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Infinite-State Energy Games

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***Abstract*—Energy games are a well-studied class of 2-player turn-based games on a finite graph where transitions are labeled with integer vectors which represent changes in a multidimensional resource (the energy). One player tries to keep the cumulative changes non-negative in every component while the other tries to frustrate this.**

We consider generalized energy games played on infinite game graphs induced by pushdown automata (modelling recursion) or their subclass of one-counter automata.

Our main result is that energy games are decidable in the case where the game graph is induced by a one-counter automaton and the energy is one-dimensional. On the other hand, every further generalization is undecidable: Energy games on one-counter automata with a 2-dimensional energy are undecidable, and energy games on pushdown automata are undecidable even if the energy is one-dimensional.

Furthermore, we show that energy games and simulation games are inter-reducible, and thus we additionally obtain several new (un)decidability results for the problem of checking simulation preorder between pushdown automata and vector addition systems.

***Index Terms*—Automata theory; Energy games.**

I. Introduction

Two-player turn-based games on transition graphs provide the mathematical foundation for the analysis of reactive systems, and they are used to solve many problems in formal verification, e.g., in model checking and semantic equivalence checking [20]. The vertices of the game graph represent states of the system, and they are partitioned into subsets that belong to Player 0 and Player 1, respectively. The game starts at an initial vertex, and in every round of the game the player who owns the current vertex chooses an outgoing transition leading to the next vertex. This yields an either finite or infinite sequence of visited vertices, called a *play* of the game.

Various types of games define different winning conditions that classify a play as winning for a given player, e.g., reachability, safety, liveness, ω-regular, or parity objectives.

A generalization of such games introduces quantitative aspects and corresponding quantitative winning conditions. Transitions are labeled with numeric values, typically integers, that are interpreted as the cost or reward of taking this transition, e.g., elapsed time, lost/gained material resources or energy, etc. The value of (a prefix of) a play is then defined as the sum of the values of the used transitions. Further generalizations use multi-dimensional labels (i.e., vectors of integers) instead of single integers.

The most commonly studied quantitative games are *energy games* and *limit-average games* (also called mean-payoff games), which differ in the quantitative winning condition.

In energy games, the objective of Player 1 is to forever keep the value of the prefix of the play non-negative (resp. non-negative in every component, for multidimensional values), while Player 0 tries to frustrate this. Intuitively, this means that the given resource (e.g., the stored energy) must never run out during the operation of the system. Clearly such games are monotone in the resource value, in the sense that higher values are always beneficial for Player 1.

There are two classic problems about energy games. In the *fixed initial credit problem* one asks whether Player 1 has a winning strategy from a given starting configuration with a fixed initial energy (resource value). In the *unknown initial credit problem* one quantifies over this initial energy and asks whether there exists a sufficiently high value for Player 1 to win. Even if the answer is positive, this does not yield any information about the minimal initial energy required.

In limit-average games, the objective is to maximize the average value per step of the play in the long run. I.e., one asks whether Player 1 has a strategy to keep the average value per transition above a given number k in the long run. Limit-average games are closely related to the unknown initial credit problem in energy games, since in both cases one tries to maximize the payoff in the long run without considering short-term fluctuations. (The fixed initial credit problem in energy games is different however, since local fluctuations matter.)

Previous work on finite game graphs. Most previous work on quantitative games has considered energy games and limit-average games on finite game graphs, sometimes combined with classic winning conditions such as parity objectives. The unknown and fixed initial credit problems for one-dimensional energy parity games were shown to be decidable in [6]. The unknown initial credit problem for multidimensional energy parity games is known to be coNP-complete [7], [8]. The fixed initial credit problem for n-dimensional energy games can be solved in n-EXPTIME [4], and the fixed initial credit problem for multidimensional energy parity games is decidable [1]. An EXPSPACE lower bound follows by a reduction from Petri

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net coverability [17]. Multidimensional limit-average games are coNP-complete [7].

Previous work on infinite game graphs. Pushdown automata have been studied extensively as a model for the analysis of recursive programs (e.g., [3], [11], [21], [22]). Due to the unbounded stack memory, they typically induce infinite transition graphs. Two-player reachability, Büchi and parity games on pushdown automata are EXPTIME-complete [18], [5], [21], [22]. A strict subclass of pushdown automata are one-counter automata, i.e., Minsky machines with a single counter. They correspond to pushdown automata with only one stack symbol (plus a non-removable bottom stack symbol). Two-player reachability, Büchi and parity games on one-counter automata are PSPACE-complete [19], [13].

Quantitative extensions of pushdown games with limit-average objectives have been studied in [9], [10]. The authors show the undecidability of limit-average pushdown games with one resource-dimension, by reduction from the non-universality problem of weighted finite automata [2]. On the other hand, they prove the decidability of limit-average pushdown games under modular strategies (a restriction on how the resources interact with the recursion).

Our contribution. We consider energy games that are played on infinite game graphs that are induced by either pushdown automata or one-counter automata. We consider both single-dimensional and multi-dimensional energies and focus on the fixed initial credit problem.

Our first observation is that energy games are closely connected to simulation games, i.e., to checking simulation preorder between transition graphs. Energy games on pushdown automata (resp. one-counter automata) with n-dimensional energy are inter-reducible with checking simulation preorder between pushdown automata (resp. one-counter automata) and n-dimensional vector addition systems with states (VASS; aka Petri nets). Using this connection, we show several (un)decidability results for infinite-state energy games.

We show that the winning sets in single-dimensional energy games on one-counter automata are semilinear by establishing semilinearity of the corresponding simulation game. This yields a positive semi-decision procedure for the fixed initial credit problem by the decidability of Presburger arithmetic. Since negative semi-decidability is easily achieved by unfolding the game tree, we obtain the decidability of the fixed initial credit problem of single-dimensional energy games on one-counter automata.

Moreover, we show that every further generalized infinite-state energy game is undecidable by reduction from the halting problem for Minsky machines. The fixed initial credit problem for 2-dimensional energy games on one-counter automata is undecidable. For energy games on pushdown automata, both the fixed and the unknown initial credit problem are undecidable, even if the energy is just single-dimensional.

II. Preliminaries

Let $\mathbb{Z}$ denote the integers and $\mathbb{N}$ the non-negative integers.

