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# KÄHLER–EINSTEIN FANO THREEFOLDS OF DEGREE 22

IVAN CHELTSOV AND CONSTANTIN SHRAMOV

**ABSTRACT.** We study the problem of existence of Kähler–Einstein metrics on smooth Fano threefolds of Picard rank one and anticanonical degree 22 that admit a faithful action of the multiplicative group  $\mathbb{C}^*$ . We prove that, with a possible exception of two explicitly described cases, all such smooth Fano threefolds are Kähler–Einstein.

All varieties are assumed to be projective and defined over the field of complex numbers.

## 1. INTRODUCTION

Smooth Fano threefolds of Picard rank 1 have been classified by Iskovskikh in [I77, I78]. Among them, he found a family missing in the original works by Fano. Threefolds in this family have the same cohomology groups as  $\mathbb{P}^3$  does. Their anticanonical degree is 22, hence they are called threefolds of type  $V_{22}$ . In fact, Iskovskikh himself missed one threefold in this family, which was later recovered by Mukai and Umemura in [MU83]. This threefold, usually called the Mukai–Umemura threefold, is an equivariant compactification of  $\mathrm{SL}_2(\mathbb{C})/\mathbf{I}$ , where  $\mathbf{I}$  denotes the icosahedral group. Its automorphism group is isomorphic to the group  $\mathrm{PGL}_2(\mathbb{C})$ .

The automorphism groups of threefolds of type  $V_{22}$  have been studied by Prokhorov in [P90]. He proved that this group is finite except for a unique threefold for which the connected component of identity of the automorphism group is isomorphic to the additive group  $\mathbb{C}^+$ ; and a one-parameter family of threefolds that admit a faithful action of the multiplicative group  $\mathbb{C}^*$ , which includes the Mukai–Umemura threefold as a special member. We refer to the latter varieties as threefolds of type  $V_{22}^*$ .

In [Ti97], Tian showed that there are threefolds of type  $V_{22}$  with trivial automorphism group that do not admit Kähler–Einstein metrics, which disproved a folklore conjecture that all smooth Fano varieties without holomorphic vector fields are Kähler–Einstein. On the other hand, Donaldson proved

**Theorem 1.1** ([D08, Theorem 3]). Let  $X$  be the Mukai–Umemura threefold, and  $G$  be its automorphism group. Then

$$\alpha_G(X) = \frac{5}{6}.$$

Here  $\alpha_G(X)$  is the  $G$ -equivariant  $\alpha$ -invariant defined by Tian in [Ti87]. If  $X$  is a smooth Fano variety, and  $G$  is a reductive subgroup in  $\mathrm{Aut}(X)$ , then Demailly’s [CS08,

Theorem A.3] gives

$$(1.2) \quad \alpha_G(X) = \sup \left\{ \epsilon \in \mathbb{Q} \left| \begin{array}{l} \text{the log pair } \left( X, \frac{\epsilon}{n} \mathcal{D} \right) \text{ is log canonical for any } n \in \mathbb{Z}_{>0} \\ \text{and every } G\text{-invariant linear system } \mathcal{D} \subset | -nK_X | \end{array} \right. \right\}.$$

Donaldson's Theorem 1.1 implies the existence of a Kähler–Einstein metric on the Mukai–Umemura threefold by famous Tian's criterion:

**Theorem 1.3** ([Ti87]). Let  $X$  be a smooth Fano variety of dimension  $n$ , and  $G$  be a reductive subgroup in  $\text{Aut}(X)$ . Suppose that

$$\alpha_G(X) > \frac{n}{n+1}.$$

Then  $X$  admits a Kähler–Einstein metric.

An example of a Kähler–Einstein threefold of type  $V_{22}$  with finite automorphism group has been constructed in [CS12]. On the other hand, there exist threefolds of this type that are not Kähler–Einstein.

**Example 1.4.** Let  $X^a$  be the unique threefold of type  $V_{22}$  such that the connected component of identity of its automorphism group is isomorphic to the additive group  $\mathbb{C}^+$ . By the Matsushima obstruction, the variety  $X^a$  is not Kähler–Einstein. It is interesting to point out that  $X^a$  is K-semistable. Indeed, it follows from [KPS18, Proposition 5.4.4] and the Mukai construction of varieties of type  $V_{22}$  (cf. [KPS18, Remark 5.4.8]) that the Mukai–Umemura threefold is a degeneration of  $X^a$ . Since the Mukai–Umemura threefold is Kähler–Einstein, it is K-polystable by [CDS15], so that in particular it is K-semistable. On the other hand, K-semistability is an open condition, see [X19, Theorem 1.4] or [BLX19, Corollary 1.2]. Hence  $X^a$  is K-semistable.

The problem of existence of Kähler–Einstein metrics on threefolds of type  $V_{22}^*$  was addressed by Donaldson in [D08, D18], by Rollin, Simanca and Tipler in [RST13], and by Dinew, Kapustka and Kapustka in [DKK17]. In particular, they proved that the set of such threefolds that are Kähler–Einstein is open in moduli in the Euclidean topology. Donaldson suggested that in fact all threefolds of type  $V_{22}^*$  are Kähler–Einstein. In [D08], he wrote

The Mukai–Umemura manifold has  $\tau = 1$ . When  $\tau$  is close to 1 we have seen that the corresponding manifold admits a Kähler–Einstein metric. It seems likely that this true for all  $\tau$  but, as far as the author is aware, this is not known. It seems an interesting test case for future developments in the existence theory.

Here  $\tau$  is a parameter in the moduli space of threefolds of type  $V_{22}^*$  that is used in [D08]. The Mukai–Umemura threefold corresponds to  $\tau = 1$ .

In [D18, §4.1], Donaldson made a more precise suggestion about which threefolds of type  $V_{22}$  are Kähler–Einstein metric and which are not. It also predicts that each threefold of type  $V_{22}^*$  must admit a Kähler–Einstein metric.

To verify Donaldson’s suggestion, Dinew, Kapustka and Kapustka estimated the  $\alpha_{\mathbb{C}^*}$ -invariants of threefolds of type  $V_{22}^*$ . It appeared that they do not exceed  $\frac{1}{2}$ , so that Tian’s Theorem 1.3 cannot be applied. However, the automorphism groups of all threefolds of type  $V_{22}^*$  are actually larger than  $\mathbb{C}^*$ . It was pointed out in [RST13, DKK17] that there exists an additional involution that anti-commutes with the  $\mathbb{C}^*$ -action, so that together they generate a subgroup isomorphic to  $\mathbb{C}^* \rtimes \mu_2$ . Here  $\mu_2$  denotes the group of order 2. In fact, by [KP17, Theorem 1.3], one has

$$\mathrm{Aut}(X) \cong \mathbb{C}^* \rtimes \mu_2$$

for every threefold  $X$  of type  $V_{22}^*$  that is not the Mukai–Umemura threefold.

Dinew, Kapustka and Kapustka posed

**Problem 1.5** ([DKK17, Problem 7.1]). Let  $X$  be a smooth Fano threefold of type  $V_{22}^*$ , and let  $G$  be a subgroup in  $\mathrm{Aut}(X)$  that is isomorphic to  $\mathbb{C}^* \rtimes \mu_2$ . Compute  $\alpha_G(X)$ .

In this paper we completely solve this problem using the description of smooth Fano threefolds of type  $V_{22}^*$  obtained recently by Kuznetsov and Prokhorov in [KP17].

Kuznetsov and Prokhorov proved that the isomorphism classes of Fano threefolds of type  $V_{22}^*$  are naturally parameterized by  $u \in \mathbb{C} \setminus \{0, 1\}$ . In §2, we present their construction in details. Note that the parameter  $u$  used by Kuznetsov and Prokhorov in [KP17] differs from the parameter  $\tau$  used by Donaldson in [D08].

To state our main result, we denote by  $V_u$  the smooth Fano threefold of type  $V_{22}^*$  that corresponds to the parameter  $u$  in the construction of [KP17]. Then the Mukai–Umemura threefold is  $V_u$  for  $u = -\frac{1}{4}$  by [KP17, Theorem 1.3]. Let  $G$  a subgroup in  $\mathrm{Aut}(V_u)$  such that

$$G \cong \mathbb{C}^* \rtimes \mu_2.$$

The main result of our paper is

**Theorem 1.6.** One has

$$\alpha_G(V_u) = \begin{cases} \frac{4}{5} & \text{if } u \neq \frac{3}{4} \text{ and } u \neq 2, \\ \frac{3}{4} & \text{if } u = \frac{3}{4}, \\ \frac{2}{3} & \text{if } u = 2. \end{cases}$$

Applying Tian’s Theorem 1.3, we obtain

**Corollary 1.7.** If  $u \neq \frac{3}{4}$  and  $u \neq 2$ , then  $V_u$  is Kähler–Einstein.

*Remark 1.8.* If  $u = \frac{3}{4}$  or  $u = 2$ , then  $V_u$  is also Kähler–Einstein. This has been recently proved by Fujita in [Fu21]. Note also that Theorem 1.6 and [ACCF<sup>+</sup>, Theorem 1.4.10] imply that  $V_u$  is Kähler–Einstein for  $u = \frac{3}{4}$ .

Let us describe the scheme of the proof of Theorem 1.6. To estimate  $\alpha_G(V_u)$ , one has to describe irreducible  $G$ -invariant subvarieties of small degree in  $V_u$ . Since  $G$  acts on  $V_u$

without fixed points, we have to deal with irreducible  $G$ -invariant curves of small degree, and  $G$ -invariant anticanonical surfaces in  $V_u$ . However, the geometry of the threefold  $V_u$  is rather complicated, and it is hard to complete these tasks in a straightforward way. Instead, we use a construction of the threefold  $V_u$  as a  $G$ -equivariant birational image of a smooth quadric hypersurface in  $\mathbb{P}^4$  found recently by Kuznetsov and Prokhorov in [KP17], see the diagram (2.5) for more details. This allows to describe irreducible  $G$ -invariant curves of small degree in  $V_u$  and  $G$ -invariant surfaces in  $|-K_{V_u}|$  in terms of the quadric, whose  $G$ -equivariant geometry is much easier to control. In particular, this description gives us an upper bound on  $\alpha_G(V_u)$ . To show that the latter bound is sharp, we have to study  $G$ -equivariant birational geometry of the threefold  $V_u$ . We do this using three explicit  $G$ -equivariant Sarkisov links that start from  $V_u$ . As a result, we obtain the formula for  $\alpha_G(V_u)$  in Theorem 1.6.

Let us describe the structure of this paper. In §2, we recall from [KP17] the explicit construction of the threefold  $V_u$  using a birational map from a three-dimensional quadric. In this section, we also describe this birational map explicitly in coordinates. In §3, we start an explicit classification of irreducible  $G$ -invariant curves of small degree in the threefold  $V_u$ , which will be used in the proof of Theorem 1.6. In §4, we complete this classification, see Proposition 4.12. In §5, we study the pencil in the linear system  $|-K_{V_u}|$  that consists of all  $G$ -invariant surfaces and describe singularities of surfaces in this pencil. In §6, we describe one Sarkisov link that plays a crucial role in the proof of Theorem 1.6. In this section, we also describe two special birational transformations of the threefold  $V_u$ , which are known as *bad Sarkisov links*. They are also used in the proof of our Theorem 1.6. Finally, in §7, we prove Theorem 1.6.

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## 2. KUZNETSOV–PROKHOROV CONSTRUCTION

Consider the projective space  $\mathbb{P}^4$  with homogeneous coordinates  $x, y, z, t$ , and  $w$ . Suppose that the group  $\mathbb{C}^*$  act on  $\mathbb{P}^4$  by

$$(2.1) \quad \lambda : (x : y : z : t : w) \mapsto (x : \lambda y : \lambda^3 z : \lambda^5 t : \lambda^6 w).$$

Furthermore, consider the involution  $\iota$  acting on  $\mathbb{P}^4$  by

$$(2.2) \quad \iota : (x : y : z : t : w) \mapsto (w : t : z : y : x).$$

This defines the action of the group  $G \cong \mathbb{C}^* \rtimes \mu_2$  on  $\mathbb{P}^4$ .

Let the quadric  $Q_u$ , where  $u \in \mathbb{C}$ , be given by equation

$$(2.3) \quad u(xw - z^2) + (z^2 - yt) = 0.$$

Then the quadric  $Q_u$  is  $G$ -invariant. Note that  $Q_u$  is smooth provided that  $u \notin \{0, 1\}$ . Therefore, until the end of the paper (with the only exception of Remark 2.12 below), we will always assume that neither  $u = 0$  nor  $u = 1$ .

Let  $\Gamma$  be the image of  $\mathbb{P}^1$  with homogeneous coordinates  $(s_0 : s_1)$  embedded into  $\mathbb{P}^4$  by

$$(s_0 : s_1) \mapsto (s_0^6 : s_0^5 s_1 : s_0^3 s_1^3 : s_0 s_1^5 : s_1^6).$$

Then  $\Gamma$  is a  $G$ -invariant curve contained in the quadric  $Q_u$ . It is the closure of the  $G$ -orbit of the point  $(1 : 1 : 1 : 1 : 1)$ . One easily checks that  $\deg(\Gamma) = 6$ , cf. Lemma 3.1 below.

Let  $\mathcal{S}$  be the complete intersection in  $\mathbb{P}^4$  that is given by

$$\begin{cases} xw - z^2 = 0, \\ z^2 - yt = 0. \end{cases}$$

Then the surface  $\mathcal{S}$  is  $G$ -invariant, and  $\Gamma \subset \mathcal{S} \subset Q_u$ .

*Remark 2.4.* The surface  $\mathcal{S}$  is a toric singular del Pezzo surface of degree 4 that has 4 ordinary double points. These points are  $(1 : 0 : 0 : 0 : 0)$ ,  $(0 : 0 : 0 : 0 : 1)$ ,  $(0 : 1 : 0 : 0 : 0)$  and  $(0 : 0 : 0 : 1 : 0)$ . The first two of them are contained in the curve  $\Gamma$ .

It was proved in [KP17, Theorem 2.2] (cf. [Ta89, (2.13.2)]) that there exists the following  $G$ -equivariant commutative diagram

$$(2.5) \quad \begin{array}{ccccc} & \tilde{Q}_u & \overset{\chi}{\dashrightarrow} & \tilde{V}_u & \\ & \searrow \alpha & & \swarrow \beta & \\ & & Y_u & & \\ \pi \swarrow & & & & \searrow \phi \\ Q_u & \overset{\gamma}{\dashrightarrow} & Y_u & \overset{\omega}{\dashrightarrow} & V_u \\ & \dashrightarrow \zeta & & & \end{array}$$

Here  $V_u$  is a smooth Fano threefold of type  $V_{22}^*$ , the morphism  $\pi$  is the blow up of the quadric  $Q_u$  along the curve  $\Gamma$ , the morphism  $\phi$  is the blow up of the threefold  $V_u$  along a (unique)  $G$ -invariant smooth rational curve  $\mathcal{C}_2$  with  $-K_{V_u} \cdot \mathcal{C}_2 = 2$ , the map  $\chi$  is a flop in two smooth rational curves, which we will describe later in Remark 2.11. The morphisms  $\alpha$  and  $\beta$  in (2.5) are small birational morphisms that are given by the linear systems  $| -nK_{\tilde{Q}_u} |$  and  $| -nK_{\tilde{V}_u} |$  for  $n \gg 0$ , respectively. By construction, the threefold  $Y_u$  is a non- $\mathbb{Q}$ -factorial Fano threefold with terminal singularities such that  $-K_{Y_u}^3 = 16$ .

*Remark 2.6.* Kuznetsov and Prokhorov showed in [KP17] that every smooth Fano threefold of type  $V_{22}^*$  can be obtained via diagram (2.5) for some  $u \in \mathbb{C} \setminus \{0, 1\}$ . Moreover, they proved that for distinct  $u$  the resulting varieties  $V_u$  are not isomorphic. Furthermore, if  $u = -\frac{1}{4}$ , then  $V_u$  is the Mukai–Umemura threefold by [KP17, Theorem 1.3]. For other descriptions of threefolds of type  $V_{22}^*$ , see [D08, §5.3], [DKK17, §2.2] and [KPS18, §5.3].

Recall from [IP99, Proposition 4.1.11] that the divisor  $-K_{V_u}$  is very ample, and the linear system  $| -K_{V_u} |$  gives an embedding  $V_u \hookrightarrow \mathbb{P}^{13}$ . In particular, the curve  $\mathcal{C}_2$  is a

conic in this embedding. Let us identify  $V_u$  with its anticanonical image in  $\mathbb{P}^{13}$  and fix the following notation.

- We denote by  $H_{Q_u}$  a hyperplane section of the quadric  $Q_u$  in  $\mathbb{P}^4$ .
- We denote by  $H_{V_u}$  a hyperplane section of the threefold  $V_u$  in  $\mathbb{P}^{13}$ .
- We denote by  $\tilde{\mathcal{S}}$  the proper transform of the surface  $\mathcal{S}$  on the threefold  $\tilde{Q}_u$ .
- We denote by  $E_{Q_u}$  the exceptional surface of the blow up  $\pi$ .
- We denote by  $E_{V_u}$  the exceptional surface of the blow up  $\phi$ .

Then  $\tilde{\mathcal{S}}$  is the proper transform of  $E_{V_u}$  on  $\tilde{Q}_u$ , which is the unique divisor in the linear system  $|2\pi^*(H_{Q_u}) - E_{Q_u}|$ . Similarly, the proper transform of  $E_{Q_u}$  on  $\tilde{V}_u$  is the unique surface in the linear system  $|2\phi^*(H_{V_u}) - 5E_{V_u}|$ . Thus, we also fix the following notation.

- We denote by  $\tilde{\mathcal{R}}$  the unique surface in the linear system  $|2\phi^*(H_{V_u}) - 5E_{V_u}|$ .
- We denote by  $\mathcal{R}$  the proper transform of the surface  $\tilde{\mathcal{R}}$  on the threefold  $V_u$ .

**Corollary 2.7.** One has  $\alpha_G(V_u) \leq \frac{4}{5}$ .

*Proof.* Let  $D = \frac{1}{2}\mathcal{R}$ . Then  $D \sim_{\mathbb{Q}} -K_{V_u}$ . Moreover, since  $\mathcal{R} \sim -2K_{V_u}$  and  $\text{mult}_{c_2}(\mathcal{R}) = 5$ , the log pair  $(V_u, \frac{4}{5}D)$  is not Kawamata log terminal. Indeed, we have

$$K_{\tilde{V}_u} + \frac{4}{5}\tilde{D} + E_{V_u} \sim_{\mathbb{Q}} \phi^*\left(K_{V_u} + \frac{4}{5}D\right).$$

This shows that  $\alpha_G(V_u) \leq \frac{4}{5}$ . □

Using the information about the classes of the exceptional divisors  $E_{Q_u}$  and  $E_{V_u}$ , one can easily check that the rational map  $\phi \circ \chi: \tilde{Q}_u \dashrightarrow V_u$  is given by the linear system  $|5\pi^*(H_{Q_u}) - 2E_{Q_u}|$ , and the rational map  $\pi \circ \chi^{-1}: \tilde{V}_u \dashrightarrow Q_u$  is given by the linear system  $|\phi^*(H_{V_u}) - 2E_{V_u}|$ .

