha}^\pm(\kappa)$, thus $P_{\lambda,\kappa}^{\pm,\alpha}u = \rho * u$. Fix any $\omega \in \mathbb{S}^{n-1}$. The key is to prove that $\|\Pi_\pm \rho\|_{L^1_{t,x}} \lesssim 1$ as well as the stronger estimate

$$\|(\Pi_\pm - \Pi_{-\omega})\rho\|_{L^1_{t,x}} \lesssim \max\{\theta(\omega, \bar{\kappa}), \beta\}. \quad (41)$$

Since assuming we have (41) and using the identity $\Pi_\pm \Pi_\omega = (\Pi_\pm - \Pi_{-\omega})\Pi_\omega$ we deduce that

$$\|P_{\lambda,\kappa}^{\pm,\alpha}\Pi_\pm \Pi_\omega u\|_{L_{t\omega}^q L_{x\omega}^r} = \|[(\Pi_\pm - \Pi_{-\omega})\rho] * (\Pi_\omega u)\|_{L_{t\omega}^q L_{x\omega}^r} \lesssim \max\{\theta(\omega, \bar{\kappa}), \beta\} \|\Pi_\omega u\|_{L_{t\omega}^q L_{x\omega}^r}.$$

Similarly the $L^1_{t,x}$ bound gives

$$\|P_{\lambda,\kappa}^{\pm,\alpha}\Pi_\pm \Pi_{-\omega} u\|_{L_{t\omega}^q L_{x\omega}^r} = \|(\Pi_\pm \rho) * (\Pi_{-\omega} u)\|_{L_{t\omega}^q L_{x\omega}^r} \lesssim \|\Pi_{-\omega} u\|_{L_{t\omega}^q L_{x\omega}^r}.$$

Applying these bounds to the relevant atoms, we obtain the boundedness of $P_{\lambda,\kappa}^{\pm,\alpha}\Pi_\pm$ on $NF^+(\bar{\kappa})$, $PW^+(\bar{\kappa})$, and $[NF^+]^*(\bar{\kappa})$.

We now prove (41). Let $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1}$. By rotating the ξ coordinates (and a reflection if needed) we may assume that $\pm = +$ and κ is centered around $(-1, 0, \dots, 0)$,

thus $\tilde{\rho}$ is supported in the set $\{\xi_1 \sim \lambda, |\xi'| \lesssim \lambda\alpha\}$. A computation gives

$$\begin{aligned} & 2(\Pi_+ - \Pi_{-\omega})\rho(t, x_1 + t, x') \\ &= \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}} \left(\omega + \frac{\xi}{|\xi|}\right) \cdot \sigma \tilde{\rho}(\tau, \xi) e^{ix \cdot \xi} e^{it(\tau + \xi_1)} d\tau d\xi \\ &= \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}} \left(\omega + \frac{\xi}{|\xi|}\right) \cdot \sigma \tilde{\rho}(\tau - \xi_1, \xi) e^{ix \cdot \xi} e^{it\tau} d\tau d\xi \\ &= \frac{(\alpha\lambda)^{n+1}}{(2\pi)^{n+1}} \int_{\mathbb{R}^n} \int_{\mathbb{R}} \left(\omega + \frac{(\xi_1, \alpha\xi')}{|(\xi_1, \alpha\xi')|}\right) \cdot \sigma \tilde{\rho}(\alpha^2\lambda\tau - \lambda\xi_1, \lambda\xi_1, \alpha\lambda\xi') e^{i\lambda(\alpha^2 t, x_1, \alpha x') \cdot (\tau, \xi)} d\tau d\xi. \end{aligned}$$

If we now rescale the (t, x) variables (which leaves the $L_{t,x}^1$ norm unchanged), it is enough to prove that

$$\partial_\tau^{N_1} \partial_{\xi_1}^{N_2} \nabla_{\xi'}^{N_3} \left(\left(\omega + \frac{(\xi_1, \alpha\xi')}{|(\xi_1, \alpha\xi')|}\right) \tilde{\rho}(\alpha^2\lambda\tau - \lambda\xi_1, \lambda\xi_1, \alpha\lambda\xi') \right) \lesssim \max\{\theta(\omega, \bar{\kappa}), \beta\}$$

where $(\tau, \xi_1, \xi') \in \{|\tau| \lesssim 1, \xi_1 \approx 1, |\xi'| \lesssim 1\}$. If we note that $\tau - \xi_1 + |\xi| = \tau + \frac{|\xi'|^2}{\xi_1 + |\xi|}$ we can write

$$\tilde{\rho}(\alpha^2\lambda\tau - \lambda\xi_1, \lambda\xi_1, \alpha\lambda\xi') = \Phi(|(\xi_1, \alpha\xi')|) \Phi_\kappa\left(-\frac{(\xi, \alpha\xi')}{|(\xi, \alpha\xi')|}\right) \Phi_0\left(c^{-1}\tau + \frac{c^{-1}|\xi'|^2}{\xi_1 + |(\xi_1, \alpha\xi')|}\right)$$

and thus whenever a derivative hits $\tilde{\rho}$, by (17), we at worst pick up a factor of $\alpha \lesssim \beta \ll 1$. Thus it remains to show that

$$\left| \partial_{\xi_1}^{N_2} \nabla_{\xi'}^{N_3} \left(\omega + \frac{(\xi_1, \alpha\xi')}{|(\xi_1, \alpha\xi')|} \right) \right| \lesssim \max\{\theta(\omega, \bar{\kappa}), \beta\}.$$

Suppose $N_2 = N_3 = 0$. Let $\eta = (\xi_1, \alpha\xi')$ and $\xi^* \in \kappa \cap \bar{\kappa} (\neq \emptyset)$. Then $-\frac{\eta}{|\eta|} \in \kappa$ and moreover

$$\left| \omega + \frac{(\xi_1, \alpha\xi')}{|(\xi_1, \alpha\xi')|} \right| \lesssim \theta(\omega, -\eta) \lesssim \theta(\omega, \xi^*) + \theta(-\eta, \xi^*) \lesssim \max\{\theta(\omega, \bar{\kappa}), \beta\}$$

since $\bar{\kappa} \in \mathcal{C}_\beta$. On the other hand for $N_1, N_2 \neq 0$, we simply note that derivatives of ξ' only add multiples of α (which is acceptable as $\alpha \lesssim \beta$), while

$$\left| \partial_{\xi_1} \left(\omega - \frac{(\xi_1, \alpha\xi')}{|(\xi_1, \alpha\xi')|} \right) \right| = \left| \frac{(\alpha^2|\xi'|^2, -\alpha\xi_1|\xi'|)}{|(\xi_1, \alpha\xi')|^3} \right| \lesssim \alpha$$

which again is clearly acceptable. Thus we obtain (41), and clearly the same argument shows that $\|\Pi_+\rho\|_{L^1} \lesssim 1$ as required.

(ii) and (iii): These are both well known, see for instance [46, Lemma 3]. □

Remark 3.15. It is clear from the proof that multipliers of the form ${}^{\sharp}P_\lambda R_\kappa^\pm C_{\ll d}^\pm$ (and other similar combinations) also satisfy the properties (i) and (ii) in the previous lemma provided $d \gtrsim \alpha^2\lambda$. In particular, if $\text{supp } \hat{u} \subset \{|\xi| \approx \lambda\}$, $\kappa \in \mathcal{C}_\alpha$, and $d \gtrsim \alpha^2\lambda$, then we can write $C_{\leq d}^\pm R_\kappa^\pm u = \rho * R_\kappa^\pm u$ with $\rho \in L_{t,x}^1(\mathbb{R}^{1+n})$. Thus

$$\|C_{\leq d}^\pm R_\kappa^\pm u\|_{PW^{\pm'}(\kappa)} \lesssim \|R_\kappa^\pm u\|_{PW^{\pm'}(\kappa)}$$

for any choice of signs \pm, \pm' .

4. LINEAR ESTIMATES

In this section we introduce the key linear estimate that we require. As similar versions of these estimates are known, at least for the related function spaces used in the waves maps case [47, 44, 29], we leave the proofs till Sections 8 and 9.

In the subcritical setting, Strichartz estimates have proven to be a key tool in the local and global well-posedness theory for the Dirac equation, especially in the $n = 3$ case, see for instance [15, 30]. We would like to show that our iteration norm F_λ^+ controls the Strichartz type norms $L_t^q L_x^r$. For the $L_t^1 L_x^2$ and $\dot{\mathcal{X}}_\pm^{-\frac{1}{2},1}$ components of our norms, we have a transference type principle, and hence, roughly speaking, any estimate satisfied by homogeneous solutions immediately holds general functions in spaces of the form $(\partial_t \pm \sigma \cdot \nabla)^{-1}(L_t^1 L_x^2 + \dot{\mathcal{X}}_\pm^{-\frac{1}{2},1})$, see for instance Section 4 in [42]. On the other hand, it is much more difficult to show that null frame component of our norms controls the Strichartz norms. In fact, in the case of the related function spaces used in the wave maps problem, initially only the ‘‘off the line’’ Strichartz estimates ($\frac{1}{q} + \frac{n-1}{2r} < \frac{n-1}{4}$) were known, see for instance [46, 27]. However, recently, it was observed by Sterbenz-Tataru [43] that the ‘‘on the line’’ Strichartz estimates also hold. In the current article, by adapting the argument used in [43], we can show that the space F_λ^+ also controls the Strichartz type norms. Note that, in the homogeneous case, the following estimates are immediate from the classical Strichartz estimates together with the L_x^2 orthogonality of the angular projections $P_{\lambda,\kappa}$.

Theorem 4.1 (F_λ^\pm controls Strichartz). *Let $2 \leq q, r \leq \infty$ with $q > 2$ and $\frac{1}{q} + \frac{n-1}{2r} \leq \frac{n-1}{4}$. Suppose $u \in F_\lambda^\pm$. Then we have the estimate*

$$\|u\|_{L_t^q L_x^r} \lesssim \lambda^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}} \|u\|_{F_\lambda^\pm}.$$

More generally, let $d \in 2\mathbb{Z}$ and suppose that \mathcal{M} be Fourier multiplier with matrix valued symbol $m(\xi)$ such that $|m(\xi)| \lesssim \delta$ for every $\xi \in \text{supp } \hat{u}$. Then with (q, r) as above

$$\|\mathcal{M}_{\leq d}^{\pm'} u\|_{L_t^q L_x^r} \lesssim \delta \lambda^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}} \|u\|_{F_\lambda^\pm}$$

where \pm and \pm' are independent choices of signs.

Proof. See Subsection 9.2 below. □

Remark 4.2. If $\text{supp } \hat{u}$ is contained in a ball of radius $\mu \leq \lambda$ in the annulus of size λ , then we can replace $\lambda^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}}$ with the smaller $(\frac{\mu}{\lambda})^{n(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}} \lambda^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}}$, see Remark 9.4 below. This small scale improvement follows from the refined Strichartz estimates of Klainerman-Tataru in [25] and can be very useful in proving bilinear estimates, particularly in the high-high frequency interaction. In the current article the high-high interaction is not particularly hard to deal with, and so this small scale refinement is not needed.

The next set of linear estimates we require are bounds involve the null frame type norms $PW^\pm(\kappa)$ and $[NF^\pm]^*(\kappa)$.

Theorem 4.3 (Null frame bounds). *Let $\alpha \ll 1$, $\lambda \in 2\mathbb{Z}$, $T > 0$, and $\rho \in C_0^\infty(\mathbb{R})$. Suppose $u \in F_\lambda^\pm$. Then we have the estimates*

$$\left(\sum_{\kappa \in C_\alpha} \|R_{\kappa, \alpha^2 \lambda}^\pm \Pi_+ u\|_{[NF^\pm]^*(\kappa)}^2 + \|R_{\kappa, \alpha^2 \lambda}^\mp \Pi_- u\|_{[NF^\pm]^*(\kappa)}^2 \right)^{\frac{1}{2}} \lesssim \|u\|_{F_\lambda^\pm} \quad (42)$$

and

$$\left(\sum_{\kappa \in \mathcal{C}_\alpha} \|R_{\kappa, \alpha^2 \lambda}^\pm \Pi_\pm [\rho(\frac{t}{T})u]\|_{PW^\mp(\kappa)}^2 + \|R_{\kappa, \alpha^2 \lambda}^\mp \Pi_- [\rho(\frac{t}{T})u]\|_{PW^\mp(\kappa)}^2 \right)^{\frac{1}{2}} \lesssim (\alpha \lambda)^{\frac{n-1}{2}} \|u\|_{F_\lambda^\pm} \quad (43)$$

where the implied constant is independent of T .

Proof. See Subsection 8.4 below. \square

Remark 4.4. Theorem 4.3 is in some sense a null form estimate, as it depends on certain cancelations involving the projections Π_\pm . In particular, if we tried to replace the $PW^\mp(\kappa)$ norm with $PW^\pm(\kappa)$, then we would only obtain (43) with the factor $(\alpha \lambda)^{\frac{n-1}{2}}$ replaced with the much larger $\alpha^{\frac{n-3}{2}} \lambda^{\frac{n-1}{2}}$.

Remark 4.5. We may replace the multipliers $R_{\kappa, \alpha^2 \lambda}^\pm$ in Theorem 4.3 with $R_{\kappa}^\pm C_{\lesssim \alpha^2 \lambda}^\pm$. In particular, it is not necessary that $|\tau \pm |\xi|| \ll \alpha^2 \lambda$, it is enough to be localised to the larger region $|\tau \pm |\xi|| \lesssim \alpha^2 \lambda$. This follows by noting that we can always reduce the later condition to the former by using the $\dot{\mathcal{X}}_\pm^{\frac{1}{2}, \infty}$ spaces. See for instance the proof of Corollary 4.6 below.

The final result in this section is a duality type estimate that helps to reduce the number of bilinear estimates we need to prove.

Corollary 4.6 (F_λ^\pm controls dual of \mathcal{N}_λ^\pm). *Let $\lambda, \mu \in 2^\mathbb{Z}$. Assume $u \in F_\lambda^\pm$ and $v \in \mathcal{N}_\mu^\pm$. Then*

$$\left| \int_{\mathbb{R}^{n+1}} u^\dagger v dx dt \right| \lesssim \|u\|_{F_\mu^\pm} \|v\|_{\mathcal{N}_\lambda^\pm}.$$

Proof. After a reflection, we may assume that $\pm = +$. The atomic definition of \mathcal{N}_λ^+ implies that it suffices to consider the case where v is an atom. If v is an energy or $\dot{\mathcal{X}}_+^{-\frac{1}{2}, 1}$ atom, then the estimate follows easily by duality together with the estimate

$$\|u\|_{L_t^\infty L_x^2} + \|u\|_{\dot{\mathcal{X}}_+^{\frac{1}{2}, \infty}} \lesssim \|u\|_{F_\mu^+}$$

which follows from Lemma 3.12. Thus it only remains to consider the case where v is a NF_λ^+ atom. By definition, there exists $\alpha > 0$ such that we have a decomposition $v = \sum_\kappa F_\kappa$ with $\text{supp } \widetilde{\Pi_\pm} F_\kappa \subset A_{\lambda, \alpha}^\pm(\kappa)$ and

$$\sum_{\kappa \in \mathcal{C}_\alpha} \|F_\kappa\|_{NF^+(\kappa)}^2 \leq 1.$$

Note that if $\kappa \in \mathcal{C}_\alpha$ and $\kappa' \in \mathcal{C}_{\frac{\alpha}{4}}$ with $\kappa \cap \kappa' \neq \emptyset$, then for every $\omega \in \mathbb{S}^{n-1}$, $\omega \notin 2\kappa$ implies $\omega \notin 2\kappa'$ and consequently $\|u\|_{[NF^+]*(\kappa)} \leq \|u\|_{[NF^+]*(\kappa')}$. Hence an application of the duality estimate (33) gives

$$\begin{aligned} \left| \int u^\dagger v \right| &\leq \sum_{\pm} \sum_{\substack{\kappa \in \mathcal{C}_\alpha, \kappa' \in \mathcal{C}_{\frac{\alpha}{4}} \\ \kappa \cap \kappa' \neq \emptyset}} \left| \int (R_{\kappa}^\pm C_{\lesssim \alpha^2 \lambda}^\pm \Pi_\pm u)^\dagger F_\kappa dx dt \right| \\ &\leq \sum_{\substack{\kappa \in \mathcal{C}_\alpha, \kappa' \in \mathcal{C}_{\frac{\alpha}{4}} \\ \kappa \cap \kappa' \neq \emptyset}} \|R_{\kappa'}^\pm \Pi_\pm u\|_{[NF^+]*(\kappa)} \|F_\kappa\|_{NF^+(\kappa)} \\ &\leq \left(\sum_{\kappa' \in \mathcal{C}_{\frac{\alpha}{4}}} \|R_{\kappa'}^\pm C_{\lesssim \alpha^2 \lambda}^\pm \Pi_\pm u\|_{[NF^+]*(\kappa')}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We now decompose $R_{\kappa'}^\pm C_{\lesssim \alpha^2 \lambda}^\pm \Pi_\pm u = R_{\kappa'}^\pm C_{\frac{\alpha^2 \lambda}{16}}^\pm \Pi_\pm u + \sum_{d \approx \alpha^2 \lambda} R_{\kappa'}^\pm C_d^\pm \Pi_\pm u$. The first term we can directly estimate by using Theorem 4.3. For the second term, we can not directly apply Theorem 4.3, as the support is not sufficiently close to the light cone. Instead, we use Lemma 3.1 to decompose into an

average of free waves, and note that as $\|\mathcal{U}_+(t)f\|_{F_\lambda^+} \lesssim \|f\|_{L_x^2}$ (for f with Fourier support in $|\xi| \approx \lambda$), we can apply Theorem 4.3 to deduce that

$$\begin{aligned} \sum_{d \approx \alpha^2 \lambda} \left(\sum_{\kappa' \in \mathcal{C}_{\frac{d}{4}}} \|R_{\kappa'}^\pm C_d^\pm \Pi_\pm u\|_{[NF+]^*(\kappa')}^2 \right)^{\frac{1}{2}} &\lesssim \int_{|\tau| \approx \alpha^2 \lambda} \left(\sum_{\kappa' \in \mathcal{C}_{\frac{d}{4}}} \|R_{\kappa'}^\pm \Pi_\pm \mathcal{U}_+(t)[f_\tau]\|_{[NF+]^*(\kappa')}^2 \right)^{\frac{1}{2}} d\tau \\ &\lesssim \int_{|\tau| \approx \alpha^2 \lambda} \|f_\tau\|_{L^2} d\tau \lesssim \|u\|_{\dot{X}_+^{\frac{1}{2}, \infty}} \lesssim \|u\|_{F_\lambda^+} \end{aligned}$$

where we used Lemma 3.12 to estimate the $\dot{X}_+^{\frac{1}{2}, \infty}$ norm. Thus result follows. \square

5. BILINEAR ESTIMATES

The key estimate we prove in this section is the following bilinear null form estimate.

Theorem 5.1 (Bilinear estimate in \mathcal{N}_λ^\pm - close cone case). *Let $\delta \lesssim \min\{\lambda, N_1\}$. Assume $v \in F_{N_1}^\mp$ has compact support in time. Suppose F is scalar valued with $\text{supp } \tilde{F} \subset \{|\xi| \lesssim \lambda, |\tau| - |\xi| \lesssim \delta\}$. Then*

$$\|S_{N_0, \ll \delta}^\pm(F\mathfrak{C}_{\ll \delta}^\mp v)\|_{\mathcal{N}_{N_0}^\pm} \lesssim (\delta \min\{\lambda, N_1\})^{\frac{n-1}{4}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^\mp}. \quad (44)$$

Remark 5.2. We should emphasize that the implied constant in (44) is *independent* of v . In particular, although the Theorem 5.1 requires v to have compact support in time, the implied constant *does not* depend on the size of the support. This is due to fact that the only place the compact support assumption is needed, is to control the $PW^\pm(\kappa)$ type norms. By Theorem 4.3, this is possible provided we can write $v = \rho(\frac{t}{T})v$ for some $\rho \in C_0^\infty(\mathbb{R})$, with the implied constant being independent of T and v , and consequently, independent of the size of the support. A similar comment applies to the bilinear estimates appearing in Corollary 5.4 and Corollary 5.5 below.

Theorem 5.1 contains the main multi-linear estimate contained in this article. In fact all other bilinear and trilinear estimates essentially follow by using Lemma 3.12 and (40) to control the region away from the cone, and Corollary 4.6 to deduce bilinear estimates in $L_{t,x}^2$ by duality.

Proof of Theorem 5.1. After a reflection in x , we may assume that $\pm = +$. Note that as $v \in F_{N_0}^-$, we have $\text{supp } \hat{v} \subset \{|\xi| \approx N_0\}$. Thus the lefthand side of (44) vanishes unless $\max\{\lambda, N_0, N_1\} \approx \text{med}\{\lambda, N_0, N_1\}$, where $\text{med}\{a, b, c\}$ is the median of $a, b, c \in \mathbb{R}$, we make use of this simple observation later. Let $\mu = \min\{\lambda, N_0, N_1\}$. We claim that it is enough to consider the case $\delta \leq \mu$. To prove the claim, note that if $\delta \geq \mu$, then using Lemma 3.12 we have the estimates

$$\|P_{N_0}(F\mathfrak{C}_{\gtrsim \mu}^- v)\|_{L_t^1 L_x^2} \lesssim \mu^{\frac{n}{2}} \|F\|_{L_{t,x}^2} \|\mathfrak{C}_{\gtrsim \mu}^- v\|_{L_{t,x}^2} \lesssim \mu^{\frac{n-1}{2}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-} \quad (45)$$

and

$$\|S_{N_0, \gtrsim \mu}^+(Fv)\|_{\dot{X}_+^{-\frac{1}{2}, 1}} \leq \mu^{-\frac{1}{2}} \|P_{N_0}(Fv)\|_{L_{t,x}^2} \lesssim \mu^{\frac{n-1}{2}} \|F\|_{L_{t,x}^2} \|v\|_{L_t^\infty L_x^2} \lesssim \mu^{\frac{n-1}{2}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-} \quad (46)$$

together with the obvious bounds (35) and (36), reduce the problem to estimating $S_{N_0, \ll \mu}^+(F\mathfrak{C}_{\ll \mu}^- v)$, this is almost the case $\delta = \mu$, but we need to restrict the support of F further. To this end, we observe the

identity¹²

$$S_{N_0, \ll \mu}^+(F\mathfrak{C}_{\ll \mu}^-v) = S_{N_0, \ll \mu}^+(C_{\lesssim a}F\mathfrak{C}_{\ll \mu}^-v)$$

where $a = \mu + \min\{N_0, N_1\}$. Then as we already have $\text{supp } \tilde{F} \subset \{|\tau| - |\xi| \lesssim \delta\}$ with $\delta \lesssim \lambda$, we deduce that $\text{supp } \widetilde{C_{\lesssim a}F} \subset \{|\tau| - |\xi| \lesssim \mu\}$. Thus we can reduce the case $\delta \geq \mu$ to $\delta \leq \mu$, hence it is enough to consider the case $\delta \leq \mu$ as claimed.

It remains to consider the case $\delta \leq \mu$. To this end we decompose $S_{N_0, \ll \delta}^+$ into terms of the form $C_{\ll \delta}^\pm P_{N_0} \Pi_\pm$ and note that, after a reflection (in t and x), it is enough to prove

$$\|C_{\ll \delta}^+ P_{N_0} \Pi_+ (FC_{\ll \delta}^\pm v^\mp)\|_{\mathcal{N}_{N_0}^+} \lesssim (\delta \min\{\lambda, N_1\})^{\frac{n-1}{4}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-}$$

where we let $v^\mp = \Pi_\mp v$. We now decompose the distance to the cone into three main interactions

$$\begin{aligned} C_{\ll \delta}^+ [(FC_{\ll \delta}^\pm v^\mp)] &= \sum_{d \lesssim \delta} C_d^+ [C_{\lesssim d}^+ FC_{\lesssim d}^\pm v^\mp] + \sum_{d \lesssim \delta} C_{\lesssim d}^+ [C_{\lesssim d}^+ FC_d^\pm v^\mp] + \sum_{d \lesssim \delta} C_{\ll d}^+ [C_d^+ FC_{\ll d}^\pm v^\mp] \\ &= A_I + A_{II} + A_{III}. \end{aligned}$$

Roughly the strategy is to use the $\dot{\mathcal{X}}_\pm^{\frac{1}{2}}$ type estimates whenever the output Fv or v is away from the light cone (the interactions A_I and A_{II}), and for the more delicate interaction, A_{III} , apply the null frame bounds in Theorem 4.3.

Case 1: A_I . The main idea is to use the close cone condition to limit the possible angular interactions. The key tool to accomplish this will be the elementary angle estimate

$$\theta(\xi + \xi', \pm \xi')^2 \approx \frac{||\xi + \xi'| \mp |\xi'| - |\xi|| \times ||\xi + \xi'| \mp |\xi'| + |\xi||}{|\xi' + \xi| |\xi'|}. \quad (47)$$

This estimate is used as follows. Suppose that

$$(\tau, \xi) \in \text{supp } \widetilde{C_{\lesssim d}F}, \quad (\tau', \xi') \in \text{supp } \widetilde{C_{\lesssim d}^\pm v^\mp}, \quad (\tau + \tau', \xi + \xi') \in \{|\xi| \approx N_0, |\tau + |\xi| \approx d\}.$$

Then using the assumption $\delta \lesssim \lambda$ we have

$$||\xi + \xi'| \mp |\xi'| - \text{sgn}(\tau)|\xi|| \leq |\tau + \tau' + |\xi + \xi'|| + |\tau' \pm |\xi'|| + |\tau| + |\xi| \lesssim \lambda$$

and

$$||\xi + \xi'| \mp |\xi'| + \text{sgn}(\tau)|\xi|| \lesssim |\tau + \tau' + |\xi + \xi'|| + |\tau' \pm |\xi'|| + ||\tau| - |\xi|| \lesssim d.$$

Therefore (47) shows that $\theta(\xi + \xi', \pm \xi') \lesssim \sqrt{\frac{\lambda d}{N_0 N_1}}$. This suggests that we should localise v and the product Fv to caps of radius $\sqrt{\frac{\lambda d}{N_0 N_1}}$, as if the Fourier support of v was contained in a cap of radius $\sqrt{\frac{\lambda d}{N_0 N_1}}$, then the Fourier support of Fv must be contained in a similar cap. More precisely, letting $\alpha = \sqrt{\frac{\lambda d}{N_0 N_1}}$, the angle estimate implies the decomposition

$$P_{N_0} C_d^+ [C_{\lesssim d} F C_{\lesssim d}^\pm v^\mp] = \sum_{\substack{\kappa, \bar{\kappa} \in C_\alpha \\ \theta(\kappa, \bar{\kappa}) \lesssim \alpha}} P_{N_0, \bar{\kappa}}^+ C_d^+ [C_{\lesssim d} F R_\kappa^\pm C_{\lesssim d}^\pm v^\mp].$$

¹² This follows by noting that if $(\tau + \tau', \xi + \xi'), (\tau', \xi') \in \{|\tau| - |\xi| \ll \mu\}$ and $|\xi + \xi'| \approx N_0, |\xi'| \approx N_1$ then the inequality

$$||\tau| - |\xi|| \leq ||\tau + \tau'| - |\xi + \xi'|| + ||\tau'| - |\xi'|| + 2 \min\{|\xi + \xi'|, |\xi'|\}$$

shows that $||\tau| - |\xi|| \lesssim \mu + \min\{|\xi + \xi'|, |\xi'|\}$ which gives the claimed identity. This method of deducing close cone information on F , from close cone information on the output Fv and v , occurs frequently in what follows.

Consequently, by orthogonality in L_x^2 , together with an application of Bernstein's inequality, and the null structure estimate $|\Pi_\omega \Pi_{\omega'}| \lesssim \theta(\omega, -\omega')$ we obtain

$$\begin{aligned} \|P_{N_0} C_d^+ \Pi_+ [C_{\lesssim d} F(t) C_{\lesssim d}^\pm v^\mp(t)]\|_{L_x^2} &\lesssim \left(\sum_{\substack{\kappa, \bar{\kappa} \in \mathcal{C}_\alpha \\ \theta(\kappa, \bar{\kappa}) \lesssim \alpha}} \left\| \int_{\mathbb{R}^n} |C_{\lesssim d} \widehat{F}(t, \xi - \eta)| \left| \Pi_{\frac{\xi}{|\xi|}} \Pi_{\mp \frac{\eta}{|\eta|}} \widehat{R_{\bar{\kappa}}^\pm} C_{\lesssim d}^\pm v(t, \eta) \right| d\eta \right\|_{L_\xi^2[A_{N_0}^+(\bar{\kappa})]}^2 \right)^{\frac{1}{2}} \\ &\lesssim \alpha (\alpha \min\{N_0, N_1\})^{\frac{n-1}{2}} (\min\{N_0, N_1\})^{\frac{1}{2}} \|C_{\lesssim d} F(t)\|_{L_x^2} \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|R_\kappa^\pm C_{\lesssim d}^\pm v(t)\|_{L_x^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim d^{\frac{n+1}{4}} (\min\{\lambda, N_0, N_1\})^{\frac{n-1}{4}} \|C_{\lesssim d} F(t)\|_{L_x^2} \|C_{\lesssim d}^\pm v(t)\|_{L_x^2}. \end{aligned}$$

Thus taking the L_t^2 norm of both sides, we that $P_{N_0} C_d^+ \Pi_+ [C_{\lesssim d} F C_{\lesssim d}^\pm v^\mp]$ is a multiple of a $\dot{\mathcal{X}}_+^{-\frac{1}{2}, 1}$ atom, in other words, the A_I term is a sum of $\dot{\mathcal{X}}_+^{-\frac{1}{2}, 1}$ atoms. Therefore, the atomic definition of $\mathcal{N}_{N_0}^+$ gives

$$\begin{aligned} \left\| \sum_{d \lesssim \delta} C_d^+ P_{N_0} \Pi_+ [C_{\lesssim d} F C_{\lesssim d}^\pm v^\mp] \right\|_{\mathcal{N}_{N_0}^+} &\leq \sum_{d \lesssim \delta} d^{-\frac{1}{2}} \left\| C_d^+ P_{N_0} \Pi_+ [C_{\lesssim d} F C_{\lesssim d}^\pm v_{N_1}^\mp] \right\|_{L_{t,x}^2} \\ &\lesssim (\min\{N_0, N_1\})^{\frac{n-1}{4}} \|F\|_{L_{t,x}^2} \sum_{d \lesssim \delta} d^{\frac{n+1}{4} - \frac{1}{2}} \|C_{\lesssim d}^\pm v\|_{L_t^\infty L_x^2} \\ &\lesssim (\delta \min\{N_0, N_1\})^{\frac{n-1}{4}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-} \end{aligned}$$

where we used an application of (iii) in Lemma 3.14 to dispose of the $C_{\lesssim d}^\pm$ multiplier, and the fact that $\frac{n-1}{4} > 0$ to control the sum over d .