Definition 1. *A* labeled transition system *is described by a triple $T = (V, Act, \longrightarrow)$ where V is a (possibly infinite) set of states, Act is a finite set of action labels and $\longrightarrow \subseteq V \times Act \times V$ is the labeled transition relation. We use the infix notation $s \xrightarrow{a} s'$ for a transition $(s, a, s') \in \longrightarrow$, in which case we say T makes an a-*step *from s to s'. In the context of games, $V = V_0 \cup V_1$ is partitioned into the subset V_0 of states that belong to player 0 and V_1 of states that belong to player 1.*

Definition 2. *A* pushdown automaton *$A = (Q, \Gamma, Act, \delta)$ is given by a finite set of control-states Q, a finite stack alphabet Γ, a finite set of action labels Act, and a finite set of transitions $\delta \subseteq Q \times \Gamma \times Act \times Q \times \Gamma^*$. It induces a transition system over $V = Q \times \Gamma^+$ by $qX\alpha \xrightarrow{a} q'\beta\alpha$ iff $(q, X, a, q', \beta) \in \delta$, for any $\alpha \in \Gamma^*$.*

Definition 3. *A* pushdown energy game *of dimension n between two players 0 and 1 is given by $G = (Q_0, Q_1, \Gamma, \delta, n)$. Q_0 and Q_1 are finite sets of control-states that belong to player 0 and 1, respectively. Γ is a finite stack alphabet and $n \in \mathbb{N}$ represents the dimension of the energy.*

The transition relation $\delta \subseteq (Q_0 \cup Q_1) \times \Gamma \times (Q_0 \cup Q_1) \times \Gamma^* \times \{-1, 0, 1\}^n$ induces a game graph over $(Q_0 \cup Q_1) \times \Gamma^+ \times \mathbb{Z}^n$ as follows: If $(q, X, q', \beta, C) \in \delta$ then $(q, X\alpha, E) \longrightarrow (q', \beta\alpha, E')$ for any $\alpha \in \Gamma^*$ and $E' = E + C$. (We don't use transition labels, since they do not influence the semantics of energy games.)

In the induced game graph, configurations with control-states in Q_0 and Q_1 belong to player 0 and 1, respectively. The player who owns the current configuration gets to choose the next step. Without restriction, we assume that every configuration has at least one outgoing transition. The game stops and Player 0 wins if a configuration (q, α, E) is reached where $E = (e_1, \ldots, e_n)$ with $e_i < 0$ for some i. Player 1 wins every infinite game.

In the special case where the stack is never used, the pushdown energy game corresponds to just an ordinary n-dimensional energy game with a finite control-graph [8], [1]. In particular, the energy dimensions are not genuine counters. They cannot be tested for zero and never influence the available transitions, but only affect the winning condition of the game.

One-counter automata can be seen as a special subclass of pushdown automata where there is only one stack symbol plus a non-removable stack bottom symbol $\perp$. In order to keep the presentation clear, we define an explicit notation for one-counter automata and games.

Definition 4. *A* one-counter automaton (OCA) *$A = (Q, Act, \delta, \delta_0)$ is given by a finite set of control-states Q, a finite set of action labels Act and transition relations $\delta \subseteq Q \times Act \times \{-1, 0, 1\} \times Q$ and $\delta_0 \subseteq Q \times Act \times \{0, 1\} \times Q$. Such an automaton is called a* one-counter net (OCN) *if $\delta_0 = \emptyset$, i.e., if the automaton cannot test if the counter is equal to 0.*

These automata induce an infinite-state labeled transition system over the stateset $Q \times \mathbb{N}$, whose elements will be written as pm, and transitions are defined as follows: $pm \xrightarrow{a} p'm'$ iff

1) $(p, a, d, p') \in \delta$ *and* $m' = m + d \geq 0$ *or*
2) $(p, a, d, p') \in \delta_0$, $m = 0$ *and* $m' = d$.

Definition 5. *A* one-counter energy game *of dimension n between two players 0 and 1 is given by $G = (Q_0, Q_1, \delta, \delta_0, n)$. Q_0 and Q_1 are finite sets of control-states that belong to player 0 and 1, respectively, and $n \in \mathbb{N}$ represents the dimension of the energy. The transition relations $\delta \subseteq (Q_0 \cup Q_1) \times \{-1, 0, 1\} \times (Q_0 \cup Q_1) \times \{-1, 0, 1\}^n$ and $\delta_0 \subseteq (Q_0 \cup Q_1) \times \{0, 1\} \times (Q_0 \cup Q_1) \times \{-1, 0, 1\}^n$ induce an infinite game graph over $(Q_0 \cup Q_1) \times \mathbb{N} \times \mathbb{Z}^n$ as follows. The number $m \in \mathbb{N}$ represents the value of the one genuine testable counter, while the $E \in \mathbb{Z}^n$ represents the available multidimensional energy. We have $(p, m, E) \longrightarrow (p', m', E')$ iff*

1) $(p, d, p', C) \in \delta$ *and* $m' = m + d \geq 0$ *and* $E' = E + C$ *or*
2) $(p, d, p', C) \in \delta_0$, $m = 0$ *and* $m' = d$ *and* $E' = E + C$.

The players choose moves depending on who owns the current control-state. The game stops and Player 0 wins if a configuration (q, k, E) is reached where $E = (e_1, \ldots, e_n)$ with $e_i < 0$ for some i. Player 1 wins every infinite game.

Definition 6. *A* vector addition system with states (VASS) *of dimension n is given by (Q, Act, δ). Q is a finite set of control-states, Act is a finite set of action labels and $\delta \subseteq Q \times Act \times Q \times \{-1, 0, 1\}^n$ is a finite transition relation. It induces an infinite-state labeled transition system over the stateset $Q \times \mathbb{N}^n$ as follows. We have $(p, C) \xrightarrow{a} (p', C + D)$ iff there is some $(p, a, p', D) \in \delta$ s.t. $C + D \in \mathbb{N}^n$.*

A VASS of dimension 1 corresponds to an OCN.

Problems about energy games. Previous works on energy games (on a finite game graph) mainly considered the following two problems [7], [6], [8], [1].

In the *fixed initial credit problem*, one considers a starting configuration with a fixed initial energy. The question is whether Player 1 has a winning strategy in the energy game, starting from this configuration with the given amount of energy.

In the *unknown initial credit problem* one instead asks whether there *exists* some level of initial energy s.t. Player 1 can win the game. Even if the answer to this question is positive, it does not necessarily yield any information about the minimal initial energy required to win the game.

Outline of the results. In the following section we show that there is a general connection between energy games and simulation games (i.e., checking simulation preorder between transition graphs). Pushdown energy games of energy dimension n are logspace inter-reducible with simulation games between a pushdown automaton and an n-dimensional vector addition system with states (VASS). A similar result holds for one-counter energy games and simulation between OCA and n-dimensional VASS.

Using this connection, we prove several decidability results for energy games on infinite game graphs.

1) For one-counter energy games of energy dimension $n = 1$ the winning sets are semilinear and the fixed initial credit problem is decidable.
2) The fixed initial credit problem is undecidable for one-counter energy games of energy dimension $n \geq 2$.
3) Both the fixed and the unknown initial credit problem are undecidable for pushdown energy games, even for energy dimension $n = 1$.

III. Energy Games vs. Simulation Games

Simulation is a semantic preorder in van Glabbeeks linear time – branching time spectrum [12]. It is used to compare the behavior of processes and is defined as follows.