*Remark 2.8.* By [IP99, Proposition 4.1.12(iii)], the threefold  $V_u$  is a scheme-theoretic intersection of quadrics in  $\mathbb{P}^{13}$ . Thus, since  $-K_{\tilde{V}_u} \sim \phi^*(H_{V_u}) - E_{V_u}$  and  $h^0(\mathcal{O}_{\tilde{V}_u}(-K_{\tilde{V}_u})) = 11$ , the linear system  $|-K_{\tilde{V}_u}|$  gives a morphism  $V_u \rightarrow \mathbb{P}^{10}$  that is birational on its image. Hence, there is a commutative diagram

$$\begin{array}{ccc} & \tilde{V}_u & \\ & \swarrow & \searrow \phi \\ \mathbb{P}^{10} & \dashleftarrow & V_u \end{array}$$

such that the dashed arrow is a linear projection from the conic  $\mathcal{C}_2$ . This implies that we can assume that the morphism  $\beta$  in (2.5) is given by the linear system  $|-K_{\tilde{V}_u}|$ . Hence, we can also assume that the morphism  $\alpha$  is given by the linear system  $|-K_{\tilde{Q}_u}|$ . Thus, the threefold  $Y_u$  is a (singular) Fano threefold anticanonically embedded into  $\mathbb{P}^{10}$ .

Let  $L_1$  and  $L_2$  be the tangent lines in  $\mathbb{P}^4$  to the curve  $\Gamma$  at the points  $(1 : 0 : 0 : 0 : 0)$  and  $(0 : 0 : 0 : 0 : 1)$ , respectively. Then  $L_1$  is given by

$$(2.9) \quad z = t = w = 0,$$

and the line  $L_2$  is given by

$$(2.10) \quad x = y = z = 0.$$

Thus, both lines  $L_1$  and  $L_2$  are contained in the surface  $\mathcal{S}$ . Denote by  $\tilde{L}_1$  and  $\tilde{L}_2$  the proper transforms of the lines  $L_1$  and  $L_2$  on the threefold  $\tilde{Q}_u$ , respectively.

*Remark 2.11.* By [KP17, Remark 5.3], the curves  $\tilde{L}_1$  and  $\tilde{L}_2$  are the flopping curves of the map  $\chi$ . The flopping curves of  $\chi^{-1}$  are described in [KP17, Proposition 5.1]. Namely, the threefold  $V_u$  contains exactly two lines that intersect the conic  $\mathcal{C}_2$ . Denote them by  $\ell_1$  and  $\ell_2$ , and denote their proper transforms on  $\tilde{V}_u$  by  $\tilde{\ell}_1$  and  $\tilde{\ell}_2$ , respectively. The lines  $\ell_1$  and  $\ell_2$  intersect the conic  $\mathcal{C}_2$  transversally, because  $V_u$  is an intersection of quadrics. Moreover, the lines  $\ell_1$  and  $\ell_2$  are contained in the surface  $\mathcal{R}$ , since  $\mathcal{R} \sim -2K_{V_u}$  and  $\text{mult}_{\mathcal{C}_2}(\mathcal{R}) = 5$ . By [KP17, Remark 5.3], the curves  $\tilde{\ell}_1$  and  $\tilde{\ell}_2$  are exactly the flopping curves of the map  $\chi^{-1}$ . Thus, the birational map  $\zeta$  in (2.5) induces an isomorphism

$$Q_u \setminus \mathcal{S} \cong V_u \setminus \mathcal{R}.$$

Without loss of generality, we may assume that  $\beta(\tilde{\ell}_1) = \alpha(\tilde{L}_1)$  and  $\beta(\tilde{\ell}_2) = \alpha(\tilde{L}_2)$ . Note that the lines  $\ell_1$  and  $\ell_2$  on the Fano threefold  $V_u$  are *special*, i.e., their normal bundles in  $V_u$  are isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ ; see the proof of [KP17, Proposition 5.1]. This implies that the normal bundles of the curves  $\tilde{\ell}_1$  and  $\tilde{\ell}_2$  in  $\tilde{V}_u$  are isomorphic to  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ , so that the flop  $\chi^{-1}$  is given by Reid's pagoda [R83, §5].

*Remark 2.12.* It follows from Theorem 1.6 and Remark 1.8 that  $V_u$  is K-polystable for every  $u \notin \{0, 1\}$ . It would be interesting to find the K-polystable limits of the threefolds  $V_u$  when  $u \rightarrow 0$ ,  $u \rightarrow 1$  and  $u \rightarrow \infty$ . In fact, we have a candidate for the limit in the case when  $u \rightarrow 1$ . Namely, if  $u = 1$ , then the quadric threefold  $Q_u$  is singular at the point  $(0 : 0 : 1 : 0 : 0)$ . This point is not contained in the surface  $\mathcal{S}$ , and it is not contained in the curve  $\Gamma$ . Thus, the commutative diagram (2.5) still makes sense in this case. The threefold  $V_1$  is a Fano threefold with one ordinary double point such that  $-K_{V_1}^3 = 22$ . By [KP17, Proposition 5.4], one has  $\text{Pic}(V_1) \cong \mathbb{Z}$  and  $\text{Cl}(V_1) \cong \mathbb{Z}^2$ , so that  $V_1$  is one of the threefolds described in [P16, Theorem 1.2]. Note also that  $\text{Cl}(V_1)^G \cong \mathbb{Z}^2$ . We expect that  $V_1$  is K-polystable, so that it is the K-polystable limit of our threefolds  $V_u$  when  $u \rightarrow 1$ .

The commutative diagram (2.5) is a Sarkisov link (that starts at  $Q_u$  and ends at  $V_u$ ). It plays a crucial role in the proof of our Theorem 1.6. In §6, we describe another  $G$ -equivariant Sarkisov link that starts at  $V_u$  and ends at another threefold of type  $V_{22}^*$  (possibly isomorphic to  $V_u$ ). This link also helps to prove Theorem 1.6.

*Remark 2.13* (cf. [CS12, CS14, CS15, CS16, CS19]). It would be interesting to study other  $G$ -Sarkisov links that start at the threefold  $V_u$  or the quadric  $Q_u$ . Such links usually arise from  $G$ -irreducible curves of small degree or  $G$ -orbits of small length. For example, the inverse of the link (2.5) arises from the conic  $\mathcal{C}_2$ , which is irreducible and  $G$ -invariant. The curve  $\ell_1 + \ell_2$  from Remark 2.11 also gives rise to a  $G$ -Sarkisov link. Namely, one can



show that there exists a  $G$ -equivariant commutative diagram

$$(2.14) \quad \begin{array}{ccccc} & \overline{V}_u & \overset{\varrho}{\dashrightarrow} & \overline{W} & \\ v \swarrow & & \searrow \varsigma & \nearrow \varphi & \searrow \nu \\ V_u & & U & & W \end{array}$$

Here  $v$  is a blow up of the lines  $\ell_1$  and  $\ell_2$ , the morphisms  $\varsigma$  and  $\varphi$  are small and birational, the map  $\varrho$  flops the curves contracted by  $\varsigma$ , the threefold  $U$  is a Fano threefold with terminal singularities such that  $-K_U^3 = 14$ , the threefold  $W$  is a smooth Fano threefold such that  $\text{Pic}(W) \cong \mathbb{Z}^2$  and  $-K_W^3 = 28$ , and  $\nu$  is a birational morphism that contracts the proper transform of the unique surface in  $| -K_{V_u} |$  which is singular along the lines  $\ell_1$  and  $\ell_2$  to a smooth rational curve of (anticanonical) degree 6. Note that  $\text{Pic}(W)^G \cong \mathbb{Z}$ , and  $W$  is the threefold No. (1.2.3) in [P13, Theorem 1.2]. It can be realized as the blow-up of a smooth quadric in  $\mathbb{P}^4$  along a twisted quartic curve. Note that unlike (2.5) the diagram (2.14) is not a Sarkisov link in the usual sense [C95], because the curve  $\ell_1 + \ell_2$  is reducible.

Now we describe the birational maps  $\gamma$  and  $\zeta$  in the diagram (2.5) explicitly using coordinates on  $\mathbb{P}^4$ . To describe the map  $\gamma$ , recall that this map is given by the restriction of the linear system of all cubic hypersurfaces in  $\mathbb{P}^4$  that pass through the curve  $\Gamma$  to the quadric  $Q_u$ . Since  $\gamma$  is  $G$ -equivariant and, in particular,  $\mathbb{C}^*$ -equivariant, we are in position to choose  $\mathbb{C}^*$ -invariant generators of this linear system. To start with, set

$$f = xw - yt,$$

so that the equation  $f = 0$  cuts out the surface  $\mathcal{S}$  on the quadric  $Q_u$ . Then we set

$$(2.15) \quad \begin{aligned} h_3 &= y^3 - x^2z, & h_5 &= x^2t - y^2z, & h_6 &= xf, & h_7 &= yf, \\ h_8 &= y^2w - xzt, & h_9 &= zf, & h_{10} &= xt^2 - yzw, & h_{11} &= tf, \\ h_{12} &= wf, & h_{13} &= yw^2 - zt^2, & h_{15} &= t^3 - zw^2. \end{aligned}$$

Then the involution  $\iota$  swaps the polynomials  $h_i$  and  $h_{18-i}$  for  $3 \leq i \leq 8$ , and it preserves the polynomial  $h_9$ . Observe also that these 11 cubic polynomials all vanish on the curve  $\Gamma$ . Moreover, the corresponding surfaces in  $Q_u$  cut out by  $h_i = 0$  are smooth at a general point of the curve  $\Gamma$ , so that their proper transforms on  $\tilde{Q}_u$  are all contained in the linear system  $| -K_{\tilde{Q}_u} | = |3\pi^*(H_{Q_u}) - E_{Q_u}|$ .

Every polynomial  $h_i$  is semi-invariant with respect to the  $\mathbb{C}^*$ -action (2.1). Moreover, the weight of the polynomial  $h_i$  equals  $i$ . This implies, in particular, that they define linearly independent sections in  $H^0(\mathcal{O}_{Q_u}(3H_{Q_u}))$ . Since  $h^0(\mathcal{O}_{\tilde{Q}_u}(-K_{\tilde{Q}_u})) = 11$  by the Riemann–Roch formula and Kawamata–Viehweg vanishing theorem, we conclude that the birational map  $\gamma$  in (2.5) is given by

$$(2.16) \quad (x : y : z : t : w) \mapsto (h_3 : h_5 : h_6 : h_7 : h_8 : h_9 : h_{10} : h_{11} : h_{12} : h_{13} : h_{15}).$$

Thus, using (2.9) and (2.10), we see that  $\gamma(L_1) = (1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0)$  and  $\gamma(L_2) = (0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 1)$ .

Now let us describe the map  $\zeta$  in (2.5). To do this, we set

$$(2.17) \quad g_{i+6} = f \cdot h_i$$

for  $i \in \{3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15\}$ . Let

$$(2.18) \quad \begin{aligned} g_{10} &= (u-1)x^2yzw - 3xy^2zt + (2-u)xyz^3 + y^4w + x^3t^2, \\ g_{20} &= (u-1)xztw^2 - 3yzt^2w + (2-u)z^3tw + xt^4 + y^2w^3, \\ g'_{15} &= (u-1)x^2t^3 + (u-1)y^3w^2 - (u+4)y^2zt^2 + (3u+2)xyztw + (4-4u)yz^3t. \end{aligned}$$

Note that the involution  $\iota$  swaps the polynomials  $g_i$  and  $g_{30-i}$  for  $9 \leq i \leq 14$ , and it preserves both polynomials  $g_{15}$  and  $g'_{15}$ . Observe that all polynomials  $g_i$  and the polynomial  $g'_{15}$  are semi-invariant with respect to the  $\mathbb{C}^*$ -action (2.1). Moreover, the weight of the polynomial  $g_i$  equals  $i$ , and the weight of the polynomial  $g'_{15}$  equals 15. Also observe that

$$g'_{15}(0, 1, 0, 0, 1) = 1 \neq 0 = g_{15}(0, 1, 0, 0, 1),$$

and the point  $(0 : 1 : 0 : 0 : 1)$  is contained in the quadric  $Q_u$ . This implies, in particular, that these 14 quintic polynomials define linearly independent sections in  $H^0(\mathcal{O}_{Q_u}(5H_{Q_u}))$ .

For every  $i \in \{9, \dots, 21\}$ , denote by  $M_i$  the surface in the quadric  $Q_u$  that is cut out by the equation  $g_i = 0$ . Similarly, denote by  $M'_{15}$  the surface in  $Q_u$  that is cut out by the equation  $g'_{15} = 0$ . It is easy to see that all these surfaces pass through the curve  $\Gamma$ .

**Lemma 2.19.** The surfaces  $M_i$  and  $M'_{15}$  are singular along  $\Gamma$ .

*Proof.* For  $i \in \{3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15\}$  this follows from the fact that the polynomials  $h_i$  and  $f$  vanish along  $\Gamma$ . To check the assertion for the surfaces  $M_{10}$ ,  $M_{20}$  and  $M'_{15}$ , one can just write down the partial derivatives of  $g_{10}$ ,  $g_{20}$  and  $g'_{15}$  at the point  $(1 : 1 : 1 : 1 : 1)$ , compare them with the partial derivatives of the left hand side of (2.3), and then use the fact that  $\Gamma$  is the closure of the orbit of the latter point.  $\square$

One can check that the multiplicities of the surfaces  $M_i$  and  $M'_{15}$  along the curve  $\Gamma$  equal 2. This also follows from the fact that the surfaces  $E_{Q_u}$  and  $\tilde{\mathcal{S}}$  generate the cone of effective divisors of the threefold  $\tilde{Q}_u$ . We conclude that the proper transforms of the surfaces  $M_i$  and  $M'_{15}$  on the threefold  $\tilde{Q}_u$  generate the linear system  $|5H_{Q_u} - 2E_{Q_u}|$ . Hence, the birational map  $\zeta$  in (2.5) is given by

$$(2.20) \quad (x : y : z : t : w) \mapsto (g_9 : g_{10} : g_{11} : g_{12} : g_{13} : g_{14} : g_{15} : g'_{15} : g_{16} : g_{17} : g_{18} : g_{19} : g_{20} : g_{21}).$$

In particular, this reproves [DKK17, Proposition 4.1].

Denote by  $T_i$  and  $T'_{15}$  the proper transforms of the surfaces  $M_i$  and  $M'_{15}$  on the threefold  $V_u$ , respectively. Then

$$T_i \sim T'_{15} \sim -K_{V_u} \sim H_{V_u}.$$

This implies that all surfaces  $T_i$  and  $T'_{15}$  are irreducible, because the group  $\text{Pic}(V_u)$  is generated by the divisor  $H_{V_u}$ . This implies that the surface  $M'_{15}$  is irreducible, since the surface  $T'_{15}$  is irreducible and  $M'_{15}$  does not contain the surface  $\mathcal{S}$ . Similarly, the surfaces  $M_{10}$  and  $M_{20}$  are also irreducible. However, the remaining surfaces  $M_i$  are reducible.

Namely, let  $N_3, N_5, N_8, N_{10}, N_{13}$  and  $N_{15}$  be the surfaces in  $Q_u$  that are cut out by the equations  $h_3 = 0, h_5 = 0, h_8 = 0, h_{10} = 0$  and  $h_{15} = 0$ , respectively. Similarly, let  $H_x, H_y, H_z, H_t$  and  $H_w$  be the hyperplane sections of the quadric  $Q_u$  that are cut out by  $x = 0, y = 0, z = 0, t = 0$  and  $w = 0$ , respectively. Then we see from (2.15) that

$$\begin{aligned} M_9 &= N_3 + \mathcal{S}, & M_{11} &= N_5 + \mathcal{S}, & M_{12} &= H_x + 2\mathcal{S}, & M_{13} &= H_y + 2\mathcal{S}, \\ M_{14} &= N_8 + \mathcal{S}, & M_{15} &= H_z + 2\mathcal{S}, & M_{16} &= N_{10} + \mathcal{S}, & M_{17} &= H_t + 2\mathcal{S}, \\ M_{18} &= H_w + 2\mathcal{S}, & M_{19} &= N_{13} + \mathcal{S}, & M_{21} &= N_{15} + \mathcal{S}. \end{aligned}$$

Thus, the surfaces  $T_9, T_{11}, T_{14}, T_{16}, T_{19}$  and  $T_{21}$  are actually the proper transforms on the threefold  $V_u$  of the surfaces  $N_3, N_5, N_8, N_{10}, N_{13}$  and  $N_{15}$ , respectively. Similarly, the surfaces  $T_{12}, T_{13}, T_{15}, T_{17}$  and  $T_{18}$  are the proper transforms on the threefold  $V_u$  of the surfaces  $H_x, H_y, H_z, H_t$  and  $H_w$ , respectively.

*Remark 2.21.* It follows from (2.20) that the conic  $\mathcal{C}_2$  is contained in the surfaces  $T_9, T_{11}, T_{12}, T_{13}, T_{14}, T_{15}, T_{16}, T_{17}, T_{18}, T_{19}$  and  $T_{21}$ , and it is not contained in the surfaces  $T_{10}, T_{20}$  and  $T'_{15}$ .

**Lemma 2.22.** The line  $\ell_1$  is contained in the surfaces  $T_{11}, T_{12}, T_{13}, T_{14}, T_{15}, T'_{15}, T_{16}, T_{17}, T_{18}, T_{19}, T_{20}, T_{21}$ , and it is not contained in the surfaces  $T_9$  and  $T_{10}$ . Similarly, the line  $\ell_2$  is contained in the surfaces  $T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}, T_{15}, T'_{15}, T_{16}, T_{17}, T_{18}, T_{19}$ , and it is not contained in the surfaces  $T_{20}$  and  $T_{21}$ .

*Proof.* Let  $P_\lambda \in \mathbb{P}^4$  be the point

$$\left( \frac{\lambda(u\lambda - \lambda + 1)}{u} : \lambda : \lambda : 1 : 1 \right),$$

where  $\lambda \in \mathbb{C}$ . Let  $C$  be the (closure of the) curve swept out by  $P_\lambda$ . Then  $C$  is contained in the quadric  $Q_u$ , and

$$C \cap L_2 = P_0 = (0 : 0 : 0 : 1 : 1).$$

Note that the point  $P_0$  is not contained in the curve  $\Gamma$ , so that the proper transforms of the curves  $C$  and  $L_2$  on the threefold  $\tilde{Q}_u$  still meet at the preimage of the point  $P_0$ . This implies that the proper transform  $C_{V_u}$  of the curve  $C$  on the threefold  $V_u$  intersects the line  $\ell_2$ . Substitute the coordinates of the point  $P_\lambda$  into (2.20), multiply the coordinates of the resulting point by  $\frac{u}{\lambda}$ , and let  $\lambda = 0$ . This gives the point

$$C_{V_u} \cap \ell_2 = (0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 1 : 1 - u).$$

Using the  $\mathbb{C}^*$ -action on  $\mathbb{P}^{13}$ , we immediately obtain the equations of the line  $\ell_2$ . The equations for the line  $\ell_1$  are obtained in a similar way. Now the required assertion follows from (2.20).  $\square$

Let us conclude this section by the following lemma.

**Lemma 2.23.** There are no  $G$ -fixed points in  $Q_u$  and  $V_u$ .

*Proof.* It follows from (2.1) that the only  $\mathbb{C}^*$ -fixed points in the quadric  $Q_u$  are the points  $(1 : 0 : 0 : 0 : 0)$ ,  $(0 : 0 : 0 : 0 : 1)$ ,  $(0 : 1 : 0 : 0 : 0)$  and  $(0 : 0 : 0 : 1 : 0)$ . Note that  $\iota$  swaps the points  $(1 : 0 : 0 : 0 : 0)$  and  $(0 : 0 : 0 : 0 : 1)$ , and it also swaps the remaining two  $\mathbb{C}^*$ -fixed points, so that there are no  $G$ -fixed points in  $Q_u$ . This also implies that there are no  $G$ -fixed points in  $\tilde{Q}_u$ .