Case 2: A_{II} . We follow a similar argument to that used to control A_I . Let $\alpha = \sqrt{\frac{\lambda d}{N_0 N_1}}$. A moments thought shows that, as in the A_I case, we have the angle estimate $\theta(\xi + \xi', \pm \xi') \lesssim \alpha$. Consequently we have the decomposition

$$P_{N_0} C_{\lesssim d}^+ [C_{\lesssim d} F C_{\lesssim d}^\pm v^\mp] = \sum_{\substack{\kappa, \bar{\kappa} \in \mathcal{C}_\alpha \\ \theta(\kappa, \bar{\kappa}) \lesssim \alpha}} P_{N_0, \bar{\kappa}}^+ C_{\lesssim d}^+ [C_{\lesssim d} F R_{\bar{\kappa}}^\pm C_{\lesssim d}^\pm v^\mp].$$

Moreover, for any caps $\kappa, \bar{\kappa} \in \mathcal{C}_\alpha$ with $\theta(\kappa, \bar{\kappa}) \lesssim \alpha$ we have by Bernstein and the null form estimate

$$\begin{aligned} \|P_{N_0, \bar{\kappa}}^+ \Pi_+ [F R_{\bar{\kappa}}^\pm v^\mp]\|_{L_x^2} &\lesssim \left\| \int_{\mathbb{R}^n} |\widehat{F}(\xi - \eta)| \left| \Pi_{\frac{\xi}{|\xi|}} \Pi_{\mp \frac{\eta}{|\eta|}} \widehat{R_{\bar{\kappa}}^\pm} v(\eta) \right| d\eta \right\|_{L_\xi^2[A_{N_0}^+(\bar{\kappa})]} \\ &\lesssim \alpha (\alpha \min\{N_0, N_1\})^{\frac{n-1}{2}} (\min\{N_0, N_1\})^{\frac{1}{2}} \|F\|_{L_x^2} \|R_{\bar{\kappa}}^\pm v\|_{L_x^2} \\ &\lesssim d^{\frac{n+1}{4}} (\min\{N_0, N_1\})^{\frac{n-1}{4}} \|F\|_{L_x^2} \|R_{\bar{\kappa}}^\pm v\|_{L_x^2} \end{aligned}$$

Therefore, by the L_x^2 orthogonality of the projections R_κ^\pm we have

$$\begin{aligned}
\left\| P_{N_0} C_{\lesssim d}^+ \Pi_+ [C_{\lesssim d} F C_d^\pm v^\mp] \right\|_{\mathcal{N}_{N_0}^+} &\leq \left\| P_{N_0} C_{\lesssim d}^+ \Pi_+ [C_{\lesssim d} F C_d^\pm v^\mp] \right\|_{L_t^1 L_x^2} \\
&\lesssim \left\| \left(\sum_{\substack{\kappa, \bar{\kappa} \in \mathcal{C}_\alpha \\ \theta(\kappa, \bar{\kappa}) \lesssim \alpha}} \| P_{N_0, \bar{\kappa}}^+ \Pi_+ [C_{\lesssim d} F C_d^\pm R_\kappa^\pm v^\mp] \|_{L_x^2}^2 \right)^{\frac{1}{2}} \right\|_{L_t^1} \\
&\lesssim d^{\frac{n+1}{4}} (\min\{N_0, N_1\})^{\frac{n-1}{4}} \left\| C_{\lesssim d} F \right\|_{L_x^2} \left(\sum_{\kappa \in \mathcal{C}_\alpha} \| C_d^\pm R_\kappa^\pm v^\mp \|_{L_x^2}^2 \right)^{\frac{1}{2}} \Big\|_{L_t^1} \\
&\lesssim d^{\frac{n+1}{4}} (\min\{N_0, N_1\})^{\frac{n-1}{4}} \| F \|_{L_{t,x}^2} \| C_d^\pm v^\mp \|_{L_{t,x}^2} \\
&\lesssim d^{\frac{n+1}{4} - \frac{1}{2}} (\min\{N_0, N_1\})^{\frac{n-1}{4}} \| F \|_{L_{t,x}^2} \| v \|_{F_{N_1}^-}
\end{aligned}$$

where we made use of the $L_{t,x}^2$ bound away from the cone (40). If we now apply the triangle inequality and sum up in $d \lesssim \delta$, we obtain the required inequality.

Case 3: A_{III} . The remaining case A_{III} is more difficult as it includes interactions where both the output and v can be concentrated very close to the cone (so it is hard to use $X^{s,b}$ type norms), while F can be (relatively) far from the light cone. Since F is just an arbitrary L^2 function, it is not easy to use the fact that F lies away from the light cone, in particular we have no weights of the form $|\tau + |\xi||$ to exploit (which is what makes $X^{s,b}$ type norms so useful away from the light cone). The key observation is that it is only possible for F and v to produce interactions close to the light cone, if the spatial Fourier supports are at an angle of $\sqrt{\frac{\lambda d}{N_0 N_1}}$. The difference to the previous cases is that we will have a bound on the angle from above *and* below. This angular separation allows us to make use of the null frame type spaces, in particular it means we can use the simple bilinear estimate in Lemma 3.4.

Another way to view this, is that we can no longer put the output Fv , or the function v , into the $L_{t,x}^2$ based $\dot{\mathcal{X}}^{-\frac{1}{2},1}$ type spaces. Thus we are essentially forced to put Fv in either $L_t^1 L_x^2$ or the null frame version $L_{t,\omega}^1 L_{x,\omega}^2$. Clearly we can not put $Fv \in L_t^1 L_x^2$ as this would require a $L_t^2 L_x^\infty$ bound on v which is out of reach. The only remaining option is to put Fv into the null frame type norms $L_{t,\omega}^1 L_{x,\omega}^2$, and then $v \in L_{t,\omega}^2 L_{x,\omega}^\infty$. This is possible, but requires that we spend powers of N_1 . Thus in the case where N_1 is very large, we need to ensure that v is localised to small caps to get the right constants.

We start by making the following observation. Suppose that

$$(\tau, \xi) \in \text{supp } \widetilde{C_d F}, \quad (\tau', \xi') \in \text{supp } \widetilde{C_{\ll d} v^\mp}, \quad (\tau + \tau', \xi + \xi') \in \{|\xi| \approx N_0, |\tau + |\xi|| \ll d\}. \quad (48)$$

Then since $|\tau| + |\xi| \approx \max\{|\tau| - |\xi|, |\xi|\} \approx \max\{d, |\xi|\}$ we have

$$||\xi + \xi'| \mp |\xi'| - \text{sgn}(\tau)|\xi|| = |(\tau + \tau' + |\xi + \xi'|) - (\tau' \pm |\xi'|) + \text{sgn}(\tau)(|\tau| + |\xi|)| \approx \max\{d, |\xi|\}.$$

Moreover, an application of (47), together with the observation that $||\xi + \xi'| \mp |\xi'| + \text{sgn}(\tau)|\xi|| \approx d$ gives

$$C \sqrt{\frac{d \max\{d, |\xi|\}}{N_0 N_1}} \leq \theta(\xi + \xi', \pm \xi') \lesssim \sqrt{\frac{d \max\{d, |\xi|\}}{N_0 N_1}}. \quad (49)$$

Note that we have a bound on the angle from above and below. This estimate suggests that we should decompose the output Fv , and v , into caps of size $\sqrt{\frac{|\xi|d}{N_0 N_1}}$. On the other hand, if we are at frequency μ , and modulation $\ll d$, then geometrically, the natural size of the caps should be $\sqrt{\frac{d}{\mu}}$ as this matches

the correct Fourier support properties used in the definition of the null frame atoms. The difficulty is that these natural sizes may not always match up, and particularly in the case $N_0 \approx N_1$, an additional decomposition is required. Consequently, unlike in the cases A_{II} and A_{III} , we cannot consider all the various frequency interactions simultaneously. Thus we separate the argument into the cases $N_0 \gg N_1$, $N_0 \ll N_1$, and $N_0 \approx N_1$.

Case 3a: A_{III} and $N_0 \gg N_1$. Note that if we have (τ, ξ) and (τ', ξ') as in (48), then $|\xi| \approx N_0$. Hence the angle estimate (49) implies that

$$C\sqrt{\frac{d}{N_1}} \leq \theta(\xi + \xi', \xi') \lesssim \sqrt{\frac{d}{N_1}}.$$

Let $\alpha = \frac{C}{100}\sqrt{\frac{d}{N_1}}$ and $\beta = \frac{C}{100}\sqrt{\frac{d}{N_0}}$, note that $\alpha \geq \beta$. By decomposing v into caps of radius α , and the output Fv into caps of size β , we have the identity

$$P_{N_0} C_{\ll d}^+ [C_d F C_{\ll d}^\pm v^\mp] = \sum_{\bar{\kappa} \in \mathcal{C}_\beta} \sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ 5\alpha \leq \theta(\kappa, \bar{\kappa}) \lesssim \alpha}} P_{N_0, \bar{\kappa}}^{+, \beta} [C_d F R_{\kappa, \alpha^2 N_1}^\pm v^\mp].$$

Since $\beta \leq \alpha$, the angle condition implies that fixing a cap $\bar{\kappa} \in \mathcal{C}_\beta$ essentially fixes the cap $\kappa \in \mathcal{C}_\alpha$. Therefore, by the orthogonality estimate in Lemma 3.4 and the $PW^\pm(\kappa)$ estimate in Theorem 4.3 (note that v has compact support in time), we have

$$\begin{aligned} & \left(\sum_{\bar{\kappa} \in \mathcal{C}_\beta} \left\| \sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ 5\alpha \leq \theta(\kappa, \bar{\kappa}) \lesssim \alpha}} P_{N_0, \bar{\kappa}}^{+, \beta} \Pi_+ [C_d F R_{\kappa, \alpha^2 N_1}^\pm v^\mp] \right\|_{NF^+(\bar{\kappa})}^2 \right)^{\frac{1}{2}} \\ & \lesssim \left(\sum_{\kappa \in \mathcal{C}_\alpha} \sum_{\substack{\bar{\kappa} \in \mathcal{C}_\beta \\ 5\alpha \leq \theta(\kappa, \bar{\kappa}) \lesssim \alpha}} \|P_{N_0, \bar{\kappa}}^{+, \beta} \Pi_+ [C_d^+ F R_{\kappa, \alpha^2 N_1}^\pm v^\mp]\|_{NF^+(\bar{\kappa})}^2 \right)^{\frac{1}{2}} \\ & \lesssim \|C_d F\|_{L_{t,x}^2} \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|R_{\kappa, \alpha^2 N_1}^\pm v^\mp\|_{PW^+(\kappa)}^2 \right)^{\frac{1}{2}} \\ & \lesssim (\alpha N_1)^{\frac{n-1}{2}} \|C_d F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-} \lesssim (d N_1)^{\frac{n-1}{4}} \|C_d F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-} \end{aligned}$$

Consequently $P_{N_0} C_{\ll d}^+ [C_d F C_{\ll d}^\pm v^\mp]$ is a multiple of a $NF_{N_0}^+$ atom, and therefore

$$\left\| \sum_{d \lesssim \delta} P_{N_0} C_{\ll d}^+ \Pi_+ [C_d F C_{\ll d}^\pm v^\mp] \right\|_{\mathcal{N}_{N_0}^+} \lesssim \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-} \sum_{d \lesssim \delta} (d N_1)^{\frac{n-1}{4}} \lesssim (\delta N_1)^{\frac{n-1}{4}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-}$$

which is acceptable as $N_1 = \min\{\lambda, N_0, N_1\}$.

Case 3b: A_{III} and $N_0 \ll N_1$. We start by observing that if we have (τ, ξ) , (τ', ξ') as in (48), then $|\xi| \approx N_1$. Consequently, from (49) we deduce that

$$C\sqrt{\frac{d}{N_0}} \leq \theta(\xi + \xi', \xi') \lesssim \sqrt{\frac{d}{N_0}}.$$

If we let $\alpha = \frac{C}{100}\sqrt{\frac{d}{N_1}}$ and $\beta = \frac{C}{100}\sqrt{\frac{d}{N_0}}$, then the angle estimate implies the decomposition

$$P_{N_0} C_{\ll d}^+ [C_d F C_{\ll d}^\pm v_{N_1}^\mp] = \sum_{\bar{\kappa} \in \mathcal{C}_\beta} \sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ 4\beta \leq \theta(\kappa, \bar{\kappa}) \lesssim \beta}} P_{N_0, \bar{\kappa}}^{+, \beta} [C_d F R_{\kappa, \alpha^2 N_1}^\pm v^\mp].$$

The key difference to the previous case, is that since we now have $\alpha \leq \beta$, if we fix a cap $\bar{\kappa} \in \mathcal{C}_\beta$, the condition $\theta(\kappa, \bar{\kappa}) \approx \beta$ no longer fixes the cap $\kappa \in \mathcal{C}_\alpha$. Thus to regain an ℓ^2 sum over the caps $\kappa \in \mathcal{C}_\alpha$, we need to exploit an additional orthogonality property. The key point is the identity

$$\theta(\operatorname{sgn}(\tau)\xi, \mp\xi')^2 \approx \frac{||\xi + \xi'| + \operatorname{sgn}(\tau)|\xi| \mp |\xi'| \times ||\xi + \xi'| - \operatorname{sgn}(\tau)|\xi| \pm |\xi'|}{|\xi||\xi'|}.$$

As in the derivation of (49), for frequencies localised as in (48), since $N_0 \ll N_1 \approx |\xi|$ we have the crude estimate $||\xi + \xi'| - \operatorname{sgn}(\tau)|\xi| \pm |\xi'| \lesssim N_1$ and hence

$$\theta(\operatorname{sgn}(\tau)\xi, \mp\xi') \lesssim \sqrt{\frac{d}{N_1}} \approx \alpha.$$

Therefore, if we also decompose F into caps of radius α , we can refine our decomposition and obtain

$$P_{N_0} C_{\ll d}^+ [C_d F C_{\ll d}^\pm v^\mp] = \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\alpha \\ \theta(\kappa, \kappa') \lesssim \alpha}} \sum_{\substack{\bar{\kappa} \in \mathcal{C}_\beta \\ 4\beta \leq \theta(\kappa, \bar{\kappa}) \lesssim \beta}} P_{N_0, \bar{\kappa}}^{+, \beta} [C_d R_{\kappa'} F R_{\kappa, \alpha^2 N_1}^\pm v^\mp]$$

which gives the required orthogonality in the κ sum. If we now apply Lemma 3.4 and Theorem 4.3 we deduce that

$$\begin{aligned} \left(\sum_{\bar{\kappa} \in \mathcal{C}_\beta} \|P_{N_0, \bar{\kappa}}^{+, \beta} \Pi_+ [C_d F C_{\ll d}^\pm v^\mp]\|_{NF^+(\bar{\kappa})}^2 \right)^{\frac{1}{2}} &\leq \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\alpha \\ \theta(\kappa, \kappa') \lesssim \alpha}} \left(\sum_{\substack{\bar{\kappa} \in \mathcal{C}_\beta \\ 4\beta \leq \theta(\kappa, \bar{\kappa}) \lesssim \beta}} \|P_{N_0, \bar{\kappa}}^{+, \beta} [C_d R_{\kappa'} F R_{\kappa, \alpha^2 N_1}^\pm v^\mp]\|_{NF^+(\bar{\kappa})}^2 \right)^{\frac{1}{2}} \\ &\lesssim \sum_{\substack{\kappa, \kappa' \in \mathcal{C}_\alpha \\ \theta(\kappa, \kappa') \lesssim \alpha}} \|R_{\kappa'} F\|_{L_{i,x}^2} \|R_{\kappa, \alpha^2 N_1}^\pm v^\mp\|_{PW^+(\kappa)} \\ &\lesssim (\alpha N_1)^{\frac{n-1}{2}} \|F\|_{L_{i,x}^2} \|v\|_{F_{N_0}^-}. \end{aligned}$$

Thus, as in the previous case, $P_{N_0} C_{\ll d}^+ [C_d F C_{\ll d}^\pm v^\mp]$ is a multiple of a $NF_{N_0}^+$ atom, and consequently by summing up over $d \lesssim \delta$ we obtain a constant of the size $(\delta N_1)^{\frac{n-1}{4}} \approx (\delta \min\{\lambda, N_1\})^{\frac{n-1}{4}}$ as required.

Case 3c: A_{III} and $N_0 \approx N_1$. Unlike the previous cases, we no longer have an estimate on $|\xi|$ from below. Thus to exploit the angle estimate (49), we need to dyadically decompose F further into $F = P_{\ll d} F + \sum_{d \lesssim \lambda' \lesssim \lambda} P_{\lambda'} F$. After an application of the triangle inequality, we reduce to estimating

$$\sum_{d \lesssim \delta} \|P_{N_0} C_{\ll d}^+ \Pi_+ (C_d P_{\ll d} F C_{\ll d}^\pm v^\mp)\|_{\mathcal{N}_{N_0}^+} + \sum_{d \lesssim \delta} \sum_{d \lesssim \lambda' \lesssim \lambda} \|P_{N_0} C_{\ll d}^+ \Pi_+ (C_d P_{\lambda'} F C_{\ll d}^\pm v^\mp)\|_{\mathcal{N}_{N_0}^+}. \quad (50)$$

We start with the first term. Assume that

$$(\tau + \tau', \xi + \xi') \in \{|\xi| \approx N_0, |\tau + |\xi|| \ll d\}, \quad (\tau, \xi) \in \{|\tau| - |\xi| \approx d\}, \quad (\tau', \xi') \in \operatorname{supp} \widetilde{C_{\ll d}^\pm v} \quad (51)$$

and $|\xi| \ll d$. Clearly, we have $|\tau| \approx d$. Moreover, the close cone condition on $(\tau + \tau', \xi + \xi')$ and (τ', ξ') implies that

$$d \approx |\tau| \approx |\tau - (\tau + \tau' + |\xi + \xi'|) + (\tau' \pm |\xi'|)| = ||\xi + \xi'| \mp |\xi'|.$$

However, as $|\xi| \ll d$, this is only possible if $\pm = -$ and $d \approx N_1$. In particular, the first term in (50) is only nonzero if we have $\pm = -$ and $d \approx N_1$. Consequently, from (49) we have $\theta(\xi + \xi', -\xi') \approx 1$ and so letting $\alpha = \frac{1}{100}$ we deduce the identity

$$\sum_{d \lesssim \delta} C_{\ll d}^+ P_{N_0} (C_d P_{\ll d} F C_{\ll d}^- v^+) = \sum_{d \approx N_1} \sum_{\substack{\kappa, \bar{\kappa} \in \mathcal{C}_\alpha \\ \alpha \ll \theta(\kappa, \bar{\kappa}) \approx \alpha}} P_{N_0, \bar{\kappa}}^{+, \alpha} (C_d P_{\ll d} F R_{\bar{\kappa}, \alpha^2 N_1}^- v^+).$$

Therefore

$$\begin{aligned}
\sum_{d \approx N_1} \|P_{N_0} C_{\ll d}^+ \Pi_+ (C_d P_{\ll d} F C_{\ll d}^\pm v^\mp)\|_{\mathcal{N}_{N_0}^+} &\leq \sum_{d \approx N_1} \left(\sum_{\substack{\kappa, \bar{\kappa} \in \mathcal{C}_\alpha \\ \alpha \ll \theta(\kappa, \bar{\kappa}) \approx \alpha}} \|P_{N_0, \bar{\kappa}}^{+, \alpha} (C_d P_{\ll d} F R_{\kappa, \alpha^2 N_1}^- v^+)\|_{NF^+(\bar{\kappa})}^2 \right)^{\frac{1}{2}} \\
&\lesssim \|F\|_{L_{t,x}^2} \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|R_{\kappa, \alpha^2 N_1}^- v^+\|_{PW^+(\kappa)}^2 \right)^{\frac{1}{2}} \\
&\lesssim (N_1)^{\frac{n-1}{2}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-} \approx (\delta \min\{\lambda, N_1\})^{\frac{n-1}{4}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-}
\end{aligned}$$

where we used the fact that $N_1 \approx d \lesssim \delta$ and the assumption $d \lesssim \delta \lesssim \lambda \lesssim N_1$, together with Lemma 3.4 and Theorem 4.3.

It remains to consider the more interesting second term in (50). Suppose that $(\tau + \tau', \xi + \xi')$ and (τ', ξ') are as in (51). If $|\xi| \approx \lambda' \gtrsim d$, then the angle estimate (49) implies that

$$C \sqrt{\frac{d\lambda'}{N_0 N_1}} \leq \theta(\xi + \xi', \pm \xi') \lesssim \sqrt{\frac{d\lambda'}{N_0 N_1}} \quad (52)$$

Let $\alpha = \frac{C}{100} \sqrt{\frac{d\lambda'}{N_0 N_1}}$. The estimate on the angle (52) implies that we should be decomposing the output Fv and v into caps of radius α . However, as $\alpha^2 N_0 \approx \frac{\lambda'}{N_1} d$, if we want to decompose Fv into $N F_{N_0}^+$ atoms, the output Fv should be at a distance $\frac{\lambda'}{N_1} d$ from the cone. Thus we need to first deal with the case where the distance to the cone is in the region $\frac{\lambda'}{N_1} d \lesssim \bullet \ll d$. To this end, by exploiting the null structure as in Case 1 above and using the decomposition

$$P_{N_0} C_{\frac{\lambda'}{N_1} d \lesssim \bullet \ll d}^+ [C_d P_{\lambda'} F C_{\ll d}^\pm v^\mp] = \sum_{\substack{\kappa, \bar{\kappa} \in \mathcal{C}_\alpha \\ \theta(\kappa, \bar{\kappa}) \approx \alpha}} P_{N_0, \bar{\kappa}}^+ C_{\frac{\lambda'}{N_1} d \lesssim \bullet \ll d}^+ [C_d P_{\lambda'} F C_{\ll d}^\pm R_{\kappa}^\pm v^\mp]$$

(which follows from (52)) we obtain by an application of Corollary 3.14

$$\begin{aligned}
\|P_{N_0} C_{\frac{\lambda'}{N_1} d \lesssim \bullet \ll d}^+ \Pi_+ [C_d P_{\lambda'} F C_{\ll d}^\pm v^\mp]\|_{\mathcal{N}_{N_0}^+} &\lesssim \left(\frac{d\lambda'}{N_1}\right)^{-\frac{1}{2}} \left\| \sum_{\substack{\kappa, \bar{\kappa} \in \mathcal{C}_\alpha \\ \theta(\kappa, \bar{\kappa}) \approx \alpha}} P_{N_0, \bar{\kappa}}^+ \Pi_+ [C_d P_{\lambda'} F C_{\ll d}^\pm R_{\kappa}^\pm v^\mp] \right\|_{L_{t,x}^2} \\
&\lesssim \left(\frac{d\lambda'}{N_1}\right)^{-\frac{1}{2}} \alpha \|F\|_{L_{t,x}^2} \left\| \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|C_{\ll d}^\pm R_{\bar{\kappa}}^\pm v\|_{L_\xi^1}^2 \right)^{\frac{1}{2}} \right\|_{L_t^\infty} \\
&\lesssim \left(\frac{d\lambda'}{N_1}\right)^{-\frac{1}{2}} \alpha (\alpha N_1)^{\frac{n-1}{2}} (N_1)^{\frac{1}{2}} \|F\|_{L_{t,x}^2} \|C_{\ll d}^\pm v\|_{L_t^\infty L_x^2} \\
&\lesssim (\lambda' d)^{\frac{n-1}{4}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-}.
\end{aligned}$$

Hence summing up in d and λ' gives the required estimate. Similarly, if we fix v to have modulation in $\frac{\lambda'}{N_0} d \lesssim \bullet \ll d$, then by following a similar argument to that used in Case 2 we obtain

$$\begin{aligned}
\|P_{N_0} C_{\ll d}^+ \Pi_+ [C_d P_{\lambda'} F C_{\frac{\lambda'}{N_0} d \lesssim \bullet \ll d}^\pm v_{N_1}^\mp]\|_{\mathcal{N}_{N_0}^+} &\lesssim \left\| \sum_{\substack{\kappa, \bar{\kappa} \in \mathcal{C}_\alpha \\ \theta(\kappa, \bar{\kappa}) \approx \alpha}} P_{N_0, \bar{\kappa}}^+ \Pi_+ [C_d P_{\lambda'} F P_{N_1, \kappa}^\pm C_{\frac{\lambda'}{N_0} d \lesssim \bullet \ll d}^\pm v_{N_1}^\mp] \right\|_{L_t^1 L_x^2} \\
&\lesssim \alpha (\alpha N_1)^{\frac{n-1}{2}} (N_1)^{\frac{1}{2}} \|F\|_{L^2} \|C_{\frac{\lambda'}{N_0} d \lesssim \bullet \ll d}^\pm v_{N_1}^\mp\|_{L_{t,x}^2} \\
&\lesssim (\lambda' d)^{\frac{n-1}{4}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-}
\end{aligned}$$

which again is acceptable.

Thus it remains to deal with the term $C_{\ll \frac{\lambda'}{N_1} d}^+ [C_d P_{\lambda'} F C_{\ll \frac{\lambda'}{N_0} d}^\pm v^\mp]$. By exploiting the angle estimate (52), we have the decomposition

$$P_{N_0} C_{\ll \frac{\lambda'}{N_1} d}^+ [C_d P_{\lambda'} F C_{\ll \frac{\lambda'}{N_0} d}^\pm v^\mp] = \sum_{\substack{\kappa, \bar{\kappa} \in \mathcal{C}_\alpha \\ 5\alpha \leq \theta(\kappa, \bar{\kappa}) \lesssim \alpha}} P_{N_0, \bar{\kappa}}^{+, \alpha} [C_d P_{\lambda'} F R_{\kappa, \alpha^2 N_1}^\pm v^\mp].$$

Therefore, as $P_{N_0} C_{\ll \frac{\lambda'}{N_1} d}^+ [C_d P_{\lambda'} F C_{\ll \frac{\lambda'}{N_0} d}^\pm v^\mp]$ is now a multiple of a $NF_{N_0}^+$ atom, by again using Lemma 3.4 together with Theorem 4.3 we obtain

$$\begin{aligned} \|C_{\ll \frac{\lambda'}{N_1} d}^+ \Pi_+ [C_d P_{\lambda'} F C_{\ll \frac{\lambda'}{N_0} d}^\pm v^\mp]\|_{\mathcal{N}_{N_0}^+} &\lesssim \left(\sum_{\substack{\kappa, \bar{\kappa} \in \mathcal{C}_\alpha \\ 5\alpha \leq \theta(\kappa, \bar{\kappa}) \lesssim \alpha}} \|P_{N_0, \bar{\kappa}}^{+, \alpha} \Pi_+ [C_d P_{\lambda'} F R_{\kappa, \alpha^2 N_1}^\pm v^\mp]\|_{NF^+(\bar{\kappa})}^2 \right)^{\frac{1}{2}} \\ &\lesssim \|F\|_{L_{t,x}^2} \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|R_{\kappa, \alpha^2 N_1}^\pm v^\mp\|_{PW^+(\kappa)}^2 \right)^{\frac{1}{2}} \\ &\lesssim (\alpha N_1)^{\frac{n-1}{2}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-} \approx (\lambda' d)^{\frac{n-1}{4}} \|F\|_{L_{t,x}^2} \|v\|_{F_{N_1}^-} \end{aligned}$$

and hence by summing up in d and λ' we obtain the required estimate. \square

Remark 5.3. It is possible to improve the factors on the righthand side, for instance we can replace $\min\{\lambda, N_1\}$ with $\min\{\lambda, N_0, N_1\}$ by a minor additional argument. Other improvements are also possible, particularly in the high-high case $N_0 \approx N_1$. However as we have no need for any further refinements here, we leave this as an exercise to the interested reader.

We now present a number of useful bilinear estimates that follow from Theorem 5.1. The first is a ‘‘far cone’’ version of Theorem 5.1.

Corollary 5.4 (Bilinear estimates in $\mathcal{N}_{N_1}^\pm$ - Far cone case). *Let $v \in F_{N_1}^\mp$ have compact support in time.*

(i) *Let $1 \leq a \leq 2$ and $\delta \gtrsim N_1$. Assume $F \in L_t^a L_x^2$ is scalar valued. Then*

$$\|P_{N_0}(Fv)\|_{\mathcal{N}_{N_0}^\pm} + \|S_{N_0, \lesssim \delta}^\pm (F \mathfrak{C}_{\lesssim \delta}^\mp v)\|_{\mathcal{N}_{N_0}^\pm} \lesssim (N_1)^{\frac{n-1}{2} + \frac{1}{a} - \frac{1}{2}} \|F\|_{L_t^a L_x^2} \|v\|_{F_{N_1}^\mp}.$$

(ii) *Let $1 \leq a < 2$, $b \geq 2$, and $\frac{1}{a} + \frac{n-1}{2b} \geq 1$. Assume $\lambda \ll N_1$ and $F \in L_{t,x}^2$ is scalar valued with $\text{supp } \hat{F} \subset \{|\xi| \lesssim \lambda\}$. Then*

$$\|P_{N_0}(Fv)\|_{\mathcal{N}_{N_0}^\pm} \lesssim \left(\lambda^{\frac{n-1}{2}} \|F\|_{L_{t,x}^2} + (N_1)^{\frac{n}{b} + \frac{1}{a} - 1} \|C_{\gtrsim N_1} F\|_{L_t^a L_x^b} \right) \|v\|_{F_{N_1}^\mp}.$$

Proof. (i): As usual, we may assume $\pm = +$. By interpolation, it is enough to considering the cases $a = 1$ and $a = 2$. The former case is simply an application of Sobolev embedding together with (35). On the other hand, the $a = 2$ case can be reduce to Theorem 5.1 by using $\dot{\mathcal{X}}_\pm^{b,q}$ spaces to deal with the region away from the cone. More precisely, if we let $\lambda \approx \max\{N_0, N_1\}$, then from the estimates (45) and (46), we reduce to estimating $S_{N_0, \ll N_1}^+ (F \mathfrak{C}_{\ll N_1}^- v)$. Noting the identity¹³

$$S_{N_0, \ll N_1}^+ (F \mathfrak{C}_{\ll N_1}^- v) = S_{N_0, \ll N_1}^+ (P_{\lesssim \lambda} C_{\lesssim N_1} F \mathfrak{C}_{\ll N_1}^- v)$$

the required estimate now follows from Theorem 5.1.