Definition 7 (Simulation)**.** *Given two labeled transition systems T and T', a relation R on the disjoint union of the sets of states of T and T' is a* simulation *if for every pair of states $(c, c') \in R$ and every step $c \xrightarrow{a} d$ there exists a step $c' \xrightarrow{a} d'$ such that $(d, d') \in R$.*

Simulations are closed under union. So there exists a unique maximal simulation $\preceq_{T,T'}$ which is a preorder, commonly called simulation preorder. *We drop the index whenever it is clear from the context and say that c'* simulates *c iff $c \preceq c'$. By simulation between M and M' or w.r.t. M, M' we mean the maximal simulation $\preceq_{T,T'}$ relative to the transition systems T and T' which are induced by M and M', respectively.*

Simulation preorder can also be characterized in terms of an interactive game between two players Spoiler (Player 0) and Duplicator (Player 1), where the latter tries to stepwise match the moves of the former. A play *is a finite or infinite sequence of pairs of states $(c_0, c'_0), (c_1, c'_1), \ldots, (c_i, c'_i) \ldots$ where the next pair (c_{i+1}, c'_{i+1}) is determined by a round of choices: First Spoiler chooses a transition $c_i \xrightarrow{a} c_{i+1}$, then Duplicator responds by choosing an equally labeled transition $c'_i \xrightarrow{a} c'_{i+1}$. If one of the players cannot move then the other wins, and Duplicator wins every infinite play. A* strategy *is a set of rules that tells a player which valid move to choose. A player plays according to a strategy if all his moves obey the rules of the strategy. A strategy is* winning *from (c, c') if every play that starts in (c, c') and which is played according to that strategy is winning. We have $c \preceq c'$ iff Duplicator has a winning strategy from (c, c').*

First we show how energy games can be reduced to simulation games.

Lemma 1. *For any n-dimensional pushdown energy game $G = (Q_0, Q_1, \Gamma, \delta, n)$ one can in logspace construct a pushdown automaton $A = (Q_0 \cup Q_1 \cup Q_A, \Gamma, Act, \delta_A)$ and a n-dimensional VASS $V = (Q_0 \cup Q_1 \cup Q_V, Act, \delta_V)$ s.t., for every $q \in Q_0 \cup Q_1$, $\gamma \in \Gamma^+$ and $E \in \mathbb{N}^n$, Player 1 wins the energy game from configuration (q, γ, E) iff $(q, \gamma) \preceq (q, E)$.*

Moreover, in the special case of a one-counter energy game, the constructed automaton A is a OCA.

Proof: Every step in the energy game on G is emulated by either one or two rounds of the simulation game between A and V. We maintain the invariant that a configuration

(q, γ, E) in the energy game corresponds to a configuration $((q, \gamma), (q, E))$ in the simulation game.

For every transition $t = (q_t, X_t, q'_t, \beta_t, C_t) \in \delta$ and every stack symbol $X \in \Gamma$ we define unique action labels $a_t, a_X \in Act$. Moreover, we add symbol a to Act. So $Act = \{a_t, a_X \mid t \in \delta, X \in \Gamma\} \cup \{a\}$.

There are two cases, depending on which player chooses the step in the energy game from the current configuration (q, γ, E), i.e., whether $q \in Q_0$ or $q \in Q_1$.

For $q \in Q_0$ and every transition $t = (q_t, X_t, q'_t, \beta_t, C_t) \in \delta$ with $q_t = q$ we add a transition $(q_t, X_t, a_t, q'_t, \beta_t)$ to δ_A and a transition (q_t, a_t, q'_t, C_t) to δ_V. Since the label a_t is unique to the transition t, Player 1 has no choice in the simulation game but to implement the effect of the transition t chosen by Player 0, and the invariant is preserved.

For every $q \in Q_1$ we add a copied auxiliary control-state $\hat{q}$ to Q_A, and we add transitions $(q, X, a_X, \hat{q}, X)$ to δ_A for every $X \in \Gamma$. I.e., in the simulation game Player 0 makes a dummy-move that announces the current top stack symbol X via the action symbol a_X. The choice of the next encoded transition t (among those that are currently enabled in the energy game) is made by Player 1 in his next move.

For every transition $t = (q_t, X_t, q'_t, \beta_t, C_t) \in \delta$ with $q_t = q \in Q_1$ and $X_t = X \in \Gamma$ we add a transition (q_t, a_X, q''_t, C_t) to δ_V where q''_t is a new auxiliary control-state that is added to Q_V. I.e., Player 1 gets to choose a transition t from the current encoded control-state of the energy game. Since he needs to use the same symbol a_X as in the previous move by Player 0, his choices are limited to transitions that are currently enabled at control-state q and top stack symbol X in the energy game. This choice of transition t by Player 1 is recorded in the new control-state q''_t. Transitions that would decrease the energy below zero (and thus be losing for Player 1 in the energy game) are disabled by the semantics of VASS.

In the next step, Player 0 will be forced to implement this chosen transition t, or else she loses the game. For every transition $t = (q_t, X_t, q'_t, \beta_t, C_t) \in \delta$ with $q_t = q \in Q_1$ we add a transition $(\hat{q}, X_t, a_t, q'_t, \beta_t)$ to δ_A. This emulates a transition t of the energy game and announces its unique identity via the action symbol a_t. It remains to check whether this transition was the same as the one chosen by Player 1 in the previous round. (If not, then Player 0 must lose the game.)

For every transition $t = (q_t, X_t, q'_t, \beta_t, C_t) \in \delta$ with $q_t = q \in Q_1$ we add a transition $(q''_t, a_t, q'_t, \vec{0})$ to δ_V, where $\vec{0} = \{0\}^n$ denotes the n-tuple with value 0 on all coordinates. This simply implements the effect of the chosen transition t in the case where Player 0 has taken the correct transition with label a_t. In the other case where Player 0 did not choose the correct transition with label a_t from state $\hat{q}$, we must ensure that Player 1 wins the simulation game. Thus we add transitions $(q''_t, b, u, \vec{0})$ to δ_V for every $b \neq a_t$ where u is a universal winning state for Player 1, i.e., we add a state u to Q_V and transitions $(u, c, u, \vec{0})$ to δ_V for every $c \in Act$.

Thus, to avoid losing the simulation game, the players emulate the effect of an enabled energy game transition t that was chosen by Player 1 in two rounds of the simulation game, and the invariant is preserved.

If Player 1 has a strategy to win the energy game, then by faithful emulation he also wins the simulation game. Since the encoded energy never drops below zero, the corresponding transitions in the VASS are never blocked by the boundary condition and the play of the simulation game is infinite. Otherwise, if Player 0 has a winning strategy in the energy game, then she can enforce that some dimension of the energy becomes negative. By faithful emulation Player 0 also wins the simulation game, since the steps that go below zero are blocked in the VASS where Player 1 plays.