By Remark 2.11, the flopping curves of  $\chi$  are disjoint and swapped by the involution  $\iota$ . Hence, there are no  $G$ -fixed points in  $\tilde{V}_u$ . Thus, if  $V_u$  contains a  $G$ -fixed point, then it must be contained in the conic  $\mathcal{C}_2$ .

Let  $\Pi \cong \mathbb{P}^2$  be the linear span of the conic  $\mathcal{C}_2$  in  $\mathbb{P}^{13}$ . Then  $\Pi$  is  $G$ -invariant. The action of  $G$  on  $\Pi$  is not faithful (indeed, it contains all elements of order 5 in  $\mathbb{C}^*$ ). However, the kernel is finite, and the automorphism  $\iota$  acts faithfully on  $\Pi$ . This implies that there is a faithful action of a quotient of  $G$  that is isomorphic to  $G$  on  $\Pi$  and thus on  $\mathcal{C}_2$ . Therefore, the conic  $\mathcal{C}_2$  does not contain  $G$ -fixed points, so that there are no  $G$ -fixed points in  $V_u$ .  $\square$

### 3. INVARIANT CURVES

In this section, we make the first steps needed for a description of irreducible  $G$ -invariant curves in  $Q_u$  and  $V_u$ . We start with

**Lemma 3.1.** Fix a point  $(a_0 : \dots : a_n) \in \mathbb{P}^n$ , and fix positive integers  $r_0 \leq \dots \leq r_n$ . Let  $Z$  be the curve in  $\mathbb{P}^n$  that is the closure of the subset

$$\left\{ (\lambda^{r_0} a_0 : \dots : \lambda^{r_n} a_n) \mid \lambda \in \mathbb{C}^* \right\} \subset \mathbb{P}^n.$$

Denote by  $\Sigma$  the set of indices  $i$  such that  $a_i \neq 0$ . Set

$$r_k = \min\{r_i \mid i \in \Sigma\}, \quad r_K = \max\{r_i \mid i \in \Sigma\}.$$

Denote by  $d$  the greatest common divisor of the numbers  $r_i - r_k$  for  $i \in \Sigma$ . Then

$$\deg(Z) = \frac{r_K - r_k}{d}.$$

Furthermore, let  $s$  be the maximal number of indices  $i$  in  $\Sigma$  with distinct  $r_i$ . Then  $Z$  is a rational normal curve if and only if  $\deg(Z) = s$ .

*Proof.* Cancelling a common factor in the homogeneous coordinates if necessary, we may assume that  $r_k = 0$ . To compute the degree of  $Z$ , note that the intersection points of  $Z$  with a general hyperplane  $\Lambda$  in  $\mathbb{P}^n$  correspond to the roots of a polynomial  $P_\Lambda(\lambda)$  of degree  $r_K$  in  $\lambda$ . Since  $P_\Lambda$  is actually a polynomial of degree  $r_K/d$  in  $\lambda^d$ , the  $r_K$  roots of  $P_\Lambda$  produce  $r_K/d$  points of  $\Lambda \cap Z$ . Thus, the degree of  $Z$  equals  $r_K/d$ . It remains to notice that the linear span of  $Z$  has dimension  $s$ , so that  $Z$  is a rational normal curve if and only if  $\deg(Z) = s$ .  $\square$

There are no  $G$ -fixed points in  $Q_u$  by Lemma 2.23. This implies, in particular, that every irreducible  $G$ -invariant curve in  $Q_u$  is rational and contains at least one  $\iota$ -fixed point. Hence, every irreducible  $G$ -invariant curve is a closure of the  $\mathbb{C}^*$ -orbit of any of its  $\iota$ -fixed points.

**Lemma 3.2.** All  $\iota$ -fixed points in  $Q_u$  are the points

$$P_{\pm} = (1 : \pm\sqrt{u} : 0 : \mp\sqrt{u} : -1)$$

and the points

$$(3.3) \quad \left( b^2 - (1-u)(a-b)^2 : u(a^2 - b^2) - a^2 : a^2 - u(a-b)^2 : u(a^2 - b^2) - a^2 : b^2 - (1-u)(a-b)^2 \right),$$

where  $(a : b) \in \mathbb{P}^1$ .

*Proof.* Using (2.2), one can see that the  $\iota$ -fixed points in  $\mathbb{P}^4$  are the points of the line

$$\begin{cases} x + w = 0, \\ y + t = 0, \\ z = 0, \end{cases}$$

and the points of the plane

$$\begin{cases} x - w = 0, \\ y - t = 0. \end{cases}$$

Intersecting the line with  $Q_u$ , we obtain the points  $P_{\pm}$ . Similarly, intersecting the plane with the quadric  $Q_u$ , we obtain the conic parameterized by (3.3).  $\square$

Observe that the  $\mathbb{C}^*$ -orbit of the point  $P_+$  is the same as the  $\mathbb{C}^*$ -orbit of the point  $P_-$ . We denote its closure by  $\Theta_{\pm}$ . Similarly, we denote the closure of the  $\mathbb{C}^*$ -orbit of the point (3.3) by  $\Theta_{a,b}$ . By construction, the curves  $\Theta_{\pm}$  and  $\Theta_{a,b}$  are all irreducible  $G$ -invariant curves contained in the quadric  $Q_u$ .

**Lemma 3.4.** The only irreducible  $G$ -invariant curves in  $\mathcal{S}$  are

$$\Gamma = \Theta_{0,1} = \Theta_{u,u-1}$$

and  $\Theta_{1,0} = \Theta_{1,1}$ . The degree of the curve  $\gamma(\Theta_{1,0})$  in  $\mathbb{P}^{10}$  is 12.

*Proof.* Recall from §2 that the surface  $\mathcal{S}$  is cut out on the quadric  $Q_u$  by the equation  $f = 0$ , where  $f = xw - yt$ . Substituting  $x = 1$ ,  $y = \pm\sqrt{u}$ ,  $z = 0$ ,  $t = \mp\sqrt{u}$  and  $w = -1$  into the polynomial  $f$ , we get  $u - 1$ , so that the curve  $\Theta_{\pm}$  is not contained in  $\mathcal{S}$ . Similarly, substituting the coordinates of the point (3.3) into  $f$ , we obtain

$$4(1-u)ab(a-b)(u(a-b)-a),$$

and the first assertion follows.

The curve  $\Theta_{1,0}$  is the closure of the  $\mathbb{C}^*$ -orbit of the point  $P = (1 : 1 : -1 : 1 : 1)$ . Thus, by (2.16), the curve  $\gamma(\Theta_{1,0})$  is the closure of the  $\mathbb{C}^*$ -orbit of the point

$$\gamma(P) = (1 : 1 : 0 : 0 : 1 : 0 : 1 : 0 : 0 : 1 : 1),$$

so that the degree of the curve  $\gamma(\Theta_{0,1})$  is 12 by Lemma 3.1.  $\square$

Let  $\Delta$  be the conic in  $Q_u$  that is cut out by

$$(3.5) \quad y = t = 0.$$

Then  $\Delta$  is  $G$ -invariant. One can check that

$$\Delta = \Theta_{\sqrt{u}, \sqrt{u-1}} = \Theta_{-\sqrt{u}, \sqrt{u-1}}.$$

Similarly, let  $\Upsilon$  be the conic in  $Q_u$  that is cut out by

$$(3.6) \quad x = w = 0$$

Then  $\Upsilon$  is  $G$ -invariant. One can check that

$$\Upsilon = \Theta_{\sqrt{1-u+1}, \sqrt{1-u}} = \Theta_{\sqrt{1-u-1}, \sqrt{1-u}}.$$

**Lemma 3.7.** The following assertions hold.

- (i) The curve  $\zeta(\Theta_{\pm})$  is a curve of degree 12. One has  $\zeta(\Theta_{\pm}) \subset T_{15} \cap T'_{15}$ .
- (ii) The curve  $\zeta(\Delta)$  is a rational normal curve of degree 4. One has  $\zeta(\Delta) \subset T_{10} \cap T_{20}$ .
- (iii) The curve  $\zeta(\Upsilon)$  is a rational normal curve of degree 6. One has  $\zeta(\Upsilon) \subset T_{10} \cap T_{20}$ .
- (iv) For every curve  $\Theta_{a,b}$  not contained in the surface  $\mathcal{S}$  and different from  $\Delta$  and  $\Upsilon$ , the degree of  $\zeta(\Theta_{a,b})$  is either 10 or 12.
- (v) If  $\Theta_{a,b}$  is not contained in the surface  $\mathcal{S}$ , then the degree of the curve  $\zeta(\Theta_{a,b})$  equals 10 if and only if the curve  $\Theta_{a,b}$  is contained in  $N_3 \cap N_{15}$ .

*Proof.* By (2.20), the curve  $\zeta(\Theta_{\pm})$  is the closure of the  $\mathbb{C}^*$ -orbit of the point  $\zeta(P_+)$  that is  $(u\sqrt{u} : -u : -\sqrt{u} : u-1 : \sqrt{u}(u-1) : -u : 0 : 0 : u : -\sqrt{u}(u-1) : -u+1 : \sqrt{u} : u : -u\sqrt{u})$ ,

which is contained in  $T_{15} \cap T'_{15}$ . Then  $\zeta(\Theta_{\pm})$  is a curve of degree 12 by Lemma 3.1, and it is contained in  $T_{15} \cap T'_{15}$ . This proves assertion (i).

To prove assertions (ii), (iii) and (iv), we need some auxiliary computations. Define the polynomial

$$q_0 = (u-1)^2 a^4 - 2(u-1)^2 a^3 b + 2(u-1)(u-2)a^2 b^2 - 6u(u-1)ab^3 + u(3u-2)b^4.$$

Furthermore, define the polynomials

$$\begin{aligned} q_1 &= (u-1)a^2 - ub^2, \\ q_2 &= (u-1)a^2 - (2u-2)ab + ub^2, \\ q_3 &= (u-1)a^2 + 2ab - (u+2)b^2, \\ q_4 &= (u-1)a^2 - (2u-2)ab + (u-2)b^2, \\ q_5 &= (u-1)a^2 - 2uab + ub^2, \\ q_6 &= (u-1)a^2 - (2u-4)ab + (u-4)b^2. \end{aligned}$$

Recall that  $u \neq 0$  and  $u \neq 1$ . Observe that  $q_i$  is coprime to  $q_j$  for  $0 \leq i < j \leq 6$  with the following exceptions:

- $q_0$  is divisible by  $q_6$  provided that  $u^2 - 2u + 2 = 0$ ;
- $q_1 = q_6$  provided that  $u = 2$ ;

- $q_3 = q_5$  provided that  $u = -1$ ;
- $q_2$  and  $q_3$  have a common linear factor provided that  $u = \frac{-1 \pm \sqrt{5}}{2}$ .

Substituting the coordinates of the point (3.3) into the polynomials  $g_i$  and  $g'_{15}$ , we obtain the polynomials  $p_i$  and  $p'_{15}$  (in  $a$  and  $b$ ), respectively. We compute

$$\begin{aligned}
p_9 &= p_{21} = -8(u-1)a^2b(a-b)((u-1)a-ub)^2q_0, \\
p_{10} &= p_{20} = 4a^2((u-1)a-ub)^2q_1q_2q_3, \\
p_{11} &= p_{19} = -8(u-1)a^2b(a-b)((u-1)a-ub)^2q_1q_4, \\
p_{12} &= p_{18} = 16(u-1)^2a^2b^2(a-b)^2((u-1)a-ub)^2q_2, \\
p_{13} &= p_{17} = 16(u-1)^2a^2b^2(a-b)^2((u-1)a-ub)^2q_1, \\
p_{14} &= p_{16} = -8(u-1)a^2b(a-b)((u-1)a-ub)^2q_1q_2, \\
p_{15} &= -16(u-1)^2a^2b^2(a-b)^2((u-1)a-ub)^2q_5, \\
p'_{15} &= 4(u-1)a^2((u-1)a-ub)^2q_1^2q_6.
\end{aligned}$$

Let us consider the curve  $\Theta_{a,b}$  not contained in the surface  $\mathcal{S}$ . By Lemma 3.4 this means that  $a \neq 0$ ,  $b \neq 0$ ,  $a - b \neq 0$  and  $(u-1)a - ub \neq 0$ . These conditions imply that

- the polynomials  $p_9$  and  $p_{21}$  vanish if and only if  $q_0$  does,
- the polynomials  $p_{10}$  and  $p_{20}$  vanish if and only if either  $q_1$ , or  $q_2$ , or  $q_3$  does,
- the polynomials  $p_{11}$  and  $p_{19}$  vanish if and only if either  $q_1$  or  $q_4$  does,
- the polynomials  $p_{12}$  and  $p_{18}$  vanish if and only if  $q_2$  does,
- the polynomials  $p_{13}$  and  $p_{17}$  vanish if and only if  $q_1$  does,
- the polynomials  $p_{14}$  and  $p_{16}$  vanish if and only if either  $q_1$  or  $q_2$  does,
- the polynomial  $p_{15}$  vanishes if and only if  $q_5$  does,
- the polynomial  $p'_{15}$  vanishes if and only if either  $q_1$  or  $q_6$  does.

Note that  $q_1 = 0$  if and only if  $\Theta_{a,b} = \Delta$ , and  $q_2 = 0$  if and only if  $\Theta_{a,b} = \Upsilon$ .

Suppose that  $\Theta_{a,b} = \Delta$ . Then  $q_1 = 0$ , so that

$$(3.8) \quad p_{10} = p_{11} = p_{13} = p_{14} = p'_{15} = p_{16} = p_{17} = p_{19} = p_{20} = 0.$$

The coprimeness properties of the polynomials  $q_i$  imply that  $p_9$ ,  $p_{12}$ ,  $p_{15}$ ,  $p_{18}$  and  $p_{21}$  do not vanish. Therefore,  $\zeta(\Delta)$  is a rational normal curve of degree 4 by (2.20) and Lemma 3.1, which proves assertion (ii).

Suppose that  $\Theta_{a,b} = \Upsilon$ . Then  $q_2 = 0$ , so that

$$(3.9) \quad p_{10} = p_{12} = p_{14} = p_{16} = p_{18} = p_{20} = 0.$$

The coprimeness properties of the polynomials  $q_i$  imply that  $p_9$ ,  $p_{11}$ ,  $p_{13}$ ,  $p_{15}$ ,  $p_{17}$ ,  $p_{19}$  and  $p_{21}$  do not vanish. Therefore, we see that  $\zeta(\Upsilon)$  is a rational normal curve of degree 6 by (2.20) and Lemma 3.1, which proves assertion (iii).

Now suppose that  $\Theta_{a,b}$  is different from  $\Delta$  and  $\Upsilon$ . This means that  $q_1 \neq 0$  and  $q_2 \neq 0$ , so that in particular  $p_{12}$  and  $p_{13}$  do not vanish. If  $q_0 \neq 0$ , then  $p_9$  and  $p_{21}$  do not vanish as well, so that the degree of the curve  $\zeta(\Theta_{a,b})$  is 12 by (2.20) and Lemma 3.1. Thus, we

may assume that  $q_0 = 0$ , so that

$$p_9 = p_{21} = 0.$$

The coprimeness properties of the polynomials  $q_i$  imply that  $p_{10}$ ,  $p_{11}$  and  $p_{20}$  do not vanish, so that the degree of the curve  $\zeta(\Theta_{a,b})$  is 10 by (2.20) and Lemma 3.1. This proves assertion (iv). The condition  $p_9 = p_{21} = 0$  means that the curve  $\Theta_{a,b}$  is contained in  $M_9$  and  $M_{21}$ . Since  $M_9 = N_3 + \mathcal{S}$  and  $M_{21} = N_{15} + \mathcal{S}$ , we see that  $\Theta_{a,b}$  is contained in  $N_3$  and  $N_{15}$ , because we assume that  $\Theta_{a,b}$  is not contained in  $\mathcal{S}$ . This proves assertion (v) and completes the proof of the lemma.  $\square$

Taking a more careful look at the proof of Lemma 3.7, one can deduce that there are only a finite number of curves among  $\zeta(\Theta_{a,b})$  that are *not* rational normal curves of degree 12. Moreover, one can explicitly describe all such curves for any given  $u$ .

*Remark 3.10.* By Lemma 3.7(i), the intersection  $T_{15} \cap T'_{15}$  contains the curve  $\zeta(\Theta_{\pm})$ , which is a curve of degree 12. Moreover, it follows from Lemma 2.22 that  $T_{15} \cap T'_{15}$  contains both lines  $\ell_1$  and  $\ell_2$ . Thus, the intersection  $T_{15} \cap T'_{15}$  does not contain irreducible  $G$ -invariant curves of degree greater than 8 that are different from the curve  $\zeta(\Theta_{\pm})$ . Note that  $T_{15} \cap T'_{15}$  does not contain the conic  $\mathcal{C}_2$  by Remark 2.21. Using (3.5), we see that  $T_{15} \cap T'_{15}$  does not contain the curve  $\mathcal{C}_4$ . Similarly, using (3.6), we see that  $T_{15} \cap T'_{15}$  does not contain the curve  $\mathcal{C}_6$ .

Let us describe explicitly the curves  $\Theta_{a,b}$  in the case when  $\zeta(\Theta_{a,b})$  is a curve of degree 10. If  $u \neq -\frac{1}{3}$ , let  $\vartheta$  be one of the roots  $\sqrt{(3u+1)(1-u)}$ . If  $u = -\frac{1}{3}$ , let  $\vartheta = 0$ . If  $u = \frac{2}{3}$ , then

$$(3u+1)(1-u) = 1.$$

In this case, we assume that  $\vartheta = 1$ . Observe that the quadric  $Q_u$  contains the point

$$(3.11) \quad \left( 1 : 1 : 1 : \frac{(u-1)(\vartheta-u-1)}{2u^2} : \frac{(u-1)(2u^2+\vartheta-u-1)}{2u^3} \right).$$

Similarly, the quadric  $Q_u$  contains the point

$$(3.12) \quad \left( 1 : 1 : 1 : \frac{(u-1)(-\vartheta-u-1)}{2u^2} : \frac{(u-1)(2u^2-\vartheta-u-1)}{2u^3} \right).$$

Let  $\Psi$  be the closure of the  $\mathbb{C}^*$ -orbit of the point (3.11), and let  $\Psi'$  be the closure of the  $\mathbb{C}^*$ -orbit of the point (3.12). Then the curve  $\Psi$  is  $G$ -invariant, since the  $\mathbb{C}^*$ -orbit of the point (3.11) contains the image of this point via the involution  $\iota$ , because

$$\begin{aligned} \left( 1 : \lambda : \lambda^3 : \lambda^5 \frac{(u-1)(\vartheta-u-1)}{2u^2} : \lambda^6 \frac{(u-1)(2u^2+\vartheta-u-1)}{2u^3} \right) &= \\ &= \left( \frac{(u-1)(2u^2+\vartheta-u-1)}{2u^3} : \frac{(u-1)(\vartheta-u-1)}{2u^2} : 1 : 1 : 1 \right) \end{aligned}$$

for  $\lambda = \frac{u(\vartheta-u-1)}{(2u^2+\vartheta-u-1)} \in \mathbb{C}^*$ . Similarly, we see that the curve  $\Psi'$  is  $G$ -invariant. Of course, the curves  $\Psi$  and  $\Psi'$  are of the form  $\Theta_{a,b}$  for certain  $a$  and  $b$ , but we will never use the values of these parameters.