¹³As in the proof of the $\delta \geq \min\{N_0, \lambda, N_2\}$ case in Theorem 5.1, the identity follows from the inequality

$$\|\tau - |\xi|\| \leq \|\tau + \tau'\| - \|\xi + \xi'\| + \|\tau'\| - \|\xi'\| + 2 \min\{|\xi + \xi'|, |\xi'|\}.$$

(ii): As in the proof of (i), we can use (45) and (46), to reduce to estimating the close cone term $S_{N_0, \ll \lambda}^+(F \mathfrak{C}_{\ll \lambda}^- v)$. We would like to now deduce that F must also be λ from the cone, and thus simply apply Theorem 5.1. This is true for when we have the output Fv and v localised close to a \pm cone, but can fail when we Fv is near a \pm cone, and v is close to a \mp cone. However in the latter case, we have the redeeming feature that F is in fact $N_1 \gg \lambda$ away from the cone, which gives the second term in (i). The details are as follows.

Write

$$S_{N_0, \ll \lambda}^+(F \mathfrak{C}_{\ll \lambda}^- v) = S_{N_0, \ll \lambda}^+(C_{\lesssim \lambda} F \mathfrak{C}_{\ll \lambda}^- v) + S_{N_0, \ll \lambda}^+(C_{\gg \lambda} F \mathfrak{C}_{\ll \lambda}^- v).$$

For the first term we can simply use Theorem 5.1 with $\delta = \lambda$. On the other hand, for the second term, we observe that if

$$(\tau, \xi) \in \text{supp } \widetilde{C_{\gg \lambda} F}, \quad (\tau', \xi') \in \text{supp } \widetilde{\mathfrak{C}_{\ll \lambda}^- v}, \quad (\tau + \tau', \xi + \xi') \in \{|\xi| \approx N_0, |\tau| - |\xi| \ll \lambda\},$$

then we have

$$\begin{aligned} |\tau| - |\xi| &\approx |(\tau - \text{sgn}(\tau)|\xi|) + (\tau' - \text{sgn}(\tau')|\xi'|) - (\tau + \tau' - \text{sgn}(\tau + \tau')|\xi + \xi'|) \\ &= |\text{sgn}(\tau + \tau')|\xi + \xi'| - \text{sgn}(\tau')|\xi'| - \text{sgn}(\tau)|\xi| \end{aligned} \quad (53)$$

Therefore, as $N_1 \gg \lambda$, and the left hand side of (53) is $\gg \lambda$, we must have $|\tau| - |\xi| \approx |\xi'|$ and consequently we have the identity

$$S_{N_0, \ll \lambda}^+(C_{\gg \lambda} F \mathfrak{C}_{\ll \lambda}^- v) = S_{N_0, \ll \lambda}^+(C_{\gtrsim N_1} F \mathfrak{C}_{\ll \lambda}^- v).$$

Note that the conditions on (a, b) imply that the pair $(\frac{a}{a-1}, \frac{2b}{b-2})$ is Strichartz admissible. Thus, by an application of Holder's inequality together with Theorem 4.1, we deduce that

$$\begin{aligned} \|S_{N_0, \ll \lambda}^+(C_{\gtrsim N_1} F \mathfrak{C}_{\ll \lambda}^- v)\|_{\mathcal{N}_{N_0}^+} &\leq \|S_{N_0, \ll \lambda}^+(C_{\gtrsim N_1} F \mathfrak{C}_{\ll \lambda}^- v)\|_{L_t^1 L_x^2} \\ &\lesssim \|C_{\gtrsim N_1} F\|_{L_t^a L_x^b} \|\mathfrak{C}_{\ll \lambda}^- v\|_{L_t^{\frac{a}{a-1}} L_x^{\frac{2b}{b-2}}} \\ &\lesssim (N_1)^{n(\frac{1}{2} - \frac{b-2}{2b}) - \frac{a-1}{a}} \|C_{\gtrsim N_1} F\|_{L_t^a L_x^b} \|v\|_{F_{N_1}^-} \\ &= (N_1)^{\frac{n}{2} + \frac{1}{a} - 1} \|C_{\gtrsim N_1} F\|_{L_t^a L_x^b} \|v\|_{F_{N_1}^-} \end{aligned}$$

as required. □

By duality, Theorem 5.1 also implies the following bilinear estimates in $L_{t,x}^2$.

Corollary 5.5 (L^2 Bilinear Estimates). *Let $u \in F_{N_1}^\pm$ and $v \in F_{N_2}^\mp$ have compact support in time.*

(i) *Let $\delta \leq \min\{N_1, N_2\}$. Then*

$$\|C_{\lesssim \delta}[(\mathfrak{C}_{\ll \delta}^\pm u)^\dagger \mathfrak{C}_{\ll \delta}^\mp v]\|_{L_{t,x}^2} \lesssim (\delta \min\{N_1, N_2\})^{\frac{n-1}{4}} \|u\|_{F_{N_1}^\pm} \|v\|_{F_{N_2}^\mp}.$$

(ii) *Let $a, b \geq 2$ and $\delta \geq \min\{N_1, N_2\}$. Then*

$$\begin{aligned} \|u^\dagger v\|_{L_t^a L_x^b} + \|(\mathfrak{C}_{\lesssim \delta}^\pm u)^\dagger \mathfrak{C}_{\lesssim \delta}^\mp v\|_{L_t^a L_x^b} \\ \lesssim (\min\{N_1, N_2\})^{\frac{n}{2} - \frac{1}{a}(\frac{1}{2} + \frac{1}{b})} (\max\{N_1, N_2\})^{(n - \frac{1}{a})(\frac{1}{2} - \frac{1}{b})} \|u\|_{F_{N_1}^\pm} \|v\|_{F_{N_2}^\mp}. \end{aligned}$$

Proof. (i): After a reflection, it is enough to consider the case $N_1 \geq N_2$ and $\pm = +$. Let $\lambda = 4N_1$ and note that $u^\dagger v = P_{\leq \lambda}(u^\dagger v)$. An application of the the duality estimate in Corollary 4.6 together with Theorem 5.1 gives

$$\begin{aligned} \|C_{\leq \delta}[(\mathfrak{C}_{\ll \delta}^+ u)^\dagger \mathfrak{C}_{\ll \delta}^- v]\|_{L_{t,x}^2} &= \sup_{\|F\|_{L_{t,x}^2} \leq 1} \left| \int_{\mathbb{R}^{n+1}} C_{\leq \delta} P_{\leq \lambda} F [(\mathfrak{C}_{\ll \delta}^+ u)^\dagger \mathfrak{C}_{\ll \delta}^- v] dt dx \right| \\ &\leq \sum_{N_0 \approx N_1} \sup_{\|F\|_{L_{t,x}^2} \leq 1} \left| \int_{\mathbb{R}^{n+1}} u^\dagger S_{N_0, \ll \delta}^+ [C_{\leq \delta} P_{\leq \lambda} F \mathfrak{C}_{\ll \delta}^- v] dt dx \right| \\ &\lesssim \|u\|_{F_{N_1}^+} \sum_{N_0 \approx N_1} \sup_{\|F\|_{L_{t,x}^2} \leq 1} \|S_{N_0, \ll \delta}^+ [C_{\leq \delta} P_{\leq \lambda} F \mathfrak{C}_{\ll \delta}^- v]\|_{\mathcal{N}_{N_0}^+} \\ &\lesssim (\delta \min\{N_1, N_2\})^{\frac{n-1}{4}} \|u\|_{F_{N_1}^+} \|v\|_{F_{N_2}^-} \end{aligned}$$

as required.

(ii): The cases $(a, b) = (\infty, \infty), (\infty, 2), (2, \infty)$ follow by Sobolev embedding and the $L_t^4 L_x^\infty$ Strichartz estimate in Theorem 4.1. Thus by interpolation we reduce to the case $(a, b) = (2, 2)$. Without loss of generality, we may assume $N_1 \geq N_2$. To deal with the far cone case, we use (40) to obtain the inequalities

$$\|(\mathfrak{C}_{\gtrsim N_2}^+ u)^\dagger v\|_{L_{t,x}^2} \lesssim \|\mathfrak{C}_{\gtrsim N_2}^+ u\|_{L_{t,x}^2} \|v\|_{L_{t,x}^\infty} \lesssim (N_2)^{\frac{n-1}{2}} \|u\|_{F_{N_1}^+} \|v\|_{F_{N_2}^-}$$

and

$$\|u^\dagger \mathfrak{C}_{\gtrsim N_2}^- v\|_{L_{t,x}^2} \leq \|u\|_{L_t^\infty L_x^2} \|\mathfrak{C}_{\gtrsim N_2}^- v\|_{L_t^2 L_x^\infty} \lesssim (N_2)^{\frac{n-1}{2}} \|u\|_{F_{N_1}^+} \|v\|_{F_{N_2}^-}.$$

Therefore we reduce to estimating $(\mathfrak{C}_{\ll N_2}^+ u)^\dagger \mathfrak{C}_{\ll N_2}^- v = C_{\leq N_2}[(\mathfrak{C}_{\ll N_2}^+ u)^\dagger \mathfrak{C}_{\ll N_2}^- v]$ in $L_{t,x}^2$, but this follows from (i) by taking $\delta = N_2$. □

6. CUBIC ESTIMATES

We now come to main trilinear estimate we require. In this case, although the bilinear estimates only required $u \in F_N^\pm$, we are forced to make use of the stronger G_N^\pm spaces. Essentially this is due to the fact that away from the light cone, we require the additional integrability in t given by the \mathcal{Y}^\pm norm.

Let N_{min} , N_{med} , and N_{max} denote, respectively, the minimum, the median, and the maximum, of the set $\{N_1, N_2, N_3\}$. Our aim is to prove the following.

Theorem 6.1. *Let $T > 0$ and assume that \pm and \pm' are independent choices of signs. There exists $\epsilon > 0$ such that if $u_1 \in G_{N_1}^{\pm'}$, $u_2 \in G_{N_2}^{\mp'}$, $u_3 \in G_{N_3}^{\mp}$ then*

$$\|\mathbb{1}_{(-T, T)}(t) P_{N_0}[(u_1^\dagger u_2) u_3]\|_{\mathcal{N}_{N_0}^\pm} \lesssim (N_{min} N_{med})^{\frac{n-1}{2}} \left(\frac{N_{min}}{N_{med}}\right)^\epsilon \|u_1\|_{G_{N_1}^{\pm'}} \|u_2\|_{G_{N_2}^{\mp'}} \|u_3\|_{G_{N_3}^{\mp}}$$

where the implied constant is independent of T .

To deal with the close cone region, the following lemma will prove crucial.

Lemma 6.2. *Let $\delta \geq N_{min}$ and $\lambda = \min\{\max\{N_1, N_2\}, \max\{N_0, N_1\}\}$. Assume supp $\tilde{u}_j \subset \{||\tau| - |\xi|| \leq \delta, |\xi| \approx N_j\}$. Then*

$$C_{\leq \delta} P_{N_0}[(u_1^\dagger u_2) u_3] = C_{\leq \delta} P_{N_0}(C_{\leq \delta} P_{\leq \lambda}[(u_1^\dagger u_2) u_3]).$$

Proof. The inequality

$$|\tau + \tau'| - |\xi + \xi'| \leq |\tau| - |\xi| + |\tau'| - |\xi'| + 2 \min\{|\xi|, |\xi'|\} \quad (54)$$

implies that if $(\tau, \xi), (\tau', \xi') \in \{|\tau| - |\xi| \leq \delta\}$ and $\delta \geq \min\{N_1, N_2\}$, then $|\tau + \tau'| - |\xi + \xi'| \lesssim \delta$. Consequently we have the identity $u_1^\dagger u_2 = C_{\lesssim \delta} P_{\lesssim \max\{N_1, N_1\}} [u_1^\dagger u_2]$. On the other hand, if $\delta \geq \min\{N_0, N_1\}$, can again use (54) to deduce the identity $C_{\lesssim \delta} P_{N_0}(Fv_3) = C_{\leq \delta} P_{N_0}(C_{\lesssim \delta} P_{\lesssim \max\{N_0, N_1\}} Fv_3)$. Thus lemma follows. \square

Proof of Theorem 6.1. After a reflection, we may assume $\pm = +$. We also fix $\pm' = +$, as the $\pm' = -$ argument is identical. Let $\rho \in C_0^\infty(\mathbb{R})$ with $\rho = 1$ on $[-1, 1]$ and fix $T^* \gg \max\{T, N_{max}\}$. An application of (ii) in Theorem 3.9, together with the identity

$$\mathbb{1}_{(-T, T)}(t) P_{N_0} [(u_1^\dagger u_2) u_3] = \mathbb{1}_{(-T, T)}(t) P_{N_0} [(\rho(\frac{t}{T^*}) u_1)^\dagger (\rho(\frac{t}{T^*}) u_2)] \rho(\frac{t}{T^*}) u_3]$$

shows that it is enough to prove

$$\|P_{N_0} [(u_1^\dagger u_2) u_3]\|_{\mathcal{N}_{N_0}^+} \lesssim (N_{min} N_{med})^{\frac{n-1}{2}} \left(\frac{N_{min}}{N_{med}}\right)^\epsilon \|u_1\|_{G_{N_1}^+} \|u_2\|_{G_{N_2}^-} \|u_3\|_{G_{N_3}^-}$$

under the additional assumption that each u_j has compact support in time (thus, in particular, we can make use of the bilinear estimates in the previous section). Let $\frac{n+1}{4n} < a < \frac{1}{2}$ and take

$$\delta = (N_{min})^{\frac{1}{2}-a} (N_{med})^{\frac{1}{2}+a}.$$

The strategy is roughly to decompose into regions δ away from the light cone, and regions within δ of the light cone. In the close cone region, we can essentially just apply the bilinear estimate Theorem 5.1. On the other hand, in the region away from the light cone, the argument is more involved and we need to exploit the bilinear estimates in Corollaries 5.5 and 5.4, together with the additional integrability in t given by the \mathcal{Y}^\pm norms.

We break the proof into three main cases, $N_3 = N_{max} \gg N_{med}$, $N_3 \approx N_{med}$, and $N_3 \approx N_{min}$.

Case 1: $N_3 = N_{max} \gg N_{med}$. We begin by decomposing

$$(u_1^\dagger u_2) u_3 = C_{\lesssim \delta} (u_1^\dagger u_2) u_3 + C_{\gg \delta} (u_1^\dagger u_2) u_3. \quad (55)$$

Note that by the $L_t^4 L_x^\infty$ Strichartz estimate in Theorem 4.1 we have the inequalities

$$\begin{aligned} \|P_{N_0} [(u_1^\dagger u_2) \mathfrak{C}_{\gtrsim \delta}^- u_3]\|_{\mathcal{N}_{N_0}^+} &\lesssim \|u_1\|_{L_t^4 L_x^\infty} \|u_2\|_{L_t^4 L_x^\infty} \|\mathfrak{C}_{\gtrsim \delta}^- u_3\|_{L_{t,x}^2} \\ &\lesssim (N_{min} N_{med})^{\frac{n-1}{2} + \frac{1}{4}} \delta^{-\frac{1}{2}} \|u_1\|_{F_{N_1}^+} \|u_2\|_{F_{N_2}^-} \|u_3\|_{F_{N_3}^-} \end{aligned} \quad (56)$$

and

$$\begin{aligned} \|\mathfrak{C}_{\gtrsim \delta}^+ P_{N_0} [(u_1^\dagger u_2) u_3]\|_{\mathcal{N}_{N_0}^+} &\lesssim \delta^{-\frac{1}{2}} \|(u_1^\dagger u_2) u_3\|_{L_{t,x}^2} \\ &\lesssim \delta^{-\frac{1}{2}} \|u_1\|_{L_t^4 L_x^\infty} \|u_2\|_{L_t^4 L_x^\infty} \|u_3\|_{L_t^\infty L_x^2} \\ &\lesssim (N_{min} N_{med})^{\frac{n-1}{2} + \frac{1}{4}} \delta^{-\frac{1}{2}} \|u_1\|_{F_{N_1}^+} \|u_2\|_{F_{N_2}^-} \|u_3\|_{F_{N_3}^-}. \end{aligned} \quad (57)$$

Since $(N_{min} N_{med})^{\frac{1}{4}} \delta^{-\frac{1}{2}} = \left(\frac{N_{min}}{N_{med}}\right)^{\frac{n}{2}}$ both estimates are acceptable. Observe that neither (56) nor (57) made any use of the structure of the product. Thus we can always control the case where the output,

$(u_1^\dagger u_2)u_3$, or the N_{max} term are δ from the cone, by putting the low frequency terms in $L_t^4 L_x^\infty$. Moreover, as the estimates (56) and (57) only made use of L_x^2 based spaces, together with Theorem 4.1, we can freely add \mathfrak{C}_δ^\pm multipliers to the left hand side without affecting the validity of (56) and (57). These observations are also used in the case $N_3 \approx N_{med}$ and $N_3 \approx N_{min}$.

The estimate for the first term in (55) is now straightforward. By (56) and (57), we reduce to considering the term $\mathfrak{C}_{\ll\delta}^+(C_{\lesssim\delta}[u_1^\dagger u_2]\mathfrak{C}_{\ll\delta}^- u_3)$. An application of Theorem 5.1 and Corollary 5.5 then gives

$$\begin{aligned} \|\mathfrak{C}_{\ll\delta}^+ P_{N_0}(C_{\lesssim\delta}[u_1^\dagger u_2]\mathfrak{C}_{\ll\delta}^- u_3)\|_{\mathcal{N}_{N_0}^+} &\lesssim (\delta N_{med})^{\frac{n-1}{4}} \|u_1^\dagger u_2\|_{L_{t,x}^2} \|u_3\|_{F_{N_3}^-} \\ &\lesssim (N_{min})^{\frac{n-1}{2}} (\delta N_{med})^{\frac{n-1}{4}} \|u_1\|_{F_{N_1}^+} \|u_2\|_{F_{N_2}^-} \|u_3\|_{F_{N_3}^-} \end{aligned}$$

which is acceptable since $(\frac{\delta}{N_{med}})^{\frac{n-1}{4}} = (\frac{N_{min}}{N_{med}})^{(\frac{1}{2}-a)\frac{n-1}{4}}$ and $\frac{n+1}{4n} < a < \frac{1}{2}$.

On the other hand, to deal with the second term in (55), assume for the moment that we have the inequalities

$$\|C_{\gg\delta}(u_1^\dagger u_2)\|_{L_{t,x}^2} \lesssim (N_{min})^{\frac{n}{2}} \delta^{-\frac{1}{2}} \|u_1\|_{F_{N_1}^+} \|u_2\|_{F_{N_2}^-} \quad (58)$$

and

$$\|C_{\gtrsim N_3}(u_1^\dagger u_2)\|_{L_t^{\frac{8n}{5n+3}} L_x^{\frac{4n}{3}}} \lesssim (N_{min} N_{med})^{\frac{n-1}{2}} \left(\frac{N_{min}}{N_{med}}\right)^{\frac{1}{4n}} (N_3)^{-\frac{3(n+1)}{8n}} \|u_1\|_{G_{N_1}^+} \|u_2\|_{G_{N_2}^-}. \quad (59)$$

Then by (ii) in Corollary 5.4 with $(a, b) = (\frac{8n}{5n+3}, \frac{4n}{3})$ (note that this pair is admissible), we have

$$\begin{aligned} \|P_{N_0}[C_{\gg\delta}(u_1^\dagger u_2)u_3]\|_{\mathcal{N}_{N_0}^+} &\lesssim \left[(N_{med})^{\frac{n-1}{2}} \|C_{\gg\delta}(u_1^\dagger u_2)\|_{L_{t,x}^2} + (N_3)^{\frac{3(n+1)}{8n}} \|C_{\gtrsim N_3}(u_1^\dagger u_2)\|_{L_t^{\frac{8n}{5n+3}} L_x^{\frac{4n}{3}}} \right] \|u_3\|_{F_{N_3}^-} \\ &\lesssim \left[(N_{min})^{\frac{n}{2}} (N_{med})^{\frac{n-1}{2}} \delta^{-\frac{1}{2}} + (N_{min} N_{med})^{\frac{n-1}{2}} \left(\frac{N_{min}}{N_{med}}\right)^{\frac{1}{4n}} \right] \|u_1\|_{G_{N_1}^+} \|u_2\|_{G_{N_2}^-} \|u_3\|_{F_{N_3}^-} \\ &\lesssim (N_{min} N_{med})^{\frac{n-1}{4}} \left(\frac{N_{min}}{N_{med}}\right)^{\min\{\frac{1}{4n}, \frac{a}{2}\}} \|u_1\|_{G_{N_1}^+} \|u_2\|_{G_{N_2}^-} \|u_3\|_{F_{N_3}^-}. \end{aligned}$$

Thus, to complete the proof of the case $N_3 \gg N_{med}$, it only remains to deduce the inequalities (58) and (59). We start with the more difficult (59). To this end, note that the inequality (54) and the assumption $N_3 \gg N_{med}$ implies the decomposition

$$C_{\gtrsim N_3}[u_1^\dagger u_2] = C_{\gtrsim N_3}[(\mathfrak{C}_{\gtrsim N_3}^+ u_1)^\dagger u_2] + C_{\gtrsim N_3}[(\mathfrak{C}_{\ll N_3}^- u_1)^\dagger \mathfrak{C}_{\gtrsim N_3}^- u_2].$$

Essentially the point is that if the output of $u_1^\dagger u_2$ is far from the cone, then it is not possible for both u_1 and u_2 to have Fourier support close the cone. If we now apply Lemma 3.14 to dispose of the $C_{\gtrsim N_3}$ multiplier, followed by the $L_t^{\frac{8n}{5-n}} L_x^\infty$ Strichartz estimate (note that $\frac{8n}{5-n} \geq 4$), by the definition of the \mathcal{Y}^\pm norm we deduce that

$$\begin{aligned} \|C_{\gtrsim N_3}[(\mathfrak{C}_{\gtrsim N_3}^+ u_1)^\dagger u_2]\|_{L_t^{\frac{8n}{5n+3}} L_x^{\frac{4n}{3}}} &\lesssim (N_1)^{\frac{n}{2}-\frac{3}{4}} \|\mathfrak{C}_{\gtrsim N_3}^+ u_1\|_{L_t^{\frac{4n}{3n-1}} L_x^2} \|u_2\|_{L_t^{\frac{8n}{5-n}} L_x^\infty} \\ &\lesssim (N_1)^{\frac{n}{2}-\frac{3}{4}} (N_2)^{\frac{n}{2}-\frac{5-n}{8n}} (N_3)^{-1} \|u_1\|_{\mathcal{Y}^+} \|u_2\|_{F_{N_2}^-} \\ &\lesssim (N_1)^{\frac{n-1}{2}+\frac{1}{4n}} (N_2)^{\frac{n-1}{2}+\frac{5(n-1)}{8n}} (N_3)^{-1} \|u_1\|_{G_{N_1}^+} \|u_2\|_{F_{N_2}^-} \end{aligned}$$

which is acceptable since, using the fact that $N_3 \geq N_1, N_2$,

$$(N_1)^{\frac{n-1}{2}+\frac{1}{4n}} (N_2)^{\frac{n-1}{2}+\frac{5(n-1)}{8n}} (N_3)^{-1} \leq (N_{min} N_{med})^{\frac{n-1}{2}} (N_3)^{-\frac{3(n+1)}{8n}} \left(\frac{N_{min}}{N_{med}}\right)^{\frac{1}{4n}}.$$

A similar argument handles the $C_{\gtrsim N_3}[(\mathbf{e}_{\ll N_3}^+ u_1)^\dagger \mathbf{e}_{\gtrsim N_3}^- u_2]$ term, we just put $\mathbf{e}_{\ll N_3}^+ u_1 \in L_t^{\frac{8n}{5-n}} L_x^\infty$ and $\mathbf{e}_{\gtrsim N_3}^- u_2 \in \mathcal{Y}^-$. Thus we obtain (59). The proof of (58) is similar, we just note that again the inequality (54) implies that

$$C_{\gg \delta}[u_1^\dagger u_2] = C_{\gg \delta}[(\mathbf{e}_{\gtrsim \delta}^+ u_1)^\dagger u_2] + C_{\gg \delta}[(\mathbf{e}_{\ll \delta}^- u_1)^\dagger \mathbf{e}_{\gtrsim \delta}^- u_2].$$

Hence putting the far cone terms in $L_{t,x}^2$ we obtain

$$\begin{aligned} \|C_{\gg \delta}[u_1^\dagger u_2]\|_{L_{t,x}^2} &\lesssim (N_{min})^{\frac{n}{2}} \left(\|\mathbf{e}_{\gtrsim \delta}^+ u_1\|_{L_{t,x}^2} \|u_2\|_{L_t^\infty L_x^2} + \|u_1\|_{L_t^\infty L_x^2} \|\mathbf{e}_{\gtrsim \delta}^- u_2\|_{L_{t,x}^2} \right) \\ &\lesssim (N_{min})^{\frac{n}{2}} \delta^{-\frac{1}{2}} \|u_1\|_{F_{N_1}^+} \|u_2\|_{F_{N_2}^-}. \end{aligned}$$

Therefore we obtain (58) and so the case $N_3 = N_{max} \gg N_{med}$ follows.

Case 2: $N_3 \approx N_{med}$. Note that $N_3 \approx N_{med}$ implies that $N_{min} \approx \min\{N_1, N_2\}$. If u_1 is δ away from the cone, then by (i) in Corollary 5.4 we have

$$\begin{aligned} \|P_{N_0}((\mathbf{e}_{\gtrsim \delta}^+ u_1)^\dagger u_2) u_3\|_{\mathcal{N}_{N_0}^+} &\lesssim (N_{med})^{\frac{n-1}{2}} \|(\mathbf{e}_{\gtrsim \delta}^+ u_1)^\dagger u_2\|_{L_{t,x}^2} \|u_3\|_{F_{N_3}^-} \\ &\lesssim (N_{min})^{\frac{n}{2}} (N_{med})^{\frac{n-1}{2}} \|\mathbf{e}_{\gtrsim \delta}^+ u_1\|_{L_{t,x}^2} \|u_2\|_{L_t^\infty L_x^2} \|u_3\|_{F_{N_3}^-} \\ &\lesssim (N_{min} N_{med})^{\frac{n-1}{2}} \left(\frac{N_{min}}{\delta}\right)^{\frac{1}{2}} \|u_1\|_{F_{N_1}^+} \|u_2\|_{F_{N_2}^-} \|u_3\|_{F_{N_3}^-} \end{aligned}$$

which is acceptable as $(\frac{N_{min}}{\delta})^{\frac{1}{2}} = (\frac{N_{min}}{N_{med}})^{\frac{1}{4} + \frac{\alpha}{2}}$. A similar argument handles the case where u_2 is δ away from the cone.

On the other hand, when u_3 is δ away from the cone, the argument is more involved as we need to make use of the \mathcal{Y}^\pm norms to gain the correct factors N_{min}, N_{med} . An application of Holder together with (ii) in Corollary 5.5 gives

$$\begin{aligned} \|P_{N_0}((\mathbf{e}_{\ll \delta}^+ u_1)^\dagger \mathbf{e}_{\ll \delta}^- u_2) \mathbf{e}_{\gtrsim \delta}^- u_3\|_{\mathcal{N}_{N_0}^+} &\leq \|P_{N_0}((\mathbf{e}_{\ll \delta}^+ u_1)^\dagger \mathbf{e}_{\ll \delta}^- u_2) \mathbf{e}_{\gtrsim \delta}^- u_3\|_{L_t^1 L_x^2} \\ &\lesssim \|(\mathbf{e}_{\ll \delta}^+ u_1)^\dagger \mathbf{e}_{\ll \delta}^- u_2\|_{L_t^{\frac{4n}{n+1}} L_x^2} \|\mathbf{e}_{\gtrsim \delta}^- u_3\|_{L_t^{\frac{4n}{3n-1}} L_x^\infty} \\ &\lesssim (N_{min})^{\frac{n}{2} - \frac{n+1}{4n}} (N_{med})^{\frac{n}{2}} \delta^{-1} \|u_1\|_{F_{N_1}^+} \|u_2\|_{F_{N_2}^-} \|u_3\|_{\mathcal{Y}^-} \\ &\lesssim (N_{min})^{\frac{n}{2} - \frac{n+1}{4n}} (N_{med})^{\frac{n}{2} + \frac{n+1}{4n}} \delta^{-1} \|u_1\|_{F_{N_1}^+} \|u_2\|_{F_{N_2}^-} \|u_3\|_{G_{N_3}^-} \end{aligned}$$

which is acceptable since

$$(N_{min})^{\frac{n}{2} - \frac{n+1}{4n}} (N_{med})^{\frac{n}{2} + \frac{n+1}{4n}} \delta^{-1} = (N_{min} N_{med})^{\frac{n-1}{2}} \left(\frac{N_{min}}{N_{med}}\right)^{a - \frac{n+1}{4n}}$$

and $\frac{n+1}{4n} < a < \frac{1}{2}$. The final far cone case is when the output is δ from the cone. However here we can simply argue as in (56) but put the low frequency terms in $L_t^4 L_x^\infty$, and the N_{max} term in $L_t^\infty L_x^2$.