Thus, for every $q \in Q_0 \cup Q_1$, Player 1 wins the energy game from configuration (q, γ, E) iff he wins the simulation game from $((q, \gamma), (q, E))$ iff $(q, \gamma) \preceq (q, E)$.

We observe that the above construction preserves the property that makes the pushdown automaton correspond to an OCA. If there is only one stack symbol plus a non-removable stack bottom symbol in the pushdown energy game G then the same property also hold for the constructed pushdown automaton A. Thus, one-counter energy games reduce to simulation games between OCA and VASS. ■

For the reverse direction we show how to reduce simulation games to energy games.

Lemma 2. *For a pushdown automaton $A = (Q_A, \Gamma, Act, \delta_A)$ and a n-dimensional VASS $V = (Q_V, Act, \delta_V)$, one can in logspace construct a n-dimensional pushdown energy game $G = (Q_0, Q_1, \Gamma, \delta, n)$ with $Q_0 = Q_A \times Q_V \times \{0\}$ and $Q_1 = Q_A \times Q_V \times Act$ s.t., for every $q_0 \in Q_A$, $q_1 \in Q_V$, $\gamma \in \Gamma^+$ and $E \in \mathbb{N}^n$, we have $(q_0, \gamma) \preceq (q_1, E)$ iff Player 1 wins the energy game from configuration $((q_0, q_1, 0), \gamma, E)$.*

Moreover, if A is a OCA then the constructed game G is a one-counter energy game.

Proof: Every round of the simulation game is emulated by two steps in the energy game, one step by Player 0 followed by one step by Player 1, and the stated invariant will be preserved.

For every transition $(q_0, X, a, q'_0, \beta) \in \delta_A$ and every state $q_1 \in Q_V$ we add a transition $((q_0, q_1, 0), X, (q'_0, q_1, a), \beta, \vec{0})$ to δ. Since $(q_0, q_1, 0) \in Q_0$, Player 0 gets to chose this move in the energy game. This move in the energy game does not change the energy, but it is enabled iff the corresponding move is enabled in the pushdown automaton and it has the same effect on the stack. The new control-state (q'_0, q_1, a) records the new state q'_0 and the symbol $a \in Act$, which forces Player 1 to emulate an a-move of the VASS in the next step. Since $(q'_0, q_1, a) \in Q_V$, Player 1 chooses the next step. For every transition $(q_1, a, q'_1, D) \in \delta_V$, $q'_0 \in Q_A$, $a \in Act$ and $X \in \Gamma$, we add a transition $((q'_0, q_1, a), X, (q'_0, q'_1, 0), X, D)$ to δ. This move is enabled regardless of the stack content, but it must match the recorded control-state and action symbol. Its effect D on the energy implements the changes in the VASS. Unlike in the VASS, moves that decrease a counter below zero are not blocked in the energy game, but they are losing for Player 1. Thus, since Player 1 chooses the moves that affect the energy, he will avoid any move that is disabled in the VASS. After this move we have emulated one round of the simulation game, the

control-state is in Q_0 again and the invariant is maintained.

Finally, to ensure deadlock-freedom in the energy game, we add transitions $(q, X, q, X, \vec{0})$ for every $q \in Q_0$ and $X \in \Gamma$ and transitions $(q, X, q, X, (-1, \ldots, -1))$ for every $q \in Q_1$ and $X \in \Gamma$. I.e., if Player 0 was deadlocked in the pushdown automaton then she will loop forever in the energy game without decreasing the energy, and thus Player 1 wins. If Player 1 is deadlocked in the VASS, then the only available moves in the energy game repeatedly decrease the energy until Player 1 loses the game.

If $(q_0, \gamma) \preceq (q_1, E)$ then Player 1 has a winning strategy in the simulation game. By the semantics of VASS he can continue forever without going below zero in any of the VASS counters. By using the same strategy in the emulating energy game he can continue forever without running out of energy and thus wins the energy game from configuration $((q_0, q_1, 0), \gamma, E)$. Conversely, if $(q_0, \gamma) \not\preceq (q_1, E)$ then Player 0 has a winning strategy in the simulation game which eventually leads to a configuration where Player 1 is blocked. By using the same strategy in the corresponding energy game, eventually a configuration is reached where only energy decreasing moves remain available to Player 1 and he loses the energy game.

We observe that the above construction preserves the property that makes the pushdown automaton correspond to an OCA. If there is only one stack symbol plus a non-removable stack bottom symbol in the pushdown automaton A then the same property also hold for the constructed pushdown energy game G. Thus, simulation games between OCA and VASS reduce to one-counter energy games. ■

IV. The main decidability result

Here we consider 1-dimensional one-counter energy games. We show that the fixed initial credit problem is decidable and that the winning sets are semilinear. The proof first shows a corresponding result for simulation preorder between a OCA and a OCN, and then applies the connection between simulation games and energy games from Section III.

The cornerstone in our argument is the following property of simulation preorder (shown below).

Theorem 3. *Simulation preorder $\preceq_{A,A'}$ between a given one-counter automaton A and a one-counter net A' is semilinear.*

It immediately yields the decidability of simulation preorder.

Theorem 4. *Simulation preorder between a OCA and a OCN is decidable.*

Proof: We use a combination of two semi-decision procedures. Since OCA/OCN define finitely-branching processes, we can apply a standard result that *non*-simulation is semidecidable. The semi-decision procedure for simulation works as follows. By Theorem 3, it suffices to enumerate semilinear sets and to check for each such set whether it is a simulation relation that moreover contains the given pair of processes. This check is effective by the definition of the simulation condition and the decidability of Presburger arithmetic. ■

By the connection between simulation games and energy games from Section III, we obtain our main result.

Theorem 5. *The fixed initial credit problem for 1-dimensional one-counter energy games is decidable, and the winning sets are semilinear.*

Proof: Directly by Theorem 3, Theorem 4, Lemma 1. ■

In the rest of this section we prove Theorem 3. We fix a one-counter automaton $A = (Q, Act, \delta, \delta_0)$ and a one-counter net $A' = (Q', Act, \delta')$. Note that a slightly more general problem, where *both* systems have zero-testing, is no longer computable. Simulation preorder between two one-counter *automata* is undecidable [16].

In our construction, we will use a previous result about a special subcase of our problem, that the maximal simulation between two one-counter *nets* is effectively semilinear [15], [14]. Ultimately, these positive results are due to the following monotonicity properties.

Proposition 6 (Monotonicity)**.** *Let p be a state of a OCA, p', q' be states of a OCN and $m, m', n', l \in \mathbb{N}$. Then,*

1) *$p'm' \xrightarrow{a} q'n'$ implies $p'(m'+l) \xrightarrow{a} q'(n'+l)$,*
2) *$p'm' \preceq p'(m'+l)$ and*
3) *if $pm \preceq p'm'$ then $pm \preceq p'(m'+l)$.*

Following [16], we interpret a binary relation $R \subseteq (Q \times \mathbb{N}) \times (Q' \times \mathbb{N})$ between the configurations of the processes of A and A' as a 2-coloring of $|Q \times Q'|$ many planes $\mathbb{N} \times \mathbb{N}$, one for every pair (p, p') of control states. The color of (m, m') on the plane for (p, p') is white if $(pm, p'm') \in R$ and black otherwise. We are particularly interested in the coloring of $\preceq$, the largest simulation w.r.t. A and A'.