It is straightforward to check that  $\Psi = \Psi'$  if and only if  $u = -\frac{1}{3}$ . Moreover, if  $u = \frac{2}{3}$ , then  $\Psi \neq \Gamma$  and  $\Psi' = \Gamma$ . This explains why we let  $\vartheta = 1$  in this case.

**Lemma 3.13.** The following assertions hold.

- (i) Both curves  $\Psi$  and  $\Psi'$  are contained in the intersection  $N_3 \cap N_{15}$ .
- (ii) The curve  $\Psi$  is not contained in  $\mathcal{S}$ . If  $u \neq \frac{2}{3}$ , then  $\Psi'$  is not contained in  $\mathcal{S}$ .
- (iii) The curve  $\zeta(\Psi)$  is a curve of degree 10.
- (iv) If  $u \neq \frac{2}{3}$ , then  $\zeta(\Psi')$  is a curve of degree 10.
- (v) If  $\Theta_{a,b} \not\subset \mathcal{S}$  and  $\zeta(\Theta_{a,b})$  is a curve of degree 10, then  $\Theta_{a,b} = \Psi$  or  $\Theta_{a,b} = \Psi'$ .
- (vi) The surfaces  $N_3$  and  $N_{15}$  are tangent along  $\Gamma$  if and only if  $u = \frac{2}{3}$ .
- (vii) If  $u = \frac{2}{3}$ , then  $N_3$  and  $N_{15}$  do not tangent  $\mathcal{S}$  at a general point of the curve  $\Gamma$ .
- (viii) If  $u = -\frac{1}{3}$ , then  $N_3$  and  $N_{15}$  are tangent along  $\Psi = \Psi'$ .

*Proof.* Using (2.3), we see that the intersection  $N_3 \cap N_{15}$  is given in  $\mathbb{P}^4$  by

$$(3.14) \quad \begin{cases} y^3 - x^2z = 0, \\ t^3 - zw^2 = 0, \\ u(xw - z^2) + (z^2 - yt) = 0. \end{cases}$$

In fact, this system of equation defines an effective one-cycle in  $Q_u$  of degree 18, which contains the curve  $\Gamma$ .

Let us show that  $N_3 \cap N_{15}$  contains the curves  $\Psi$  and  $\Psi'$ . To do this, we may consider the subset where  $x \neq 0$ , so that we let  $x = 1$ . Substituting  $z = y^3$  and

$$w = \frac{yt}{u} + \frac{u-1}{u}z^2$$

into  $t^3 - zw^2 = 0$ , we obtain the equation

$$(t - y^5) \left( t^2u^2 + (u^2 - 1)ty^5 + (u - 1)^2y^{10} \right) = 0.$$

If  $t = y^5$ , we get the curve  $\Gamma$ . Thus, the remaining part of the subset (3.14) consists of the  $\mathbb{C}^*$ -orbits of the points

$$\left( 1 : 1 : 1 : t : \frac{t + u - 1}{u} \right)$$

where  $t$  is a solution of the quadratic equation

$$u^2t^2 + (u^2 - 1)t + (u - 1)^2 = 0.$$

Solving this equation, we obtain exactly the points (3.11) and (3.12). This shows that (3.14) contains the curves  $\Psi$  and  $\Psi'$ . This proves assertion (i).

Observe that the intersection  $\mathcal{S} \cap N_3$  consists of the curve  $\Gamma$ , the line  $L_2$ , and the line  $y = z = w = 0$ . Similarly, the intersection  $\mathcal{S} \cap N_{15}$  consists of the curve  $\Gamma$ , the line  $L_1$ , and the line  $x = z = t = 0$ . Thus, the curve  $\Psi$  is contained in  $\mathcal{S}$  if and only if  $\Psi = \Gamma$ . Since  $\mathcal{S}$  is cut out on  $Q_u$  by the equation  $xw = yt$ , we see that if  $\Psi$  is contained in  $\mathcal{S}$ , then

$$\frac{(u-1)(\vartheta - u - 1)}{2u^2} = \frac{(u-1)(2u^2 + \vartheta - u - 1)}{2u^3}.$$

Simplifying this equation, we get  $\vartheta = \frac{3u^2-1}{u-1}$ , which implies that  $u = \frac{2}{3}$ , so that  $\vartheta = 1$  by assumption, which implies that the point (3.11) is not contained in  $\mathcal{S}$ . Hence, we see that  $\Psi$  is not contained in  $\mathcal{S}$ . Similarly, we see that  $\Psi'$  is contained in  $\mathcal{S}$  if and only if  $u = \frac{2}{3}$ . This proves assertion (ii).

Since  $\Psi$  is not contained in  $\mathcal{S}$ , we see that  $\zeta(\Psi)$  is a curve of degree 10 by Lemma 3.7(v). Similarly, if  $u \neq \frac{2}{3}$ , then  $\Psi'$  is not contained in  $\mathcal{S}$ , so that  $\zeta(\Psi')$  is a curve of degree 10 by Lemma 3.7(v) as well. This proves assertions (iii) and (iv).

If  $\Theta_{a,b}$  is not contained in the surface  $\mathcal{S}$  and  $\zeta(\Theta_{a,b})$  is a curve of degree 10, then  $\Theta_{a,b}$  is contained in  $N_3 \cap N_{15}$  by Lemma 3.7(v). On the other hand, the intersection  $N_3 \cap N_{15}$  is given by (3.14). We just proved that this system of equation defines the union  $\Gamma \cup \Psi \cup \Psi'$ , so that either  $\Theta_{a,b} = \Psi$  or  $\Theta_{a,b} = \Psi'$ . This proves assertion (v).

To prove assertions (vi) and (vii), let us find the local equations of the surfaces  $N_3$ ,  $N_{15}$  and  $\mathcal{S}$  at the point  $(1 : 1 : 1 : 1 : 1)$ . We may work in a chart  $x \neq 0$ , so that we let  $x = 1$ . Substituting  $w = \frac{yt}{u} + \frac{u-1}{u}z^2$  into the equation  $t^3 - w^2z = 0$  and multiplying the resulting equation by  $u^2$ , we obtain the equation

$$t^3u^2 - t^2y^2z + 2(1-u)tyz^3 - (u-1)^2z^5 = 0.$$

Similarly, the surface  $\mathcal{S}$  is given by  $ty = z^2$ , and the surface  $N_3$  is given by  $z = y^3$ . Now introducing new coordinates  $\bar{y} = y - 1$ ,  $\bar{z} = z - 1$  and  $\bar{t} = t - 1$ , we see that  $N_{15}$  is given by

$$2\bar{y} + (5u - 4)\bar{z} + (2 - 3u)\bar{t} + \text{higher order terms} = 0.$$

Similarly, the surface  $\mathcal{S}$  is given by

$$(3.15) \quad \bar{y} - 2\bar{z} + \bar{t} + \text{higher order terms} = 0,$$

while the linear term of the defining equation of the surface  $N_3$  is  $3\bar{y} - \bar{z}$ . Hence, the surface  $N_3$  is not tangent to  $\mathcal{S}$  at the point  $(1 : 1 : 1 : 1 : 1)$ . Similarly, we see that the surface  $N_3$  is tangent to  $N_{15}$  at the point  $(1 : 1 : 1 : 1 : 1)$  if and only if  $u = \frac{2}{3}$ . This proves assertions (vi) and (vii).

To prove assertion (viii), we assume that  $u = -\frac{1}{3}$ . Then  $\Psi = \Psi'$ , and the point (3.11) is the point  $(1 : 1 : 1 : 4 : -8)$ . Arguing as above, we see that the local equations of the surfaces  $N_3$  and  $N_{15}$  at the point  $(1 : 1 : 1 : 4 : -8)$  have the same linear part (in coordinates  $\bar{y} = y - 1$ ,  $\bar{z} = z - 1$  and  $\bar{t} = t - 4$ ). Hence, the surface  $N_3$  is tangent to  $N_{15}$  at the point  $(1 : 1 : 1 : 4 : -8)$ . This proves assertion (viii) and completes the proof of the lemma.  $\square$

Recall from Remark 2.11 that the birational map  $\zeta$  in (2.5) induces an isomorphism

$$Q_v \setminus \mathcal{S} \cong V_u \setminus \mathcal{R}.$$

Therefore, from (2.20) and Lemmas 3.7 and 3.13, we obtain an explicit description of all irreducible  $G$ -invariant curves in the Fano threefold  $V_u$  that are not contained in the surface  $\mathcal{R}$ . Thus, to classify all such curves in  $V_u$ , we need to describe those of them that are contained in  $\mathcal{R}$ . This will be done in the next section.

4. INVARIANT CURVES IN THE SURFACE  $\mathcal{R}$ 

In this section we describe irreducible  $G$ -invariant curves in the surface  $\mathcal{R}$ , and complete the classification of irreducible  $G$ -invariant curves in the threefold  $V_u$  (see Proposition 4.12). We will show that  $\mathcal{R}$  contains exactly two irreducible  $G$ -invariant curves, one of which is the conic  $\mathcal{C}_2$ . To describe the other curve, we analyze all irreducible  $G$ -invariant curves in surface  $E_{Q_u}$ . We start with

*Remark 4.1.* Recall from Remark 2.4 that the surface  $\mathcal{S}$  is smooth at every point of the curve  $\Gamma$  except for the points  $(1 : 0 : 0 : 0 : 0)$  and  $(0 : 0 : 0 : 0 : 1)$ , which are isolated ordinary double singularities. This implies that

$$\tilde{\mathcal{S}}|_{E_{Q_u}} = \tilde{\Gamma} + \mathbf{l}_1 + \mathbf{l}_2$$

for some section  $\tilde{\Gamma}$  of the projection  $E_{Q_u} \rightarrow \Gamma$ , where  $\mathbf{l}_1$  and  $\mathbf{l}_2$  are the fibers of this projection over the points  $(1 : 0 : 0 : 0 : 0)$  and  $(0 : 0 : 0 : 0 : 1)$ , respectively. The curve  $\tilde{\Gamma}$  is irreducible and  $G$ -invariant. Since  $\tilde{\Gamma}$  is contained in  $\tilde{\mathcal{S}}$ , its image in  $V_u$  is the conic  $\mathcal{C}_2$ .

Now let us show that  $E_{Q_u}$  contains exactly two irreducible  $G$ -invariant curves.

**Lemma 4.2.** The surface  $E_{Q_u}$  contains exactly two irreducible  $G$ -invariant curves. One of them is the curve  $\tilde{\Gamma}$  from Remark 4.1. The second one is also a section of the projection  $E_{Q_u} \rightarrow \Gamma$ .

*Proof.* Let  $\mathbf{l}$  be the fiber of the natural projection  $E_{Q_u} \rightarrow \Gamma$  over the point  $(1 : 1 : 1 : 1 : 1)$ . Then  $\mathbf{l} \cong \mathbb{P}^1$  and the curve  $\mathbf{l}$  is  $\iota$ -invariant. Thus, either  $\iota$  fixes every point in  $\mathbf{l}$ , or  $\iota$  fixes exactly two points in  $\mathbf{l}$ . Let us show that the former case is impossible. To do this, recall from §2 that

$$\Gamma \subset N_3 \cap N_5 \cap N_8 \cap N_{10} \cap N_{13} \cap N_{15},$$

and the surfaces  $N_3, N_5, N_8, N_{10}, N_{13}, N_{15}$  are smooth at a general point of the curve  $\Gamma$ . Denote by  $\tilde{N}_3, \tilde{N}_5, \tilde{N}_8, \tilde{N}_{10}, \tilde{N}_{13}$  and  $\tilde{N}_{15}$  the proper transforms of the surfaces  $N_3, N_5, N_8, N_{10}, N_{13}$  and  $N_{15}$  on the threefold  $\tilde{Q}_u$ , respectively. Then each intersection

$$\tilde{N}_3 \cap \mathbf{l}, \quad \tilde{N}_5 \cap \mathbf{l}, \quad \tilde{N}_8 \cap \mathbf{l}, \quad \tilde{N}_{10} \cap \mathbf{l}, \quad \tilde{N}_{13} \cap \mathbf{l}, \quad \tilde{N}_{15} \cap \mathbf{l}$$

consists of a single point. Moreover, if  $u \neq \frac{2}{3}$ , then  $N_3$  is not tangent to  $N_{15}$  at a general point of  $\Gamma$  by Lemma 3.13(vi). Hence, in this case, we have

$$\tilde{N}_3 \cap \mathbf{l} \neq \tilde{N}_{15} \cap \mathbf{l},$$

so that the involution  $\iota$  swaps these two points, since  $\iota(N_3) = N_{15}$ . Thus, if  $u \neq \frac{2}{3}$ , then the involution  $\iota$  acts on the curve  $\mathbf{l}$  non-trivially.

Recall that  $\iota(N_5) = N_{13}$ , the surface  $N_5$  is cut out on  $Q_u$  by  $x^2t - y^2z = 0$ , and the surface  $N_{13}$  is cut out on  $Q_u$  by  $yw^2 - zt^2 = 0$ . Let us find out when  $N_5$  is tangent to  $N_{13}$  at a general point of  $\Gamma$ . To do this, let us describe the local equations of the surfaces  $N_5$

and  $N_{13}$  at the point  $(1 : 1 : 1 : 1 : 1)$ . We may work in a chart  $x \neq 0$ , so that we let  $x = 1$ . Substituting

$$w = \frac{yt}{u} + \frac{u-1}{u}z^2$$

into  $yw^2 - zt^2 = 0$  and multiplying the resulting equation by  $u^2$ , we obtain the equation

$$t^2y^3 - u^2t^2z + 2(u-1)ty^2z^2 + (u-1)^2yz^4 = 0.$$

This is the equation of  $N_{13}$ . The equation of the surface  $N_5$  is simply  $t = y^2z$ . Now introducing new coordinates  $\bar{y} = y - 1$ ,  $\bar{z} = z - 1$  and  $\bar{t} = t - 1$ , we see that  $N_{13}$  is given by

$$(u+2)\bar{y} + (3u-4)\bar{z} + 2(1-u)\bar{t} + \text{higher order terms} = 0.$$

Similarly, the surface  $N_{13}$  is given by

$$2\bar{y} + \bar{z} - \bar{t} + \text{higher order terms} = 0.$$

This implies that  $N_5$  is tangent to  $N_{13}$  at the point  $(1 : 1 : 1 : 1 : 1)$  if and only if  $u = 2$ .

Recall from Lemma 3.13(vi) that  $N_3$  is tangent to  $N_{15}$  at a general point of the curve  $\Gamma$  if and only if  $u = \frac{2}{3}$ . We see that  $N_5$  is tangent to the surface  $N_{13}$  at a general point of the curve  $\Gamma$  if and only if  $u = 2$ . The same arguments imply that  $N_8$  is never tangent to  $N_{10}$  at a general point of the curve  $\Gamma$ . Arguing as above, we see that  $\iota$  acts on  $\mathbf{1}$  non-trivially as claimed.

Since  $\iota$  acts non-trivially on the fiber  $\mathbf{1}$ , it fixes two points in  $\mathbf{1}$ . One of them is the point  $\mathbf{1} \cap \tilde{\mathcal{S}}$ . It is contained in  $\tilde{\Gamma}$ , so that  $\tilde{\Gamma}$  is the closure of the  $\mathbb{C}^*$ -orbit of the point  $\mathbf{1} \cap \tilde{\mathcal{S}}$ . Similarly, the closure of the  $\mathbb{C}^*$ -orbit of the second fixed point of the involution  $\iota$  is another irreducible  $G$ -invariant curve in  $E_{Q_u}$ . Then every irreducible  $G$ -invariant curve in  $E_{Q_u}$  must be one of these two curves. Indeed, an irreducible  $G$ -invariant curve in  $E_{Q_u}$  cannot be contracted by  $\pi$ , since  $Q_u$  does not have  $G$ -fixed points. Moreover, since all  $\mathbb{C}^*$ -orbits in  $E_{Q_u}$  that are not contained in the fibers of the projection  $E_{Q_u} \rightarrow \Gamma$  are its sections, we conclude that an intersection of any irreducible  $G$ -invariant curve in  $E_{Q_u}$  with  $\mathbf{1}$  must consist of a  $\iota$ -invariant point, which in turn uniquely determines this curve. Since we proved that  $\mathbf{1}$  contains exactly two  $\iota$ -fixed points, an irreducible  $G$ -invariant curve in  $E_{Q_u}$  must be the closure of the  $\mathbb{C}^*$ -orbit of one of these two points. This completes the proof of the lemma.  $\square$

Thus, the surface  $E_{Q_u}$  contains exactly two irreducible  $G$ -invariant curves. One of them is the curve  $\tilde{\Gamma}$  from Remark 4.1. The second curve can be described rather explicitly.

*Remark 4.3.* Let us use the notation of the proof of Lemma 4.2. Recall from this proof that  $\iota$  fixes exactly two points in  $\mathbf{1}$ . One of them is the point  $\mathbf{1} \cap \tilde{\mathcal{S}}$ . To describe the second  $\iota$ -fixed point in  $\mathbf{1}$ , denote by  $M_{15}^\mu$  the surface in  $Q_u$  that is cut out by the equation

$$g'_{15} + \mu g_{15} = 0,$$

where  $\mu \in \mathbb{C}$ . Denote by  $\tilde{M}_{15}^\mu$  the proper transform of the surface  $M_{15}^\mu$  on the threefold  $\tilde{Q}_u$ . Then  $M_{15}^\mu$  is singular along  $\Gamma$  by Lemma 2.19. Moreover, it has a double point at a general point of  $\Gamma$ . To determine its type, let us describe the local equation of the surface  $M_{15}^\mu$

at the point  $(1 : 1 : 1 : 1 : 1)$ . We may work in the chart  $x \neq 0$ , so that we let  $x = 1$ . Substituting  $x = 1$  and  $w = \frac{yt}{u} + \frac{u-1}{u}z^2$  into  $g'_{15} + \mu g_{15}$  and multiplying the result by  $u^2$ , we obtain the polynomial

$$\begin{aligned} & u^2t^3 + t^2y^5 + (u^2\mu - 2u\mu + \mu + u - 4)t^2y^2z + \\ & + 2(u-1)ty^4z^2 + (8 - 2u^2\mu + 4u\mu - 3u^2 - 2\mu - 4u)tyz^3 + \\ & + (u-1)^2y^3z^4 + (u^2\mu - 2u\mu + u^2 + \mu + 3u - 4)z^5. \end{aligned}$$

Then introducing new coordinates  $\bar{y} = y - 1$ ,  $\bar{z} = z - 1$  and  $\bar{t} = t - 1$ , we rewrite this polynomial as

$$(4.4) \quad \begin{aligned} & (\mu u^2 - 2\mu u + 3u^2 + \mu + u - 3)\bar{t}^2 + \\ & + (2\mu u^2 - 4\mu u - 3u^2 + 2\mu + 8u - 6)\bar{t}\bar{y} + (12 - 4\mu u^2 + 8\mu u - 9u^2 - 4\mu - 6u)\bar{t}\bar{z} + \\ & + (\mu u^2 - 2\mu u + 3u^2 + \mu + 7u - 3)\bar{y}^2 + (12 - 4\mu u^2 + 8\mu u + 3u^2 - 4\mu - 18u)\bar{y}\bar{z} + \\ & + (4\mu u^2 - 8\mu u + 7u^2 + 4\mu + 8u - 12)\bar{z}^2 + \text{higher order terms.} \end{aligned}$$

If  $\mu \neq -\frac{3u^2+16u-16}{4(u-1)^2}$ , then the surface  $M_{15}^\mu$  has an non-isolated ordinary double point at a general point of  $\Gamma$ . Vice versa, if  $\mu = -\frac{3u^2+16u-16}{4(u-1)^2}$ , then the quadratic part of the polynomial (4.4) simplifies as

$$\frac{1}{4} \left( (2+3u)\bar{y} + 4(u-1)\bar{z} + (2-3u)\bar{t} \right)^2.$$

Comparing it with (3.15), we see that the intersection  $\widetilde{M}_{15}^\mu \cap \mathbf{I}$  consists of a single point that is not contained in  $\widetilde{\mathcal{S}}$ . This is the second point fixed in  $\mathbf{I}$  by the involution  $\iota$ .