It remains to deal with the close cone case $S_{N_0, \ll \delta}^+((\mathbf{e}_{\ll \delta}^+ u_1)^\dagger \mathbf{e}_{\ll \delta}^- u_2) \mathbf{e}_{\ll \delta}^- u_3$. To this end, as in the $N_3 \approx N_{max}$ case, we apply Lemma 6.2, Theorem 5.1, and Corollary 5.5 to obtain

$$\begin{aligned} \|S_{N_0, \ll \delta}^+((\mathbf{e}_{\ll \delta}^+ u_1)^\dagger \mathbf{e}_{\ll \delta}^- u_2) \mathbf{e}_{\ll \delta}^- u_3\|_{\mathcal{N}_{N_0}^+} &= \|S_{N_0, \ll \delta}^+(C_{\lesssim \delta}[(\mathbf{e}_{\ll \delta}^+ u_1)^\dagger \mathbf{e}_{\ll \delta}^- u_2] \mathbf{e}_{\ll \delta}^- u_3)\|_{\mathcal{N}_{N_0}^+} \\ &\lesssim (\delta N_{med})^{\frac{n-1}{4}} \|C_{\lesssim \delta}[(\mathbf{e}_{\ll \delta}^+ u_1)^\dagger \mathbf{e}_{\ll \delta}^- u_2]\|_{L_{t,x}^2} \|u_3\|_{F_{N_3}^-} \\ &\lesssim (N_{min})^{\frac{n-1}{2}} (\delta N_{med})^{\frac{n-1}{4}} \|u_1\|_{F_{N_1}^+} \|u_2\|_{F_{N_2}^-} \|u_3\|_{F_{N_3}^-}. \end{aligned}$$

which is acceptable as $(\frac{\delta}{N_{med}})^{\frac{n-1}{4}} = (\frac{N_{min}}{N_{med}})^{\frac{n-1}{4}(\frac{1}{2}-a)}$. Therefore we obtain the case $N_3 \approx N_{med}$.

Case 3: $N_3 \approx N_{min}$. Without loss of generality, we assume that $N_1 \geq N_2$, thus $N_1 \approx N_{max}$. The argument to control the case $N_3 \approx N_{min}$ is very similar to the previous case, essentially the only difference is that we need to reverse the order in which we estimate the far cone case to avoid having to estimate the multiplier $\mathfrak{C}_{\ll\delta}^\pm$ in F_N^\pm with $\delta \ll N$. As before, we start by dealing with the far cone case. An application of Corollary 5.5 gives the bound

$$\begin{aligned} \|P_{N_0}([u_1^\dagger u_2] \mathfrak{C}_{\gtrsim\delta}^- u_3)\|_{\mathcal{N}_{N_0}^+} &\leq \| [u_1^\dagger u_2] \mathfrak{C}_{\gtrsim\delta}^- u_3 \|_{L_t^1 L_x^2} \lesssim (N_{min})^{\frac{n}{2}} \|u_1^\dagger u_2\|_{L_{t,x}^2} \| \mathfrak{C}_{\gtrsim\delta}^- u_3 \|_{L_{t,x}^2} \\ &\lesssim (N_{med})^{\frac{n-1}{2}} (N_{min})^{\frac{n}{2}} \delta^{-\frac{1}{2}} \|u_1\|_{F_{N_1}^+} \|u_2\|_{F_{N_2}^-} \|u_3\|_{F_{N_3}^-} \end{aligned}$$

which is as before is acceptable. Together with (56), (but with the minor difference that we put the low frequency terms u_2 and u_3 in $L_t^4 L_x^\infty$), we may assume that the output is within δ of the cone. Similarly, when u_1 is δ away from the cone, we follow (57) and put $u_2, u_3 \in L_t^4 L_x^\infty$ by using Theorem 4.1. Finally, if u_2 is δ away from the cone, then we use (i) in Corollary 5.4 together with the \mathcal{Y}^\pm norm to deduce that

$$\begin{aligned} \|S_{N_0, \ll\delta}^+([(\mathfrak{C}_{\ll\delta}^+ u_1)^\dagger \mathfrak{C}_{\gtrsim\delta}^- u_2] \mathfrak{C}_{\ll\delta}^- u_3)\|_{\mathcal{N}_{N_0}^+} &\lesssim (N_{min})^{\frac{n}{2} - \frac{n+1}{4n}} \|(\mathfrak{C}_{\ll\delta}^+ u_1)^\dagger \mathfrak{C}_{\gtrsim\delta}^- u_2\|_{L_t^{\frac{4n}{3n-1}} L_x^2} \|u_3\|_{F_{N_3}^-} \\ &\lesssim (N_{min})^{\frac{n}{2} - \frac{n+1}{4n}} (N_{med})^{\frac{n}{2}} \| \mathfrak{C}_{\ll\delta}^+ u_1 \|_{L_t^\infty L_x^2} \| \mathfrak{C}_{\gtrsim\delta}^- u_2 \|_{L_t^{\frac{4n}{3n-1}} L_x^2} \|u_3\|_{F_{N_3}^-} \\ &\lesssim (N_{min})^{\frac{n}{2} - \frac{n+1}{4n}} (N_{med})^{\frac{n}{2}} \delta^{-1} \|u_1\|_{F_{N_1}^+} \|u_2\|_{\mathcal{Y}^-} \|u_3\|_{F_{N_3}^-} \\ &\leq (N_{min} N_{med})^{\frac{n-1}{2}} \left(\frac{N_{min}}{N_{med}}\right)^{a - \frac{n+1}{4n}} \|u_1\|_{F_{N_1}^+} \|u_2\|_{G_{N_2}^-} \|u_3\|_{F_{N_3}^-} \end{aligned}$$

which again is acceptable.

The final case is the close cone term $S_{N_0, \ll\delta}^+([(\mathfrak{C}_{\ll\delta}^+ u_1)^\dagger \mathfrak{C}_{\ll\delta}^- u_2] \mathfrak{C}_{\ll\delta}^- u_3)$. As previously, by applying Lemma 6.2 and Corollaries 5.4 and 5.5, we obtain

$$\begin{aligned} \|S_{N_0, \ll\delta}^+([(\mathfrak{C}_{\ll\delta}^+ u_1)^\dagger \mathfrak{C}_{\ll\delta}^- u_2] \mathfrak{C}_{\ll\delta}^- u_3)\|_{\mathcal{N}_{N_0}^+} &= \|S_{N_0, \ll\delta}^+(C_{\lesssim\delta} [(\mathfrak{C}_{\ll\delta}^+ u_1)^\dagger \mathfrak{C}_{\ll\delta}^- u_2] \mathfrak{C}_{\ll\delta}^- u_3)\|_{\mathcal{N}_{N_0}^+} \\ &\lesssim (N_{min})^{\frac{n-1}{2}} \|C_{\lesssim\delta} [(\mathfrak{C}_{\ll\delta}^+ u_1)^\dagger \mathfrak{C}_{\ll\delta}^- u_2]\|_{L_{t,x}^2} \|u_3\|_{F_{N_3}^-} \\ &\lesssim (N_{min})^{\frac{n-1}{2}} (\delta N_{med})^{\frac{n-1}{4}} \|u_1\|_{F_{N_1}^+} \|u_2\|_{F_{N_2}^-} \|u_3\|_{F_{N_3}^-} \end{aligned}$$

which is acceptable. Therefore we obtain the case $N_3 \approx N_{min}$ and hence theorem follows. \square

To control the \mathcal{Y}^\pm component of the G_λ^\pm norm, we use the following.

Theorem 6.3. *Let $T > 0$ and \pm, \pm' be independent choices of signs. There exists $\epsilon > 0$ such that for $u_1 \in F_{N_1}^{\pm'}$, $u_2 \in F_{N_2}^{\mp'}$, and $u_3 \in F_{N_3}^{\mp}$ we have*

$$\|\mathbb{1}_{(-T, T)}(t)(u_1^\dagger u_2)u_3\|_{L_t^{\frac{4n}{3n-1}} L_x^2} \lesssim (N_{min} N_{med})^{\frac{n-1}{2}} \left(\frac{N_{min}}{N_{med}}\right)^\epsilon (N_{max})^{\frac{n+1}{4n}} \|u_1\|_{F_{N_1}^{\pm'}} \|u_2\|_{F_{N_2}^{\mp'}} \|u_3\|_{F_{N_3}^{\mp}}$$

where the implied constant is independent of T .

Proof. As in the proof of Theorem 6.1, we only consider the case $\pm = \pm' = +$ as the remaining cases are essentially identical. The required estimate follows by an application of the Strichartz estimates in Theorem 4.1, together with the bilinear estimates in Corollary 5.5. More precisely, if $N_3 \approx N_{max}$, then

as $(q, r) = (\frac{4n}{n-1}, \frac{2n}{n-1})$ is Strichartz admissible, by an application of Holder, followed by Theorem 4.1 and (ii) in Corollary 5.5 with $(a, b) = (2, 2n)$, we deduce that

$$\begin{aligned} \|(u_1^\dagger u_2)u_3\|_{L_t^{\frac{4n}{3n-1}} L_x^2} &\leq \|u_{N_1}^\dagger u_2\|_{L_t^2 L_x^{2n}} \|u_3\|_{L_t^{\frac{4n}{n-1}} L_x^{\frac{2n}{n-1}}} \\ &\lesssim (\min\{N_1, N_2\})^{\frac{n-1}{2} + \frac{n-1}{4n}} (\max\{N_1, N_2\})^{\frac{n-1}{2} - \frac{n-1}{4}} (N_3)^{\frac{n+1}{4n}} \|u\|_{F_{N_1}^+} \|v\|_{F_{N_2}^-} \|v\|_{F_{N_3}^-} \\ &\lesssim (N_{min} N_{med})^{\frac{n-1}{2}} \left(\frac{N_{min}}{N_{med}}\right)^{\frac{n-1}{4n}} (N_{max})^{\frac{n+1}{4n}} \|u\|_{F_{N_1}^+} \|v\|_{F_{N_2}^-} \|v\|_{F_{N_3}^-} \end{aligned}$$

as required. On the other hand, if $N_3 \ll N_{max}$, then we put $u_3 \in L_t^4 L_x^\infty$ and again apply (ii) in Corollary 5.5 with $(a, b) = (\frac{4n}{2n-1}, 2)$ to obtain

$$\begin{aligned} \|(u_1^\dagger u_2)u_3\|_{L_t^{\frac{4n}{3n-1}} L_x^2} &\leq \|u_{N_1}^\dagger u_2\|_{L_t^{\frac{4n}{2n-1}} L_x^2} \|u_3\|_{L_t^4 L_x^\infty} \\ &\lesssim (\min\{N_1, N_2\})^{\frac{n-1}{2} + \frac{1}{4n}} (N_3)^{\frac{n-1}{2} + \frac{1}{4}} \|u\|_{F_{N_1}^+} \|v\|_{F_{N_2}^-} \|v\|_{F_{N_3}^-} \\ &\leq (N_{min} N_{med})^{\frac{n-1}{2}} \left(\frac{N_{min}}{N_{med}}\right)^{\frac{1}{4n}} (N_{max})^{\frac{n+1}{4}} \|u\|_{F_{N_1}^+} \|v\|_{F_{N_2}^-} \|v\|_{F_{N_3}^-}. \end{aligned}$$

□

Combining the previous results, we deduce the following corollary.

Corollary 6.4. *Let $s \geq \frac{n-1}{2}$, $T > 0$ and suppose \pm and \pm' are independent choices of signs. Assume $u_1 \in G^{\frac{n-1}{2}, \pm'}$, $u_2 \in G^{\frac{n-1}{2}, \mp'}$, $u_3 \in G^{\frac{n-1}{2}, \mp}$. If we let*

$$\Gamma = \|u_1\|_{G^{\frac{n-1}{2}, \pm'}} + \|u_2\|_{G^{\frac{n-1}{2}, \mp'}} + \|u_3\|_{G^{\frac{n-1}{2}, \mp}}$$

then

$$\|\mathbb{1}_{(-T, T)}(t)(u_1^\dagger u_2)u_3\|_{(\mathcal{N} \cap \mathcal{Y})^{s, \pm}} \lesssim (\|u_1\|_{G^{s, \pm'}} + \|u_2\|_{G^{s, \mp'}} + \|u_3\|_{G^{s, \mp}}) \Gamma^2$$

where the implied constant is independent of T .

Proof. As previously, we may assume that $\pm' = \pm = +$. After dyadically decomposing u_j , an application of Theorems 6.1 and 6.3 gives

$$\begin{aligned} &\lambda^{\frac{n+1}{4n}} \|\mathbb{1}_{(-T, T)}(t)P_\lambda[(u_1^\dagger u_2)u_3]\|_{\mathcal{N}_\lambda^+} + \|\mathbb{1}_{(-T, T)}(t)P_\lambda[(u_1^\dagger u_2)u_3]\|_{L_t^{\frac{4n}{3n-1}} L_x^2} \\ &\lesssim \sum_{N_{max} \approx \max\{\lambda, N_{med}\}} (N_{min} N_{med})^{\frac{n-1}{2}} \left(\frac{N_{min}}{N_{med}}\right)^\epsilon (N_{max})^{\frac{n+1}{4n}} \|P_{N_1} u_1\|_{G_{N_1}^+} \|P_{N_2} u_2\|_{G_{N_2}^-} \|P_{N_3} u_3\|_{G_{N_3}^-}. \end{aligned}$$

The required estimate now follows by summing up over λ , and exploiting the $(\frac{N_{min}}{N_{med}})^\epsilon$ factor by using the inequality

$$\sum_{\substack{\lambda_1, \lambda_2 \in 2^{\mathbb{Z}} \\ \lambda_1 \leq \lambda_2}} \left(\frac{\lambda_1}{\lambda_2}\right)^\epsilon a_{\lambda_1} b_{\lambda_2} \lesssim \left(\sum_{\lambda_1} (a_{\lambda_1})^2\right)^{\frac{1}{2}} \left(\sum_{\lambda_2} (b_{\lambda_2})^2\right)^{\frac{1}{2}}.$$

In more detail, as we have now exploited all the structural properties of the product, we may assume that $N_1 \geq N_2 \geq N_3$. We consider separately the cases $N_1 \gg N_2$, and $N_1 \approx N_2$. In the former case, by

summing up in λ we obtain

$$\begin{aligned}
& \left(\sum_{\lambda} \lambda^{2s} \left(\|\mathbb{1}_{(-T,T)}(t) P_{\lambda}[(u_1^{\dagger} u_2) u_3]\|_{\mathcal{N}_{\lambda}^+} + \lambda^{-\frac{n+1}{4n}} \|\mathbb{1}_{(-T,T)}(t) P_{\lambda}[(u_1^{\dagger} u_2) u_3]\|_{L_t^{\frac{4n}{3n-1}} L_x^2} \right)^2 \right)^{\frac{1}{2}} \\
& \lesssim \left(\sum_{\lambda} \left(\sum_{N_1 \approx \lambda, N_2 \geq N_3} (N_2 N_3)^{\frac{n-1}{2}} \left(\frac{N_3}{N_2} \right)^{\epsilon} (N_1)^s \|P_{N_1} u_1\|_{G_{N_1}^+} \|P_{N_2} u_2\|_{G_{N_2}^-} \|P_{N_3} u_3\|_{G_{N_3}^-} \right)^2 \right)^{\frac{1}{2}} \\
& \lesssim \sum_{N_2 \geq N_3} (N_2 N_3)^{\frac{n-1}{2}} \left(\frac{N_3}{N_2} \right)^{\epsilon} \|P_{N_2} u_2\|_{G_{N_2}^-} \|P_{N_3} u_3\|_{G_{N_3}^-} \left(\sum_{N_1} (N_1)^{2s} \|P_{N_1} u_1\|_{G_{N_1}^+}^2 \right)^{\frac{1}{2}} \\
& \lesssim \|u_1\|_{G^{s,+}} \|u_2\|_{G^{\frac{n-1}{2},-}} \|u_3\|_{G^{\frac{n-1}{2},-}}.
\end{aligned}$$

On the other hand, if $N_1 \approx N_2$, then as $s - \frac{n+1}{4n} > 0$, we deduce that

$$\begin{aligned}
& \left(\sum_{\lambda} \lambda^{2s} \left(\|\mathbb{1}_{(-T,T)}(t) P_{\lambda}[(u_1^{\dagger} u_2) u_3]\|_{\mathcal{N}_{\lambda}^+} + \lambda^{-\frac{n+1}{4n}} \|\mathbb{1}_{(-T,T)}(t) P_{\lambda}[(u_1^{\dagger} u_2) u_3]\|_{L_t^{\frac{4n}{3n-1}} L_x^2} \right)^2 \right)^{\frac{1}{2}} \\
& \lesssim \sum_{N_1 \approx N_2 \geq N_3} (N_2 N_3)^{\frac{n-1}{2}} \left(\frac{N_3}{N_2} \right)^{\epsilon} (N_1)^{\frac{n+1}{4n}} \|P_{N_1} u_1\|_{G_{N_1}^+} \|P_{N_2} u_2\|_{G_{N_2}^-} \|P_{N_3} u_3\|_{G_{N_3}^-} \left(\sum_{\lambda \leq N_1} \lambda^{2(s-\frac{n+1}{4})} \right)^{\frac{1}{2}} \\
& \lesssim \sum_{N_1 \approx N_2} (N_1)^s (N_2)^{\frac{n-1}{2}} \|P_{N_1} u_1\|_{G_{N_1}^+} \|P_{N_2} u_2\|_{G_{N_2}^-} \sum_{N_3 \leq N_2} \left(\frac{N_3}{N_2} \right)^{\epsilon} (N_3)^{\frac{n-1}{2}} \|P_{N_3} u_3\|_{G_{N_3}^-} \\
& \lesssim \|u_1\|_{G^{s,+}} \|u_2\|_{G^{\frac{n-1}{2},-}} \|u_3\|_{G^{\frac{n-1}{2},-}}.
\end{aligned}$$

Therefore result follows. \square

7. PROOF OF GLOBAL WELL-POSEDNESS

The proof of global existence and scattering follows from a more or less standard argument from the energy type inequality in Theorem 3.9, together with the crucial trilinear estimate in Corollary 6.4. We begin by considering the smooth case.

Theorem 7.1. *Let $n = 2, 3$, $m = 0$, and $s \geq \frac{n-1}{2}$. Let $\rho \in C_0^{\infty}(\mathbb{R})$. There exists $\epsilon > 0$ such that if $f, g \in C_0^{\infty}(\mathbb{R}^n)$ with*

$$\|f\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s} + \|g\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s} < \epsilon \quad (60)$$

then we have a global solution $(u, v) \in C^{\infty}(\mathbb{R}^{1+n})$ to (5) such that $(u, v)(0) = (f, g)$ and

$$\|u\|_{L_t^{\infty} \dot{H}^s} + \|v\|_{L_t^{\infty} \dot{H}^s} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^s}.$$

Moreover, if $f', g' \in C_0^{\infty}(\mathbb{R}^n)$ also satisfies (60) and (u', v') denotes the corresponding solution to (5) with $(u', v')(0) = (f', g')$, then we have the Lipschitz bound

$$\sup_{T>0} \left(\|\rho(\frac{\cdot}{T})(u - u')\|_{F^{s,+}} + \|\rho(\frac{\cdot}{T})(v - v')\|_{F^{s,-}} \right) \lesssim \|f - f'\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s} + \|g - g'\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s}.$$

Proof. Let $(f, g) \in C_0^{\infty}(\mathbb{R}^n)$ satisfy (60). A standard fixed point argument in¹⁴ $L_t^{\infty} H^N$ with $N > \frac{n}{2}$ shows that there exists $T^* > 0$ and a smooth solution $(u, v) \in C((-T^*, T^*), H^N(\mathbb{R}^n))$ with $(u, v)(0) = (f, g)$. Let $T < T^*$ and define (u_T, v_T) as the solution to

$$\begin{aligned}
(\partial_t + \sigma \cdot \nabla) u_T &= \mathbb{1}_{(-T,T)} (B_1(u, v)v + B_2(u, v)\beta u) \\
(\partial_t - \sigma \cdot \nabla) v_T &= \mathbb{1}_{(-T,T)} (B_3(u, v)u + B_4(u, v)\beta v)
\end{aligned}$$

¹⁴Alternatively we could appeal to work of Pecher [36, 37] in the $n = 2$ case, and Escobedo-Vega [15] in the $n = 3$ case, although these results do not *directly* deal with the system (5).

with $(u_T(0), v_T(0)) = (f, g)$. Note that (u_T, v_T) is the extension of (u, v) from $(-T, T) \times \mathbb{R}^n$ to \mathbb{R}^{1+n} by a linear solution, in particular, we have $(u_T, v_T) = (u, v)$ on $(-T, T) \times \mathbb{R}^n$. Define

$$a_s(T) = \|u_T\|_{G^{s,+}} + \|v_T\|_{G^{s,-}}.$$

The bound

$$\|F\|_{(\mathcal{N} \cap \mathcal{Y})^{s,\pm}} \lesssim \|F\|_{L_t^1 \dot{H}_x^s} + \|F\|_{L_t^{\frac{4n}{3n-1}} \dot{H}_x^{s-\frac{n+1}{4n}}}$$

together with the equation for (u_T, v_T) , implies that for $T, T' \leq T^*$

$$|a_s(T) - a_s(T')| \lesssim_{u,v,T^*} |T - T'|^{\frac{3n-1}{4n}}.$$

In particular, $a_s(T)$ is a continuous function of T . Moreover, an application of Corollary 6.4 gives

$$a_s(T) \leq \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^s} + C(a_{\frac{n-1}{2}}(T))^2 a_s(T). \quad (61)$$

Thus as we clearly have $a_s(0) \leq \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^s}$, a continuity argument shows that provided $\epsilon > 0$ is sufficiently small (independent of T and T^*) we have for every $T < T^*$

$$a_s(T) \leq 2\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^s}. \quad (62)$$

Hence we have the bound

$$\|u\|_{L_t^\infty \dot{H}^s((-T^*, T^*) \times \mathbb{R}^n)} + \|v\|_{L_t^\infty \dot{H}^s((-T^*, T^*) \times \mathbb{R}^n)} \leq \sup_{T < T^*} a_s(T) \leq 2(\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^s}).$$

If we apply this with $s > \frac{n}{2}$, then the classical local existence theory shows that the solution (u, v) exists globally in time, i.e we may take $T^* = \infty$.

It only remains to show the Lipschitz bound. To this end, let $f', g' \in C_0^\infty(\mathbb{R}^n)$ satisfy (60) and let (u', v') denote the corresponding solution. Another application of the cubic estimate in Corollary 6.4 together with the bound (62) shows that for any $T < \infty$

$$\|u_T - u'_T\|_{G^{\frac{n-1}{2},+}} + \|v_T - v'_T\|_{G^{\frac{n-1}{2},-}} \leq \|f - f'\|_{\dot{H}^{\frac{n-1}{2}}} + \|g - g'\|_{\dot{H}^{\frac{n-1}{2}}} + C\epsilon^2 \left(\|u_T - u'_T\|_{G^{\frac{n-1}{2},+}} + \|v_T - v'_T\|_{G^{\frac{n-1}{2},-}} \right).$$

Hence as $\epsilon > 0$ is small, for any $T > 0$ we obtain the Lipschitz bound

$$\|u_T - u'_T\|_{G^{\frac{n-1}{2},+}} + \|v_T - v'_T\|_{G^{\frac{n-1}{2},-}} \leq 2 \left(\|f - f'\|_{\dot{H}^{\frac{n-1}{2}}} + \|g - g'\|_{\dot{H}^{\frac{n-1}{2}}} \right).$$

Similarly, for higher regularities $s > \frac{n-1}{2}$, we can use a similar argument to show that

$$\begin{aligned} & \|u_T - u'_T\|_{G^{s,+}} + \|v_T - v'_T\|_{G^{s,-}} \\ & \lesssim \|f - f'\|_{\dot{H}^s} + \|g - g'\|_{\dot{H}^s} + C\epsilon^2 \left(\|u_T - u'_T\|_{G^{s,+}} + \|v_T - v'_T\|_{G^{s,-}} \right. \\ & \quad \left. + \|u_T - u'_T\|_{G^{\frac{n-1}{2},+}} + \|v_T - v'_T\|_{G^{\frac{n-1}{2},-}} \right) \\ & \lesssim \|f - f'\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s} + \|g - g'\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s} + C\epsilon^2 \left(\|u_T - u'_T\|_{G^{s,+}} + \|v_T - v'_T\|_{G^{s,-}} \right) \end{aligned}$$

and hence

$$\|u_T - u'_T\|_{G^{s,+}} + \|v_T - v'_T\|_{G^{s,-}} \leq 2 \left(\|f - f'\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s} + \|g - g'\|_{\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s} \right). \quad (63)$$

Let $\rho \in C_0^\infty(\mathbb{R})$ and note that provided we choose δ sufficiently large,

$$\rho\left(\frac{\cdot}{T}\right)(u, v) = \rho\left(\frac{\cdot}{T}\right)(u_{\delta T}, v_{\delta T}).$$

Therefore by (ii) in Theorem 3.9 we have

$$\begin{aligned}
& \|\rho(\frac{t}{T})(u - u')\|_{F^{s,+}} + \|\rho(\frac{t}{T})(v - v')\|_{F^{s,-}} \\
&= \|\rho(\frac{t}{T})(u_{\delta T} - u'_{\delta T})\|_{F^{s,+}} + \|\rho(\frac{t}{T})(v_{\delta T} - v'_{\delta T})\|_{F^{s,-}} \\
&\lesssim \|u_{\delta T} - u'_{\delta T}\|_{F^{s,+}} + \|v_{\delta T} - v'_{\delta T}\|_{F^{s,-}} \\
&\lesssim \|u_{\delta T} - u'_{\delta T}\|_{G^{s,+}} + \|v_{\delta T} - v'_{\delta T}\|_{G^{s,-}}.
\end{aligned}$$

Thus the required Lipschitz bound follows from (63) and noting that all constants are independent of $T > 0$. \square

The proof of Theorem 1.5 is now straightforward.

Proof of Theorem 1.5. Let $s \geq \frac{n-1}{2}$ with $f, g \in \dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s$ and

$$\|f\|_{\dot{H}^{\frac{n-1}{2}}} + \|g\|_{\dot{H}^{\frac{n-1}{2}}} < \epsilon$$

where ϵ is the constant in Theorem 7.1. By rescaling, we may assume that (60) holds (if $s = \frac{n-1}{2}$ this is already true, if $s > \frac{n-1}{2}$ then the \dot{H}^s norm is subcritical so we can rescale it to be small without changing the size of the data in $\dot{H}^{\frac{n-1}{2}}$). Choose a sequence $f_j, g_j \in C_0^\infty(\mathbb{R}^n)$ satisfying (60) such that $(f_j, g_j) \rightarrow (f, g)$ in $\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s$ and let (u_j, v_j) denote the corresponding solution given by Theorem 7.1. Let $\rho \in C_0^\infty(\mathbb{R})$ with $\rho = 1$ on $[-1, 1]$. Then as

$$\|\phi\|_{L_t^\infty \dot{H}^b} \leq \sup_{T>0} \|\rho(\frac{t}{T})\phi\|_{L_t^\infty H_x^b} \leq \sup_T \|\rho(\frac{t}{T})\phi\|_{F^{b,\pm}}$$

the Lipschitz bound in Theorem 7.1 shows that (u_j, v_j) is a Cauchy sequence in $L_t^\infty \dot{H}^{\frac{n-1}{2}} \cap L_t^\infty \dot{H}^s$ and hence converges to a solution $(u, v) \in C(\mathbb{R}, \dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s)$. Moreover, for every $T > 0$, $\rho(\frac{t}{T})(u_j, v_j)$ forms a Cauchy sequence in $F^{b,+} \times F^{b,-}$ for $b = \frac{n-1}{2}, s$. Consequently we must have $\rho(\frac{t}{T})(u, v) \in F^{b,+} \times F^{b,-}$ with

$$\sup_{T>0} \left(\|\rho(\frac{t}{T})u\|_{F^{b,-}} + \|\rho(\frac{t}{T})v\|_{F^{b,-}} \right) \lesssim \|f\|_{\dot{H}^b} + \|g\|_{\dot{H}^b}.$$

Therefore, by (iii) in Theorem 3.9, we see that (u, v) scatters to a homogeneous solution in $\dot{H}^{\frac{n-1}{2}} \cap \dot{H}^s$ as required. Thus Theorem 7.1 follows. \square

Remark 7.2. If we have positive mass $m > 0$, then we can prove local existence up to times $T \ll m^{-1}$ essentially by just treating the mass term as an additional perturbation. To see this, note that by rescaling, we may assume that $m = 1$. Then instead of (61) we would have

$$a_s(T) \leq \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^s} + C \left(T (\|u\|_{L_t^\infty \dot{H}^s} + \|v\|_{L_t^\infty \dot{H}^s}) + T^{\frac{3n-1}{4n}} (\|u\|_{L_t^\infty \dot{H}^{s-\frac{n+1}{4n}}} + \|v\|_{L_t^\infty \dot{H}^{s-\frac{n+1}{4n}}}) + \epsilon^2 a_s(T) \right).$$

If we now note that

$$\|\phi\|_{\dot{H}^{s-\frac{n+1}{4n}}} \lesssim \|\phi\|_{L_x^2} + \|\phi\|_{\dot{H}^s}$$

and use the fact that the charge (i.e. the L_x^2 norm) is conserved¹⁵, then provided $T \ll 1$ we obtain

$$a_s(T) \leq 2(\|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^s}) + \|f\|_{L_x^2} + \|g\|_{L_x^2}.$$

We can follow a similar minor modification of the remainder of Theorem 7.1 to deduce an equivalent result with the restriction $T \ll 1$, which after undoing the scaling, corresponds to $T \ll m^{-1}$.

¹⁵Strictly speaking, the charge is not necessarily conserved for the general system as written in (5). However, charge is conserved for the original system (1). Thus for the versions of (5) we are interested in, the charge is certainly conserved.

8. NULL FRAME BOUNDS

The proof of the null frame bounds is based on a transference type argument to reduce to the linear case. For the $L_t^1 L_x^2$ and $\dot{\mathcal{X}}^{-\frac{1}{2},1}$ components of our iteration space, this is not so difficult. On the other hand the NF_λ^+ case is more challenging and requires some theory on how the Dirac equation behaves in null coordinates. In particular we rely on a version of the Duhamel formula for the Dirac equation in null coordinates. The results in this section are based on related arguments in the work of Tataru [47] and Tao [46].

8.1. Preliminary Results on Null Frames. We start with a number of results on the geometry of null frames. These results are more or less implicit in [47, 46], but for the readers convenience, we include the statements and proofs here.