Definition 8. *Consider the* coloring $\mathbb{C}$ *defined by $\mathbb{C}_{p,p'}(m, m') = white$ iff $pm \preceq p'm'$. We write $\mathbb{C}_{p,p'}(i, -)$ for the vertical line at level i on the plane for states (p, p'). That is, $\mathbb{C}_{p,p'}(i, -) : \mathbb{N} \to \{white, black\}$ with $\mathbb{C}_{p,p'}(i, -)(n) = \mathbb{C}_{p,p'}(i, n)$. We say that this line is black iff $\mathbb{C}_{p,p'}(i, -)(n) = black$ for all $n \in \mathbb{N}$, i.e., if every point on the line is colored black.*

By monotonicity of simulation preorder (Proposition 6.3), for every line that is not *black there is a minimal value $W_{p,p'}(i)$ such that $\mathbb{C}_{p,p'}(i, -)(n) = white$ for all $n \geq W_{p,p'}(i)$. We define $W_{p,p'}(i) = \infty$ if no such value exists (the line is black) and write $W(i)$ for the maximal finite value $W_{p,p'}(i)$ over all pairs $(p, p') \in (Q \times Q')$.*

We show that the distribution of black lines in the coloring of $\preceq$ follows a regular pattern.

Definition 9 (Safe strategies)**.** *Let σ be a winning strategy for Spoiler in the simulation game from position $(pm, p'm')$ for some $m, m', l \in \mathbb{N}$ such that $m \geq l$. A strategy σ is called* l-safe*, if whenever a play according to σ reaches a position of the form $(ql, q'n')$ for the first time, then the line $\mathbb{C}_{q,q'}(l, -)$ is black.*

Lemma 7. *Let $m \geq l \in \mathbb{N}$ such that $\mathbb{C}_{p,p'}(m, -)$ is black. For all values $m' \in \mathbb{N}$, Spoiler can win the simulation game from*

position $(pm, p'm')$ using a l-safe strategy.

Proof: Fix any position $(pm, p'm')$. As the line $\mathbb{C}_{p,p'}(m, -)$ is black, we have $pm \not\preceq pn'$ for all $n' \in \mathbb{N}$. In particular, Spoiler has a winning strategy σ_l from position $(pm, p'(m' + W(l)))$. We see that Spoiler may re-use this strategy also in the game that starts from position $(pm, p'm')$, maintaining an offset of $W(l)$ in her opponents counter value to the corresponding position in σ_l. Let σ be the so constructed strategy for the game from $(pm, p'm')$.

We argue that σ is winning and l-safe. Indeed, let $(ql, q'n')$ be some position on a branch of σ. The same branch in σ_l leads to position $(ql, q'(n' + W(l)))$. Clearly, $n' + W(l) \geq W_{q,q'}(l)$ and the color of this point is black, because it is a position on a winning strategy for Spoiler. Recall that black point on the line $\mathbb{C}_{p,p'}(l, -)$ above $W(l)$ means that $\mathbb{C}_{p,p'}(l, -)$ is black. Together this means that the whole line $\mathbb{C}_{q,q'}(l, -)$ must be black. ∎

Lemma 8. *There exist $l, K \in \mathbb{N}$ such that for any pair $(p, p') \in (Q \times Q')$ of control-states and any $i \geq l$ it holds that*

1) *The line $\mathbb{C}_{p,p'}(i, -)$ is black iff $\mathbb{C}_{p,p'}(i + K, -)$ is black*
2) *$W_{p,p'}(i) \leq W_{p,p'}(i + K)$.*

Proof: We consider the patterns $Pat_i : Q \times Q' \to \{black, white\}$ indicating the colors of all lines at level $i \in N$: $Pat_i(p, p') = black$ iff $\mathbb{C}_{p,p'}(i, -)$ is black. Naturally, with increasing index i, this pattern eventually repeats and we can extract an infinite sequence of indices with the same pattern. This prescribes a sequence of vectors in which each component corresponds to $W_{q,q'}(i) < \infty$ for some $(q, q') \in Q \times Q'$. Dickson's Lemma allows us to pick indices l and $l + K \in \mathbb{N}$ that satisfy both conditions in the claim of the lemma (for $i = l$). It remains to show that both claims hold also for all $i > l$.

For the first claim, assume towards a contradiction that for some pair (p, p'), the color of $\mathbb{C}_{p,p'}(i, -)$ is different from that of $\mathbb{C}_{p,p'}(i + K, -)$. W.l.o.g. assume further $\mathbb{C}_{p,p'}(i, -)$ is white (the other case is symmetric). Then,

1) $\forall m'.\ pi + K \not\preceq p'm'$, and
2) $\forall m' \geq W(i).\ pi \preceq p'm'$.

We fix $m' \geq W(i)$, that is, $pi \preceq p'm'$ and $pi + K \not\preceq p'm'$. By Lemma 7, Spoiler has a $(l + K)$-safe winning strategy σ for the simulation game from position $(p(i + K), p'm')$. Figure 1 illustrates this scenario.

We claim that Spoiler can reuse this strategy and win also from position $(pi, p'm')$. Consider the simulation game from $(pi, p'm')$ in which Spoiler initially plays according to σ. Then on any branch, either no position visits level l or there is some first position which does. Fix some branch on this partial strategy. In the first case, the corresponding branch on σ never visits level $l + K$ and ends in some position $(q(n + K), q'n')$ which is immediately winning for Spoiler. This means our branch ends in position $(qn, q'n')$ where $n > l \geq 0$ and from which Spoiler wins immediately because she can mimic the attack in σ. Alternatively, this branch visits some position $(ql, q'n')$ at level l for the first time and the corresponding

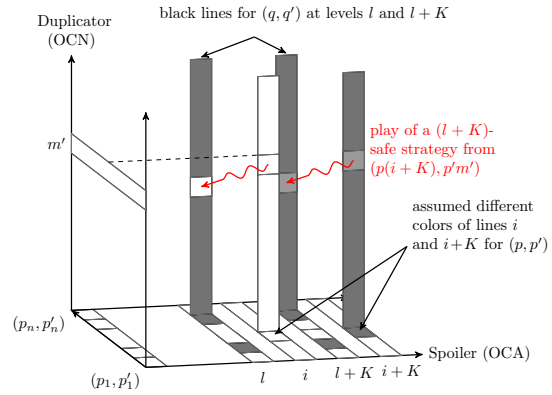

Fig. 1. Illustrates part 1) of the proof of Lemma 8

branch in σ visits $(q(l + K), q'n')$. Since σ is $(l + K)$-safe, we know that the line $\mathbb{C}_{q,q'}(l + K, -)$ is black. Because of our assumption that the pattern at levels l and $l + K$ agree, the line $\mathbb{C}_{q,q'}(l, -)$ must also be black. Therefore in particular we have $ql \not\preceq q'n'$, so Spoiler may continue the game from position $(ql, q'n')$ using some winning strategy. We have shown that there is a winning strategy for Spoiler from position $(pi, p'm')$. As m' was chosen arbitrarily, the line $\mathbb{C}_{p,p'}(i, -)$ is black, which contradicts our assumption and thus completes the proof of the first claim.