*Remark 4.5.* Suppose that  $u = \frac{2}{3}$ . Let  $\widetilde{Z}$  be an irreducible  $G$ -invariant curve contained in the surface  $E_{Q_u}$  that is different from the curve  $\widetilde{\Gamma}$ . Denote by  $\widetilde{\Psi}$  the proper transform of the curve  $\Psi$  on the threefold  $\widetilde{Q}_u$ . Let us use the notation from the proof of Lemma 4.2 and Remark 4.3. Then

$$\widetilde{N}_3 \cap \widetilde{N}_{15} = \widetilde{Z} \cup \widetilde{\Psi}$$

by Lemma 3.13(vi), because  $N_3$  is smooth at the point  $(1 : 0 : 0 : 0 : 0)$ , and  $N_{15}$  is smooth at the point  $(0 : 0 : 0 : 0 : 1)$ . Observe also that the curve  $\widetilde{L}_1$  is contained in  $\widetilde{N}_3$ , and it is not contained in  $\widetilde{N}_{15}$ . Similarly, the curve  $\widetilde{L}_2$  is contained in  $\widetilde{N}_{15}$ , and it is not contained in  $\widetilde{N}_3$ . Thus, since  $\widetilde{N}_{15} \cdot \widetilde{L}_1 = 0$  and  $\widetilde{N}_3 \cdot \widetilde{L}_2 = 0$ , we see that  $\widetilde{L}_1$  is disjoint from  $\widetilde{N}_{15}$ , and  $\widetilde{L}_2$  is disjoint from  $\widetilde{N}_3$ . Using (2.5) and (2.20), we see that

$$T_9 \cap T_{21} = \mathcal{C}_2 \cup \zeta(\Psi) \cup \phi \circ \chi(\widetilde{Z}).$$

Moreover, the surfaces  $T_9$  and  $T_{21}$  intersect transversally at a general point of the conic  $\mathcal{C}_2$ , since the surface  $\widetilde{\mathcal{S}}$  does not contain the curves  $\widetilde{Z}$  and  $\widetilde{\Psi}$ . Furthermore, the curve  $\zeta(\Psi)$  has degree 10 by Lemma 3.13(iii). Thus  $\phi \circ \chi(\widetilde{Z})$  is also a curve of degree 10.

*Remark 4.6.* Suppose that  $u = 2$ . Let  $\tilde{Z}$  be an irreducible  $G$ -invariant curve contained in the surface  $E_{Q_u}$  that is different from the curve  $\tilde{\Gamma}$ . Let us use the notation from the proof of Lemma 4.2 and Remark 4.3. In the proof of Lemma 4.2, we showed that both surfaces  $\tilde{N}_5$  and  $\tilde{N}_{13}$  contain the curve  $\tilde{Z}$ . On the other hand, we have

$$N_5 \cap N_{13} = \Gamma \cup \Delta \cup L_1 \cup L_2.$$

Moreover, the surfaces  $N_5$  and  $N_{13}$  are not tangent at a general point of the conic  $\Delta$ . This can be checked, for example, using local equations of the surfaces  $N_5$  and  $N_{13}$  at the point  $(1 : 0 : 2 : 0 : 2)$ . Observe also that the surface  $N_5$  is smooth at the point  $(0 : 0 : 0 : 0 : 1)$ , and the surface  $N_{13}$  is smooth at the point  $(1 : 0 : 0 : 0 : 0)$ . Hence, we deduce that

$$\tilde{N}_5 \cap \tilde{N}_{13} = \tilde{Z} \cup \tilde{\Delta} \cup \tilde{L}_1 \cup \tilde{L}_2,$$

where  $\tilde{\Delta}$  is the proper transform of the conic  $\Delta$ . Moreover, the surfaces  $\tilde{N}_5$  and  $\tilde{N}_{13}$  intersect transversally at a general point of the curve  $\tilde{Z}$ . Indeed, otherwise the curve  $\Gamma$  would be contained in the one-cycle  $N_5 \cdot N_{13}$  with multiplicity at least 3, which is impossible, since  $H_{Q_u} \cdot N_5 \cdot N_{13} = 18$ , and the one-cycle  $N_5 \cdot N_{13}$  also contains the conic  $\Delta$  and the lines  $L_1$  and  $L_2$ . Thus, keeping in mind that the curves  $\tilde{L}_1$  and  $\tilde{L}_2$  are contracted by  $\alpha$ , we conclude that

$$\alpha(\tilde{N}_5) \cap \alpha(\tilde{N}_{13}) = \alpha(\tilde{Z}) \cup \gamma(\Delta).$$

On the other hand, the degree of the curve  $\gamma(\Delta)$  is 4, one has  $-K_{Y_u}^3 = 16$  and

$$\alpha(\tilde{N}_5) \sim \alpha(\tilde{N}_{13}) \sim -K_{Y_u}.$$

This implies that  $\alpha(\tilde{Z})$  is a curve of degree 12, because  $\alpha(\tilde{N}_5)$  and  $\alpha(\tilde{N}_{13})$  intersect transversally at general points of the curves  $\alpha(\tilde{Z})$  and  $\gamma(\Delta)$ . Denote by  $\tilde{C}$  the proper transform of the curve  $\tilde{Z}$  on the threefold  $\tilde{V}_u$ . Then

$$\begin{aligned} 12 = \deg(\alpha(\tilde{Z})) &= -K_{\tilde{Q}_u} \cdot \tilde{Z} = -K_{Y_u} \cdot \alpha(\tilde{Z}) = -K_{Y_u} \cdot \beta(\tilde{C}) = -K_{\tilde{V}_u} \cdot \tilde{C} = \\ &= \left( \phi^*(H_{V_u}) - E_{V_u} \right) \cdot \tilde{C} \leq \phi^*(H_{V_u}) \cdot \tilde{C} = H_{V_u} \cdot \tilde{C} = \deg(\phi(\tilde{C})). \end{aligned}$$

We conclude our investigation of irreducible  $G$ -invariant curves in  $E_{Q_u}$  by the following result, which also completes the description of irreducible  $G$ -invariant curves in  $V_u$  of degree 10 started in Lemma 3.13 and Remark 4.5.

**Lemma 4.7.** Let  $\tilde{Z}$  be an irreducible  $G$ -invariant curve contained in the surface  $E_{Q_u}$ . Then one of the following two possibilities holds.

- The curve  $\tilde{Z}$  is the curve  $\tilde{\Gamma}$  from Remark 4.1. The curve  $\phi \circ \chi(\tilde{Z})$  is the conic  $\mathcal{C}_2$ . The degree of the curve  $\alpha(\tilde{Z})$  is at least 12.
- The curve  $\tilde{Z}$  is the unique irreducible  $G$ -invariant curve in  $E_{Q_u}$  not contained in  $\tilde{\mathcal{S}}$ . If  $u \neq \frac{2}{3}$ , then  $\deg(\phi \circ \chi(\tilde{Z})) \geq 12$ . If  $u = \frac{2}{3}$ , then  $\deg(\phi \circ \chi(\tilde{Z})) = 10$ , and the curve  $\phi \circ \chi(\tilde{Z})$  is contained in  $T_9 \cap T_{21}$ .

*Proof.* The normal bundle of the smooth rational curve  $\Gamma$  in  $Q_u$  is isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(p) \oplus \mathcal{O}_{\mathbb{P}^1}(q)$  for some integers  $p$  and  $q$  such that  $p \geq q$  and  $p + q = 16$ . Thus, the exceptional surface  $E_{Q_u}$  is a Hirzebruch surface  $\mathbb{F}_n$  for  $n = p - q \geq 0$ . Denote by  $\mathbf{s}$  the section of the natural projection  $E_{Q_u} \rightarrow \Gamma$  such that  $\mathbf{s}^2 = -n$ . Then  $-E_{Q_u}|_{E_{Q_u}} \sim \mathbf{s} + \kappa \mathbf{1}$  for some integer  $\kappa$ . One has

$$-16 = E_{Q_u}^3 = (\mathbf{s} + \kappa \mathbf{1})^2 = -n + 2\kappa,$$

so that  $\kappa = \frac{n-16}{2}$ . This implies that  $\tilde{\mathcal{S}}|_{E_{Q_u}} \sim \mathbf{s} + \frac{n+8}{2}\mathbf{1}$ . On the other hand, it follows from Remark 4.1 that  $\tilde{\mathcal{S}}|_{E_{Q_u}} = \tilde{\Gamma} + \mathbf{l}_1 + \mathbf{l}_2$ , where  $\mathbf{l}_1$  and  $\mathbf{l}_2$  are the fibers of the natural projection  $E_{Q_u} \rightarrow \Gamma$  over the points  $(1 : 0 : 0 : 0 : 0)$  and  $(0 : 0 : 0 : 0 : 1)$ , respectively. This gives  $\tilde{\Gamma} \sim \mathbf{s} + \frac{n+4}{2}\mathbf{1}$ , which implies, in particular, that  $\tilde{\Gamma} \neq \mathbf{s}$ . Hence, we have

$$0 \leq \tilde{\Gamma} \cdot \mathbf{s} = \left(\mathbf{s} + \frac{n+4}{2}\mathbf{1}\right) \cdot \mathbf{s} = \frac{4-n}{2},$$

which implies that  $n \leq 4$ . Thus, we compute

$$(4.8) \quad \deg(\alpha(\tilde{Z})) = -K_{\tilde{Q}_u} \cdot \tilde{Z} = \left(3\pi^*(H_{Q_u}) - E_{Q_u}\right) \cdot \tilde{Z} = \left(\mathbf{s} + \frac{n+20}{2}\mathbf{1}\right) \cdot \tilde{Z}.$$

In particular, if  $\tilde{Z} = \tilde{\Gamma}$ , then (4.8) gives

$$\deg(\alpha(\tilde{Z})) = \left(\mathbf{s} + \frac{n+20}{2}\mathbf{1}\right) \cdot \left(\mathbf{s} + \frac{n+4}{2}\mathbf{1}\right) = 12.$$

Let  $\tilde{C}$  be the proper transform of the curve  $\tilde{Z}$  on the threefold  $\tilde{V}_u$ , and let  $C = \phi(\tilde{C})$ . If  $\tilde{Z} \neq \tilde{\Gamma}$ , then

$$(4.9) \quad \begin{aligned} \deg(\alpha(\tilde{Z})) &= -K_{\tilde{Q}_u} \cdot \tilde{Z} = -K_{Y_u} \cdot \alpha(\tilde{Z}) = -K_{Y_u} \cdot \beta(\tilde{C}) = -K_{\tilde{V}_u} \cdot \tilde{C} = \\ &= \left(\phi^*(H_{V_u}) - E_{V_u}\right) \cdot \tilde{C} \leq \phi^*(H_{V_u}) \cdot \tilde{C} = H_{V_u} \cdot \tilde{C} = \deg(C). \end{aligned}$$

Now let us use the notation from the proof of Lemma 4.2 and Remark 4.3. To complete the proof, we may assume that  $\tilde{Z}$  is the closure of the  $\mathbb{C}^*$ -orbit of the point  $\tilde{M}_{15}^\mu \cap \mathbf{1}$ . Then  $\tilde{Z}$  is contained in  $\tilde{M}_{15}^\mu$ , it is a section of the natural projection  $E_{Q_u} \rightarrow \Gamma$ , and it is not contained in  $\tilde{\mathcal{S}}$ . In particular, we have  $\tilde{Z} \neq \tilde{\Gamma}$ .

By Remarks 4.5 and 4.6, we may assume that  $u \neq \frac{2}{3}$  and  $u \neq 2$ . This implies that  $n = 0$ , cf. Remark 4.10 below. Indeed, suppose that  $n > 0$ . Then  $\tilde{Z} = \mathbf{s}$  by Lemma 4.2, because the curve  $\mathbf{s}$  is clearly  $G$ -invariant. Then it follows from (4.8) that

$$\deg(\alpha(\tilde{Z})) = -K_{\tilde{Q}_u} \cdot \tilde{Z} = \frac{20-n}{2} < 10.$$

Hence, at least one surface among  $\tilde{N}_3, \tilde{N}_5, \tilde{N}_8, \tilde{N}_{10}, \tilde{N}_{13}$  and  $\tilde{N}_{15}$  contains the curve  $\tilde{Z}$ . Since  $\iota(\tilde{N}_3) = \tilde{N}_{15}$ ,  $\iota(\tilde{N}_5) = \tilde{N}_{13}$  and  $\iota(\tilde{N}_8) = \tilde{N}_{10}$ , this implies that  $\tilde{Z}$  is contained in at least one of the intersections  $\tilde{N}_3 \cap \tilde{N}_{15}, \tilde{N}_5 \cap \tilde{N}_{13}, \tilde{N}_8 \cap \tilde{N}_{10}$ . On the other hand, it follows

from Lemma 3.13(vi) that  $N_3$  is tangent to  $N_{15}$  at a general point of the curve  $\Gamma$  if and only if  $u = \frac{2}{3}$ . Since we assumed that  $u \neq \frac{2}{3}$ , we see that

$$\tilde{Z} \not\subset \tilde{N}_3 \cap \tilde{N}_{15}.$$

Likewise, the surface  $N_5$  is tangent to the surface  $N_{13}$  at a general point of the curve  $\Gamma$  if and only if  $u = 2$ . We showed this in the proof of Lemma 4.2. Similar computations imply that the surface  $N_8$  is not tangent to  $N_{10}$  at a general point of the curve  $\Gamma$ . Therefore, the curve  $\tilde{Z}$  is contained neither in  $\tilde{N}_5 \cap \tilde{N}_{13}$  nor in  $\tilde{N}_8 \cap \tilde{N}_{10}$ . The obtained contradiction shows that the case  $n > 0$  is impossible, so that  $n = 0$ .

Since  $n = 0$ , one has  $E_{Q_u} \cong \mathbb{P}^1 \times \mathbb{P}^1$ . By (4.8), we have

$$-K_{\tilde{Q}_u} \cdot \tilde{Z} = (\mathbf{s} + 10\mathbf{l}) \cdot \tilde{Z} \geq (\mathbf{s} + 10\mathbf{l}) \cdot \mathbf{s} = 10.$$

This also shows that  $-K_{\tilde{Q}_u} \cdot \tilde{Z} = 10$  if and only if  $\tilde{Z} \sim \mathbf{s}$ . However, this case is impossible. Indeed, if  $\tilde{Z} \sim \mathbf{s}$ , then the linear system  $|\mathbf{s}|$  contains at least two irreducible  $G$ -invariant curves. On the other hand, we already know from Lemma 4.2 that  $\tilde{Z}$  and  $\tilde{\Gamma} \sim \mathbf{s} + 2\mathbf{l}$  are the only irreducible  $G$ -invariant curves in the surface  $E_{Q_u}$ . Hence, using (4.9) we conclude that  $\deg(C) \geq -K_{\tilde{Q}_u} \cdot \tilde{Z} \geq 11$ .

Using Lemma 3.7, we see that  $V_u$  does not contain irreducible  $G$ -invariant curves of degree 1, 3, 5, 7, 8 and 9. In particular, the threefold  $V_u$  does not contain  $G$ -invariant lines, which also follows from [KP17, Lemma 4.1(i)].

By Remark 3.10, there exists a unique surface in the pencil generated by  $T_{15}$  and  $T'_{15}$  that contains  $C$ . In fact, we know this surface from Remark 4.3. It is the image of the surface  $\tilde{M}_{15}^\mu$  from Remark 4.3, where  $\mu = -\frac{3u^2+16u-16}{4(u-1)^2}$ . Thus, if  $\deg(C) = 11$ , there should be at least one surface among  $T_9, T_{10}, T_{11}, T_{12}, T_{13}, T_{14}, T_{16}, T_{17}, T_{18}, T_{19}, T_{20}, T_{21}$  that also contains  $C$ . But we proved above that none of the surfaces  $\tilde{N}_3, \tilde{N}_5, \tilde{N}_8, \tilde{N}_{10}, \tilde{N}_{13}, \tilde{N}_{15}$  contains the curve  $\tilde{Z}$ , so that the surfaces  $T_9, T_{11}, T_{14}, T_{16}, T_{19}$  and  $T_{21}$  do not contain  $C$  either. Similarly, the surfaces  $T_{12}, T_{13}, T_{17}$  and  $T_{18}$  do not contain the curve  $C$ , because the surfaces  $H_x, H_y, H_z, H_t$  and  $H_w$  do not contain the curve  $\Gamma$ . Thus, to complete the proof, we may assume that either  $T_{10}$  or  $T_{20}$  contains the curve  $C$ . Actually, this assumption implies that both surfaces  $T_{10}$  and  $T_{20}$  contain the curve  $C$ , since  $\iota(T_{10}) = T_{20}$ . Note that this case is indeed possible when  $u = -2$  by Remark 4.11 below.

By Lemma 3.7, both surfaces  $T_{10}$  and  $T_{20}$  contain the curves  $\zeta(\Delta)$  and  $\zeta(\Upsilon)$ , the degree of the curve  $\zeta(\Delta)$  is 4, and the degree of the curve  $\zeta(\Upsilon)$  is 6. Since we already know that  $\deg(C) \geq 11$ , we see that the  $G$ -invariant one-cycle  $T_{10} \cdot T_{20}$  consists of the curves  $\zeta(\Delta)$ ,  $\zeta(\Upsilon)$ ,  $C$  and a  $G$ -invariant curve of degree  $12 - \deg(C)$ . Since  $V_u$  does not contain  $G$ -invariant lines, we see that

$$T_{10} \cdot T_{20} = \zeta(\Delta) + \zeta(\Upsilon) + C,$$

so that  $\deg(C) = 12$ . This completes the proof of the lemma.  $\square$

*Remark 4.10.* If  $u \neq \frac{2}{3}$  and  $u \neq 2$ , then  $E_{Q_u} \cong \mathbb{P}^1 \times \mathbb{P}^1$ , so that the normal bundle of the curve  $\Gamma$  in the quadric  $Q_u$  is isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(8) \oplus \mathcal{O}_{\mathbb{P}^1}(8)$ . We showed this in the proof



of Lemma 4.7. Vice versa, if  $u = \frac{2}{3}$  or  $u = 2$ , then, arguing as in the proof of Lemma 4.7, one can show that  $E_{Q_u} \cong \mathbb{F}_4$ , so that the normal bundle of the curve  $\Gamma$  is  $\mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(10)$  in this case. However, we will not use this information in the sequel.