For a set $A \subset \mathbb{R}^{n+1}$ we define $\mathcal{P}r_\omega(A) = A + (1, \omega)\mathbb{R}$ to be the projection along the null direction $(1, \omega)$. Note that the projected sets $\mathcal{P}r_\omega(A)$ depend only on the ξ_ω coordinate. More precisely, since $\mathbb{1}_{\mathcal{P}r_\omega(A)}(\tau, \xi) = \mathbb{1}_{\mathcal{P}r_\omega(A)}(0, \xi - \tau\omega)$ we have

$$(\mathbb{1}_{\mathcal{P}r_\omega(A)})^*(\tau_\omega, \xi_\omega) = \mathbb{1}_{\mathcal{P}r_\omega(A)}(0, \xi_\omega). \quad (64)$$

Moreover, we have the following.

Lemma 8.1 (Geometric Properties of Null Projections).

(i) Let $\alpha, \beta \ll 1$, $\kappa \in \mathcal{C}_\alpha$, $\bar{\kappa} \in \mathcal{C}_\beta$, and¹⁶ $\omega \notin 2\kappa$. Then

$$\{|\tau| = |\xi|\} \cap \mathcal{P}r_\omega[\mathfrak{h}A_{\alpha,\lambda}(\kappa) \cap \mathfrak{h}A_{\beta,\lambda}(\bar{\kappa})] \subset \left[\mathfrak{h}A_\lambda\left(\frac{3}{2}\kappa\right) \cap \mathfrak{h}A_\lambda\left(\frac{3}{2}\bar{\kappa}\right) \right].$$

(ii) Let $\alpha \ll 1$ and $\omega \in \mathbb{S}^{n-1}$. Let $J \subset \mathcal{C}_\alpha$ be a collection of caps with $\omega \notin 2\kappa$ for every $\kappa \in J$. Then the sets $\Omega_\alpha(\kappa) = \mathcal{P}r_\omega[\mathfrak{h}A_{\alpha,\lambda}(\kappa)]$ have finite overlap in the sense that

$$\sum_{\kappa \in J} \mathbb{1}_{\Omega_\alpha(\kappa)}(t, x) \lesssim 1.$$

Proof. (i): Let $(\tau, \xi) \in \{|\tau| = |\xi|\} \cap \mathcal{P}r_\omega[\mathfrak{h}A_{\alpha,\lambda}(\kappa) \cap \mathfrak{h}A_{\beta,\lambda}(\bar{\kappa})]$, in other words there exists $a \in \mathbb{R}$ such that

$$(\tau, \xi) - (a, a\omega) \in \mathfrak{h}A_{\alpha,\lambda}(\kappa) \cap \mathfrak{h}A_{\beta,\lambda}(\bar{\kappa}), \quad |\tau| = |\xi|.$$

Let ξ^\perp denote the component of ξ orthogonal to ω . Then since

$$|\xi^\perp|^2 + (\tau - \xi \cdot \omega)^2 = |\xi|^2 + \tau^2 - 2\tau\xi \cdot \omega = 2\operatorname{sgn}(\tau)|\xi|(\tau - \xi \cdot \omega)$$

together with the fact that $|\xi^\perp| = |(\xi - a\omega)^\perp| \approx \lambda\theta(\omega, \kappa)\theta(\omega, -\bar{\kappa})$ and the estimate

$$\tau - \xi \cdot \omega = \tau - a - \operatorname{sgn}(\tau - a)|\xi - a\omega| + \operatorname{sgn}(\tau - a)|\xi - a\omega| - \omega \cdot (\xi - a\omega) \approx \lambda\theta(\omega, \kappa)^2 \quad (65)$$

we obtain $|\xi| \approx \lambda$.

It remains to show that $\operatorname{sgn}(\tau) \frac{\xi}{|\xi|} \in \frac{3}{2}\kappa \cap \frac{3}{2}\bar{\kappa}$. To this end, by noting that $(\tau - a)^2 - |\xi - a\omega|^2 = -2a(\tau - \xi \cdot \omega)$ we get

$$|a| = \frac{||\tau - a|^2 - |\xi - a\omega|^2|}{2|\tau - \xi \cdot \omega|} \approx \frac{||\tau - a| - |\xi - a\omega||}{\theta(\omega, \kappa)^2}$$

¹⁶An important point here is that we can take *any* $\bar{\kappa} \in \mathcal{C}_\beta$, there is no condition needed on ω with respect to $\bar{\kappa}$.

Consequently, using the assumption $\tau = \text{sgn}(\tau)|\xi|$, we have

$$\begin{aligned} \left| \text{sgn}(\tau) \frac{\xi}{|\xi|} - \text{sgn}(\tau - a\omega) \frac{\xi - a\omega}{|\xi - a\omega|} \right|^2 &= 2 - 2 \frac{\text{sgn}(\tau)\xi}{|\xi|} \cdot \frac{\text{sgn}(\tau - a\omega)(\xi - a\omega)}{|\xi - a\omega|} \\ &= \frac{2}{\tau} (\tau - a - \text{sgn}(\tau - a\omega)|\xi - a\omega|) + \frac{2a}{\tau} \left(1 - \omega \cdot \frac{\xi - a\omega}{|\xi - a\omega|} \right) \\ &\lesssim \frac{|\tau - a| - |\xi - a\omega|}{\lambda} \leq \frac{1}{100} (\min\{\alpha, \beta\})^2 \end{aligned} \quad (66)$$

(provided we choose the close cone constant in the definition of $\mathfrak{h}A_{\alpha, \lambda}(\kappa)$ to be sufficiently small). Let ω' denote the centre of κ . Since $\kappa \in \mathcal{C}_\alpha$, we have $\theta(\text{sgn}(\tau - a)(\xi - a\omega), \omega') \leq \frac{101}{100}\alpha$. Thus from (66) we deduce that

$$\begin{aligned} \left| \text{sgn}(\tau) \frac{\xi}{|\xi|} - \omega' \right| &\leq \left| \text{sgn}(\tau) \frac{\xi}{|\xi|} - \text{sgn}(\tau - a\omega) \frac{\xi - a\omega}{|\xi - a\omega|} \right| + \left| \text{sgn}(\tau - a\omega) \frac{\xi - a\omega}{|\xi - a\omega|} - \omega' \right| \\ &\leq \frac{1}{10} \min\{\alpha, \beta\} + \frac{101}{100}\alpha \leq \frac{6}{5}\alpha. \end{aligned}$$

Therefore, using (13) we obtain

$$\theta(\text{sgn}(\tau)\xi, \omega') \leq \frac{50}{49} \left(\frac{6}{5}\alpha \right) = \frac{60}{49}\alpha < \frac{3}{2}\alpha$$

and so $\text{sgn}(\tau)\xi \in \frac{3}{2}\kappa$. Similarly, since $(\tau, \xi) - a(1, \omega) \in \mathfrak{h}A_\lambda(\bar{\kappa})$, then using $\bar{\omega}$ to denote the centre of $\bar{\kappa}$, we get

$$\begin{aligned} \theta(\text{sgn}(\tau)\xi, \bar{\omega}) &\leq \frac{50}{49} \left| \text{sgn}(\tau) \frac{\xi}{|\xi|} - \text{sgn}(\tau - a\omega) \frac{\xi - a\omega}{|\xi - a\omega|} \right| + \frac{50}{49} \left| \text{sgn}(\tau - a\omega) \frac{\xi - a\omega}{|\xi - a\omega|} - \bar{\omega} \right| \\ &\leq \frac{50}{49} \left(\frac{1}{10} \min\{\alpha, \beta\} + \frac{101}{100}\beta \right) \leq \frac{3}{2}\beta \end{aligned}$$

as required.

(ii): It is enough to show that if $\kappa_1, \kappa_2 \in J$ with $\Omega_\alpha(\kappa_1) \cap \Omega_\alpha(\kappa_2) \neq \emptyset$, then $\theta(\kappa_1, \kappa_2) \lesssim \alpha$. As if this holds, then the result follows by using the bounded overlap of the collection \mathcal{C}_α . So suppose $(\tau, \xi) \in \Omega_\alpha(\kappa_1) \cap \Omega_\alpha(\kappa_2)$. Note that for every $a \in \mathbb{R}$

$$(\tau, \xi) \in \Omega_\alpha(\kappa_1) \cap \Omega_\alpha(\kappa_2) \iff (\tau, \xi) + a(1, \omega) \in \Omega_\alpha(\kappa_1) \cap \Omega_\alpha(\kappa_2)$$

In particular, since $\tau - \omega \cdot \xi \neq 0$ by¹⁷ (65), we may take $a = \frac{\tau^2 - |\xi|^2}{2(\tau - \omega \cdot \xi)}$ and consequently

$$(\tau, \xi) + a(1, \omega) \in \Omega_\alpha(\kappa_1) \cap \Omega_\alpha(\kappa_2) \cap \{|\tau| = |\xi|\}.$$

Therefore, by the first half of the lemma, we must have $\mathfrak{h}A_\lambda(\frac{3}{2}\kappa_1) \cap \mathfrak{h}A_\lambda(\frac{3}{2}\kappa_2) \neq \emptyset$, which by the finite overlap of the collection \mathcal{C}_α implies that $\theta(\kappa_1, \kappa_2) \lesssim \alpha$ as required. \square

The previous geometric lemma implies the following important orthogonality properties.

Corollary 8.2 (Orthogonality in Null frames). *Let $\alpha, \beta \ll 1$ and $\bar{\kappa} \in \mathcal{C}_\beta$.*

¹⁷Note that the derivation of (65) did not make use of the assumption $|\tau| = |\xi|$.

(i) Assume $\text{supp } \tilde{u}_\kappa \subset \mathfrak{A}_{\alpha,\lambda}(\kappa)$ for $\kappa \in \mathcal{C}_\alpha$. Then

$$\left\| \sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ \kappa \cap \bar{\kappa} \neq \emptyset}} u_\kappa \right\|_{[NF^\pm]^*(\bar{\kappa})} \lesssim \left(\sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ \kappa \cap \bar{\kappa} \neq \emptyset}} \|u_\kappa\|_{[NF^\pm]^*(\bar{\kappa})}^2 \right)^{\frac{1}{2}}$$

(ii) Let $F \in NF^\pm(\bar{\kappa})$ with $\text{supp } \widehat{F} \subset \{|\xi| \approx \lambda\}$. Then

$$\left(\sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ \kappa \cap \bar{\kappa} \neq \emptyset}} \|R_{\kappa,\alpha^2\lambda}^\pm \Pi_\pm F\|_{NF^\pm(\bar{\kappa})}^2 + \|R_{\kappa,\alpha^2\lambda}^\mp \Pi_\mp F\|_{NF^\pm(\bar{\kappa})}^2 \right)^{\frac{1}{2}} \lesssim \|F\|_{NF^\pm(\bar{\kappa})}.$$

(iii) Let $\omega \in \mathbb{S}^{n-1}$. Let $J \subset \mathcal{C}_\alpha$ be a collection of caps with $\omega \notin 2\kappa$ and $\theta(\omega, \kappa) \gtrsim \delta$ for every $\kappa \in J$. Then

$$\left(\sum_{\kappa \in J} \|P_{\lambda,\kappa}^{\pm,\alpha} \Pi_\pm F\|_{NF^\pm(\kappa)}^2 + \|P_{\lambda,\kappa}^{\pm,\alpha} \Pi_\mp F\|_{NF^\pm(\kappa)}^2 \right)^{\frac{1}{2}} \lesssim \|\Pi_{\pm\omega} F\|_{L_{t_\omega}^1 L_{x_\omega}^2} + \delta^{-1} \|\Pi_{\mp\omega} F\|_{L_{t_\omega}^1 L_{x_\omega}^2}.$$

Proof. (i): If $\alpha \gtrsim \beta$ the sum only contain $\mathcal{O}(1)$ terms and thus the inequality follows by the triangle inequality. It remains to consider the case $\alpha \ll \beta$. Let $\omega \notin 2\bar{\kappa}$. Then for every $\kappa \in \mathcal{C}_\alpha$ with $\kappa \cap \bar{\kappa} \neq \emptyset$ we have $\omega \notin 2\kappa$. Note that the ξ_ω support of u_κ lies in the set $\Omega_\alpha(\kappa) = \mathcal{P}r_\omega[\mathfrak{A}_{\alpha,\lambda}(\kappa)]$. By Lemma 8.1 these sets are essentially disjoint and thus we deduce that

$$\left\| \sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ \kappa \cap \bar{\kappa} \neq \emptyset}} \widehat{u}^*(t_\omega, \xi_\omega) \right\|_{L_{\xi_\omega}^2}^2 \lesssim \sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ \kappa \cap \bar{\kappa} \neq \emptyset}} \|\widehat{u}^*(t_\omega, \xi_\omega)\|_{L_{\xi_\omega}^2}^2.$$

By taking $L_{t_\omega}^\infty$ norms of both sides, inserting the relevant Π_ω projections, and then taking the sup over $\omega \notin 2\bar{\kappa}$ we obtain (i).

(ii): It is enough to consider the case $\pm = +$. As in the proof of (i), if $\alpha \gtrsim \beta$ then the sum only contains $\mathcal{O}(1)$ terms and so the required inequality follows by (i) in Lemma 3.14. Thus we may assume that $\alpha \ll \beta$. Let $\omega \notin 2\bar{\kappa}$. Note that this implies that $\omega \notin 2\kappa$ for every $\kappa \in \mathcal{C}_\alpha$ with $\kappa \cap \bar{\kappa} \neq \emptyset$. As in Lemma 8.1, we let $\Omega_\alpha(\kappa) = \mathcal{P}r_\omega[\mathfrak{A}_{\alpha,\lambda}(\kappa)]$. Define $\widehat{P_{\Omega(\kappa)}^\omega G} = \mathbb{1}_{\Omega_\alpha(\kappa)}(\xi_\omega) \widehat{G}$, clearly $R_{\kappa,\alpha^2\lambda}^\pm F = P_{\Omega_\alpha(\kappa)}^\omega R_{\kappa,\alpha^2\lambda}^\pm F$. Thus an application of Lemma 3.14 to dispose of the $R_{\kappa,\alpha^2\lambda}^\pm$ multiplier, followed by an application of Plancherel gives

$$\begin{aligned} \sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ \kappa \cap \bar{\kappa} \neq \emptyset}} \|R_{\kappa,\alpha^2\lambda}^\pm \Pi_\pm F\|_{NF^+(\bar{\kappa})}^2 &\lesssim \sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ \kappa \cap \bar{\kappa} \neq \emptyset}} \|P_{\Omega(\kappa)}^\omega F\|_{NF^+(\bar{\kappa})}^2 \\ &\lesssim \sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ \kappa \cap \bar{\kappa} \neq \emptyset}} \|P_{\Omega(\kappa)}^\omega \Pi_\omega F\|_{L_{t_\omega}^1 L_{x_\omega}^2}^2 + \theta(\omega, \bar{\kappa})^{-2} \|P_{\Omega(\kappa)}^\omega \Pi_\mp F\|_{L_{t_\omega}^1 L_{x_\omega}^2}^2. \end{aligned}$$

We now observe that, by the finite overlap of the sets $\Omega_\alpha(\kappa)$ in Lemma 8.1 together with the identity (64), we have

$$\sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ \kappa \cap \bar{\kappa} \neq \emptyset}} \|P_{\Omega(\kappa)}^\omega \Pi_\pm F\|_{L_{t_\omega}^1 L_{x_\omega}^2}^2 \leq \left\| \left(\int_{\mathbb{R}^n} \sum_{\substack{\kappa \in \mathcal{C}_\alpha \\ \kappa \cap \bar{\kappa} \neq \emptyset}} |\mathbb{1}_{\Omega_\alpha(\kappa)}(0, \xi_\omega) \widehat{\Pi_{\pm\omega} F}^*(t_\omega, \xi_\omega)|^2 d\xi_\omega \right)^{\frac{1}{2}} \right\|_{L_{t_\omega}^1}^2 \lesssim \|\Pi_{\pm\omega} F\|_{L_{t_\omega}^1 L_{x_\omega}^2}^2.$$

Applying these inequalities to $NF^+(\bar{\kappa})$ atoms, we obtain (ii).

(iii): We follow a similar argument to (ii). The properties of the collection J imply that the sets $\Omega_\alpha(\kappa)$ finitely overlap for $\kappa \in J$. Hence, after an application of Lemma 3.14 to dispose of the multipliers

$P_{\lambda, \kappa}^{\pm, \alpha}$, we obtain

$$\begin{aligned} \sum_{\kappa \in J} \|P_{\lambda, \kappa}^{\pm, \alpha} \Pi_{\pm} F\|_{NF^{\pm}(\kappa)}^2 + \|P_{\lambda, \kappa}^{\mp, \alpha} \Pi_{\mp} F\|_{NF^{\pm}(\kappa)}^2 &\lesssim \sum_{\kappa \in J} \|P_{\Omega(\kappa)}^{\omega} F\|_{NF^{\pm}(\kappa)}^2 \\ &\lesssim \sum_{\kappa \in J} \|P_{\Omega(\kappa)}^{\omega} \Pi_{\pm\omega} F\|_{L_{t\omega}^1 L_{x\omega}^2}^2 + \theta(\omega, \kappa)^{-2} \|P_{\Omega(\kappa)}^{\omega} \Pi_{\mp\omega} F\|_{L_{t\omega}^1 L_{x\omega}^2}^2 \\ &\lesssim \|\Pi_{\pm\omega} F\|_{L_{t\omega}^1 L_{x\omega}^2}^2 + \delta^{-2} \|\Pi_{\mp\omega} F\|_{L_{t\omega}^1 L_{x\omega}^2}^2 \end{aligned}$$

as required. \square

8.2. The Dirac Equation in Null coordinates. We want to write the equation

$$(\partial_t \pm \sigma \cdot \nabla) u = F \tag{67}$$

in null coordinates (t_ω, x_ω) . A computation shows that

$$\begin{aligned} \sqrt{2} \partial_{t_\omega} \Pi_{\pm\omega} u \pm (\sigma \cdot \nabla_{x_\omega}^\perp) \Pi_{\mp\omega} u &= \Pi_{\pm\omega} F \\ -(\omega \cdot \nabla_{x_\omega}) \Pi_{\mp\omega} u \pm (\sigma \cdot \nabla_{x_\omega}^\perp) \Pi_{\pm\omega} u &= \Pi_{\mp\omega} F. \end{aligned}$$

Rearranging, and assuming that we can divide by $\omega \cdot \nabla_{x_\omega}$ (i.e. we assume that \tilde{u} and \tilde{F} are supported away from the null plane $\omega \cdot \xi_\omega = 0 \Leftrightarrow \tau = \omega \cdot \xi$), we see that u satisfies

$$\begin{aligned} \left(\sqrt{2} \partial_{t_\omega} + \frac{|\nabla_{x_\omega}^\perp|^2}{\omega \cdot \nabla_{x_\omega}} \right) \Pi_{\pm\omega} u &= \Pi_{\pm\omega} F \pm \frac{\sigma \cdot \nabla_{x_\omega}^\perp}{\omega \cdot \nabla_{x_\omega}} \Pi_{\mp\omega} F \\ \Pi_{\mp\omega} u &= \pm \frac{\sigma \cdot \nabla_{x_\omega}^\perp}{\omega \cdot \nabla_{x_\omega}} \Pi_{\pm\omega} u - \frac{1}{\omega \cdot \nabla_{x_\omega}} \Pi_{\mp\omega} F. \end{aligned} \tag{68}$$

The equation (68) is interesting as it shows that, in null coordinates (t_ω, x_ω) , the $\Pi_{\mp\omega} u$ component of u solves what is essentially an *elliptic* equation. This observation is the motivation for building the projections $\Pi_{\pm\omega}$ into the definition of the null frame spaces $PW^\pm(\kappa)$ and $NF^\pm(\kappa)$, as it allows us to isolate the “elliptic” and dispersive components of the evolution.

The equation (68) also shows that we can write the forward fundamental solution in null coordinates as

$$(E_\omega^\pm * F)^*(t_\omega, x_\omega) = \pm \frac{\sigma \cdot \nabla_{x_\omega}}{\sqrt{2} \omega \cdot \nabla_{x_\omega}} \int_{-\infty}^{t_\omega} e^{-(t_\omega - a)} \frac{|\nabla_{x_\omega}^\perp|^2}{\sqrt{2} \omega \cdot \nabla_{x_\omega}} \left(\Pi_{\pm\omega} \pm \frac{\sigma \cdot \nabla_{x_\omega}^\perp}{\omega \cdot \nabla_{x_\omega}} \Pi_{\mp\omega} \right) F^*(a) da - \frac{1}{\omega \cdot \nabla_{x_\omega}} \Pi_{\mp\omega} F.$$

It is easy to check that $u = E_\omega^\pm * F$ gives us a solution to (67), and moreover, that

$$\left(\frac{\sigma \cdot \nabla_{x_\omega}}{\sqrt{2} \omega \cdot \nabla_{x_\omega}} e^{-t_\omega} \frac{|\nabla_{x_\omega}^\perp|^2}{\sqrt{2} \omega \cdot \nabla_{x_\omega}} f \right)^*(x_\omega) = \int_{\mathbb{R}^n} \frac{\sigma \cdot \xi_\omega}{-2\xi_\omega^1} e^{it_\omega \frac{|\xi_\omega^\perp|^2}{2\xi_\omega^1}} \hat{f}^*(\xi_\omega) e^{ix_\omega \cdot \xi_\omega} d\xi_\omega$$

is a solution to the equation $(\partial_t + \sigma \cdot \nabla) u = 0$ (lets assume that f has support away from $\xi_\omega^1 = 0$ for simplicity). The fundamental solution operator E_ω^\pm plays a crucial role in the proof of Theorem 4.3 as it gives a suitable substitute for the missing transference type principle. In other words, using the following lemma, we are able to essentially deduce the required null frame bounds from their homogeneous counterparts.

Lemma 8.3 (Decomposition of $E_\omega^\pm * F$ into free waves, cf. [47, Proposition 3.4]). *Let $\alpha, \beta \ll 1$ and $\kappa \in \mathcal{C}_\alpha$, $\bar{\kappa} \in \mathcal{C}_\beta$. Fix $\omega \notin 2\kappa$. Assume $\text{supp } \tilde{F} \subset \mathfrak{h}A_{\alpha, \lambda}(\kappa) \cap \mathfrak{h}A_{\beta, \lambda}(\bar{\kappa})$. Then we can write*

$$E_\omega^\pm * F = \int_{-\infty}^{t_\omega} \psi_a da + \Pi_{\mp\omega} G$$

where ψ_a satisfies $(\partial_t \pm \sigma \cdot \nabla)\psi_a = 0$ and

$$\text{supp } \widehat{\Pi_{\pm}\psi_a} \subset \mathfrak{h}A_{\lambda}^{\pm}(\tfrac{3}{2}\kappa) \cap \mathfrak{h}A_{\lambda}^{\pm}(\tfrac{3}{2}\bar{\kappa}), \quad \text{supp } \widehat{\Pi_{\mp}\psi_a} \subset \mathfrak{h}A_{\lambda}^{\mp}(\tfrac{3}{2}\kappa) \cap \mathfrak{h}A_{\lambda}^{\mp}(\tfrac{3}{2}\bar{\kappa}),$$

and $G \in L_{t,x}^2$ with $\text{supp } \tilde{G} = \text{supp } \tilde{F}$. Moreover we have the bound

$$\int_{\mathbb{R}} \|\psi_a\|_{L_t^{\infty} L_x^2} da + \|G\|_{\dot{X}_{\pm}^{\frac{1}{2},1}} \lesssim \|\Pi_{\pm\omega} F\|_{L_{t\omega}^1 L_{x\omega}^2} + \theta(\omega, \kappa)^{-1} \|\Pi_{\mp\omega} F\|_{L_{t\omega}^1 L_{x\omega}^2}. \quad (69)$$

Proof. If we let

$$(\psi_a)^*(t_{\omega}, x_{\omega}) = \pm \frac{\sigma \cdot \nabla_{x_{\omega}}}{\sqrt{2\omega \cdot \nabla_{x_{\omega}}}} e^{-(t_{\omega}-a) \frac{|\nabla_{x_{\omega}}^{\perp}|^2}{\sqrt{2\omega \cdot \nabla_{x_{\omega}}}}} \left(\Pi_{\pm\omega} \pm \frac{\sigma \cdot \nabla_{x_{\omega}}^{\perp}}{\omega \cdot \nabla_{x_{\omega}}} \Pi_{\mp\omega} \right) F^*(a)$$

and $G = -\frac{1}{\omega \cdot \nabla_{x_{\omega}}} \Pi_{\mp\omega} F$, then by definition of E_{ω}^{\pm} , we have a decomposition

$$E_{\omega}^{\pm} * F = \int_{-\infty}^{t_{\omega}} \psi_a da + G.$$

A calculation using (68) shows that $(\partial_t \pm \sigma \cdot \nabla)\psi_a = 0$ and thus $\tilde{\psi}_a$ is supported on the light cone. On the other hand, if $(\tau, \xi) \in \text{supp } \tilde{\psi}_a$, then

$$\xi - \tau\omega \in \bigcup_{a \in \mathbb{R}} \text{supp } \hat{\psi}^*(a) \subset \bigcup_{a \in \mathbb{R}} \text{supp } \hat{F}^*(a)$$

and so $(\tau, \omega) \in \mathcal{P}r_{\omega}(\text{supp } \tilde{F})$. Consequently the claim on the support of ψ_a follows from Lemma 8.1. Thus it only remains to prove the bound (69). To this end, note that

$$\begin{aligned} \left[e^{-t_{\omega} \frac{|\nabla_{x_{\omega}}^{\perp}|}{\sqrt{2\omega \cdot \nabla_{x_{\omega}}}}} f^* \right](t, x) &= \int_{\mathbb{R}^n} \hat{f}^*(\xi_{\omega}) e^{-i(t+x \cdot \omega) \frac{|\xi_{\omega}^{\perp}|^2}{2\omega \cdot \xi_{\omega}}} e^{i(x - \frac{1}{2}(t+x \cdot \omega)\omega) \cdot \xi_{\omega}} d\xi_{\omega} \\ &= \int_{\mathbb{R}^n} \hat{f}^*(\xi_{\omega}) e^{-it \frac{|\xi_{\omega}^{\perp}|^2}{2\omega \cdot \xi_{\omega}}} e^{ix \cdot (\xi_{\omega} - \frac{1}{2}\xi_{\omega} \omega)} d\xi_{\omega} \\ &= \int_{\mathbb{R}^n} \left[\hat{f}^*(\xi_{\omega}) e^{-it \frac{|\xi_{\omega}^{\perp}|^2}{2\omega \cdot \xi_{\omega}}} J^{-1}(\xi_{\omega}) \right](y) e^{ix \cdot y} dy \end{aligned}$$

where $dy = J(\xi_{\omega}) d\xi_{\omega}$ and¹⁸ the Jacobian is given by $J(\xi_{\omega}) = \frac{1}{2} \left(\frac{|\xi_{\omega}^{\perp}|}{\omega \cdot \xi_{\omega}} \right)^2$. Thus by an application of Plancherel we get

$$\begin{aligned} \left\| \left[e^{-t_{\omega} \frac{|\nabla_{x_{\omega}}^{\perp}|}{\sqrt{2\omega \cdot \nabla_{x_{\omega}}}}} f^* \right](t, x) \right\|_{L_x^2} &= \left\| \left[\hat{f}^*(\xi_{\omega}) e^{-it \frac{|\xi_{\omega}^{\perp}|^2}{2\omega \cdot \xi_{\omega}}} J^{-1}(\xi_{\omega}) \right](y) \right\|_{L_y^2} \\ &= 2 \left\| \frac{(\omega \cdot \xi_{\omega})}{|\xi_{\omega}|} f^*(\xi_{\omega}) \right\|_{L_{\xi_{\omega}}^2}. \end{aligned}$$

If we now observe that $\text{supp } \hat{F}^*(a) \subset \{ |\xi_{\omega} \cdot \omega| \approx \theta(\omega, \kappa)^2 \lambda, |\xi_{\omega}^{\perp}| \lesssim \theta(\omega, \kappa) \lambda \}$ we obtain

$$\begin{aligned} \|\psi_a\|_{L_t^{\infty} L_x^2} &\approx \left\| \frac{\omega \cdot \xi_{\omega}}{|\xi_{\omega}|} \frac{\sigma \cdot \xi_{\omega}}{\omega \cdot \xi_{\omega}} \left(\Pi_{\pm\omega} \pm \frac{\sigma \cdot \xi_{\omega}^{\perp}}{\omega \cdot \xi_{\omega}} \Pi_{\mp\omega} \right) \hat{F}^*(a, \xi_{\omega}) \right\|_{L_{\xi_{\omega}}^2} \\ &\lesssim \|\Pi_{\pm\omega} F^*(a)\|_{L_{x_{\omega}}^2} + \theta(\omega, \kappa)^{-1} \|\Pi_{\mp\omega} F^*(a)\|_{L_{\xi_{\omega}}^2}. \end{aligned}$$

Integrating over a then controls the ψ_a component. To estimate G , we write $G = \frac{1}{\omega \cdot \nabla_{x_{\omega}}} \Pi_{-\omega} \Pi_{+} F + \frac{1}{\omega \cdot \nabla_{x_{\omega}}} \Pi_{-\omega} \Pi_{-} F$. Note that for $(\tau, \xi) \in \text{supp } \widehat{\Pi_{\pm} F} \subset \mathfrak{h}A_{\lambda, \alpha}^{\pm}(\kappa)$, we have $|\omega \cdot \xi_{\omega}| \approx |\xi_{\omega}^{\perp}| \approx \lambda \theta(\omega, \kappa)^2$ as

¹⁸One way to see this is to note that we are only changing ξ_{ω} in the ω direction, thus if we let $a = \omega \cdot \xi_{\omega}$ and $b = |\xi_{\omega}^{\perp}|$, then we are effectively computing the Jacobian for the change of variables $a' = a - \frac{a^2 + b^2}{2a}$ which is $\frac{1}{2} + \frac{b^2}{2a^2} = \frac{a^2 + b^2}{2a^2} = \frac{1}{2} \left(\frac{|\xi_{\omega}^{\perp}|}{\omega \cdot \xi_{\omega}} \right)^2$.

well as the null form estimate $\Pi_{\mp \frac{\kappa}{|\xi|}} \Pi_{-\omega} \lesssim \theta(\omega, \kappa)$. Therefore, using the bound (37) and the fact that $\alpha \lesssim \theta(\omega, \kappa)$, we have

$$\begin{aligned} \left\| \frac{1}{\omega \cdot \nabla_{x_\omega}} \Pi_{-\omega} \Pi_{\pm} F \right\|_{\dot{\mathcal{X}}_{\mp}^{\frac{1}{2}, 1}} &\lesssim \frac{1}{\lambda \theta(\omega, \kappa)^2} \left(\sum_{d \lesssim \alpha^2 \lambda} d^{\frac{1}{2}} \|C_d^{\pm} \Pi_{\pm} \Pi_{-\omega} \Pi_{\pm} F\|_{L_{t,x}^2} + \sum_{d \lesssim \lambda} d^{\frac{1}{2}} \|C_d^{\mp} \Pi_{\mp} \Pi_{-\omega} \Pi_{\pm} F\|_{L_{t,x}^2} \right) \\ &\lesssim \frac{\lambda^{\frac{1}{2}} \alpha + \lambda^{\frac{1}{2}} \theta(\omega, \kappa)}{\lambda \theta(\omega, \kappa)^2} \|\Pi_{\pm} F\|_{L_x^2} \lesssim \|\Pi_{\omega} F\|_{L_{t\omega}^1 L_{x\omega}^2} + \theta(\omega, \kappa)^{-1} \|\Pi_{-\omega} F\|_{L_{t\omega}^1 L_{x\omega}^2} \end{aligned}$$

as required. \square

Remark 8.4. The bound for the ‘‘elliptic’’ term $G = \frac{1}{\omega \cdot \nabla_{x_\omega}} \Pi_{\mp \omega} F$ can be improved somewhat. For instance, by a similar argument, we could replace $\|G\|_{\dot{\mathcal{X}}_{\pm}^{\frac{1}{2}, 1}}$ with the larger $\lambda^{\frac{1}{2}} \frac{\theta(\omega, \kappa)^2}{\min\{\alpha, \beta\}} \|\Pi_{\mp \omega} G\|_{L_{t,x}^2}$. However, the use of the $\dot{\mathcal{X}}_{\pm}^{\frac{1}{2}, 1}$ norm is technically convenient, and slightly simpler to state.