For the second claim, it suffices to show that for all pairs $(p, p') \in Q \times Q'$, $i \geq l$ and $m' \in \mathbb{N}$ it holds that

$$p(i + K) \preceq p'm' \implies pi \preceq p'm' \tag{1}$$

Let σ be a winning strategy for Duplicator in the simulation game from position $(p(i + K), p'm')$ and consider a first position $(q(l + K), q'n')$ on a play of σ where Spoiler's counter drops to $l + K$. Since this is a position on a winning strategy, n' must be greater or equal $W_{q,q'}(l+K)$. If Duplicator plays according to the same strategy from position $(pi, p'm')$, then this partial play ends in $(ql, q'n')$, which is winning for Duplicator because $n' \geq W_{q,q'}(l + K) \geq W_{q,q'}(l)$ due to our choice of l and K. We conclude that σ must be a winning strategy for Duplicator in the simulation game from $(pi, p'm')$. ∎

Lemma 8 allows us to fix a value $l \in \mathbb{N}$, that marks the column in the coloring of $\preceq$ from which on the distribution of black lines is repetitive (with period K). We split the simulation relation into two infinite subsets (see Figure 2):

$$S_{<l} = \ \preceq \cap\ Q \times \{n \in \mathbb{N} \mid n < l\} \times Q' \times \mathbb{N}$$
$$S_{\geq l} = \ \preceq \cap\ Q \times \{n \in \mathbb{N} \mid n \geq l\} \times Q' \times \mathbb{N}.$$

Note that the set $S_{<l}$ is semilinear because it is the upward closure of the minimal white positions $W_{p,p'}(i)$ for the finitely many pairs (p, p') and $i < l$. It remains to show that $S_{\geq l}$ is semilinear. For this, we recall the following result on strong simulation over one-counter nets.

Theorem 9 ([14])**.** *Let B and B' be two one-counter nets. The maximal simulation relation $\preceq_{B,B'}$ relative to B and B' is semilinear and one can effectively construct a semilinear representation.*

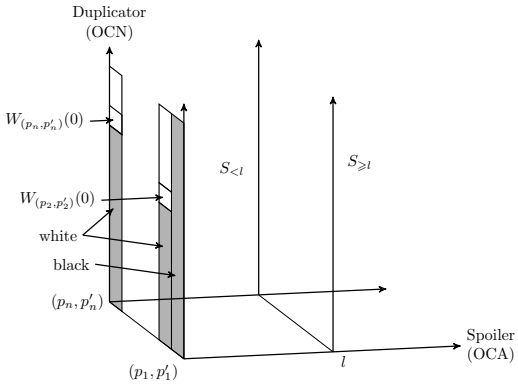

Fig. 2. The coloring of $\leq$ is cut into two sets $S_{<l}$ and $S_{\geq l}$.

In order to compute $S_{\geq l}$, we construct two one-counter nets B and B', such that there is a direct, and Presburger definable, correspondence between simulation in B, B' and simulation in A, A'. These nets are parameterized by A, A', l, K, the minimal values $W_{p,p'}(l)$ at level l and the patterns Pat_i which determine the distribution of black lines for indices $l \leq i \leq l + K$.

Lemma 10. *There exist two one-counter nets B and B' with state-sets R and R' respectively, and Presburger definable functions $F : (Q \times Q' \times \mathbb{N}) \to R$ and $G : (Q \times Q' \times \mathbb{N}) \to R'$ such that for all $p \in Q$, $p' \in Q'$ and $m, m' \in \mathbb{N}$,*

$$p(m+l) \preceq_{A,A'} p'm' \iff F(p,p',m)m \preceq_{B,B'} G(p,p',m)m'.$$

Proof: We construct nets $B = (Q \times Q' \times \mathbb{N}_{<K} \cup R_B, Act', \delta_B)$ and $B' = ((Q \times Q' \times \mathbb{N}_{<K}) \cup R'_B, Act', \delta'_B)$ with actions $Act' = \delta_A \cup \delta'_A \cup \{\$\}$. One round of the simulation game w.r.t. A and A' will be emulated in two rounds of the game w.r.t. B and B'. Apart from auxiliary states in R_B and R'_B, each state encodes a pair of states of A and A' respectively, together with the counter value of Spoiler modulo K, so that an original position $(p(m+l), p'm')$ corresponds to the position $((p,p',m \mod K)m, (p,p',m \mod K)m')$. Unless the parameter lets us immediately derive a winner for the current position (for example the game reaches a position on a black line) the new game will continue to emulate the old game and end in a position of the above form every other round.

The net B contains the following transitions for every $(p,p',m) \in (Q \times Q' \times \mathbb{N}_{<K})$ and every $t = (p,a,d,q) \in \delta$ where $n = m + d \mod K$.

$$(p,p',m) \xrightarrow{t,d} (t,p',n), \tag{2}$$

$$(t,p',n) \xrightarrow{t',0} (q,q',n) \quad \text{for every } t' = (p',a,d',q') \in \delta \tag{3}$$

The net B' contains a universal state $u \in R'_B$ such that $u \xrightarrow{a,0} u$ for every $a \in Act'$ and moreover, the following transitions for every $(p,p',m) \in (Q \times Q' \times \mathbb{N}_{<K})$ and every $t = (p,a,d,q) \in \delta$ where $n = m + d \mod K$.

$$(p,p',m) \xrightarrow{t,d'} (t,t',n) \quad \text{for every } t' = (p',a,d',q') \in \delta' \tag{4}$$

$$(t,t',n) \xrightarrow{t',0} (q,q',n) \tag{5}$$

$$(q,t',q',n) \xrightarrow{s,0} u \qquad \text{for every } s \neq t' \in \delta'. \tag{6}$$

The transitions above allow to emulate one round of the original game in two rounds: Spoiler announces the transition t she chooses in the original game, then Duplicator responds by recording his chosen transition t' and remembers both choices in his control-state. In the next round Spoiler, who needs to prevent her opponent from becoming universal, must faithfully announce her opponents original response. Afterwards, Duplicator has no choice but to also update his state to (q,q',n), which reflects the new original pair of states and Spoiler's new counter value modulo K.

So far, this new game is in favor of Duplicator, because it is essentially the original game where Spoiler is deprived of her zero-testing transitions. We now correct this imbalance, using the additional (given) information about the values $W_{p,p'}(l)$, as well as the knowledge about which lines are completely black.