*Remark 4.11.* Denote by  $\widetilde{M}_{10}$  and  $\widetilde{M}_{20}$  the proper transform of the surfaces  $M_{10}$  and  $M_{20}$  on the threefold  $\widetilde{Q}_u$ , respectively. Recall that both  $M_{10}$  and  $M_{20}$  has quadratic singularity at the point  $(1 : 1 : 1 : 1 : 1)$ . Substituting  $x = 1$  and  $w = \frac{yt}{u} + \frac{u-1}{u}z^2$  into the polynomial  $ug_{10}$ , we obtain the polynomial  $ut^2 + ty^5 - (2u+1)y^2zt + (u-1)y^4z^2 + yz^3$ . The quadratic part of its local expansion at the point  $(1 : 1 : 1 : 1 : 1)$  is

$$u\bar{t}^2 + (3-4u)\bar{y}\bar{t} - (2u+1)\bar{t}\bar{z} + (4u+3)\bar{y}^2 + (4u-7)\bar{y}\bar{z} + (u+2)\bar{z}^2,$$

where  $\bar{y} = y-1$ ,  $\bar{z} = z-1$  and  $\bar{t} = t-1$ . Similarly, substituting  $x = 1$  and  $w = \frac{yt}{u} + \frac{u-1}{u}z^2$  into the polynomial  $u^3g_{20}$ , we obtain the polynomial

$$\begin{aligned} u^3t^4 + t^3y^5 - (2u^2+u)t^3y^2z + (3u-3)t^2y^4z^2 + (-2u^3+u^2+2u)t^2yz^3 + \\ + (3u^2-6u+3)ty^3z^4 + (u^2-u)tz^5 + (u^3-3u^2+3u-1)y^2z^6. \end{aligned}$$

Then the quadratic part of the local expansion of the polynomial  $u^2g_{20}$  is

$$\begin{aligned} (4u^2-5u+2)\bar{t}^2 + (4-4u^2-u)\bar{y}\bar{t} - (12u^2-17u+8)\bar{t}\bar{z} + \\ + (u^2+4u+2)\bar{y}^2 + (6u^2-u-8)\bar{y}\bar{z} + (9u^2-14u+8)\bar{z}^2. \end{aligned}$$

Both these quadric forms are degenerate, so that they define reducible conics in  $\mathbb{P}_{\bar{y},\bar{z},\bar{t}}^2$ . If  $u \neq -2$ , then these conics do not have common components. However, if  $u = -2$ , then the former quadratic form is  $(\bar{t} - 5\bar{y})(\bar{y} + 3\bar{z} - 2\bar{t})$ , and the latter quadratic form is  $4(\bar{y} - 12\bar{z} + 7\bar{t})(\bar{y} + 3\bar{z} - 2\bar{t})$ . Note that the quadratic part of the polynomial (4.4) is a multiple of  $(\bar{y} + 3\bar{z} - 2\bar{t})^2$ . Thus, if  $u = -2$ , then  $\widetilde{M}_{10} \cap \widetilde{M}_{20}$  contains the irreducible  $G$ -invariant curve in  $E_{Q_u}$  that is different from the curve  $\widetilde{\Gamma}$ , see Remark 4.1.

Recall that  $\zeta(\mathcal{S}) = \mathcal{C}_2$ . Denote the curves  $\zeta(\Delta)$  and  $\zeta(\Upsilon)$  by  $\mathcal{C}_4$  and  $\mathcal{C}_6$ , respectively. Similarly, if  $u \neq \frac{2}{3}$ , let  $\mathcal{C}_{10} = \zeta(\Psi)$  and  $\mathcal{C}'_{10} = \zeta(\Psi')$ . Finally, if  $u = \frac{2}{3}$ , let  $\mathcal{C}_{10} = \zeta(\Psi)$  and let  $\mathcal{C}'_{10} = \phi \circ \chi(\widetilde{Z})$ , where  $\widetilde{Z}$  is the irreducible  $G$ -invariant curve in  $E_{Q_u}$  that is different from the curve  $\widetilde{\Gamma}$ .

**Proposition 4.12.** Let  $C$  be an irreducible  $G$ -invariant curve in  $V_u$  with  $\deg(C) < 12$ . Then either  $C = \mathcal{C}_2$ , or  $C = \mathcal{C}_4$ , or  $C = \mathcal{C}_6$ , or  $C = \mathcal{C}_{10}$ , or  $C = \mathcal{C}'_{10}$ .

*Proof.* We may assume that  $C \neq \mathcal{C}_2$ . Denote by  $\widetilde{C}$  the proper transform of the curve  $C$  on the threefold  $\widetilde{V}_u$ . By Remark 2.11, the curve  $\widetilde{C}$  is not flopped by  $\chi^{-1}$ . Denote by  $\widetilde{Z}$  the proper transform of the curve  $\widetilde{C}$  on the threefold  $\widetilde{Q}_u$ . Then  $\widetilde{Z}$  is not contracted by  $\pi$ , since  $Q_u$  does not have  $G$ -fixed points by Lemma 2.23.

Let  $Z = \pi(\widetilde{Z})$ . Then  $Z$  is an irreducible  $G$ -invariant curve. Hence, the curve  $Z$  is either the curve  $\Theta_{\pm}$ , or the curve  $\Theta_{a,b}$  for some  $(a : b) \in \mathbb{P}^1$ . Therefore, if  $Z$  is not contained in  $\mathcal{S}$ , the required assertion follows from Lemmas 3.7 and 3.13. Thus, we may assume that  $Z \subset \mathcal{S}$ , which implies that  $Z = \Gamma$ , because  $C \neq \mathcal{C}_2$  by assumption. This simply

means that  $\tilde{Z}$  is contained in the exceptional surface  $E_{Q_u}$ . Then  $u = \frac{2}{3}$  and  $Z = \mathcal{C}'_{10}$  by Lemma 4.7.  $\square$

Using Remark 2.21 and Lemmas 3.13 and 4.7, we see that

$$(4.13) \quad T_9 \cdot T_{21} = \mathcal{C}_{10} + \mathcal{C}'_{10} + \mathcal{C}_2.$$

## 5. ANTICANONICAL PENCIL

Let  $\mathcal{P}_{Q_u}$  be the pencil of surfaces in  $|5H_{Q_u}|$  that are cut out on  $Q_u$  by

$$\mu_0 g_{15} + \mu_1 g'_{15} = 0,$$

where  $(\mu_0 : \mu_1) \in \mathbb{P}^1$ . Here  $g_{15}$  is the polynomial of weight 15 in (2.17), and  $g'_{15}$  is the polynomial of weight 15 in (2.18). Then the pencil  $\mathcal{P}_{Q_u}$  is free from base components.

Denote by  $\mathcal{P}_{V_u}$  the proper transform of the pencil  $\mathcal{P}_{Q_u}$  on the threefold  $V_u$ . Then  $\mathcal{P}_{V_u}$  is generated by the irreducible surfaces  $T_{15}$  and  $T'_{15}$ , and it contains all  $G$ -invariant surfaces in the linear system  $|-K_{V_u}|$ . This follows from (2.20).

By Lemma 2.22, the base locus of the pencil  $\mathcal{P}_{V_u}$  contains the lines  $\ell_1$  and  $\ell_2$  from Remark 2.11. Similarly, we know from Lemma 3.7(i) that the base locus of the pencil  $\mathcal{P}_{V_u}$  contains the curve  $\zeta(\Theta_{\pm})$ . Thus, using Remark 3.10 and Proposition 4.12, we obtain

**Corollary 5.1.** The curve  $\zeta(\Theta_{\pm})$  is the only irreducible  $G$ -invariant curve in  $V_u$  which is contained in the base locus of the pencil  $\mathcal{P}_{V_u}$ .

Therefore, for every irreducible  $G$ -invariant curve in  $V_u$  that is different from  $\zeta(\Theta_{\pm})$ , there exists a unique surface in the pencil  $\mathcal{P}_{V_u}$  that contains this curve. In particular, the pencil  $\mathcal{P}_{V_u}$  contains a unique surface that passes through  $\mathcal{C}_4$ , and it contains a unique surface that passes through  $\mathcal{C}_6$ . Below we describe both of them.

**Lemma 5.2.** The curve  $\mathcal{C}_6$  is not contained in  $T'_{15}$ . On the other hand, the curve  $\mathcal{C}_4$  is contained in  $T'_{15}$ . Moreover, the surface  $T'_{15}$  is singular along the curve  $\mathcal{C}_4$ . If  $u \neq 2$ , then  $T'_{15}$  has a non-isolated ordinary double point at a general point of the curve  $\mathcal{C}_4$ . If  $u = 2$ , then  $T'_{15}$  has a non-isolated ordinary triple point at general point of the curve  $\mathcal{C}_4$ .

*Proof.* Recall from (2.18) that

$$g'_{15} = (u-1)x^2t^3 + (u-1)y^3w^2 - (u+4)y^2zt^2 + (3u+2)xyztw + (4-4u)yz^3t.$$

Substituting (3.6) into  $g'_{15}$ , we see that  $\Upsilon$  is not contained in  $M'_{15}$ , so that  $\mathcal{C}_6$  is not contained in  $T'_{15}$ . Similarly, substituting (3.5) into  $g'_{15}$ , we see that  $\Delta$  is contained in  $M'_{15}$ , so that  $\mathcal{C}_4$  is contained in  $T'_{15}$ .

To describe the singularity of the surface  $T'_{15}$  at a general point of the curve  $\mathcal{C}_4$ , it is enough to describe the singularity of the surface  $M'_{15}$  at a general point of the curve  $\Delta$ . The latter point has the form  $(\frac{u-1}{u}\tau^2 : 0 : \tau : 0 : 1)$  with  $\tau \in \mathbb{C}^*$ . Substituting  $w = 1$  and  $x = z^2 + \frac{ty-z^2}{u}$  into  $g'_{15} = 0$  and multiplying the resulting equation by  $\frac{u^2}{u-1}$ , we obtain

$$(5.3) \quad -u(u-2)tyz^3 + u^2y^3 + (u-1)^2t^3z^4 - u(u+2)t^2y^2z + 2(u-1)t^4yz^2 + t^5y^2 = 0.$$

Thus, at a general point of the curve  $\mathcal{C}_4$ , the surface  $M'_{15}$  has singularity locally isomorphic to the product of  $\mathbb{C}$  and the germ of the curve singularity given by

$$-u(u-2)ty + u^2y^3 + (u-1)^2t^3 - u(u+2)t^2y^2 + 2(u-1)t^4y + t^5y^2 = 0.$$

If  $u \neq 2$ , the quadratic part  $-u(u-2)ty$  of the left hand side is non-degenerate, so that  $M'_{15}$  has a non-isolated ordinary double point at  $P$ . If  $u = 2$ , the above equation becomes  $t^3 + 4y^3 - 8t^2y^2 + 2t^4y + t^5y^2 = 0$ , which defines an ordinary triple point (also known as curve singularity of type  $\mathbf{D}_4$ ), and the assertion follows.  $\square$

**Corollary 5.4.** If  $u = 2$ , then  $\alpha_G(V_u) \leq \frac{2}{3}$ .

Let  $g''_{15} = ug_{15} + g'_{15}$ . Then

$$g''_{15} = (u-1)x^2t^3 + (u-1)y^3w^2 - 4y^2zt^2 + (u+2)xyztw - 4(u-1)yz^3t + ux^2zw^2.$$

Denote by  $M''_{15}$  the surface in the quadric  $Q_u$  that is cut out by  $g''_{15} = 0$ . Let  $T''_{15}$  be its proper transform on the threefold  $V_u$ . Then  $T''_{15}$  is an irreducible surface in  $\mathcal{P}_{V_u}$ .

**Lemma 5.5.** The curve  $\mathcal{C}_4$  is not contained in  $T''_{15}$ . On the other hand, the curve  $\mathcal{C}_6$  is contained in  $T''_{15}$ . Moreover, the surface  $T''_{15}$  is singular along the curve  $\mathcal{C}_6$ . If  $u \neq \frac{3}{4}$ , then  $T''_{15}$  has a non-isolated ordinary double point at a general point of the curve  $\mathcal{C}_6$ . If  $u = \frac{3}{4}$ , then  $T''_{15}$  has a non-isolated tacnodal singularity at a general point of the curve  $\mathcal{C}_6$ .

*Proof.* Substituting (3.5) into  $g''_{15}$ , we see that  $\Delta \not\subset M''_{15}$ , so that  $\mathcal{C}_4 \not\subset T''_{15}$ . Similarly, substituting (3.6) into  $g''_{15}$ , we see that  $\Upsilon \subset M''_{15}$ , so that  $\mathcal{C}_6 \subset T''_{15}$ .

To describe the singularity of the surface  $T''_{15}$  at a general point of the curve  $\mathcal{C}_6$ , it is enough to describe the singularity of the surface  $M''_{15}$  at a general point of the curve  $\Upsilon$ . The latter point has the form  $P = (0 : (1-u)\tau^2 : \tau : 1 : 0)$  with  $\tau \in \mathbb{C}^*$ .

Substituting  $t = 1$  and  $y = z^2 + u(wx - z^2)$  into  $g''_{15} = 0$  and dividing the resulting equation by  $(u-1)$ , we obtain

$$x^2 + (3u-2)z^3xw - (u-1)^3w^2z^6 + 3u(u-1)^2z^4xw^3 - 3uw^2x^2z - 3u^2(u-1)z^2x^2w^4 + u^3w^5x^3 = 0.$$

Thus, at a general point of the curve  $\mathcal{C}_6$ , the surface  $M''_{15}$  has singularity locally isomorphic to the product of  $\mathbb{C}$  and the germ of the curve singularity given by

$$x^2 + (3u-2)xw - (u-1)^3w^2 + 3u(u-1)^2xw^3 - 3uw^2x^2 - 3u^2(u-1)x^2w^4 + u^3w^5x^3 = 0.$$

If  $u \neq \frac{3}{4}$ , the quadratic part  $x^2 + (3u-2)xw - (u-1)^3w^2$  of the left hand side is non-degenerate, so that  $M''_{15}$  has a non-isolated ordinary double point at  $P$ . If  $u = \frac{3}{4}$ , the above equation becomes  $w^2 + 16wx + 64x^2 + 9w^3x - 144w^2x^2 + 27w^4x^2 + 27w^5x^3 = 0$ . So, introducing new auxiliary coordinates  $w = v - 8x$ , we get

$$\begin{aligned} &v^2 - 13824x^4 + 4032vx^3 + 110592x^6 - 360v^2x^2 + \\ &+ 9v^3x - 55296vx^5 + 10368v^2x^4 - 884736x^8 + 552960vx^7 - 864v^3x^3 + \\ &+ 27v^4x^2 - 138240v^2x^6 + 17280v^3x^5 - 1080v^4x^4 + 27v^5x^3 = 0. \end{aligned}$$

This equation defines a tacnodal point (also known as curve singularity of type  $\mathbf{A}_3$ ), and the assertion follows.  $\square$

**Corollary 5.6.** If  $u = \frac{3}{4}$ , then  $\alpha_G(V_u) \leq \frac{3}{4}$ .

*Proof.* Suppose that  $u = \frac{3}{4}$ . Recall that  $T''_{15} \sim -K_{V_u}$ . Since  $T''_{15}$  has a tacnodal singularity at a general point of the curve  $\mathcal{C}_6$  by Lemma 5.5, the log pair  $(V_u, \frac{3}{4}T''_{15})$  is not Kawamata log terminal. Hence  $\alpha_G(V_u) \leq \frac{3}{4}$ .  $\square$

## 6. SARKISOV LINKS

Let  $\mathcal{C}$  be one of the irreducible  $G$ -invariant curves  $\mathcal{C}_4$  or  $\mathcal{C}_6$  in the threefold  $V_u$ , let  $\sigma: \widehat{V}_u \rightarrow V_u$  be the blow up of the curve  $\mathcal{C}$ , and let  $E_\sigma$  be the exceptional surface of  $\sigma$ . Denote by  $\widehat{T}_i, \widehat{T}'_{15}, \widehat{T}''_{15}$  the proper transforms on  $\widehat{V}_u$  of the surfaces  $T_i, T'_{15}, T''_{15}$ , respectively.

*Remark 6.1.* Suppose that  $\mathcal{C} = \mathcal{C}_4$ . Then  $\widehat{T}'_{15} \sim \sigma^*(H_{V_u}) - m'E_\sigma$ , where  $m' = \text{mult}_{\mathcal{C}}(T'_{15})$ . By Lemma 5.2, one has

$$m' = \begin{cases} 2 & \text{if } u \neq 2, \\ 3 & \text{if } u = 2. \end{cases}$$

Moreover, if  $u \neq 2$ , then  $T'_{15}$  has a non-isolated ordinary double point at a general point of the curve  $\mathcal{C}$ . In this case, one has

$$\widehat{T}'_{15}|_{E_\sigma} = \widehat{\mathcal{C}} + \varkappa(\mathbf{l}_1 + \mathbf{l}_2),$$

where  $\widehat{\mathcal{C}}$  is a 2-section of the natural projection  $E_\sigma \rightarrow \mathcal{C}_4$ , the curves  $\mathbf{l}_1$  and  $\mathbf{l}_2$  are the fibers of this projection over two  $\mathbb{C}^*$ -fixed points in  $\mathcal{C}_4$ , respectively, and  $\varkappa$  is a non-negative integer. Moreover, it can be seen from (5.3) that the curve  $\widehat{\mathcal{C}}$  is reducible, so that it consists of two sections of the projection  $E_\sigma \rightarrow \mathcal{C}$ . However, the curve  $\widehat{\mathcal{C}}$  is  $G$ -irreducible. This follows from (2.2) and (5.3).

Let us show that the divisor  $-K_{\widehat{V}_u} \sim \sigma^*(H_{V_u}) - E_\sigma$  is nef.

**Lemma 6.2.** Suppose that  $\mathcal{C} = \mathcal{C}_4$ . Then  $\sigma^*(H_{V_u}) - E_\sigma$  is nef.

*Proof.* Recall from (3.5) that the conic  $\Delta$  is the scheme-theoretic intersection of the surfaces  $H_y$  and  $H_t$ . Moreover, it follows from (3.8) that  $\mathcal{C}_4$  is contained in the intersection

$$(6.3) \quad T_{10} \cap T_{11} \cap T_{13} \cap T_{14} \cap T'_{15} \cap T_{16} \cap T_{17} \cap T_{19} \cap T_{20}.$$

Recall also that  $T_{13}$  is the proper transform on  $V_u$  of the surface  $H_y$ , and the surface  $T_{17}$  is the proper transform on  $V_u$  of the surface  $H_t$ . Thus, using Remark 2.21 and Lemma 2.22, we see that the intersection  $T_{13} \cap T_{17}$  consists of the curve  $\mathcal{C}_4$ , the conic  $\mathcal{C}_2$ , the lines  $\ell_1$  and  $\ell_2$  from Remark 2.11, and the proper transform on  $V_u$  of the fibers of  $\pi$  over the points  $(1 : 0 : 0 : 0 : 0)$  and  $(0 : 0 : 0 : 0 : 1)$ .

Recall that  $T_{11}$  is the proper transform on  $V_u$  of the surface  $N_5$ , and the surface  $T_{19}$  is the proper transform on  $V_u$  of the surface  $N_{13}$ . Since  $N_5$  contains  $\Gamma$  and is smooth at the point  $(1 : 0 : 0 : 0 : 0)$ , the surface  $\widetilde{N}_5$  does not contain the fiber of  $\pi$  over this point. Similarly, the surface  $\widetilde{N}_{13}$  does not contain the fiber of  $\pi$  over the point  $(0 : 0 : 0 : 0 : 1)$ .