As an application of the previous result, we obtain control the solution $E_{\omega}^{\pm} * F$ in $L_t^{\infty} L_x^2$.

Corollary 8.5 (*$L_t^{\infty} L_x^2$ Control of Fundamental solution in Null Frames*). *Let $\alpha \ll 1$ and $\kappa \in \mathcal{C}_{\alpha}$. Assume $\text{supp } \tilde{F} \subset \mathfrak{h}A_{\alpha, \lambda}(\kappa)$ and $\omega \notin 2\kappa$. Then*

$$\|E_{\omega}^{\pm} * F\|_{L_t^{\infty} L_x^2} \lesssim \|\Pi_{\pm \omega} F\|_{L_{t\omega}^1 L_{x\omega}^2} + \theta(\omega, \kappa)^{-1} \|\Pi_{\mp \omega} F\|_{L_{t\omega}^1 L_{x\omega}^2}.$$

Proof. By Lemma 8.3, we can write

$$E_{\omega}^{\pm} * F = \int_{-\infty}^{t\omega} \psi_a da + G$$

where ψ_a, G are as in the statement of the Lemma. Thus by an application of Minkowski’s inequality and (24) we have

$$\|E_{\omega}^{\pm} * F\|_{L_t^{\infty} L_x^2} \leq \int_{\mathbb{R}} \|\psi_a\|_{L_t^{\infty} L_x^2} da + \|G\|_{L_t^{\infty} L_x^2} \lesssim \int_{\mathbb{R}} \|\psi_a\|_{L_t^{\infty} L_x^2} da + \|G\|_{\dot{\mathcal{X}}_{\pm}^{\frac{1}{2}, 1}}.$$

Hence result follows by Lemma 8.3. \square

We also have the following crucial energy type inequality.

Corollary 8.6. *Suppose $F \in \mathcal{N}_{\lambda}^{\pm}$ and $(\partial_t \pm \sigma \cdot \nabla)u = F$. Then*

$$\|u\|_{L_t^{\infty} L_x^2} \leq \|u(0)\|_{L_x^2} + C\|F\|_{\mathcal{N}_{\lambda}^{\pm}}$$

for some constant C (independent of u and F).

Proof. It is enough to consider the case $\pm = +$. By writing $u = u - \mathcal{U}_+(t)[u(0)] + \mathcal{U}_+(t)[u(0)]$ and using the homogeneous energy estimate, we reduce to the case $u(0) = 0$. By definition of \mathcal{N}_{λ}^+ , we reduce to considering the case where F is an $L_t^1 L_x^2$ atom, a $\dot{\mathcal{X}}_+^{-\frac{1}{2}, 1}$ atom, or a NF_{λ}^+ atom.

The case $F \in L_t^1 L_x^2$ is immediate by the standard energy inequality. On the other hand, if F is an $\dot{\mathcal{X}}_+^{-\frac{1}{2}, 1}$ atom, then we write

$$\Pi_{\pm} u = (\partial_t \pm i|\nabla|)^{-1} \Pi_{\pm} F - e^{\mp i t |\nabla|} [(\partial_t \pm i|\nabla|)^{-1} \Pi_{\pm} F(0)].$$

Then

$$\|\Pi_{\pm} u\|_{L_t^{\infty} L_x^2} \lesssim \|(\partial_t \pm i|\nabla|)^{-1} \Pi_{\pm} F\|_{L_t^{\infty} L_x^2} \lesssim d^{\frac{1}{2}} \|(\partial_t \pm i|\nabla|)^{-1} \Pi_{\pm} F\|_{L_{t,x}^2} \approx d^{-\frac{1}{2}} \|F\|_{L_{t,x}^2} \lesssim 1$$

as required.

Finally, if F is a NF_λ^+ atom, then there exists a decomposition $F = \sum_{\kappa \in \mathcal{C}_\alpha} F_\kappa$ with $\text{supp } \Pi_\pm F_\kappa \subset A_{\lambda, \alpha}^\pm(\kappa)$. Let $\omega \notin 2\kappa$. Then by Corollary 8.5 we obtain

$$\begin{aligned} \left\| \int_0^t \mathcal{U}_+(t-s) F_\kappa(s) ds \right\|_{L_t^\infty L_x^2} &= \|E_\omega^+ * F_\kappa - \mathcal{U}_+(t)[E_\omega^+ * F_\kappa(0)]\|_{L_t^\infty L_x^2} \\ &\lesssim \|E_\omega^+ * F_\kappa\|_{L_t^\infty L_x^2} \\ &\lesssim \|\Pi_\omega F_\kappa\|_{L_{t_\omega}^1 L_{x_\omega}^2} + \theta(\omega, \kappa)^{-1} \|\Pi_{-\omega} F_\kappa\|_{L_{t_\omega}^1 L_{x_\omega}^2}. \end{aligned}$$

If we apply this estimate to $NF^+(\kappa)$ atoms (and using an application of Lemma 3.14 to retain the support properties) we deduce that

$$\left\| \int_0^t \mathcal{U}_+(t-s) F_\kappa(s) ds \right\|_{L_t^\infty L_x^2} \lesssim \|F_\kappa\|_{NF^+(\kappa)}.$$

Thus, as the F_κ are essentially orthogonal in L_x^2 , we have

$$\|u\|_{L_t^\infty L_x^2} \leq \left(\sum_{\kappa \in \mathcal{C}_\alpha} \left\| \int_0^t \mathcal{U}_+(t-s) F_\kappa(s) ds \right\|_{L_t^\infty L_x^2}^2 \right)^{\frac{1}{2}} \lesssim \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|F_\kappa\|_{NF^+(\kappa)}^2 \right)^{\frac{1}{2}}$$

as required. \square

8.3. Null Frame Bounds - The Homogeneous Case. In this section we prove a number of preliminary bounds that are used in the proof of Theorem 4.3. We start by proving the following preliminary estimate.

Proposition 8.7 ($\dot{\mathcal{X}}_\pm^{\frac{1}{2}, 1}$ controls $PW^-(\kappa)$). *Let $\beta \leq \alpha$, $\kappa \in \mathcal{C}_\alpha$, $\bar{\kappa} \in \mathcal{C}_\beta$, and $2\kappa \cap 2\bar{\kappa} \neq \emptyset$. For every $s \in \mathbb{R}$, let $b_s \in L_{t,x}^\infty$ be a scalar valued function, and let $\psi_s \in L_{t,x}^2$ with the support conditions*

$$\text{supp } \widehat{\Pi_+ \psi_s} \subset A_\lambda^\pm(2\bar{\kappa}), \quad \text{supp } \widehat{\Pi_- \psi_s} \subset A_\lambda^\mp(2\bar{\kappa}).$$

Then

$$\left\| \int_{\mathbb{R}} b_s(t, x) \psi_s(t, x) ds \right\|_{PW^\mp(\kappa)} \lesssim (\beta\lambda)^{\frac{n-1}{2}} \int_{\mathbb{R}} \|b_s\|_{L_{t,x}^\infty} \|\psi_s\|_{\dot{\mathcal{X}}_\pm^{\frac{1}{2}, 1}} ds.$$

Proof. We only prove the case $\pm = +$, the remain case follows by a reflection in x . The assumption $\psi_s \in L_{t,x}^2$ implies that $\psi_s = \sum_d C_d^\pm \psi_s$ and so after an application of Holder is is enough to prove

$$\left\| \int_{\mathbb{R}} b_s(t, x) \Pi_\pm C_d^\pm \psi_s(t, x) ds \right\|_{PW^-(\kappa)} \lesssim (\beta\lambda)^{\frac{n-1}{2}} d^{\frac{1}{2}} \int_{\mathbb{R}} \|b_s\|_{L_{t,x}^\infty} \|\Pi_\pm C_d^\pm \psi_s\|_{L_{t,x}^2} ds.$$

If $d \gtrsim \beta^2 \lambda$, then the support assumptions on ψ_s together with (18) imply that for every $\omega \in 2\bar{\kappa}$, $|\xi_\omega^\pm| \lesssim \lambda\beta$ and $|\xi_\omega^\pm| \lesssim d$. Hence, by taking $\omega \in 2\bar{\kappa} \cap 2\kappa$ and using the null form estimate $|\Pi_{-\omega} \Pi_{\frac{\xi}{|\xi}}| \lesssim \theta(\omega, \xi)$, the definition of the norm $\|\cdot\|_{PW(\kappa)}$ gives

$$\begin{aligned} &\left\| \int_{\mathbb{R}} b_s(t, x) \Pi_\pm C_d^\pm \psi_s(t, x) ds \right\|_{PW^-(\kappa)} \\ &\leq \left\| \int_{\mathbb{R}} b_s(t, x) \Pi_{-\omega} \Pi_\pm C_d^\pm \psi_s(t, x) ds \right\|_{L_{t_\omega}^2 L_{x_\omega}^\infty} + \alpha^{-1} \left\| \int_{\mathbb{R}} b_s(t, x) \Pi_\omega \Pi_\pm C_d^\pm \psi_s(t, x) ds \right\|_{L_{t_\omega}^2 L_{x_\omega}^\infty} \\ &\lesssim \int_{\mathbb{R}} \|b_s\|_{L_{t,x}^\infty} \left(\|\Pi_\pm C_d^\pm \psi_s\|_{L_{t_\omega}^2 L_{x_\omega}^\infty} + \alpha^{-1} \|\Pi_{-\omega} \Pi_\pm C_d^\pm \psi_s\|_{L_{t_\omega}^2 L_{x_\omega}^\infty} \right) ds \\ &\lesssim (\beta\lambda)^{\frac{n-1}{2}} d^{\frac{1}{2}} \int_{\mathbb{R}} \|b_s\|_{L_{t,x}^\infty} \left(1 + \frac{\theta(\omega, \mp \xi)}{\alpha} \right) \|\Pi_\pm \widetilde{C_d^\pm} \psi_s\|_{L_{\tau, \xi}^2} ds \\ &\lesssim (\beta\lambda)^{\frac{n-1}{2}} d^{\frac{1}{2}} \int_{\mathbb{R}} \|b_s\|_{L_{t,x}^\infty} \|\Pi_\pm C_d^\pm \psi_s\|_{L_{t,x}^2} ds \end{aligned}$$

as required. On the other hand, if $d \ll \beta^2 \lambda$, then we decompose $\Pi_{\pm} \psi_s = \sum_{\substack{\kappa' \in \mathcal{C}_{\beta'} \\ \kappa' \cap 2\bar{\kappa} \neq \emptyset}} \Pi_{\pm} R_{\kappa'}^{\pm} \psi_s$ where $\beta' = \sqrt{\frac{\lambda}{d}}$. Now as $d \gtrsim (\beta')^2 \lambda$ we can repeat the previous argument (with β and $\bar{\kappa}$ replaced with β' and κ') together with the orthogonality of the projections $R_{\kappa'}^{\pm}$ in $L_{t,x}^2$ to obtain

$$\begin{aligned} \left\| \int_{\mathbb{R}} b_s(t, x) \Pi_{\pm} C_d^{\pm} \psi_s(t, x) ds \right\|_{PW^-(\kappa)} &\leq \sum_{\substack{\kappa' \in \mathcal{C}_{\beta'} \\ \kappa' \cap 2\bar{\kappa} \neq \emptyset}} \left\| \int_{\mathbb{R}} b_s(t, x) \Pi_{\pm} R_{\kappa'}^{\pm} C_d^{\pm} \psi_s(t, x) ds \right\|_{PW^-(\kappa)} \\ &\lesssim (\beta' \lambda)^{\frac{n-1}{2}} d^{\frac{1}{2}} \sum_{\substack{\kappa' \in \mathcal{C}_{\beta'} \\ \kappa' \cap 2\bar{\kappa} \neq \emptyset}} \int_{\mathbb{R}} \|b_s\|_{L_{t,x}^{\infty}} \|\Pi_{\pm} R_{\kappa'}^{\pm} C_d^{\pm} \psi_s\|_{L_{t,x}^2} ds \\ &\lesssim (\beta' \lambda)^{\frac{n-1}{2}} d^{\frac{1}{2}} \left(\sum_{\substack{\kappa' \in \mathcal{C}_{\beta'} \\ \kappa' \cap 2\bar{\kappa} \neq \emptyset}} \right)^{\frac{1}{2}} \int_{\mathbb{R}} \|b_s\|_{L_{t,x}^{\infty}} \|\Pi_{\pm} C_d^{\pm} \psi_s\|_{L_{t,x}^2} ds \\ &\lesssim (\beta \lambda)^{\frac{n-1}{2}} d^{\frac{1}{2}} \int_{\mathbb{R}} \|b_s\|_{L_{t,x}^{\infty}} \|\Pi_{\pm} C_d^{\pm} \psi_s\|_{L_{t,x}^2} ds \end{aligned}$$

where we used the fact that the number of small caps $\kappa' \in \mathcal{C}_{\beta'}$ required to cover the larger cap $\bar{\kappa} \in \mathcal{C}_{\beta}$ is bounded above by $(\frac{\beta}{\beta'})^{n-1}$. \square

We can now prove the homogeneous case of Theorem 4.3.

Corollary 8.8 (Null frame bounds - homogeneous case).

(i) Let $\alpha \ll 1$ and $f \in L_x^2$ with $\text{supp } \widehat{\Pi_+ f} \subset A_{\lambda}^{\pm}(\frac{3}{2}\kappa)$ and $\text{supp } \widehat{\Pi_- f} \subset A_{\lambda}^{\mp}(\frac{3}{2}\kappa)$. Then

$$\|\mathcal{U}_{\pm}(t)f\|_{[NF^{\pm}]^*(\kappa)} \lesssim \|f\|_{L_x^2}. \quad (70)$$

(ii) Let $\beta \leq \alpha \ll 1$, $\kappa \in \mathcal{C}_{\alpha}$, and $\bar{\kappa} \in \mathcal{C}_{\beta}$ with $2\kappa \cap 2\bar{\kappa} \neq \emptyset$. Let $\rho \in C_0^{\infty}(\mathbb{R})$ and $T > 0$. Assume $f \in L_x^2$ with $\text{supp } \widehat{\Pi_+ f} \subset A_{\lambda}^{\pm}(2\bar{\kappa})$ and $\text{supp } \widehat{\Pi_- f} \subset A_{\lambda}^{\mp}(2\bar{\kappa})$. Then

$$\|\rho(\frac{\cdot}{T})\mathcal{U}_{\pm}(t)f\|_{PW^{\mp}(\kappa)} \lesssim (\beta\lambda)^{\frac{n-1}{2}} \|f\|_{L_x^2} \quad (71)$$

with constant independent of T .

Proof. We start by proving (i). By a reflection in the x variable, we may assume that $\pm = +$. The estimate (11) gives

$$\|e^{\mp it|\nabla|} \Pi_{\pm} f\|_{L_{t\omega}^{\infty} L_{x\omega}^2} = \|e^{\mp i(t-x\cdot\omega)|\nabla|} \Pi_{\pm} f(x)\|_{L_t^{\infty} L_x^2} \approx \|\theta(\omega, \mp\xi)^{-1} \widehat{\Pi_{\pm} f}\|_{L_{\xi}^2} \lesssim \theta(\omega, \kappa)^{-1} \|\Pi_{\pm} f\|_{L_x^2}.$$

Note that for $\omega \notin 2\kappa$ and $\xi \in \text{supp } \widehat{\Pi_{\pm} f}$, we have the null form estimate $|\Pi_{\omega} \Pi_{\pm \frac{\xi}{|\xi}}| \lesssim \theta(\omega, \mp\xi) \approx \theta(\omega, \kappa)$.

Therefore, by decomposing $\mathcal{U}_+(t) = e^{-it|\nabla|} \Pi_+ + e^{it|\nabla|} \Pi_-$, we deduce that

$$\begin{aligned} \|\Pi_{\omega} \mathcal{U}_+(t)f\|_{L_{t\omega}^{\infty} L_{x\omega}^2} + \theta(\omega, \kappa) \|\Pi_{-\omega} \mathcal{U}_+(t)f\|_{L_{t\omega}^{\infty} L_{x\omega}^2} &\lesssim \sum_{\pm} \|e^{\mp it|\nabla|} \Pi_{\omega} \Pi_{\pm} f\|_{L_{t\omega}^{\infty} L_{x\omega}^2} + \theta(\omega, \kappa) \|e^{\mp it|\nabla|} \Pi_{-\omega} \Pi_{\pm} f\|_{L_{t\omega}^{\infty} L_{x\omega}^2} \\ &\lesssim \sum_{\pm} \theta(\omega, \kappa)^{-1} \|\Pi_{\omega} \widehat{\Pi_{\pm} f}\|_{L_{\xi}^2} + \|\Pi_{-\omega} \widehat{\Pi_{\pm} f}\|_{L_{\xi}^2} \\ &\lesssim \|f\|_{L_x^2}. \end{aligned}$$

Taking the sup over $\omega \notin 2\kappa$ then gives (70).

On the other hand, (ii) follows directly from Proposition 8.7 (with $\psi_s(t, x) = \rho(\frac{\cdot}{T})\mathcal{U}_{\pm}(t)f$ and $b_s(t, x) = \mathbb{1}_{[0,1]}(s)$) and Lemma 3.2. \square

Finally, we need to be able to commute the projections $C_{\ll\alpha^2\lambda}^\pm$ and the time cutoff $\rho(\frac{t}{T})$.

Lemma 8.9. *Let $\rho \in C_0^\infty(\mathbb{R})$ and $T > 0$. If $u \in F_\lambda^\pm$ then*

$$\left(\sum_{\kappa \in \mathcal{C}_\alpha} \|R_{\kappa, \alpha^2\lambda}^\pm \Pi_\pm(\rho(\frac{t}{T})u)\|_{PW^-(\kappa)}^2 \right)^{\frac{1}{2}} \lesssim \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|\rho(\frac{t}{T})R_{\kappa, \alpha^2\lambda}^\pm \Pi_\pm u\|_{PW^-(\kappa)}^2 \right)^{\frac{1}{2}} + (\alpha\lambda)^{\frac{n-1}{2}} \|u\|_{F_\lambda^+} \quad (72)$$

with constant independent of T .

Proof. The idea is to decompose $u = C_{\ll\alpha^2\lambda}^\pm u + C_{\gtrsim\alpha^2\lambda}^\pm u$ into a component close to the cone, and a component far from the cone. For the close cone term, after an application of Lemma 3.14 to dispose of the outer multiplier, we deduce that

$$\begin{aligned} \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|R_{\kappa, \alpha^2\lambda}^\pm \Pi_\pm(\rho(\frac{t}{T})C_{\ll\alpha^2\lambda}^\pm u)\|_{PW^-(\kappa)}^2 \right)^{\frac{1}{2}} &= \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|C_{\ll\alpha^2\lambda}^\pm(\rho(\frac{t}{T})\Pi_\pm R_{\kappa, \alpha^2\lambda}^\pm u)\|_{PW^-(\kappa)}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|\rho(\frac{t}{T})R_{\kappa, \alpha^2\lambda}^\pm \Pi_\pm u\|_{PW^-(\kappa)}^2 \right)^{\frac{1}{2}} \end{aligned}$$

which gives the first term on the righthand side of (72). On the other hand, for the far cone term, an application of the $X^{s,b}$ estimate in Proposition 8.7 together with the L_x^2 orthogonality of the $R_{\kappa, \alpha^2\lambda}^\pm$ multipliers gives

$$\begin{aligned} \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|R_{\kappa, \alpha^2\lambda}^\pm \Pi_\pm(\rho(\frac{t}{T})C_{\gtrsim\alpha^2\lambda}^\pm u)\|_{PW^-(\kappa)}^2 \right)^{\frac{1}{2}} &\lesssim (\alpha\lambda)^{\frac{n-1}{2}} \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|R_{\kappa, \alpha^2\lambda}^\pm \Pi_\pm(\rho(\frac{t}{T})C_{\gtrsim\alpha^2\lambda}^\pm u)\|_{\dot{\mathcal{X}}_+^{\frac{1}{2},1}}^2 \right)^{\frac{1}{2}} \\ &\lesssim (\alpha\lambda)^{\frac{n-1}{2}} \|\Pi_\pm C_{\ll\alpha^2\lambda}^\pm(\rho(\frac{t}{T})C_{\gtrsim\alpha^2\lambda}^\pm u)\|_{\dot{\mathcal{X}}_+^{\frac{1}{2},1}} \\ &\lesssim (\alpha\lambda)^{\frac{n-1}{2}} (\alpha^2\lambda)^{\frac{1}{2}} \|\rho(\frac{t}{T})\Pi_\pm C_{\gtrsim\alpha^2\lambda}^\pm u\|_{L_{t,x}^2} \\ &\lesssim (\alpha\lambda)^{\frac{n-1}{2}} \|\rho\|_{L^\infty} (\alpha^2\lambda)^{\frac{1}{2}} \|C_{\gtrsim\alpha^2\lambda}^\pm \Pi_\pm u\|_{L_{t,x}^2} \\ &\lesssim (\alpha\lambda)^{\frac{n-1}{2}} \|u\|_{F_\lambda^+} \end{aligned}$$

where the last inequality followed from (40). □

8.4. Proof of Theorem 4.3.

We now turn to the proof of Theorem 4.3.

Proof of Theorem 4.3. The proof proceeds by essentially reducing the problem to the homogeneous case, at which point we may apply Corollary 8.8. This type of argument is fairly straightforward if we are in $L_t^1 L_x^2$ or $X^{s,b}$, but is more involved in the null frame case as we need to use the Duhamel formula in null coordinates, together with the decomposition into free waves contained in Lemma 8.3.

We begin by noting that after a reflection in the x variable, we may assume $\pm = +$. An application of Corollary 8.2 gives the orthogonality bound¹⁹

$$\left(\sum_{\kappa \in \mathcal{C}_\alpha} \|R_{\kappa, \alpha^2\lambda}^\pm \Pi_\pm F\|_{\mathcal{N}_\lambda^+}^2 \right)^{\frac{1}{2}} \lesssim \|F\|_{\mathcal{N}_\lambda^+}.$$

¹⁹This follows by decomposing F into atoms. For energy and $\dot{\mathcal{X}}_+^{-\frac{1}{2},1}$ atoms, we can just use the orthogonality in L_x^2 of R_κ^\pm . For NF_λ^+ atoms we just use (ii) in Corollary 8.2.

Consequently, after an application of the homogeneous case Corollary 8.8 (together with Lemma 8.9 in the $PW^-(\kappa)$ case) it suffices to prove

$$\|u\|_{[NF^+]*(\kappa)} \lesssim \|(\partial_t + \sigma \cdot \nabla)R_{\kappa, \alpha^2 \lambda}^\pm \Pi_\pm u\|_{\mathcal{N}_\lambda^+}$$

and

$$\|\rho(\frac{t}{T})u\|_{PW^-(\kappa)} \lesssim (\alpha\lambda)^{\frac{n-1}{2}} \|(\partial_t + \sigma \cdot \nabla)R_{\kappa, \alpha^2 \lambda}^\pm \Pi_\pm u\|_{\mathcal{N}_\lambda^+}$$

If we now decompose $(\partial_t + \sigma \cdot \nabla)R_{\kappa, \alpha^2 \lambda}^\pm \Pi_\pm u$ into atoms, we reduce to showing that

$$\|u\|_{[NF^+]*(\kappa)} + (\alpha\lambda)^{-\frac{n-1}{2}} \|\rho(\frac{t}{T})u\|_{PW^-(\kappa)} \lesssim 1 \quad (73)$$

where u is the solution to $(\partial_t + \sigma \cdot \nabla)u = F$ with $u(0) = 0$, and $F = \Pi_\pm F$ is a \mathcal{N}_λ^+ atom. Note that we may assume $\text{supp } \widetilde{\Pi_\pm F} \subset \mathfrak{h}A_{\lambda, \alpha}^\pm(\kappa)$. We now separately consider the three possible cases; F is an energy atom, F is a $\dot{\mathcal{X}}_+^{-\frac{1}{2}, 1}$ atom, or F is a NF_λ^+ atom.

Case 1: F is a energy atom. If we write u using the Duhamel formula we have

$$u(t, x) = \int_0^t \mathcal{U}_+(t-s)F(s)ds.$$

Note that for each fixed s , the ξ support of $\mathcal{U}_+(t-s)F(s)$ is contained in the set $\mathfrak{h}A_\lambda^\pm(\kappa)$. Therefore, as $\|\cdot\|_{[NF^+]*(\kappa)}$ satisfies Minkowski's inequality, we have by Corollary 8.8 (for a fixed cap)

$$\begin{aligned} \|u\|_{[NF^+]*(\kappa)} &\lesssim \int_{\mathbb{R}} \|\mathbb{1}_{[0, T]}(s)\mathcal{U}_+(t-s)F(s)\|_{[NF^+]*(\kappa)} ds \\ &\lesssim \|F\|_{L_t^1 L_x^2} \leq 1. \end{aligned}$$

Similarly, an application of Proposition 8.7 followed by Lemma 3.2 gives

$$\begin{aligned} \|\rho(\frac{t}{T})u\|_{PW^-(\kappa)} &= \left\| \int_{\mathbb{R}} \mathbb{1}_{[0, T]}(s) \rho(\frac{t}{T})\mathcal{U}_+(t-s)F(s)ds \right\|_{PW^-(\kappa)} \\ &\lesssim (\alpha\lambda)^{\frac{n-1}{2}} \int_{\mathbb{R}} \|\rho(\frac{t}{T})\mathcal{U}_+(t-s)F(s)\|_{\dot{\mathcal{X}}_+^{\frac{1}{2}, 1}} ds \\ &\lesssim (\alpha\lambda)^{\frac{n-1}{2}} \|F\|_{L_t^1 L_x^2} \leq 1 \end{aligned}$$

Therefore (73) follows in the case where F is a $L_t^1 L_x^2$ atom.

Case 2: F is a $\dot{\mathcal{X}}_+^{-\frac{1}{2}, 1}$ atom. Assume F is a $\dot{\mathcal{X}}_+^{-\frac{1}{2}, 1}$ atom, thus

$$\text{supp } \widetilde{F} = \text{supp } \widetilde{\Pi_\pm F} \subset \{|\tau \pm |\xi|| \approx d\} \cap \mathfrak{h}A_{\lambda, \alpha}^\pm(\kappa)$$

and $\|F\|_{L_{t,x}^2} \leq d^{\frac{1}{2}}$. Note that an application of Lemma 3.1 together with Corollary 8.8 shows that, provided $v \in L_{t,x}^2$ with $\text{supp } \widetilde{\Pi_\pm v} \subset \mathfrak{h}A_\lambda^\pm(\frac{3}{2}\kappa)$, we have

$$\|v\|_{[NF^+]*(\kappa)} \lesssim \|v\|_{\dot{\mathcal{X}}_+^{\frac{1}{2}, 1}}. \quad (74)$$

Therefore, by writing

$$u = \Pi_\pm u = (\partial_t \pm i|\nabla|)^{-1} \Pi_\pm F - \mathcal{U}_+(t)[(\partial_t \pm i|\nabla|)^{-1} \Pi_\pm F(0)]$$

we have by Corollary 8.8 and (74)

$$\begin{aligned} \|u\|_{[NF^+]*(\kappa)} &\leq \|(\partial_t \pm i|\nabla|)^{-1}\Pi_{\pm}F\|_{[NF^+]*(\kappa)} + \|\mathcal{U}_+(t)[(\partial_t \pm i|\nabla|)^{-1}\Pi_{\pm}F(0)]\|_{[NF^+]*(\kappa)} \\ &\lesssim \|(\partial_t \pm i|\nabla|)^{-1}\Pi_{\pm}F\|_{\dot{X}_+^{\frac{1}{2},1}} + \|(\partial_t \pm i|\nabla|)^{-1}\Pi_{\pm}F\|_{L_t^{\infty}L_x^2} \\ &\lesssim d^{-\frac{1}{2}}\|F\|_{L_{t,x}^2} \leq 1 \end{aligned}$$

as required. The $PW^-(\kappa)$ estimate is similar, we just replace the estimate (74) with an application of Proposition 8.7 (where $b_s(t, x) = \mathbb{1}_{[0,1]}(s)$, $\psi_s(t, x) = (\partial_t \pm i|\nabla|)^{-1}\Pi_{\pm}F$, and $\kappa = \bar{\kappa}$).