Recall that for all $i > l$, the line $\mathbb{C}_{p,p'}(i,-)$ is black iff the line $\mathbb{C}_{p,p'}(i+K,-)$ is. Every state (p,p',m) of B' has a \$-labeled self-loop with effect -1. Moreover, a state (p,p',m) of B has a non-decreasing \$-labeled self-loop if the line $\mathbb{C}_{p,p'}(m+l,-)$ is *black*. This ensures that in the new game, Spoiler can win from positions $(p,p',m \mod K)m, (p,p',m \mod K)m'$ if the color of $\mathbb{C}_{p,p'}(l+m,-)$ is black, regardless of the actual value m' of Duplicator's counter.

Lastly, we add the possibility for Spoiler to successfully end the game if a position $((p,p',0)l, (p,p',0)m')$ is reached where her counter value equals l and Duplicator's value m' is below $W_{p,p'}(l)$. For each $(p,p') \in Q \times Q'$, the control graph of net B contains a path

$$(p,p',0) \xrightarrow{\$,0} s_k \xrightarrow{\$,0} s_{k-1} \xrightarrow{\$,0} \cdots \xrightarrow{\$,0} s_0 \tag{7}$$

of length $k = W_{p,p'}(l)$. We argue, assuming that nets B and B' are correctly parameterized, that $p(m+l) \preceq_{A,A'} p'm'$ iff $(p,p',m \mod K)m \preceq_{B,B'} (p,p',m \mod K)m'$. Observe that this implies the claim of the lemma, as projections and multiplication (and division) by fixed values K are definable in Presburger Arithmetic.

Assume $p(m+l) \not\preceq_{A,A'} p'm'$ and consider the game on B and B' from position $((p,p',m \mod K)m, (p,p',m \mod K)m')$. Spoiler moves according to her original winning strategy, preventing her opponent from reaching state u and thus faithfully emulates a play of the game on A and A'. One of three things must eventually happen:

1) Duplicator is forced to reduce his counter below 0 and up to then, Spoiler's counter always remains strictly above level l and no visited position corresponds to a point on a black line. Such a play is losing for Duplicator in both games.

2) A position $((q,q',n \mod K)n, (q,q',n \mod K)0)$ is reached and the line $\mathbb{C}_{q,q'}(l+n,-)$ is black. This means the

states $(q, q', n \mod K)$ of both nets have a \$-labeled self loop, where only the one in B' is decreasing. Spoiler wins from such a position by iterating this loop.

3) A position $((q, q', 0)0, (q, q', 0)m')$ is reached where Spoiler's counter equals 0 and $m' < W_{q,q'}(l)$. In this case, Spoiler wins by moving along the \$-labeled path described by Equation (7), which allows her to make exactly $W_{q,q'}(l) + 1$ many steps which are decreasing Duplicator's counter.

Conversely, assume $(p, p', m \mod K)m \not\leq_{B,B'} (p, p', m \mod K)m'$ and consider a play along which Spoiler plays according to some winning strategy. Until Spoiler makes use of a \$-labeled step, the play in the game on B, B' directly corresponds to a play on A, A'. This is because Spoiler must prevent her opponent from reaching the universal state u. A play that Spoiler wins without \$-steps thus yields a winning play in the game on A, A'. Otherwise, consider a first position $((q, q', n \mod K)n, (q, q', n \mod K)n')$ from which Spoiler plays a \$-step. The fact that the state $(q, q', n \mod K)$ has a \$-labeled outgoing transition means that either $\mathbb{C}_{q,q'}(n+l, -)$ is black or this line is white but $n \mod K = 0$. The former means that in particular that $q(n+l) \not\leq_{A,A'} q'n'$. In the latter case, Spoiler moves from state $(q, q', 0)$ to the initial state s_k of a chain of length $k = W_{q,q'}(l)$, at the end of which she would deadlock. This does not happen because the play is a win for Spoiler. Therefore, the counter value n' of Duplicator must be strictly below $W_{q,q'}(l)$. Since K divides n, point 2 in Lemma 8 implies $n' < W_{q,q'}(l) \leq W_{q,q'}(l+n)$ which means that $q(l+n) \not\leq_{A,A'} q'n'$.

We conclude that a winning strategy from $((p, p', m \mod K)m, (p, p', m \mod K)m')$ in the game on B, B' prescribes a strategy for Spoiler in the game on A, A' from position $(p(m+l), p'm')$ such that each play eventually leads to a winning position and therefore, $p(m+l) \not\leq_{A,A'} p'm'$ ■

The semilinearity of $S_{\geq l}$ and thus of $\leq$ now follows from Theorem 9 and Lemma 10. This completes the proof of Theorem 3.

V. Undecidability Results

A. Pushdown Energy Games

We show that the fixed as well as the unknown initial credit problem for pushdown energy games is undecidable. This holds even with a single energy dimension. Consequently, we may use Lemma 2 to deduce the undecidability of the simulation problem between pushdown automata and OCN.

The undecidability is established by a reduction from the halting problem for Minsky's 2-counter machines. We begin by recalling the definition. The machine is equipped with two counters, c_1 and c_2, that take values over natural numbers. There are two types of transitions. The first kind simply increments one of the counters. The second kind checks the value of a counter and decrements it iff it is > 0.

Definition 10. *A 2-counter machine (MCM) M is a tuple $(Q, q_{init}, q_{halt}, \delta)$ where Q is a finite set of control-states, q_{init} is the initial state, q_{halt} is the halting state, and δ is a finite set of transitions of the following two forms.*

- $(q, [c_i{}^+; q'])$, *$i \in \{1, 2\}$ and $q \neq q_{halt}$. Increment counter c_i unconditionally and go to state q'.*
- $(q, [\textbf{if } (c_i = 0) \textbf{ then } c_i; q' \textbf{ else } c_i{}^-; q''])$, *$i \in \{1, 2\}$ and $q \neq q_{halt}$. Check the value of counter c_i, go to state q' if it is 0 or go to state q'' after decrementing c_i if it is > 0.*

A configuration *of such a machine is an element of $Q \times \mathbb{N}^2$. The initial configuration is $(q_{init}, (0, 0))$. We say that the configuration $(q, (m_1, m_2))$ moves to $(q', (m_1', m_2'))$ in one step, written $(q, (m_1, m_2)) \longrightarrow (q', (m_1', m_2'))$, iff*

1) *$(q, [c_i{}^+; q']) \in \delta$, $m_i' = m_i + 1$ and $m_{3-i}' = m_{3-i}$*
2) *$(q, [\textbf{if } (c_i = 0) \textbf{ then } c_i; q' \textbf{ else } c_i{}^-; q'']) \in \delta$, $m_i = m_i' = 0$ and $m_{3-i}' = m_{3-i}$.*
3) *$(q, [\textbf{if } (c_i = 0) \textbf{ then } c_i; q'' \textbf{ else } c_i{}^-; q']) \in \delta$, $m_i > 0$, $m_i' = m_i - 1$ and $m_{3-i}' = m_{3-i}$.*

A run *is a finite or infinite sequence of steps between configurations. A run is* maximal *if it is either infinite or ends at a configuration where no move is possible. W.l.o.g. we assume that at least one move is possible in any configuration whose control state is not q_{halt}.*

A 2-counter machine is said to be deterministic *if there is a unique maximal run starting from the initial configuration $(q_{init}, (0, 0))$. Notice that this run either reaches the state q_{halt} (and halts) or is infinite.*

We now prove the following lemma which immediately implies that both the fixed and unknown initial credit problem for pushdown games are undecidable.