Hence, using Remark 2.21 again, we see that the only curves contained in the intersection  $T_{11} \cap T_{13} \cap T_{17} \cap T_{19}$  are the conic  $\mathcal{C}_2$ , the curve  $\mathcal{C}_4$ , and the lines  $\ell_1$  and  $\ell_2$ .

By Remark 2.21, the surface  $T'_{15}$  does not contain the conic  $\mathcal{C}_2$ . Similarly, it follows from Lemma 2.22 that the intersection  $T_{10} \cap T_{20}$  contains neither  $\ell_1$  nor  $\ell_2$ . Thus, we see that  $\mathcal{C}_4$  is the only curve contained in the intersection (6.3).

The base locus of the linear system  $|\sigma^*(H_{V_u}) - E_\sigma|$  does not contain any curves outside the exceptional surface  $E_\sigma$ . Moreover, the surfaces  $T_{13}$  and  $T_{17}$  intersect transversally at a general point of the curve  $\mathcal{C}_4$ , because the surfaces  $H_y$  and  $H_t$  intersect transversally at every point of the conic  $\Delta$ . Hence, the base locus of the linear system  $|\sigma^*(H_{V_u}) - E_\sigma|$  does not contain curves, with the only possible exception of finitely many fibers of the projection  $E_\sigma \rightarrow \mathcal{C}_4$ . This implies the required assertion.  $\square$

**Lemma 6.4.** Suppose that  $\mathcal{C} = \mathcal{C}_6$ . Then  $\sigma^*(H_{V_u}) - E_\sigma$  is nef.

*Proof.* Recall from (3.6) that the conic  $\Upsilon$  is the scheme-theoretic intersection of the surfaces  $H_x$  and  $H_w$ . Moreover, it follows from (3.9) that  $\mathcal{C}_6$  is contained in the intersection

$$(6.5) \quad T_{10} \cap T_{12} \cap T_{14} \cap T''_{15} \cap T_{16} \cap T_{18} \cap T_{20}.$$

Recall also that  $T_{12}$  is the proper transform on  $V_u$  of the surface  $H_x$ , and the surface  $T_{18}$  is the proper transform on  $V_u$  of the surface  $H_w$ . Moreover, the surface  $H_x$  does not contain the point  $(1 : 0 : 0 : 0 : 0)$ , and the surface  $H_w$  does not contain the point  $(0 : 0 : 0 : 0 : 1)$ . Thus, using Remark 2.21 and Lemma 2.22, we see that the intersection  $T_{12} \cap T_{18}$  consists of the curve  $\mathcal{C}_6$ , the conic  $\mathcal{C}_2$ , and the lines  $\ell_1$  and  $\ell_2$  from Remark 2.11.

By Remark 2.21, the surface  $T''_{15}$  does not contain the conic  $\mathcal{C}_2$ . Similarly, it follows from Lemma 2.22 that the intersection  $T_{10} \cap T_{20}$  contains neither  $\ell_1$  nor  $\ell_2$ . Thus, the curve  $\mathcal{C}_6$  is the only curve contained in the intersection (6.5).

The base locus of the linear system  $|\sigma^*(H_{V_u}) - E_\sigma|$  does not contain any curves outside the exceptional surface  $E_\sigma$ . Moreover, the surfaces  $T_{13}$  and  $T_{18}$  intersect transversally at a general point of the curve  $\mathcal{C}_6$ , because the surfaces  $H_x$  and  $H_w$  intersect transversally at every point of the conic  $\Upsilon$ . Therefore, the base locus of the linear system  $|\sigma^*(H_{V_u}) - E_\sigma|$  does not contain curves with the only possible exception of finitely many fibers of the projection  $E_\sigma \rightarrow \mathcal{C}_6$ . This implies the required assertion.  $\square$

We see that  $-K_{\widehat{V}_u}$  is nef. Since  $E_\sigma^3 = -\deg(\mathcal{C}) + 2$  and  $\sigma^*(H_{V_u}) \cdot E^2 = -\deg(\mathcal{C})$ , we compute

$$-K_{\widehat{V}_u}^3 = \begin{cases} 12 & \text{if } \mathcal{C} = \mathcal{C}_4, \\ 8 & \text{if } \mathcal{C} = \mathcal{C}_6. \end{cases}$$

Therefore, the divisor  $-K_{\widehat{V}_u}$  is also big. Thus, it follows from Basepoint-free Theorem that the linear system  $|-nK_{\widehat{V}_u}|$  is free from base points for  $n \gg 0$ , see [KM98, Theorem 3.3]. This linear system gives a crepant birational morphism  $\eta: \widehat{V}_u \rightarrow Y$ , so that  $Y$  is a Fano threefold with at most canonical singularities such that  $-K_Y^3 = -K_{\widehat{V}_u}^3$ . Observe that according to the classification of smooth Fano threefolds with Picard rank 2, the threefold  $\widehat{V}_u$  is not Fano. In other words,  $\eta$  is not an isomorphism, and  $Y$  is indeed singular.

**Lemma 6.6.** Suppose that  $\mathcal{C} = \mathcal{C}_4$ . Then  $\eta$  is small if and only if  $u \neq 2$ .

*Proof.* If  $u = 2$ , then  $\text{mult}_{\mathcal{C}}(T'_{15}) = 3$  by Lemma 5.2, so that

$$\begin{aligned} 0 \leq -K_{\widehat{V}_u}^2 \cdot \widehat{T}'_{15} &= \left(\sigma^*(H_{V_u}) - E_\sigma\right)^2 \cdot \left(\sigma^*(H_{V_u}) - 3E_\sigma\right) = \\ &= 22 + 3\sigma^*(H_{V_u}) \cdot E_\sigma^2 + 4\sigma^*(H_{V_u}) \cdot E_\sigma^2 - 3E_\sigma^3 = 0, \end{aligned}$$

which implies that  $\widehat{T}'_{15}$  is contracted by  $\eta$ .

We may assume that  $u \neq 2$ . Then  $\text{mult}_{\mathcal{C}}(T'_{15}) = 2$  by Lemma 5.2. Let  $F$  be an irreducible surface in  $\widehat{V}_u$ . Then  $F \sim \sigma^*(nH_{V_u}) - mE_\sigma$  for some integers  $n$  and  $m$ . We compute

$$\begin{aligned} -K_{\widehat{V}_u}^2 \cdot F &= \left(\sigma^*(H_{V_u}) - E_\sigma\right)^2 \cdot \left(\sigma^*(nH_{V_u}) - mE_\sigma\right) = \\ &= 22n + n\sigma^*(H_{V_u}) \cdot E_\sigma^2 + 2m\sigma^*(H_{V_u}) \cdot E_\sigma^2 - mE_\sigma^3 = 18n - 6m, \end{aligned}$$

so that  $F$  is contracted by  $\eta$  if and only if  $m = 3n$ . In particular, the surface  $\widehat{T}'_{15}$  is not contracted by  $\eta$ . On the other hand, if  $F \neq \widehat{T}'_{15}$ , then

$$\begin{aligned} 0 \leq \left(\sigma^*(H_{V_u}) - E_\sigma\right) \cdot F \cdot \widehat{T}'_{15} &= \left(\sigma^*(H_{V_u}) - E_\sigma\right) \cdot \left(\sigma^*(nH_{V_u}) - mE_\sigma\right) \cdot \left(\sigma^*(H_{V_u}) - 2E_\sigma\right) = \\ &= 22n + 2n\sigma^*(H_{V_u}) \cdot E_\sigma^2 + 3m\sigma^*(H_{V_u}) \cdot E_\sigma^2 - 2mE_\sigma^3 = 14n - 8m, \end{aligned}$$

so that  $m \neq 3n$ , which implies that  $F$  is also not contracted by  $\eta$ .  $\square$

Therefore, if  $\mathcal{C} = \mathcal{C}_4$  and  $u \neq 2$ , then it follows from standard computations as in [IP99, §4.1] or [Ta89, ACM17, CM13] that there exists a  $G$ -equivariant commutative diagram

$$(6.7) \quad \begin{array}{ccccc} & \widehat{V}_u & \overset{\rho}{\dashrightarrow} & \widehat{V}_{u'} & \\ \sigma \swarrow & & & & \searrow \sigma' \\ V_u & & \eta \searrow & \swarrow \eta' & V_{u'} \\ & & Y & & \end{array}$$

where  $\rho$  is the flop in the curves contracted by  $\eta$ , and the variety  $V_{u'}$  is a smooth Fano threefold of type  $V_{22}^*$  that corresponds to (some) parameter  $u'$ , which is possibly different from  $u$ . Here the map  $\sigma'$  is a birational morphism that contracts the proper transform of the surface  $\widehat{T}'_{15}$  to a unique irreducible  $G$ -invariant (rational normal) curve  $\mathcal{C}'_4$  of degree 4 in  $V_{u'}$ . The diagram (6.7) is Sarkisov link No. 104 in [CM13].

*Remark 6.8.* It would be interesting to know whether the threefold  $V_{u'}$  in (6.7) is isomorphic to the threefold  $V_u$  or not, that is, whether  $u = u'$  or not.

**Lemma 6.9.** Suppose that  $\mathcal{C} = \mathcal{C}_4$  and  $u \neq 2$ . Then  $\eta$  does not contract curves in  $E_\sigma$ .

*Proof.* The normal bundle of the curve  $\mathcal{C}_4$  in  $V_u$  is isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(p) \oplus \mathcal{O}_{\mathbb{P}^1}(q)$  for some integers  $p$  and  $q$  such that  $p \geq q$  and  $p + q = 2$ . Thus, the exceptional surface  $E_\sigma$  is a Hirzebruch surface  $\mathbb{F}_n$  for  $n = p - q \geq 0$ . Denote by  $\mathbf{s}$  a section of the natural

projection  $E_\sigma \rightarrow \mathcal{C}_4$  such that  $\mathbf{s}^2 = -n$ , and denote by  $\mathbf{l}$  a fiber of this projection. Then  $-E_\sigma|_{E_\sigma} \sim \mathbf{s} + \kappa\mathbf{l}$  for some integer  $\kappa$ . One has

$$-2 = E_\sigma^3 = (\mathbf{s} + \kappa\mathbf{l})^2 = -n + 2\kappa,$$

so that  $\kappa = \frac{n-2}{2}$ . By Remark 6.1, one has

$$\widehat{T}'_{15}|_{E_\sigma} = \widehat{\mathcal{C}} + \varkappa(\mathbf{l}_1 + \mathbf{l}_2),$$

where  $\widehat{\mathcal{C}}$  is a reducible  $G$ -irreducible 2-section of the projection  $E_\sigma \rightarrow \mathcal{C}_4$ , the curves  $\mathbf{l}_1$  and  $\mathbf{l}_2$  are the fibers of this projection over two  $\mathbb{C}^*$ -fixed points in  $\mathcal{C}_4$ , respectively, and  $\varkappa$  is a non-negative integer. This gives

$$\widehat{\mathcal{C}} \sim 2\mathbf{s} + (n + 2 - 2\varkappa)\mathbf{l}.$$

Since  $\widehat{\mathcal{C}} \neq \mathbf{s}$ , we have  $0 \leq \widehat{\mathcal{C}} \cdot \mathbf{s} = 2 - n - 2\varkappa$ , which gives  $n \leq 2$ . This implies that the divisor

$$-K_{\widehat{V}_u}|_{E_\sigma} \sim \mathbf{s} + \frac{n+6}{2}\mathbf{l}$$

is ample, and the assertion follows.  $\square$

If  $\mathcal{C} = \mathcal{C}_6$ , then the morphism  $\eta$  is never small, since it contracts the surface  $\widehat{T}''_{15}$ . Indeed, in this case, we have  $\widehat{T}''_{15} \sim \sigma^*(H_{V_u}) - 2E_\sigma$  by Lemma 5.5, which implies that

$$K_{\widehat{V}_u}^2 \cdot \widehat{T}''_{15} = (\sigma^*(H_{V_u}) - E_\sigma)^2 \cdot (\sigma^*(H_{V_u}) - 2E_\sigma) = 22 + 5\sigma^*(H_{V_u}) \cdot E_\sigma^2 - 2E_\sigma^3 = 0.$$

This is a so-called *bad link* (cf. Sarkisov link No. 93 in [ACM17]).

## 7. THE PROOF

In this section, we prove Theorem 1.6. Let

$$\varepsilon(u) = \begin{cases} \frac{4}{5} & \text{if } u \neq \frac{3}{4} \text{ and } u \neq 2, \\ \frac{3}{4} & \text{if } u = \frac{3}{4}, \\ \frac{2}{3} & \text{if } u = 2. \end{cases}$$

By Corollaries 2.7, 5.4 and 5.6, we know that  $\alpha_G(V_u) \leq \varepsilon(u)$ . Thus, by (1.2), to prove Theorem 1.6, we have to show that the log pair  $(V_u, \frac{\varepsilon(u)}{n}\mathcal{D})$  has log canonical singularities for every  $G$ -invariant linear system  $\mathcal{D} \subset |-nK_{V_u}|$  and for every positive integer  $n$ . For basic properties of singularities of such log pairs, we refer the reader to [Ko97, Theorem 4.8].

*Remark 7.1.* Let  $\mathcal{D}$  be a non-empty  $G$ -invariant linear subsystem in  $|-nK_{V_u}|$  for some  $n \in \mathbb{Z}_{>0}$ . Fix a positive rational number  $\varepsilon$ . Suppose that the log pair  $(V_u, \frac{\varepsilon}{n}\mathcal{D})$  is strictly log canonical, i.e., log canonical but not Kawamata log terminal. Let  $Z$  be a center of log canonical singularities of the log pair  $(V_u, \frac{\varepsilon}{n}\mathcal{D})$  (see [Ka97, Definition 1.3]).

Then  $Z$  is  $\mathbb{C}^*$ -invariant. This follows from the existence of an equivariant strong resolution of singularities (see [RY02, Ko07]).

*Remark 7.2.* In the assumptions of Remark 7.1, let  $\mathcal{F}$  be the fixed part of the linear system  $\mathcal{D}$ , and let  $\mathcal{M}$  be its mobile part, so that

$$\mathcal{D} = \mathcal{F} + \mathcal{M}.$$

Since  $\text{Pic}(V_u) = \mathbb{Z}[-K_{V_u}]$ , one has  $\mathcal{F} \sim -n_1 K_{V_u}$  and  $\mathcal{M} \sim -n_2 K_{V_u}$  for some non-negative integers  $n_1$  and  $n_2$  such that  $n_1 + n_2 = n$ . Then  $Z$  is a center of log canonical singularities of either  $(V_u, \frac{\epsilon}{n_1} \mathcal{F})$  or  $(V_u, \frac{\epsilon}{n_2} \mathcal{M})$ , see [CS09, Remark 2.9] and the proof of [CS09, Lemma 2.10].

*Remark 7.3.* In the assumptions of Remark 7.2, there is a  $\mathbb{C}^*$ -invariant divisor  $D \in \mathcal{D}$ . Then  $Z$  is a center of log canonical singularities of the log pair  $(V_u, \frac{\epsilon}{2n}(D + \iota(D)))$ .

Hence, to prove Theorem 1.6, it is enough to show that the log pair  $(V_u, \varepsilon(u)D)$  is log canonical for every  $G$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on the threefold  $V_u$  such that

$$D \sim_{\mathbb{Q}} -K_{V_u}.$$

Moreover, if necessary, we may assume that  $D = \frac{1}{n}S$  for some irreducible surface  $S$  in the linear system  $|-nK_{V_u}|$ . This follows from

*Remark 7.4.* Let  $D$  be a  $G$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on the threefold  $V_u$  such that  $D \sim_{\mathbb{Q}} -K_{V_u}$ , and let  $Z$  be an irreducible subvariety in  $V_u$  such that  $Z$  is a center of log canonical singularities of the log pair  $(V_u, \epsilon D)$ , where  $\epsilon$  is a positive rational number. Suppose that

$$D = D_1 + D_2$$

for two non-zero effective  $G$ -invariant  $\mathbb{Q}$ -divisors  $D_1 \sim_{\mathbb{Q}} -\epsilon_1 K_{V_u}$  and  $D_2 \sim_{\mathbb{Q}} -\epsilon_2 K_{V_u}$ . Here  $\epsilon_1$  and  $\epsilon_2$  are positive rational numbers such that  $\epsilon_1 + \epsilon_2 = 1$ . Then either  $Z$  is a center of log canonical singularities of the log pair  $(V_u, \frac{\epsilon}{\epsilon_1} D_1)$ , or  $Z$  is a center of log canonical singularities of the log pair  $(V_u, \frac{\epsilon}{\epsilon_2} D_2)$  (or both). This is well known and easy to prove. See, for instance, [CS08, Remark 2.22] or [CP16, Lemma 2.2].

The key point in the proof of Theorem 1.6 is the following

**Proposition 7.5.** Let  $D$  be a  $G$ -invariant effective  $\mathbb{Q}$ -divisor on  $V_u$  such that  $D \sim_{\mathbb{Q}} -K_{V_u}$ . Suppose that  $(V_u, \varepsilon(u)D)$  is not log canonical. Then  $(V_u, \varepsilon(u)D)$  is not log canonical at a general point of one of the curves  $\mathcal{C}_2, \mathcal{C}_4$  or  $\mathcal{C}_6$ .

*Proof.* Let  $\epsilon$  be a positive rational number such that  $(V_u, \epsilon D)$  is strictly log canonical. Then  $\epsilon < \varepsilon(u)$ . Let  $Z$  be a minimal center of log canonical singularities of the log pair  $(V_u, \epsilon D)$ . Since  $\text{Pic}(V_u)$  is generated by  $-K_{V_u}$  and  $\epsilon < 1$ , the center  $Z$  is either a point or a curve. Recall from Remark 7.1 that  $Z$  is  $\mathbb{C}^*$ -invariant. Observe that  $\iota(Z)$  is also a minimal center of log canonical singularities of the log pair  $(V_u, \frac{\epsilon}{n} D)$ .

Now we will use the so-called *perturbation trick*. For details, see [CS16, Lemma 2.4.10], and the proofs of [Ka97, Theorem 1.10] and [Ka98, Theorem 1]. Observe that there



exists a mobile  $G$ -invariant linear system  $\mathcal{B}$  on the threefold  $V_u$ , and there are rational numbers  $1 \gg \epsilon_1 \geq 0$  and  $1 \gg \epsilon_2 \geq 0$  such that

$$(\epsilon - \epsilon_1)D + \epsilon_2\mathcal{B} \sim_{\mathbb{Q}} -\theta K_{V_u},$$

for some positive rational number  $\theta < \varepsilon(u)$ , the log pair

$$(7.6) \quad \left( V_u, (\epsilon - \epsilon_1)D + \epsilon_2\mathcal{B} \right)$$

has strictly log canonical singularities, and the only centers of log canonical singularities of the log pair (7.6) are  $Z$  and  $\iota(Z)$ .

Observe that the divisor  $-(K_{V_u} + (\epsilon - \epsilon_1)D + \epsilon_2\mathcal{B})$  is ample, since  $\theta < \varepsilon(u) < 1$ . Thus, the locus of log canonical singularities of the pair (7.6) is connected by the Kollár–Shokurov connectedness principle [KM98, Corollary 5.49]. Since there are no  $G$ -fixed points on  $V_u$  by Lemma 2.23, the center  $Z$  is not a point, so that  $Z$  is a curve.

By [Ka97, Proposition 1.5], either  $Z = \iota(Z)$ , or the centers  $Z$  and  $\iota(Z)$  are disjoint. Using the Kollár–Shokurov connectedness, we see that  $Z = \iota(Z)$ , so that  $Z$  is  $G$ -invariant.