Case 3: F is a NF_{λ}^+ atom. By definition, we have a decomposition $F = \sum_{\bar{\kappa} \in \mathcal{C}_{\beta}} F_{\bar{\kappa}}$ where we may assume $F_{\bar{\kappa}} = \Pi_{\pm}F_{\kappa}$,

$$\text{supp } \widetilde{\Pi_{\pm}F_{\bar{\kappa}}} \subset \natural A_{\lambda,\alpha}^{\pm}(\kappa) \cap A_{\lambda,\beta}^{\pm}(\bar{\kappa})$$

and

$$\left(\sum_{\bar{\kappa}} \|F_{\bar{\kappa}}\|_{NF^+(\bar{\kappa})}^2 \right)^{\frac{1}{2}} \leq 1.$$

Define $u_{\bar{\kappa}}$ as the solution to $(\partial_t + \sigma \cdot \nabla)u_{\bar{\kappa}} = F_{\bar{\kappa}}$ with $u_{\bar{\kappa}}(0) = 0$. Assume for the moment that we have the cap localised estimates

$$\|u_{\bar{\kappa}}\|_{[NF^+]*(\kappa)} \lesssim \|F_{\bar{\kappa}}\|_{NF^+(\bar{\kappa})} \quad (75)$$

and

$$\|\rho(\frac{t}{T})u_{\bar{\kappa}}\|_{PW^-(\kappa)} \lesssim (\min\{\alpha, \beta\}\lambda)^{\frac{n-1}{2}} \|F_{\bar{\kappa}}\|_{NF^+(\bar{\kappa})}. \quad (76)$$

Then, writing

$$u = \sum_{\bar{\kappa} \in \mathcal{C}_{\beta}} u_{\bar{\kappa}} = \sum_{\substack{\bar{\kappa} \in \mathcal{C}_{\beta} \\ \natural \kappa \cap \bar{\kappa} \neq \emptyset}} u_{\bar{\kappa}}$$

and using the orthogonality given by (i) in Corollary 8.2, together with (75) we deduce that

$$\|u\|_{[NF^+]*(\kappa)} \lesssim \left(\sum_{\bar{\kappa} \in \mathcal{C}_{\beta}} \|u_{\bar{\kappa}}\|_{[NF^+]*(\kappa)}^2 \right)^{\frac{1}{2}} \lesssim \left(\sum_{\bar{\kappa} \in \mathcal{C}_{\beta}} \|F_{\bar{\kappa}}\|_{NF^+(\bar{\kappa})}^2 \right)^{\frac{1}{2}} \leq 1$$

as required. For the $PW^-(\kappa)$ estimate, the argument is slightly different as we don't have any orthogonality in the $\bar{\kappa}$ sum due to the fact that the PW^- norm is built up of $L_{x\omega}^{\infty}$ terms. This is not a problem in the case $\alpha \lesssim \beta$ as the sum only contains $\mathcal{O}(1)$ terms. On the other hand, if $\alpha \gg \beta$, then the estimate (76) has a much better constant than what is needed, since we want to end up with $(\alpha\lambda)^{\frac{n-1}{2}}$ and have $(\beta\lambda)^{\frac{n-1}{2}}$. Thus, in place of any orthogonality argument, we use the triangle inequality to reduce to a single cap $\bar{\kappa}$, followed by Holder to regain the square sum over the caps $\bar{\kappa}$. In more detail, from the cap

localised estimate (76), and an application of Holder in the $\bar{\kappa}$ sum, we obtain

$$\begin{aligned}
\|\rho(\frac{t}{T})u\|_{PW^-(\kappa)} &\lesssim \sum_{\substack{\bar{\kappa} \in \mathcal{C}_\beta \\ \natural_{\kappa \cap \bar{\kappa}} \neq \emptyset}} \|\rho(\frac{t}{T})u_{\bar{\kappa}}\|_{PW^-(\kappa)} \\
&\lesssim (\min\{\alpha, \beta\}\lambda)^{\frac{n-1}{2}} \sum_{\substack{\bar{\kappa} \in \mathcal{C}_\beta \\ \natural_{\kappa \cap \bar{\kappa}} \neq \emptyset}} \|F_{\bar{\kappa}}\|_{NF^+(\bar{\kappa})} \\
&\lesssim (\min\{\alpha, \beta\}\lambda)^{\frac{n-1}{2}} \left(\frac{\alpha}{\min\{\alpha, \beta\}}\right)^{\frac{n-1}{2}} \left(\sum_{\bar{\kappa} \in \mathcal{C}_\beta} \|F_{\bar{\kappa}}\|_{NF^+(\bar{\kappa})}^2\right)^{\frac{1}{2}} \\
&\lesssim (\alpha\lambda)^{\frac{n-1}{2}}
\end{aligned}$$

where we used the fact that, for a fixed $\kappa \in \mathcal{C}_\alpha$, $\#\{\bar{\kappa} \in \mathcal{C}_\beta \mid \natural_{\kappa \cap \bar{\kappa}} \neq \emptyset\} \lesssim \left(\frac{\alpha}{\min\{\alpha, \beta\}}\right)^{n-1}$.

It remains to proof the cap localised estimates (75) and (76). The atomic definition of $NF^+(\bar{\kappa})$, shows that it is enough to consider the case where $F_{\bar{\kappa}}$ is an atom, in other words there exists $\omega \notin 2\bar{\kappa}$ such that

$$\|\Pi_\omega F_{\bar{\kappa}}\|_{L^1_{t_\omega} L^2_{x_\omega}} + \theta(\omega, \bar{\kappa})^{-1} \|\Pi_{-\omega} F_{\bar{\kappa}}\|_{L^1_{t_\omega} L^2_{x_\omega}} \leq 1$$

and (by Lemma 3.14) we may assume that $F_{\bar{\kappa}} = \Pi_\pm F_{\bar{\kappa}}$ and $\text{supp } \widetilde{\Pi_\pm F_{\bar{\kappa}}} \subset \natural_{\lambda, \alpha}^\pm(\kappa) \cap \natural_{\lambda, \beta}^\pm(\bar{\kappa})$. As $u_{\bar{\kappa}}$ satisfies $(\partial_t + \sigma \cdot \nabla)u_{\bar{\kappa}} = F_{\bar{\kappa}}$ with $u_{\bar{\kappa}}(0) = 0$, we have

$$u_{\bar{\kappa}} = \Pi_\pm E_\omega^+ * F_{\bar{\kappa}} - \Pi_\pm \mathcal{U}_+(t)[E_\omega^+ * F_{\bar{\kappa}}(0)].$$

The homogeneous term can be controlled by²⁰ Corollary 8.8 followed by Corollary 8.5. For the $E_\omega^+ * F_{\bar{\kappa}}$ term, we use an application of Lemma 8.3 to write

$$E_\omega^+ * F_{\bar{\kappa}} = \int_{-\infty}^{t_\omega} \psi_a da + \Pi_{-\omega} G$$

where ψ_a is a homogeneous solution with $\text{supp } \widetilde{\Pi_\pm \psi_a} \subset \natural_{\lambda}^\pm(\frac{3}{2}\kappa) \cap \natural_{\lambda}^\pm(\frac{3}{2}\bar{\kappa})$, $\text{supp } \tilde{G} = \text{supp } \tilde{F}_{\bar{\kappa}}$, and we have the bound

$$\begin{aligned}
&\int_{\mathbb{R}} \|\psi_a\|_{L_t^\infty L_x^2} da + \|G\|_{\dot{\chi}_+^{\frac{1}{2}, 1}} \\
&\lesssim \|\Pi_\omega F_{\bar{\kappa}}\|_{L^1_{t_\omega} L^2_{x_\omega}} + \theta(\omega, \kappa)^{-1} \|\Pi_{-\omega} F_{\bar{\kappa}}\|_{L^1_{t_\omega} L^2_{x_\omega}} \leq 1.
\end{aligned} \tag{77}$$

The integral term is easy to control via Proposition 8.7 and Corollary 8.8. For instance, using (77) and Corollary 8.8 we have

$$\left\| \int_{-\infty}^{t_\omega} \psi_a da \right\|_{[NF^+(\kappa)]} \leq \int_{\mathbb{R}} \|\psi_a\|_{[NF^+(\kappa)]} da \lesssim \int_{\mathbb{R}} \|\psi_a\|_{L_t^\infty L_x^2} da \lesssim 1.$$

Similarly, using Proposition 8.7 and Lemma 3.2, gives

$$\begin{aligned}
\left\| \rho(\frac{t}{T}) \int_{-\infty}^{t_\omega} \psi_a da \right\|_{PW^-(\kappa)} &\lesssim (\min\{\alpha, \beta\}\lambda)^{\frac{n-1}{2}} \int_{\mathbb{R}} \|\rho(\frac{t}{T})\psi_a\|_{\dot{\chi}_+^{\frac{1}{2}, 1}} da \\
&\lesssim (\min\{\alpha, \beta\}\lambda)^{\frac{n-1}{2}} \int_{\mathbb{R}} \|\psi_a\|_{L_t^\infty L_x^2} da \lesssim (\min\{\alpha, \beta\}\lambda)^{\frac{n-1}{2}}.
\end{aligned}$$

Finally, the estimate for the G term simply follows from (74) (in the $[NF^+(\kappa)]$ case) and Proposition 8.7 (in the $PW^-(\kappa)$ case). This completes the proof of the NF_λ^+ case, and hence Theorem 4.3 follows. \square

²⁰Note that $\text{supp } [\widetilde{E_\omega^+ * F_{\bar{\kappa}}}] = \text{supp } \tilde{F}_{\bar{\kappa}} \subset \natural_{\lambda, \alpha}^\pm(\kappa) \cap \natural_{\lambda, \beta}^\pm(\bar{\kappa})$.

9. STRICHARTZ TYPE ESTIMATES

In this section our aim is to prove Theorem 4.1, i.e. we want to show that the norm F_λ^\pm controls the Strichartz norms $L_t^q L_x^r$. The observation that it is possible to control the Strichartz norms by the null frame type norms was first observed by Sterbenz-Tataru in [43, Lemma 5.8] in work related to the wave maps equation. The proof use a version of the $X^{s,b}$ spaces, with the derivative in the “time” direction, replaced with spaces of bounded variation. Spaces of this type have been used in the work of Koch-Tataru [26], and Hadac-Herr-Koch [19]. The crucial point is an atomic decomposition contained in [26, Lemma 6.4]. The argument presented below is based heavily on the arguments used by Sterbenz-Tataru in [43], although it has been slightly simplified, compressed, and adapted to our context.

9.1. The Spaces V^p . We define the p -variation of a function $u : \mathbb{R} \rightarrow L_x^2$ as

$$|u|_{V^p}^p = \sup_{(t_k) \in \mathcal{Z}} \sum_{k \in \mathbb{Z}} \|u(t_{k+1}) - u(t_k)\|_{L_x^2}^p$$

where $\mathcal{Z} = \{(t_k)_{k \in \mathbb{Z}} \mid t_k \leq t_{k+1}\}$ denotes the set of all increasing sequences on \mathbb{R} . If $|u|_{V^p} < \infty$, then u has at most countable discontinuities, and its left and right limits exist everywhere. In particular $\lim_{t \rightarrow \pm\infty} u(t)$ exists in L_x^2 . These properties are all classical results, but for completeness we sketch the proof here.

Lemma 9.1. *Let $0 < p < \infty$ and $u : \mathbb{R} \rightarrow L_x^2(\mathbb{R}^n)$ with $|u|_{V^p} < \infty$. Then u has left and right limits everywhere, and in particular $u(t)$ converges to some function $f_{\pm\infty} \in L_x^2(\mathbb{R}^n)$ as $t \rightarrow \pm\infty$.*

Proof. Let

$$\rho(t) = \sup \left\{ \sum_{k=1}^{N-1} \|u(t_{k+1}) - u(t_k)\|_{L_x^2}^p \mid -\infty < t_1 < t_2 < \dots < t_N = t \right\}.$$

We claim that ρ is increasing, and for $s < t$ we have the inequality

$$\|u(t) - u(s)\|_{L_x^2}^p \leq \rho(t) - \rho(s). \quad (78)$$

This follows by observing that we have a sequence $-\infty < s_1 < \dots < s_N = s$ such that

$$\rho(s) \leq \epsilon + \sum_{k=1}^{N-1} \|u(s_{k+1}) - u(s_k)\|_{L_x^2}^p$$

and consequently

$$\begin{aligned} \|u(t) - u(s)\|_{L_x^2}^p &\leq \sum_{k=1}^{N-1} \|u(s_{k+1}) - u(s_k)\|_{L_x^2}^p + \|u(t) - u(s)\|_{L_x^2}^p - \left(\sum_{k=1}^{N-1} \|u(s_{k+1}) - u(s_k)\|_{L_x^2}^p \right) \\ &\leq \rho(t) - (\rho(s) - \epsilon) = \rho(t) - \rho(s) + \epsilon \end{aligned}$$

since this is true for every $\epsilon > 0$, we obtain (78) and consequently ρ must be increasing.

Now as ρ is increasing and bounded (we clearly have $0 \leq \rho \leq |u|_{V^p}^p$), its left and right limits must exist everywhere. Hence, given any sequence $t_k \rightarrow t$ from below, $\rho(t_k)$ forms a Cauchy sequence in \mathbb{R} , which implies by (78), that $u(t_k)$ also forms a Cauchy sequence. Thus $u(t_k)$ must converge in L^2 , and consequently, must have limits from the left and right everywhere. \square

We now define V^p to be the set of all right continuous functions from \mathbb{R} into L_x^2 with norm

$$\|u\|_{V^p}^p = \|u\|_{L_t^\infty L_x^2}^p + |u|_{V^p}^p,$$

the L_t^∞ term is needed to ensure that $\|\cdot\|_{V^p}$ is a norm. The key property of the V^p spaces that we require is the following.

Lemma 9.2 (Lemma 6.4 in [26]). *Assume $u \in V^p$. Then we have a decomposition $u = \sum_{j=1}^\infty v_j$, where the sum converges in $L_t^\infty L_x^2$, and moreover*

(i) *For each j , we have a partition \mathcal{I}_j of \mathbb{R} , into intervals*

$$I_0 = (-\infty, t_0^{(j)}), \quad I_1 = [t_0^{(j)}, t_1^{(j)}), \quad \dots, \quad I_N = [t_{N-1}^{(j)}, \infty),$$

and we can write

$$v_j(t) = \sum_{I_k \in \mathcal{I}_j} \mathbb{1}_{I_k}(t) f_k^{(j)}$$

for functions $f_k^{(j)} \in L_x^2$.

(ii) *We have the bounds*

$$\#\mathcal{I}_j \lesssim 2^{pj}, \quad \sup_k \|f_k^{(j)}\|_{L_x^2} \lesssim 2^{-j} \|u\|_{V^p}.$$

Proof. The proof follows from minor modifications of the argument in Lemma 6.4 in [26]. \square

Recall that $\mathcal{U}_\pm(t)$ denotes the forward solution operator for $(\partial_t \pm \sigma \cdot \nabla)u = 0$ with data at $t = 0$. With this notation in hand, the previous lemma then has the following important corollary.

Corollary 9.3. *Let $2 \leq q, r \leq \infty$ with $q > 2$ and $\frac{1}{q} + \frac{n-1}{2r} \leq \frac{n-1}{4}$. Assume $\mathcal{U}_\pm(-t)u \in V^2$ with $\text{supp } \hat{u} \subset \{|\xi| \approx \lambda\}$. Let \mathcal{M} denote a spatial Fourier multiplier with matrix valued symbol $m(\xi)$ such that $|m(\xi)| \lesssim \delta$ for all $\xi \in \text{supp } \hat{u}$. Then*

$$\|\mathcal{M}u\|_{L_t^q L_x^r} \lesssim \delta \lambda^{n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}} \|\mathcal{U}_\pm(-t)u\|_{V^2}.$$

Proof. Since $\mathcal{U}_\pm(-t)u \in V^2$, an application of Lemma 9.2 gives a decomposition

$$u = \mathcal{U}_\pm(t) \mathcal{U}_\pm(-t)u = \sum_j \mathcal{U}_\pm(t) v_j$$

with v_j satisfying the properties in (i) and (ii) in Lemma 9.2. We may assume that $\text{supp } \hat{v}_j = \text{supp } \hat{u}$, and hence the same holds for the L_x^2 functions $f_k^{(j)}$ making up the sum in v^j . Then recalling that

$$\mathcal{U}_\pm(t) = e^{\pm it|\nabla|} \Pi_\pm + e^{\mp it|\nabla|} \Pi_\mp$$

we obtain from Lemma 9.2

$$\begin{aligned} \|\mathcal{M}u\|_{L_t^q L_x^r} &\leq \sum_j \|\mathcal{M} \mathcal{U}_\pm(t) v_j\|_{L_t^q L_x^r} \\ &\leq \sum_j \left(\sum_{I_k \in \mathcal{I}_j} \|e^{\pm it|\nabla|} (\mathcal{M} \Pi_\pm f_k^{(j)})\|_{L_t^q L_x^r(I_k \times \mathbb{R}^n)}^q + \|e^{\mp it|\nabla|} (\mathcal{M} \Pi_\mp f_k^{(j)})\|_{L_t^q L_x^r(I_k \times \mathbb{R}^n)}^q \right)^{\frac{1}{q}} \\ &\lesssim \lambda^{n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}} \sum_j \left(\sum_{I_k \in \mathcal{I}_j} (\|\mathcal{M} \Pi_\pm f_k^{(j)}\|_{L_x^2} + \|\mathcal{M} \Pi_\mp f_k^{(j)}\|_{L_x^2})^q \right)^{\frac{1}{q}} \\ &\leq \delta \lambda^{n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}} \sum_j \left(\sum_{I_k \in \mathcal{I}_j} \|f_k^{(j)}\|_{L_x^2}^q \right)^{\frac{1}{q}}. \end{aligned}$$

Now using the properties $\sup_k \|f_k^{(j)}\|_{L_x^2} \lesssim 2^{-j} \|\mathcal{U}_\pm(-t)u\|_{V^2}$ and $\#\mathcal{I}_j \lesssim 2^{2j}$ we obtain

$$\sum_j \left(\sum_{I_k \in \mathcal{I}_j} \|f_k^{(j)}\|_{L_x^2}^q \right)^{\frac{1}{q}} \lesssim \|\mathcal{U}_\pm(-t)u\|_{V^2} \sum_j 2^{-j} (2^{2j})^{\frac{1}{q}} \lesssim \|\mathcal{U}_\pm(-t)u\|_{V^2}$$

where we needed $q > 2$ to ensure that the sum converges. \square

Remark 9.4. If we restrict the support of u further to a ball of radius μ in the annulus $\{|\xi| \approx \lambda\}$, i.e. assume that $\hat{u} \subset \{|\xi - \xi^*| \leq \mu\}$ for some $\mu \leq \lambda$ and $|\xi^*| \approx \lambda$, then the refined Strichartz estimate of Klainerman-Tataru [25] implies that

$$\|\mathcal{M}u\|_{L_t^q L_x^r} \lesssim \delta \left(\frac{\mu}{\lambda} \right)^{n(\frac{1}{2}-\frac{1}{r})-\frac{2}{q}} \lambda^{n(\frac{1}{2}-\frac{1}{r})-\frac{1}{q}} \|\mathcal{U}_\pm(-t)u\|_{V^2}.$$

We have no need for this additional refinement here, but it may prove useful elsewhere.

The final result we need for the V^2 spaces is the crucial fact that our iteration norm F_λ^\pm controls V^2 , this theorem (together with the previous corollary) is the key reason why the V^2 norms are so useful.

Theorem 9.5. *Let $u \in F_\lambda^\pm$. Then we have*

$$\|\mathcal{U}_\pm(-t)u\|_{V^2} \lesssim \|u\|_{F_\lambda^\pm}.$$

Proof. As usual, by a reflection, we may assume that $\pm = +$. Let $F = (\partial_t + \sigma \cdot \nabla)u$, by the definition of F_λ^\pm , it is enough to show that

$$\sum_{j \in \mathbb{Z}} \|\mathcal{U}_+(-t_{j+1})u(t_{j+1}) - \mathcal{U}_+(-t_j)u(t_j)\|_{L_x^2}^2 \lesssim \|F\|_{N_\lambda^+}^2$$

with the constant independent of the sequence $(t_j)_{j \in \mathbb{Z}} \in \mathcal{Z}$. If we observe that

$$\begin{aligned} \mathcal{U}_+(-t_{j+1})u(t_{j+1}) - \mathcal{U}_+(-t_j)u(t_j) &= \int_{t_j}^{t_{j+1}} \mathcal{U}_+(-s)F(s)ds \\ &= \int_{t_j}^{t_{j+1}} \mathcal{U}_+(-s)\mathbb{1}_{I_j}F(s)ds \\ &= \int_0^{t_{j+1}} \mathcal{U}_+(-s)\mathbb{1}_{I_j}F(s)ds - \int_0^{t_j} \mathcal{U}_+(-s)\mathbb{1}_{I_j}F(s)ds \end{aligned}$$

where $I_j = [t_j, t_{j+1})$, then we have

$$\begin{aligned} \sum_j \|\mathcal{U}_+(-t_j)u(t_j) - \mathcal{U}_+(-t_{j+1})u(t_{j+1})\|_{L_x^2}^2 &= \sum_j \left\| \int_0^{t_{j+1}} \mathcal{U}_+(-s)\mathbb{1}_{I_j}F(s)ds - \int_0^{t_j} \mathcal{U}_+(-s)\mathbb{1}_{I_j}F(s)ds \right\|_{L_x^2}^2 \\ &\leq 2 \sum_j \left\| \int_0^{t_j} \mathcal{U}_+(-s)\mathbb{1}_{I_j}F(s)ds \right\|_{L_t^\infty L_x^2}^2 \end{aligned}$$

An application of Corollary 8.6 shows that

$$\left\| \int_0^t \mathcal{U}_+(-s)\mathbb{1}_{I_j}F(s)ds \right\|_{L_t^\infty L_x^2} = \left\| \int_0^t \mathcal{U}_+(t-s)\mathbb{1}_{I_j}F(s)ds \right\|_{L_t^\infty L_x^2} \lesssim \|\mathbb{1}_{I_j}F\|_{N_\lambda^+}$$

and so we reduce to proving the inequality

$$\left(\sum_j \|\mathbb{1}_{I_j}F\|_{N_\lambda^+}^2 \right)^{\frac{1}{2}} \lesssim \|F\|_{N_\lambda^+}. \quad (79)$$

By the atomic definition of N_λ^+ , it suffices to consider separately the cases where F is a $L_t^1 L_x^2$ atom, F is a $\dot{X}_+^{-\frac{1}{2},1}$ atom, and F is a NF_λ^+ atom.

Case 1: F is a $L_t^1 L_x^2$ atom. This is the easiest case as we in fact have the stronger estimate

$$\sum_j \|\mathbb{1}_{I_j} F\|_{\mathcal{N}_\lambda^+} \leq \sum_j \|\mathbb{1}_{I_j} F\|_{L_t^1 L_x^2} \leq \|F\|_{L_t^1 L_x^2}.$$

Case 2: F is a $\dot{\mathcal{X}}_+^{-\frac{1}{2},1}$ atom. By definition, there exists $d \in 2^{\mathbb{Z}}$ such that

$$\text{supp } \widetilde{\Pi_\pm F} \subset \{|\xi| \approx \lambda, |\tau \pm |\xi|| \approx d\}$$

and $\|F\| \leq d^{-\frac{1}{2}}$. We start by decomposing $\mathbb{1}_{I_j} F$ into close cone and far cone terms, and estimate

$$\begin{aligned} \|\mathbb{1}_{I_j} F\|_{\mathcal{N}_\lambda^+} &\leq \|\mathfrak{C}_{\ll d}^+(\mathbb{1}_{I_j} F)\|_{\mathcal{N}_\lambda^+} + \|\mathfrak{C}_{\gtrsim d}^+(\mathbb{1}_{I_j} F)\|_{\mathcal{N}_\lambda^+} \\ &\leq \|\mathfrak{C}_{\ll d}^+(\mathbb{1}_{I_j} F)\|_{L_t^1 L_x^2} + \sum_{d' \gtrsim d} (d')^{-\frac{1}{2}} \|\mathfrak{C}_{d'}^+(\mathbb{1}_{I_j} F)\|_{L_{t,x}^2} \end{aligned} \quad (80)$$

$$\lesssim \|\mathfrak{C}_{\ll d}^+(\mathbb{1}_{I_j} F)\|_{L_t^1 L_x^2} + d^{-\frac{1}{2}} \|\mathbb{1}_{I_j} F\|_{L_{t,x}^2}. \quad (81)$$

The second term is easy to control by simply summing up in j . On the other hand, for the first term in (81) the argument is more complicated. By rescaling²¹ it is enough to consider the case $d = 1$. We start by decomposing the intervals I_j into those intervals which are smaller than 1, and those that are larger than 1, i.e. we write $\{I_j\} = \{K_j\} \cup \{K'_j\}$ where $|K_j| < 1$ and $|K'_j| \geq 1$. For the intervals smaller than 1, we can simply discard the outer multipliers, apply Holder in time, and sum up in j

$$\sum_j \|\mathfrak{C}_{\ll 1}^+(\mathbb{1}_{K_j} F)\|_{L_t^1 L_x^2}^2 \lesssim \sum_j \|\mathbb{1}_{K_j} F\|_{L_t^1 L_x^2}^2 \leq \sum_j \|\mathbb{1}_{K_j} F\|_{L_{t,x}^2}^2 \leq \|F\|_{L_{t,x}^2}^2.$$

To deal with the intervals greater than 1, we note that since $|\tau - \tau'| = |\tau \pm |\xi| - (\tau' \pm |\xi|)|$, for any $|\tau \pm |\xi|| \ll 1$ we have the identity

$$(\Pi_\pm(\widetilde{\mathbb{1}_{K'_j} F}))(\tau, \xi) = \int_{|\tau - \tau'| \approx 1} \widehat{\mathbb{1}_{K'_j}}(\tau - \tau') \widetilde{\Pi_\pm F}(\tau', \xi) d\tau' = \int_{\mathbb{R}} \widehat{\rho}_j(\tau - \tau') \widetilde{F}(\tau', \xi) d\tau' \quad (82)$$

where $\widehat{\rho}_j(\tau) = \sigma(\tau) \widehat{\mathbb{1}_{K'_j}}(\tau)$ and σ has support in the set $\{|\tau| \approx 1\}$. Now, as $\frac{\sigma(\tau)}{\tau}$ is smooth and bounded, we can use integration by parts to deduce that for any $N > 0$

$$\rho_j(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \sigma(\tau) \widehat{\mathbb{1}_{K'_j}}(\tau) e^{it\tau} d\tau = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{i\sigma(\tau)}{\tau} (e^{i\tau(t-b_j)} - e^{i\tau(t-a_j)}) d\tau \lesssim \frac{1}{(1+|t-b_j|)^N} + \frac{1}{(1+|t-a_j|)^N}$$

where we let $K'_j = [a_j, b_j]$. Hence, applying Holder in t and assuming N large,

$$\begin{aligned} \sum_j \|\mathfrak{C}_{\ll 1}^+(\mathbb{1}_{K'_j} F)\|_{L_t^1 L_x^2}^2 &= \sum_j \|\mathfrak{C}_{\ll 1}^+(\rho_j F)\|_{L_t^1 L_x^2}^2 \\ &\lesssim \sum_j \|\rho_j F\|_{L_t^1 L_x^2}^2 \\ &\lesssim \sum_j \left\| |\rho_j|^{\frac{1}{2}} \|F\|_{L_x^2} \right\|_{L_t^2}^2 \\ &\lesssim \|F\|_{L_{t,x}^2} \sum_j \left(\frac{1}{(1+|t-b_j|)^{\frac{N}{2}}} + \frac{1}{(1+|t-a_j|)^{\frac{N}{2}}} \right) \Big|_{L_t^2}^2 \lesssim \|F\|_{L_{t,x}^2} \end{aligned}$$

where the sum converges since $|K'_j| \geq 1 \Rightarrow |a_j - a_{j+1}|, |b_j - b_{j+1}| \geq 1$.

²¹I.e. use the identity

$$C_{\ll d}^\pm(\mathbb{1}_{I_j} C_d^\pm F)(t, x) = d^{n+1} C_{\ll 1}^\pm(\mathbb{1}_{dI_j} C_1^\pm F_d)(dt, dx)$$

where $\widetilde{F}_d(\tau, \xi) = \widetilde{F}(d\tau, d\xi)$. Note that the rescaled intervals dI_j satisfy the same properties as the original intervals...

Case 3: F a NF_λ^+ atom. By definition, we have $\alpha \ll 1$ and a decomposition $F = \sum_{\kappa \in \mathcal{C}_\alpha} F_\kappa$ with $\text{supp } \widetilde{\Pi_\pm F_\kappa} \subset A_{\lambda, \alpha}^\pm(\kappa)$ and $\sum_{\kappa \in \mathcal{C}_\alpha} \|F_\kappa\|_{NF^+(\kappa)}^2 \leq 1$. Our aim is to deduce that

$$\sum_j \|\mathbb{1}_{I_j} F\|_{\mathcal{N}_\lambda^+}^2 \lesssim 1.$$

To this end we decompose $\mathbb{1}_{I_j} F$ into regions close to the cone, and far from the cone

$$\mathbb{1}_{I_j} F = \sum_{\pm} \left(\Pi_\pm C_{\ll \alpha^2 \lambda}^\pm(\mathbb{1}_{I_j} F) + \Pi_\pm C_{\gtrsim \alpha^2 \lambda}^\pm(\mathbb{1}_{I_j} F) \right). \quad (83)$$

For the close cone case, since the spatial Fourier projections commute with $\mathbb{1}_{I_j}(t)$, $\Pi_\pm C_{\ll \alpha^2 \lambda}^\pm(\mathbb{1}_{I_j} F)$ forms a (perhaps scalar multiple) of a NF_λ^+ atom. Therefore we can write

$$\begin{aligned} \left\| \sum_{\pm} \Pi_\pm C_{\ll \alpha^2 \lambda}^\pm(\mathbb{1}_{I_j} F) \right\|_{\mathcal{N}_\lambda^+} &\leq \left(\sum_{\kappa \in \mathcal{C}_\alpha} \left\| \sum_{\pm} \Pi_\pm C_{\ll \alpha^2 \lambda}^\pm(\mathbb{1}_{I_j} F_\kappa) \right\|_{NF^+(\kappa)}^2 \right)^{\frac{1}{2}} \\ &\lesssim \left(\sum_{\kappa \in \mathcal{C}_\alpha} \|\mathbb{1}_{I_j} F_\kappa\|_{NF^+(\kappa)}^2 \right)^{\frac{1}{2}} \end{aligned}$$

where we used Lemma 3.14 to dispose of the outer multipliers. Since we have an ℓ^2 sum in j , we can swap the j and κ summations, and reduce to proving the inequality

$$\sum_j \|\mathbb{1}_{I_j} F_\kappa\|_{NF^+(\kappa)}^2 \lesssim \|F_\kappa\|_{NF^+(\kappa)}^2. \quad (84)$$

To this end, we note that for every $\omega \notin 2\kappa$, we have

$$\begin{aligned} \sum_j \|\mathbb{1}_{I_j} G\|_{NF^+(\kappa)}^2 &\leq \sum_j \left(\|\mathbb{1}_{I_j} \Pi_\omega G\|_{L_{t_\omega}^1 L_{x_\omega}^2} + \theta(\omega, \kappa)^{-1} \|\mathbb{1}_{I_j} \Pi_{-\omega} G\|_{L_{t_\omega}^1 L_{x_\omega}^2} \right)^2 \\ &\leq \left\| \left(\sum_j \mathbb{1}_{I_j} \right)^{\frac{1}{2}} \Pi_\omega G \right\|_{L_{t_\omega}^1 L_{x_\omega}^2}^2 + \theta(\omega, \kappa)^{-2} \left\| \left(\sum_j \mathbb{1}_{I_j} \right)^{\frac{1}{2}} \Pi_{-\omega} G \right\|_{L_{t_\omega}^1 L_{x_\omega}^2}^2 \\ &= \|\Pi_\omega G\|_{L_{t_\omega}^1 L_{x_\omega}^2}^2 + \theta(\omega, \kappa)^{-2} \|\Pi_{-\omega} G\|_{L_{t_\omega}^1 L_{x_\omega}^2}^2. \end{aligned}$$

Taking infimum over $\omega \notin 2\kappa$, and then applying the previous inequality to $NF^+(\kappa)$ atoms, then gives (84).

For the remaining far cone term in (83), if we put the left hand side into $\mathcal{X}_{\lambda, +}^{-\frac{1}{2}, 1}$, and use Lemma 3.8 to control the resulting $L_{t, x}^2$ norm of the atom F , we deduce that

$$\begin{aligned} \sum_j \left\| \sum_{\pm} \Pi_\pm C_{\gtrsim \alpha^2 \lambda}^\pm(\mathbb{1}_{I_j} F) \right\|_{\mathcal{N}_\lambda^+}^2 &\leq \sum_j \left(\sum_{\pm} \sum_{d \gtrsim \alpha^2 \lambda} d^{-\frac{1}{2}} \|\Pi_\pm C_d^\pm(\mathbb{1}_{I_j} F)\|_{L_{t, x}^2} \right)^2 \\ &\lesssim (\alpha^2 \lambda)^{-1} \sum_j \|\mathbb{1}_{I_j} F\|_{L_{t, x}^2}^2 \\ &\lesssim (\alpha^2 \lambda)^{-1} \|F\|_{L_{t, x}^2}^2 \lesssim 1 \end{aligned}$$

as required. \square

By repeating the proof of the previous theorem, we have the following corollary which will prove useful when we come to the proof of (ii) in Theorem 3.9.

Corollary 9.6. *Let $\rho \in \dot{B}_{2, \infty}^{\frac{1}{2}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $F \in \mathcal{N}_\lambda^\pm$. Then*

$$\|\rho(t)F\|_{\mathcal{N}_\lambda^\pm} \lesssim \left(\|\rho\|_{\dot{B}_{2, \infty}^{\frac{1}{2}}(\mathbb{R})} + \|\rho\|_{L^\infty(\mathbb{R})} \right) \|F\|_{\mathcal{N}_\lambda^\pm}.$$

Similarly, if $u \in G_\lambda^\pm$, then

$$\lambda^{-\frac{n+1}{4n}} \|\rho(t)u\|_{\mathcal{Y}^\pm} \lesssim \left(\|\rho\|_{L^\infty(\mathbb{R})} + \lambda^{-\frac{n+1}{4n}} \|\partial_t \rho\|_{L^{\frac{4n}{3n-1}}(\mathbb{R})} \right) \|u\|_{G_\lambda^\pm}.$$

Proof. Fix $\pm = +$, the $\pm = -$ case follows by a reflection in x . As usual, we decompose F into atoms. If F is a $L_t^1 L_x^2$ atom, we clearly have

$$\|\rho(t)F\|_{\mathcal{N}_\lambda^\pm} \leq \|\rho(t)F\|_{L_t^1 L_x^2} \leq \|\rho\|_{L_t^\infty}.$$

On the other hand, if F is a $\dot{\mathcal{X}}_+^{-\frac{1}{2},1}$ atom with $\text{supp } \widetilde{\Pi_\pm F} \subset \{|\tau \pm |\xi|| \approx d\}$, then from (81) we have

$$\|\rho(t)F\|_{\mathcal{N}_\lambda^\pm} \lesssim \|\mathfrak{C}_{\ll d}^\pm(\rho(t)F)\|_{L_t^1 L_x^2} + d^{-\frac{1}{2}} \|\rho(t)F\|_{L_{t,x}^2} \leq \|\mathfrak{C}_{\ll d}^\pm(\rho(t)F)\|_{L_t^1 L_x^2} + \|\rho\|_{L_t^\infty(\mathbb{R})}.$$

To control the first term we use the identity (82) to deduce that

$$\|\mathfrak{C}_{\ll d}^\pm(\rho(t)F)\|_{L_t^1 L_x^2} \lesssim \|\widehat{\rho}\|_{L_\tau^2(|\tau| \approx d)} \|F\|_{L_{t,x}^2} \leq d^{\frac{1}{2}} \|\widehat{\rho}\|_{L_\tau^2(|\tau| \approx d)} \leq \|\rho\|_{\dot{B}_{2,\infty}^{\frac{1}{2}}}$$

and hence the required estimate is true whenever F is a $\dot{\mathcal{X}}_+^{-\frac{1}{2},1}$ atom. Finally, suppose $F = \sum_{\kappa \in \mathcal{C}_\alpha} F_\kappa$ is a NF_λ^+ atom. As in the NF_λ^+ case above, we write

$$\rho(t)F = \mathfrak{C}_{\ll \alpha^2 \lambda}^+(\rho(t)F) + \mathfrak{C}_{\gtrsim \alpha^2 \lambda}^+(\rho(t)F)$$

The first term is a scalar multiple of a NF_λ^+ atom, and hence via Lemma 3.14 we obtain

$$\|\mathfrak{C}_{\ll \alpha^2 \lambda}^+(\rho(t)F)\|_{\mathcal{N}_\lambda^+} \leq \left(\sum_{\kappa} \|\mathfrak{C}_{\ll \alpha^2 \lambda}^+(\rho(t)F)\|_{NF^+(\kappa)}^2 \right)^{\frac{1}{2}} \lesssim \left(\sum_{\kappa} \|\rho(t)F\|_{NF^+(\kappa)}^2 \right)^{\frac{1}{2}} \leq \|\rho\|_{L_x^\infty}$$

where we made use of the obvious bound $\|\rho F_\kappa\|_{NF^+(\kappa)} \leq \|\rho\|_{L_t^\infty} \|F_\kappa\|_{NF^+(\kappa)}$. For the remaining term, we estimate \mathcal{N}_λ^+ by $\dot{\mathcal{X}}_+^{-\frac{1}{2},1}$, and use the $L_{t,x}^2$ bound for Null Frame atoms in Lemma 3.8 to deduce

$$\|\mathfrak{C}_{\gtrsim \alpha^2 \lambda}^+(\rho(t)F)\|_{\mathcal{N}_\lambda^+} \lesssim (\alpha^2 \lambda)^{-\frac{1}{2}} \|\rho(t)F\|_{L_{t,x}^2} \leq (\alpha^2 \lambda)^{-\frac{1}{2}} \|\rho\|_{L_t^\infty} \|F\|_{L_{t,x}^2} \lesssim \|\rho\|_{L_t^\infty}.$$

It only remains to prove the \mathcal{Y}^\pm estimate. We again make use of a similar argument to that used to control the $\dot{\mathcal{X}}_\pm^{-\frac{1}{2},1}$ case above. We start by observing that

$$\|\rho(t)u\|_{\mathcal{Y}^\pm} \leq \sup_d d \|\mathfrak{C}_d^\pm(\rho(t)\mathfrak{C}_{\gtrsim d}^\pm u)\|_{L_t^{\frac{4n}{3n-1}} L_x^2} + \sup_d d \|\mathfrak{C}_d^\pm(\rho(t)\mathfrak{C}_{\ll d}^\pm u)\|_{L_t^{\frac{4n}{3n-1}} L_x^2}.$$

To control the first term, we discard the outer multiplier and put $\rho \in L^\infty(\mathbb{R})$

$$\sup_d d \|\mathfrak{C}_d^\pm(\rho(t)\mathfrak{C}_{\gtrsim d}^\pm u)\|_{L_t^{\frac{4n}{3n-1}} L_x^2} \lesssim \|\rho\|_{L^\infty} \sum_{d' \gtrsim d} d \|\mathfrak{C}_{d'}^\pm u\|_{L_t^{\frac{4n}{3n-1}} L_x^2} \lesssim \|\rho\|_{L^\infty} \|u\|_{\mathcal{Y}^\pm} \sum_{d' \gtrsim d} \frac{d}{d'} \lesssim \lambda^{\frac{n+1}{4n}} \|\rho\|_{L^\infty} \|u\|_{G_\lambda^\pm}.$$

For the second term, the identity (82) allows us to replace ρ with $P_{\approx d} \rho$ where $P_{\approx d}$ restricts the Fourier support of ρ to the region $\tau \approx d$. We now use some standard Harmonic analysis to deduce that

$$\sup_d d \|\mathfrak{C}_d^\pm(\rho(t)\mathfrak{C}_{\ll d}^\pm u)\|_{L_t^{\frac{4n}{3n-1}} L_x^2} \lesssim \|u\|_{L_t^\infty L_x^2} \sup_d d \|P_{\approx d} \rho\|_{L_t^{\frac{4n}{3n-1}}} \lesssim \|\partial_t \rho\|_{L_t^{\frac{4n}{3n-1}}} \|u\|_{G_\lambda^\pm}$$

as required. \square

9.2. Proof of Theorem 4.1. We now come to the proof of Theorem 4.1.

Proof of Theorem 4.1. By Corollary 9.3 and Theorem 9.5, it is enough to show that the $C_{\leq d}^{\pm}$ multipliers are disposable in V^2 . If we use the boundedness of $C_{\leq d}^{\pm}$ on $L_t^{\infty} L_x^2$ (which follows from (iii) in Lemma 3.14), it is enough to show that $|C_{\leq d}^{\pm} u|_{V^2} \lesssim |u|_{V^2}$. To this end, by noting the identity

$$\widehat{C_{\leq d}^{\pm} u}(t) = \int_{\mathbb{R}} \Phi_0\left(\frac{\tau \pm |\xi|}{d}\right) \tilde{u}(\tau, \xi) e^{it\tau} d\tau = \int_{\mathbb{R}} \widehat{\Phi}_0(a) e^{\pm i \frac{a}{d} |\xi|} \widehat{u}\left(t + \frac{a}{d}, \xi\right) da$$

we have for any sequence $(t_k) \in \mathcal{Z}$,

$$\begin{aligned} \left(\sum_k \|C_{\leq d}^{\pm} u(t_{k+1}) - C_{\leq d}^{\pm} u(t_k)\|_{L_x^2}^2 \right)^{\frac{1}{2}} &\leq \int_{\mathbb{R}} |\widehat{\Phi}_0(a)| \left(\sum_k \|\widehat{u}(t_{k+1} + \frac{a}{d}) - \widehat{u}(t_k + \frac{a}{d})\|_{L_{\xi}^2}^2 \right)^{\frac{1}{2}} da \\ &\leq \int_{\mathbb{R}} |\widehat{\Phi}_0(a)| |u|_{V^2} da \lesssim |u|_{V^2}. \end{aligned}$$

Taking the sup over $(t_k) \in \mathcal{Z}$, then gives $|C_{\leq d}^{\pm} u|_{V^2} \lesssim |u|_{V^2}$ as required. \square

10. THE ENERGY INEQUALITY

Here we give the proof of Theorem 3.9.

Proof of Theorem 3.9. (i) We start by noting that F_{λ}^{\pm} is a Banach space, since if $u_j \in F_{\lambda}^{\pm}$ is a Cauchy sequence with respect to $\|\cdot\|_{F_{\lambda}^{\pm}}$, then it is Cauchy with respect to $\|\cdot\|_{L_t^{\infty} L_x^2}$ and hence converges to some $u \in L_t^{\infty} L_x^2$ with $\text{supp } u \subset \{|\xi| \approx \lambda\}$. On the other hand, as $\mathcal{N}_{\lambda}^{\pm}$ is a Banach space, there exists $F \in \mathcal{N}_{\lambda}^{\pm}$ such that $(\partial_t \pm \sigma \cdot \nabla) u_j$ converges to F (with respect to $\|\cdot\|_{\mathcal{N}_{\lambda}^{\pm}}$). Consequently, $(\partial_t \pm \sigma \cdot \nabla) u_j$ converges to F in \mathcal{S}' , and hence by uniqueness of limits, we must have $(\partial_t \pm \sigma \cdot \nabla) u = F \in \mathcal{N}_{\lambda}^{\pm}$. Therefore $u \in F_{\lambda}^{\pm}$ as required and so F_{λ}^{\pm} is a Banach space.

To prove $F^{s,\pm}$ is a Banach space follows a similar argument, namely, if we have a Cauchy sequence in $F^{s,\pm}$, then it is also Cauchy in $L_t^{\infty} \dot{H}_x^s$ and hence converges to some $u \in L_t^{\infty} \dot{H}_x^s$. By uniqueness of limits, and the fact that F_{λ}^{\pm} and ℓ^2 are Banach spaces, we then deduce that $u \in F^{s,\pm}$ as required. Thus $F^{s,\pm}$ is a Banach space.

To prove the energy inequality for $F^{s,\pm}$, we clearly have

$$\|u\|_{F^{s,\pm}} \leq \left(\sum_{\lambda} \lambda^{2s} \|P_{\lambda} u\|_{L_t^{\infty} L_x^2}^2 \right)^{\frac{1}{2}} + \|(\partial_t \pm \sigma \cdot \nabla) u\|_{\mathcal{N}^{s,\pm}}$$

thus it is enough to prove that if $(\partial_t \pm \sigma \cdot \nabla) u = F$ with $u(0) = 0$, then

$$\|P_{\lambda} u\|_{L_t^{\infty} L_x^2} \lesssim \|P_{\lambda} F\|_{\mathcal{N}_{\lambda}^{\pm}}$$

but this is just Corollary 8.6. Similarly, to prove the energy inequality for $G^{s,\pm}$, again using Corollary 8.6, it suffices to show that

$$d \|C_d^{\pm} u\|_{L_t^{\frac{4n}{3n-1}} L_x^2} \lesssim \|(\partial_t \pm i|\nabla|) u\|_{L_t^{\frac{4n}{3n-1}} L_x^2}.$$

But this is straightforward by writing

$$\widehat{(C_d^{\pm} u)}(t, \xi) = \frac{e^{\mp it|\xi|}}{2\pi} \int_{\mathbb{R}} \Phi\left(\frac{\tau}{d}\right) \tilde{u}(\tau \mp |\xi|, \xi) e^{i\tau t} d\tau = \frac{e^{\mp it|\xi|}}{d} [\rho_d *_{\mathbb{R}} \widehat{v}(\xi)](t)$$

where $\widehat{\rho}_d(\tau) = \frac{d}{\tau}\Phi(\frac{\tau}{d}) \in C_0^\infty$ and $\tilde{v}(\tau, \xi) = \tau\tilde{u}(\tau \mp |\xi|, \xi)$. If we now apply Plancheral, Holder, and a change of variables, we deduce that

$$d\|C_d^\pm u\|_{L_t^{\frac{4n}{3n-1}}L_x^2} \lesssim \|\rho_d\|_{L_t^1(\mathbb{R})}\|\widehat{v}\|_{L_t^{\frac{4n}{3n-1}}L_\xi^2} \lesssim \|(\partial_t \pm |\nabla|)u\|_{L_t^{\frac{4n}{3n-1}}L_x^2}$$

as required.

(ii) Let $\phi \in C_0^\infty$. An application of Corollary 9.6 gives

$$\begin{aligned} \|(\partial_t \pm \sigma \cdot \nabla)[\phi(t)u]\|_{\mathcal{N}_\lambda^\pm} &\leq \|\phi(t)(\partial_t \pm \sigma \cdot \nabla)u\|_{\mathcal{N}_\lambda^\pm} + \|(\partial_t \phi)(t)u\|_{L_t^1L_x^2} \\ &\lesssim \left(\|\phi\|_{L_t^\infty} + \|\phi\|_{\dot{B}_{2,\infty}^{\frac{1}{2}}}\right)\|(\partial_t \pm \sigma \cdot \nabla)u\|_{\mathcal{N}_\lambda^\pm} + \|\partial_t \phi\|_{L_t^1}\|u\|_{L_t^\infty L_x^2} \\ &\lesssim \left(\|\phi\|_{L_t^\infty} + \|\phi\|_{\dot{B}_{2,\infty}^{\frac{1}{2}}} + \|\partial_t \phi\|_{L_t^1}\right)\|u\|_{F_\lambda^\pm} \end{aligned}$$

and hence

$$\|\phi(t)u\|_{F_\lambda^\pm} \lesssim \left(\|\phi\|_{L_t^\infty} + \|\phi\|_{\dot{B}_{2,\infty}^{\frac{1}{2}}} + \|\partial_t \phi\|_{L_t^1}\right)\|u\|_{F_\lambda^\pm}.$$

Applying this inequality with $\phi(t) = \rho(\frac{t}{T})$ and noting that $\|\rho(\frac{t}{T})\|_{\dot{B}_{2,\infty}^{\frac{1}{2}}} \approx \|\rho(t)\|_{\dot{B}_{2,\infty}^{\frac{1}{2}}}$, we deduce that $\|\rho(\frac{t}{T})u\|_{F_\lambda^\pm} \lesssim \|u\|_{F_\lambda^\pm}$ as required. The G_λ^\pm version follows from the F_λ^\pm estimate together with another application of Corollary 9.6 to deduce that

$$\lambda^{-\frac{n+1}{4n}}\|\rho(\frac{t}{T})u\|_{\mathcal{Y}^\pm} \lesssim \left(\|\rho\|_{L_t^\infty} + \lambda^{-\frac{n+1}{4n}}T^{\frac{3n-1}{4n}-1}\|\partial_t \rho\|_{L_t^{\frac{4n}{3n-1}}}\right)\|u\|_{G_\lambda^\pm} \lesssim \|u\|_{G_\lambda^\pm}$$

provided $T\lambda \geq 1$. Finally, the \mathcal{N}_λ^\pm estimate follows by again applying Corollary 9.6 and noting that since $\mathbb{1}_{(-1,1)} \in \dot{B}_{2,\infty}^{\frac{1}{2}}$, by rescaling we have

$$\|\mathbb{1}_{(-T,T)}(t)\|_{\dot{B}_{2,\infty}^{\frac{1}{2}}(\mathbb{R})} \approx \|\mathbb{1}_{(1,1)}(t)\|_{\dot{B}_{2,\infty}^{\frac{1}{2}}(\mathbb{R})} < \infty.$$

(iii) There are a number of ways to prove this, for instance it is possible to argue directly using the definition of \mathcal{N}_λ^\pm see [47, Prop 6.2]. Here we use an alternative argument based on Theorem 9.5. For maps $\phi : \mathbb{R} \rightarrow \dot{H}^s(\mathbb{R}^n)$ we let

$$|\phi|_{V^{s,p}}^p = \sup_{(t_j) \in \mathcal{Z}} \sum_j \|\phi(t_{j+1}) - \phi(t_j)\|_{\dot{H}^s}^p.$$

Arguing as in Lemma 9.1, we deduce that if $|\phi|_{V^{s,2}} < \infty$, then there exists $\phi_{\pm\infty} \in \dot{H}^s$ such that $\lim_{t \rightarrow \pm\infty} \|\phi(t) - \phi_{\pm\infty}\|_{\dot{H}^s} = 0$. In particular, the scattering result we require would follow by showing that $|\mathcal{U}_\pm(-t)u|_{V^{s,2}} < \infty$. To this end note that an application of Theorem 9.5 together with the fact that multiplication by $\rho(t)$ commutes with the homogeneous solution operator $\mathcal{U}_\pm(-t)$ gives

$$|\rho(\frac{t}{T})\mathcal{U}_\pm(-t)u|_{V^{s,2}} \lesssim \|\rho(\frac{t}{T})u\|_{F^{s,\pm}}.$$

Consequently it is enough to show that $|\phi|_{V^{s,2}} \leq \sup_T |\rho(\frac{t}{T})\phi|_{V^{s,2}}$. To this end, take any increasing sequence $(t_j) \in \mathcal{Z}$. Let $N \in \mathbb{N}$ and choose $T > \max_{|j| \leq N+1} |t_j|$. Then

$$\sum_{|j| \leq N} \|\phi(t_{j+1}) - \phi(t_j)\|_{\dot{H}^s}^2 = \sum_{|j| \leq N} \|\rho(\frac{t_{j+1}}{T})\phi(t_{j+1}) - \rho(\frac{t_j}{T})\phi(t_j)\|_{\dot{H}^s}^2 \leq \sup_{T>0} |\rho(\frac{t}{T})\phi|_{V^{s,2}}.$$

Hence letting $N \rightarrow \infty$ and taking the sup over $(t_j) \in \mathcal{Z}$, we get $|\phi|_{V^{s,2}} \leq \sup_{T>0} |\rho(\frac{t}{T})\phi|_{V^{s,2}}$ as required. \square

REFERENCES

1. M. Beals, *Self-spreading and strength of singularities for solutions to semilinear wave equations*, Ann. of Math. (2) **118** (1983), no. 1, 187–214.
2. I. Bejenaru and S. Herr, *The cubic Dirac equation: Small initial data in $H^1(\mathbb{R}^3)$* , Communications in Mathematical Physics (2014), 1–40.
3. F. F. Bonsall, *A general atomic decomposition theorem and Banach’s closed range theorem*, Quart. J. Math. Oxford Ser. (2) **42** (1991), no. 165, 9–14.
4. J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*, Geom. Funct. Anal. **3** (1993), no. 2, 107–156.
5. N. Bournaveas and T. Candy, *Local well-posedness for the space-time Monopole equation in Lorenz gauge*, Nonlinear Diff. Equations and Applications **19** (2012), no. 1, 67–78.
6. M. Bownik, *Boundedness of operators on Hardy spaces via atomic decompositions*, Proc. Amer. Math. Soc. **133** (2005), no. 12, 3535–3542.
7. T. Candy, *Global existence for an L^2 critical nonlinear Dirac equation in one dimension*, Adv. Differential Equations **16** (2011), no. 7-8, 643–666.
8. ———, *Bilinear estimates and applications to global well-posedness for the Dirac-Klein-Gordon equation*, J. Hyper. Differential Equations **10** (2013), no. 1, 1–35.
9. T. Cazenave and L. Vázquez, *Existence of localized solutions for a classical nonlinear Dirac field*, Comm. Math. Phys. **105** (1986), no. 1, 35–47.
10. A. Contreras, D. E. Pelinovsky, and Y. Shimabukuro, *L^2 orbital stability of Dirac solitons in the massive Thirring model*, arXiv:1312.1019 (2013).
11. P. D’Ancona, D. Foschi, and S. Selberg, *Null structure and almost optimal local regularity for the Dirac-Klein-Gordon system*, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 4, 877–899.
12. ———, *Null structure and almost optimal local well-posedness of the Maxwell-Dirac system*, Amer. J. Math. **132** (2010), no. 3, 771–839.
13. V. Delgado, *Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac and other nonlinear Dirac equations in one space dimension*, Proc. Amer. Math. Soc. **69** (1978), no. 2, 289–296.
14. J.-P. Dias and M. Figueira, *Global existence of solutions with small initial data in H^s for the massive nonlinear Dirac equations in three space dimensions*, Boll. Un. Mat. Ital. B (7) **1** (1987), no. 3, 861–874.
15. M. Escobedo and L. Vega, *A semilinear Dirac equation in $H^s(\mathbb{R}^3)$ for $s > 1$* , SIAM J. Math. Anal. **28** (1997), no. 2, 338–362.
16. M. Fierz, *Zur fermischen theorie des β -zerfalls*, Zeitschrift für Physik **104** (1937), no. 7-8, 553–565.
17. R. Finkelstein, C. Fronsdal, and P. Kaus, *Nonlinear Spinor Field*, Physical Review **103** (1956), 1571–1579.
18. D. Foschi and S. Klainerman, *Bilinear space-time estimates for homogeneous wave equations*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 2, 211–274.
19. M. Hadac, S. Herr, and H. Koch, *Well-posedness and scattering for the KP-II equation in a critical space*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 3, 917–941.
20. D. J. Kaup and A. C. Newell, *On the Coleman correspondence and the solution of the massive Thirring model*, Lett. Nuovo Cimento (2) **20** (1977), no. 9, 325–331.
21. C. E. Kenig, G. Ponce, and L. Vega, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J. **71** (1993), no. 1, 1–21.
22. S. Klainerman and M. Machedon, *Smoothing estimates for null forms and applications*, Duke Math. J. **81** (1995), no. 1, 99–133 (1996), A celebration of John F. Nash, Jr.
23. S. Klainerman and I. Rodnianski, *Bilinear estimates on curved space-times*, J. Hyperbolic Differ. Equ. **2** (2005), no. 2, 279–291.
24. S. Klainerman and S. Selberg, *Bilinear estimates and applications to nonlinear wave equations*, Commun. Contemp. Math. **4** (2002), no. 2, 223–295.
25. S. Klainerman and D. Tataru, *On the optimal local regularity for Yang-Mills equations in \mathbb{R}^{4+1}* , J. Amer. Math. Soc. **12** (1999), no. 1, 93–116.

26. H. Koch and D. Tataru, *Dispersive estimates for principally normal pseudodifferential operators*, *Comm. Pure Appl. Math.* **58** (2005), no. 2, 217–284.
27. J. Krieger, *Global regularity of wave maps from \mathbf{R}^{3+1} to surfaces*, *Comm. Math. Phys.* **238** (2003), no. 1-2, 333–366.
28. ———, *Null-form estimates and nonlinear waves*, *Adv. Differential Equations* **8** (2003), no. 10, 1193–1236.
29. J. Krieger and W. Schlag, *Concentration compactness for critical wave maps*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2012.
30. S. Machihara, M. Nakamura, K. Nakanishi, and T. Ozawa, *Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation*, *J. Funct. Anal.* **219** (2005), no. 1, 1–20.
31. S. Machihara, K. Nakanishi, and T. Ozawa, *Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation*, *Rev. Mat. Iberoamericana* **19** (2003), no. 1, 179–194.
32. F. Merle, *Existence of stationary states for nonlinear Dirac equations*, *J. Differential Equations* **74** (1988), no. 1, 50–68.
33. J. F. Nieves and P. B. Pal, *Generalized Fierz identities*, *Amer. J. Phys.* **72** (2004), no. 8, 1100–1108.
34. T. Ortín, *Gravity and strings*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2004.
35. H. Pecher, *Low regularity well-posedness for the one-dimensional Dirac-Klein-Gordon system*, *Electron. J. Differential Equations* (2006), no. 150, 13 pp.
36. ———, *Local well-posedness for the nonlinear Dirac equation in two space dimensions*, *Commun. Pure Appl. Anal.* **13** (2014), no. 2, 673 – 685.
37. ———, *Corrigendum of “Local well-posedness for the nonlinear Dirac equation in two space dimensions”*, *Commun. Pure Appl. Anal.* **14** (2015), no. 2, 737–742.
38. D. E. Pelinovsky and Y. Shimabukuro, *Orbital stability of Dirac solitons*, *Lett. Math. Phys.* **104** (2014), no. 1, 21–41.
39. M. Reed, *Abstract non-linear wave equations*, Lecture Notes in Mathematics, Vol. 507, Springer-Verlag, Berlin-New York, 1976.
40. S. Selberg and A. Tesfahun, *Low regularity well-posedness for some nonlinear Dirac equations in one space dimension*, *Differential Integral Equations* **23** (2010), no. 3-4, 265–278.
41. M. Soler, *Classical, stable, nonlinear spinor field with positive rest energy*, *Phys. Rev. D* **1** (1970), 2766–2769.
42. J. Sterbenz, *Global regularity for general non-linear wave equations. I. $(6 + 1)$ and higher dimensions*, *Comm. Partial Differential Equations* **29** (2004), no. 9-10, 1505–1531.
43. J. Sterbenz and D. Tataru, *Energy dispersed large data wave maps in $2 + 1$ dimensions*, *Comm. Math. Phys.* **298** (2010), no. 1, 139–230.
44. ———, *Regularity of wave-maps in dimension $2 + 1$* , *Comm. Math. Phys.* **298** (2010), no. 1, 231–264.
45. W. A. Strauss and L. Vázquez, *Stability under dilations of nonlinear spinor fields*, *Phys. Rev. D* (3) **34** (1986), no. 2, 641–643.
46. T. Tao, *Global regularity of wave maps. II. Small energy in two dimensions*, *Comm. Math. Phys.* **224** (2001), no. 2, 443–544.
47. D. Tataru, *On global existence and scattering for the wave maps equation*, *Amer. J. Math.* **123** (2001), no. 1, 37–77.
48. B. Thaller, *The Dirac equation*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1992.
49. W. E. Thirring, *A soluble relativistic field theory*, *Annals of Physics* **3** (1958), no. 1, 91 – 112.
50. N. Tzvetkov, *Existence of global solutions to nonlinear massless Dirac system and wave equation with small data*, *Tsukuba J. Math.* **22** (1998), no. 1, 193–211.
51. V. E. Zakharov and A. V. Mikhailov, *On the integrability of classical spinor models in two-dimensional space-time*, *Comm. Math. Phys.* **74** (1980), no. 1, 21–40.

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