Since  $(\theta - \varepsilon(u))K_{V_u}$  is an ample  $\mathbb{Q}$ -divisor, using Kawamata subadjunction theorem [Ka98, Theorem 1], we see that  $Z$  is smooth and

$$(1 - \varepsilon(u))K_{V_u} \Big|_Z \sim_{\mathbb{Q}} \left( K_{V_u} + (\epsilon - \epsilon_1)D + \epsilon_2\mathcal{B} + (\theta - \varepsilon(u))K_{V_u} \right) \Big|_Z \sim_{\mathbb{Q}} K_Z + D_Z$$

for some ample divisor  $D_Z$  on the curve  $Z$ . In particular, we see that  $Z$  is rational and

$$(\varepsilon(u) - 1)\deg(Z) > -2,$$

which implies that  $\deg(Z) < \frac{2}{1 - \varepsilon(u)} \leq 10$ , so that  $\deg(Z) \leq 9$ . Thus, by Proposition 4.12, the curve  $Z$  is one of the curves  $\mathcal{C}_2$ ,  $\mathcal{C}_4$  or  $\mathcal{C}_6$ , which is exactly what we need.  $\square$

In the remaining part of this section, we will show that  $(V_u, \varepsilon(u)D)$  is log canonical at general points of the curves  $\mathcal{C}_2$ ,  $\mathcal{C}_4$  or  $\mathcal{C}_6$  for every  $G$ -invariant effective  $\mathbb{Q}$ -divisor  $D$  on the threefold  $V_u$  such that  $D \sim_{\mathbb{Q}} -K_{V_u}$ . We start with the conic  $\mathcal{C}_2$ .

**Lemma 7.7.** Let  $D$  be a  $G$ -invariant effective  $\mathbb{Q}$ -divisor on  $V_u$  such that  $D \sim_{\mathbb{Q}} -K_{V_u}$ . Then the log pair  $(V_u, \frac{4}{5}D)$  is log canonical at a general point of the curve  $\mathcal{C}_2$ .

*Proof.* The normal bundle of the conic  $\mathcal{C}_2$  in  $V_u$  is either isomorphic to  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ , or isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ . Thus, the exceptional surface  $E_{V_u}$  is either  $\mathbb{P}^1 \times \mathbb{P}^1$  or the Hirzebruch surface  $\mathbb{F}_2$ .

If  $E_{V_u} \cong \mathbb{P}^1 \times \mathbb{P}^1$ , we denote by  $\mathbf{s}$  the section of the natural projection  $E_{V_u} \rightarrow \mathcal{C}_2$  such that  $\mathbf{s}^2 = 0$ . Similarly, if  $E_{V_u} \cong \mathbb{F}_2$ , we denote by  $\mathbf{s}$  the section of the projection  $E_{V_u} \rightarrow \mathcal{C}_2$  such that  $\mathbf{s}^2 = -2$ . If  $E_{V_u} \cong \mathbb{P}^1 \times \mathbb{P}^1$ , then  $-E_{V_u}|_{E_{V_u}} \sim \mathbf{s}$ . Similarly, if  $E_{V_u} \cong \mathbb{F}_2$ , then

$$-E_{V_u}|_{E_{V_u}} \sim \mathbf{s} + \mathbf{l},$$

where  $\mathbf{l}$  is the fiber of the natural projection  $E_{V_u} \rightarrow \mathcal{C}_2$ .

Denote by  $\tilde{D}$  the proper transform of the divisor  $D$  on the threefold  $\tilde{V}_u$ . Then

$$\tilde{D} \sim_{\mathbb{Q}} \phi^*(H_{V_u}) - mE_{V_u},$$

where  $m = \text{mult}_{\mathcal{C}_2}(D)$ . On the other hand, we know that  $\mathcal{R} \sim 2\phi^*(H_{V_u}) - 5E_{V_u}$ , so that

$$\tilde{D} \sim_{\mathbb{Q}} \frac{1}{2}\mathcal{R} + \left(\frac{5}{2} - m\right)E_{V_u},$$

which implies that  $m \leq \frac{5}{2}$ , because  $E_{Q_u}$  is the proper transform of the surface  $\mathcal{R}$  on the threefold  $\tilde{Q}_u$ .

Suppose that the log pair  $(V_u, \frac{4}{5}D)$  is not log canonical at a general point of the curve  $\mathcal{C}_2$ . Then  $m > \frac{5}{4}$ . Moreover, the surface  $E_{V_u}$  contains a  $G$ -irreducible curve  $\tilde{C}$  such that  $\phi(\tilde{C}) = \mathcal{C}_2$ , and the log pair

$$(7.8) \quad \left( \tilde{V}_u, \frac{4}{5}\tilde{D} + \left(\frac{4m}{5} - 1\right)E_{V_u} \right)$$

is not log canonical at a general point of the curve  $\tilde{C}$ . Furthermore, since we know that  $m \leq \frac{5}{2}$ , the curve  $\tilde{C}$  must be a section of the natural projection  $E_{V_u} \rightarrow \mathcal{C}_2$ . This fact is well-known. See for instance [CP16, Remark 2.5]. Thus, the curve  $\tilde{C}$  is irreducible.

Applying [KM98, Theorem 5.50] to (7.8), we see that the log pair  $(E_{V_u}, \frac{4}{5}\tilde{D}|_{E_{V_u}})$  is also not log canonical at a general point of the curve  $\tilde{C}$ . This simply means that

$$\frac{4}{5}\tilde{D}|_{E_{V_u}} = \theta\tilde{C} + \Omega$$

for some rational number  $\theta > 1$  and some effective  $\mathbb{Q}$ -divisor  $\Omega$  on the surface  $E_{V_u}$ .

One has  $\tilde{C} \sim \mathbf{s} + \kappa\mathbf{l}$  for some non-negative integer  $\kappa$ . If  $E_{V_u} \cong \mathbb{P}^1 \times \mathbb{P}^1$ , then

$$\theta\mathbf{s} + \theta\kappa\mathbf{l} + \Omega \sim_{\mathbb{Q}} \theta\tilde{C} + \Omega = \frac{4}{5}\tilde{D}|_{E_{V_u}} \sim_{\mathbb{Q}} \frac{4m}{5}\mathbf{s} + \frac{8}{5}\mathbf{l},$$

so that either  $\kappa = 0$  or  $\kappa = 1$ . Thus, in this case we have

$$-K_{\tilde{V}_u} \cdot \tilde{C} = -K_{\tilde{V}_u}|_{E_{V_u}} \cdot \tilde{C} = (\mathbf{s} + 2\mathbf{l}) \cdot (\mathbf{s} + \kappa\mathbf{l}) = 2 + \kappa \leq 3.$$

Similarly, if  $E_{V_u} \cong \mathbb{F}_2$ , then

$$\theta\mathbf{s} + \theta\kappa\mathbf{l} + \Omega \sim_{\mathbb{Q}} \theta\tilde{C} + \Omega = \frac{4}{5}\tilde{D}|_{E_{V_u}} \sim_{\mathbb{Q}} \frac{4m}{5}\mathbf{s} + \frac{8 + 4m}{5}\mathbf{l},$$

so that  $\kappa \leq 3$ , which gives

$$-K_{\tilde{V}_u} \cdot \tilde{C} = -K_{\tilde{V}_u}|_{E_{V_u}} \cdot \tilde{C} = (\mathbf{s} + 3\mathbf{l}) \cdot (\mathbf{s} + \kappa\mathbf{l}) = 1 + \kappa \leq 4.$$

We proved that  $-K_{\tilde{V}_u} \cdot \tilde{C} \leq 4$ . Then the degree of the curve  $\beta(\tilde{C})$  is  $-K_{\tilde{V}_u} \cdot \tilde{C} \leq 4$ . This is impossible by Lemmas 3.4 and 4.7.  $\square$

Now we deal with the curve  $\mathcal{C}_6$ .

**Lemma 7.9.** Let  $D$  be an effective  $\mathbb{Q}$ -divisor on the threefold  $V_u$  such that  $D \sim_{\mathbb{Q}} -K_{V_u}$ . Suppose that  $\text{Supp}(D)$  does not contain  $T''_{15}$ . Then the log pair  $(V_u, D)$  is log canonical at a general point of the curve  $\mathcal{C}_6$ .

*Proof.* Let us use the notation of §6 with  $\mathcal{C} = \mathcal{C}_6$ . Denote by  $\widehat{T}_{15}''$  the proper transform of the surface  $T_{15}''$  on the threefold  $\widehat{V}_u$ . Then

$$\widehat{T}_{15}'' \sim \sigma^*(H_{V_u}) - 2E_\sigma$$

by Lemma 5.5.

Denote by  $\widehat{D}$  the proper transform on  $\widehat{V}_u$  of the divisor  $D$ . We also let  $m = \text{mult}_{\mathcal{C}_6}(D)$ . Using  $E_\sigma^3 = -4$  and  $\sigma^*(H_{V_u}) \cdot E^2 = -6$ , we compute

$$\begin{aligned} (\sigma^*(H_{V_u}) - E_\sigma) \cdot \widehat{D} \cdot \widehat{T}_{15}'' &= (\sigma^*(H_{V_u}) - E_\sigma) \cdot (\sigma^*(H_{V_u}) - mE_\sigma) \cdot (\sigma^*(H_{V_u}) - 2E_\sigma) = \\ &= 22 + 2\sigma^*(H_{V_u}) \cdot E_\sigma^2 + 3m\sigma^*(H_{V_u}) \cdot E_\sigma^2 - 2mE_\sigma^3 = 10 - 10m. \end{aligned}$$

On the other hand, the divisor  $\sigma^*(H_{V_u}) - E_\sigma$  is nef by Lemma 6.4. Thus, we have  $m \leq 1$ , and the assertion follows.  $\square$

**Corollary 7.10.** Let  $D$  be an effective  $\mathbb{Q}$ -divisor on  $V_u$  such that  $D \sim_{\mathbb{Q}} -K_{V_u}$ . If  $u = \frac{3}{4}$ , then the log pair  $(V_u, \frac{3}{4}D)$  is log canonical at a general point of the curve  $\mathcal{C}_6$ . If  $u \neq \frac{3}{4}$ , then the log pair  $(V_u, D)$  is log canonical at a general point of the curve  $\mathcal{C}_6$ .

*Proof.* If  $u = \frac{3}{4}$ , then  $(V_u, \frac{3}{4}T_{15}'')$  is log canonical at a general point of  $\mathcal{C}_6$  by Lemma 5.5. Likewise, if  $u \neq \frac{3}{4}$ , then the pair  $(V_u, T_{15}'')$  is log canonical at a general point of the curve  $\mathcal{C}_6$ . Thus, by Remark 7.4, we may assume that  $\text{Supp}(D)$  does not contain the surface  $T_{15}''$ . Now the assertion follows from Lemma 7.9.  $\square$

Combining Proposition 7.5, Lemma 7.7 and Corollary 7.10, we obtain

**Corollary 7.11.** Let  $D$  be a  $G$ -invariant effective  $\mathbb{Q}$ -divisor on  $V_u$  such that  $D \sim_{\mathbb{Q}} -K_{V_u}$ . Suppose that the log pair  $(V_u, \varepsilon(u)D)$  is log canonical at a general point of the curve  $\mathcal{C}_4$ . Then the log pair  $(V_u, \varepsilon(u)D)$  is log canonical.

Finally, we deal with the curve  $\mathcal{C}_4$  using Corollary 7.11.

**Lemma 7.12.** Let  $D$  be a  $G$ -invariant effective  $\mathbb{Q}$ -divisor on  $V_u$  such that  $D \sim_{\mathbb{Q}} -K_{V_u}$ . Suppose that  $\text{Supp}(D)$  does not contain  $T_{15}'$ . Then the log pair  $(V_u, \frac{5}{6}D)$  is log canonical at a general point of the curve  $\mathcal{C}_4$ .

*Proof.* Let us use the notation of §6 with  $\mathcal{C} = \mathcal{C}_4$ . Then  $\sigma^*(H_{V_u}) - E_\sigma$  is nef by Lemma 6.2. Denote by  $\widehat{D}$  the proper transform on  $\widehat{V}_u$  of the divisor  $D$ . We also let  $m = \text{mult}_{\mathcal{C}_4}(D)$ . If  $u = 2$ , then  $\text{mult}_{\mathcal{C}_4}(T_{15}') = 3$  by Remark 6.1, so that

$$\begin{aligned} 0 \leq (\sigma^*(H_{V_u}) - E_\sigma) \cdot \widehat{D} \cdot \widehat{T}_{15}' &= (\sigma^*(H_{V_u}) - E_\sigma) \cdot (\sigma^*(H_{V_u}) - mE_\sigma) \cdot (\sigma^*(H_{V_u}) - 3E_\sigma) = \\ &= 22 + 3\sigma^*(H_{V_u}) \cdot E_\sigma^2 + 4m\sigma^*(H_{V_u}) \cdot E_\sigma^2 - 3mE_\sigma^3 = 10 - 10m, \end{aligned}$$

so that  $m \leq 1$ , which implies that the log pair  $(V_u, D)$  is log canonical at a general point of the curve  $\mathcal{C}_4$ .

Hence, we may assume that  $u \neq 2$ , so that  $\text{mult}_{\mathcal{C}_4}(T'_{15}) = 2$  by Remark 6.1. Then

$$\begin{aligned} 0 \leq (\sigma^*(H_{V_u}) - E_\sigma) \cdot \widehat{D} \cdot \widehat{T}'_{15} &= (\sigma^*(H_{V_u}) - E_\sigma) \cdot (\sigma^*(H_{V_u}) - mE_\sigma) \cdot (\sigma^*(H_{V_u}) - 2E_\sigma) = \\ &= 22 + 2\sigma^*(H_{V_u}) \cdot E_\sigma^2 + 3m\sigma^*(H_{V_u}) \cdot E_\sigma^2 - 2mE_\sigma^3 = 14 - 8m, \end{aligned}$$

which gives  $m \leq \frac{7}{4}$ . Let us show that this implies that  $(V_u, \frac{5}{6}D)$  is log canonical at a general point of the curve  $\mathcal{C}_4$ .

Let  $\epsilon = \frac{5}{6}$ . Suppose that  $(V_u, \epsilon D)$  is not log canonical at a general point of the curve  $\mathcal{C}_4$ . Then the surface  $E_\sigma$  contains a  $G$ -irreducible curve  $\widehat{Z}$  such that  $\sigma(\widehat{Z}) = \mathcal{C}_4$ , and the log pair

$$(7.13) \quad \left( \widehat{V}_u, \epsilon \widehat{D} + (\epsilon m - 1)E_\sigma \right)$$

is not log canonical at a general point of the curve  $\widehat{Z}$ . Moreover, since  $\epsilon m = \frac{5m}{6} \leq \frac{35}{24} < 2$ , the curve  $\widehat{Z}$  must be a section of the natural projection  $E_\sigma \rightarrow \mathcal{C}_4$ . This is well-known. See for instance [CP16, Remark 2.5].

We see that  $\widehat{Z}$  is irreducible. Thus, the curve  $\widehat{Z}$  is not contained in  $\widehat{T}'_{15}$  by Remark 6.1. Moreover, it follows from Lemma 6.9 that the curve  $\widehat{Z}$  is not contracted by  $\eta$ , so that  $\widehat{Z}$  is not flopped by  $\rho$ .

Denote by  $D'$  the proper transform of the divisor  $D$  on the threefold  $V_{u'}$ , and denote by  $T'$  the proper transform of the exceptional surface  $E_\sigma$  on the threefold  $V_{u'}$ . Then the log pair

$$(7.14) \quad \left( V_{u'}, \epsilon D' + (\epsilon m - 1)T' \right)$$

is not log canonical, because the log pair (7.13) is not log canonical at a general point of the curve  $\widehat{Z}$ .

Let us compute the class of the divisor  $D'$  in the group  $\text{Pic}(V_{u'})$ , and the multiplicity of the divisor  $D'$  at a general point of the curve  $\mathcal{C}'_4$ . Recall from (6.7) that  $\mathcal{C}'_4$  is the unique irreducible  $G$ -invariant curve of degree 4 in the threefold  $V_{u'}$ . We have

$$\widehat{D} + (m - 1)E_\sigma \sim_{\mathbb{Q}} -K_{\widehat{V}_u}.$$

This implies that  $D' + (m - 1)T' \sim_{\mathbb{Q}} -K_{V_{u'}}$ , where  $T'$  is the unique surface in the linear system  $| -K_{V_{u'}} |$  that is singular along the curve  $\mathcal{C}'_4$ . Thus, we have

$$D' \sim_{\mathbb{Q}} -(2 - m)K_{V_{u'}}.$$

Similar arguments applied to the divisor  $\frac{1}{2-m}D'$  give

$$-\frac{1}{2-m}K_V \sim_{\mathbb{Q}} \frac{1}{2-m}D \sim_{\mathbb{Q}} -\left( 2 - \frac{\text{mult}_{\mathcal{C}'_4}(D')}{2-m} \right) K_V,$$

so that  $\text{mult}_{\mathcal{C}'_4}(D') = 3 - 2m$ .

Observe that  $\text{mult}_{\mathcal{C}'_4}(T') = 2$ . Thus, we have

$$\text{mult}_{\mathcal{C}'_4}(\epsilon D' + (\epsilon m - 1)T') = 3\epsilon - 2 < 1,$$

so that (7.14) is log canonical at a general point of the curve  $\mathcal{C}'_4$ . On the other hand, we have

$$\epsilon D' + (\epsilon m - 1)T' \sim_{\mathbb{Q}} -(2\epsilon - 1)K_{V_{u'}}$$

and  $2\epsilon - 1 = \frac{2}{3} \leq \varepsilon(u)$ . Thus, the log pair (7.14) must be log canonical by Corollary 7.11 applied to  $V_{u'}$ . The obtained contradiction completes the proof of the lemma.  $\square$

**Corollary 7.15.** Let  $D$  be an effective  $\mathbb{Q}$ -divisor on  $V_u$  such that  $D \sim_{\mathbb{Q}} -K_{V_u}$ . If  $u = 2$ , then the log pair  $(V_u, \frac{2}{3}D)$  is log canonical at a general point of the curve  $\mathcal{C}_4$ . If  $u \neq 2$ , then the log pair  $(V_u, \frac{5}{6}D)$  is log canonical at a general point of the curve  $\mathcal{C}_4$ .

*Proof.* If  $u = 2$ , then  $(V_u, \frac{2}{3}T'_{15})$  is log canonical at a general point of  $\mathcal{C}_4$  by Lemma 5.2. Similarly, if  $u \neq 2$ , then the pair  $(V_u, T'_{15})$  is log canonical at a general point of the curve  $\mathcal{C}_4$ . Thus, by Remark 7.4, we may assume that  $\text{Supp}(D)$  does not contain the surface  $T'_{15}$ . Now the assertion follows from Lemma 7.12.  $\square$

Combining Corollaries 7.11 and 7.15, we obtain the assertion of Theorem 1.6. Indeed, let  $D$  be an effective  $\mathbb{Q}$ -divisor on the threefold  $V_u$  such that  $D \sim_{\mathbb{Q}} -K_{V_u}$ . As we already mentioned, we have to show that the log pair  $(V_u, \varepsilon(u)D)$  is log canonical. But the log pair  $(V_u, \varepsilon(u)D)$  is log canonical at a general point of the curve  $\mathcal{C}_4$  by Corollary 7.15, so that it is log canonical everywhere by Corollary 7.11.

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