Lemma 11. *Given a deterministic MCM $M = (Q, q_{init}, q_{halt}, \delta)$, one can effectively construct a 1-dimensional pushdown energy game $G = (Q_0, Q_1, \Gamma, \delta_G, 1)$ s.t. M halts iff Player 0 wins the energy game from every initial energy credit. Moreover, if Player 0 wins the energy game for some initial energy credit then she wins from every initial energy credit.*

Proof: If M does not terminate then it diverges i.e., the only valid run is infinite. The overall idea is to let Player 1 propose this infinite run by pushing the corresponding sequence $\tau_1, \tau_2, \ldots$ of transitions onto the stack. If M actually terminates and there is no infinite run, Player 1 must eventually "cheat" and announce a next step that is not a valid continuation of the run committed to the stack. After each such move, Player 0 can choose to either accept the last MCM step and let Player 1 continue to push the next one, or she can choose to challenge its validity and move the game to a test. There is a test for every type of error that can be spotted by Player 0 and each such test has one of three possible outcomes.

1) The energy level goes below zero and Player 0 wins.
2) The game returns to a position with initial state and empty stack, but the energy level is smaller than it was before.
3) The game returns to a position with initial state and empty stack, but the energy level is greater or equal than it was before.

The first outcome is obviously good for Player 0 and so is outcome 2, because eventually, after sufficiently many such outcomes, the energy has to run out and she will win. Outcome

3 is good for Player 1, because it makes his position at least as good as it was before. We will implement test gadgets that Player 0 may choose to invoke. If Player 0 has correctly spotted an error then the outcome of the gadget will be 1 or 2, and if she was wrong then the outcome will be 3.

We now describe the construction formally, starting with the part of the game in which Player 1 writes a run of the MCM onto the stack. The first part of the construction guarantees that the ending state of each transition τ_i matches the starting state of the next transition τ_{i+1}. Given M, we build a finite graph as follows. Vertices are states of M and for every transition of the first type $\tau_l = (q, [c_i{}^+; q'])$, we add a directed edge q, q' labeled with τ_{l,c_i} and for every transition of the second type $\tau_l = (q, [\textbf{if } (c_i = 0) \textbf{ then } c_i; q' \textbf{ else } c_i{}^-; q''])$ we add a pair of directed edges q, q' and q, q'' labeled $\tau_{l>0,c_i}$ and $\tau_{l=0,c_i}$, respectively. Every path in this graph corresponds to some correct or incorrect run of M. Incorrectness may come from the fact that paths in the graph do not care about the values of the counters.

Now we encode this graph into a part of the energy game. Vertices becomes states owned by Player 1, and each edge q, q' labeled with τ_X is encoded by two sets of transitions. In the first set we have transitions $(q, Y, s_{\tau_X}, \tau_X Y, 1)$ for every stack symbol Y, and s_{τ_X} is an intermediate state which belongs to Player 0. These push a record of the transition τ_X onto the stack and increment the energy by 1. In the second set we have transitions $(s_{\tau_X}, Y, q', Y, 0)$ for every stack symbol Y. These do not change the stack and energy. Additionally, in s_{τ_X}, Player 0 can decide if she wants to follow the edge to q' or if she would rather invoke some testing gadget (see below).

In the halting state we add a self-loop which decrements the energy; this guarantees that if M halts then Player 0 wins.

The crucial properties are that (except in the situation when the game reaches the halting state):

- on the stack we have a word which remembers the path from the beginning up to now,
- the energy level is equal to the initial value of the energy + the number of elements on the stack.

So what are the possible errors? Player 1 can cheat only when he has a choice. Since M is deterministic, we know that in the game choice comes only from the translation of the transitions of the second type $((q, [\textbf{if } (c_i = 0) \textbf{ then } c_i; q' \textbf{ else } c_i{}^-; q'']))$. There are two types of errors.

1) Player 1 tries to go from q to q' when $c_i > 0$, or
2) Player 1 tries to go from q to q'' when $c_i = 0$.

To detect these errors, we define four gadgets; two types times two counters. The gadget that we have to use is determined by the last transition; if it pushes to the stack $\tau_{l=0,c_i}$ then we use the first type gadget for the counter c_i and if it pushes $\tau_{l>0,c_i}$ then we use the second type gadget for c_i.

In both gadgets we pop stored transitions from the stack, and reduce the energy level accordingly, in order to check the conditions $c_i > 0$ and $c_i = 0$, respectively.

- In the first type of error, Player 0 should gain (outcome 1 or 2) if $c_i > 0$, i.e., if in the stored history the number of increments of counter c_i is greater than the number of decrements of c_i. Otherwise, we should get outcome 3.
- In the second type of error, Player 0 should gain if $c_i = 0$, i.e., we need to check if the number of decrements is equal to the number of increments. Observe that we can safely assume that it is not greater, because this would mean $c_i < 0$ and then there was an error earlier. So Player 0 should gain (outcome 1 or 2) if the number of decrements is equal to the number of increments, but lose (outcome 3) if the number of decrements is less than the number of increments.

In the first type of gadget, we allow Player 0 to pop the content of the stack according to following rules.

1) In removing a transition not affecting c_i she reduces the energy level by 1.
2) In removing a transition incrementing c_i she reduces the energy level by 2.
3) In removing a transition decrementing c_i she leaves the energy level unchanged.

If the number of increments was greater than the number of decrements then in the end of the gadget the energy level drops below the initial energy level (outcome 1 or 2). This is because, before we enter the gadget, the energy level is equal to the initial energy level + the number of elements on the stack, and in the gadget we decrease it by more than the number of elements on the stack. On the other hand, if the numbers of decrements and increments are equal, then the energy level is equal to the initial one (outcome 3).

In the second type of gadgets, the roles are reversed.

1) In removing a transition not affecting c_i she reduces the energy level by 1.
2) In removing a transition incrementing c_i she leaves the energy level unchanged.
3) In removing a transition decrementing c_i she reduces the energy level by 2.

Finally, in the end we decrement the energy by 1.

If the number of decrements is equal to the number of increments then we end with an energy level which is smaller than the initial energy level (outcome 1 or 2), due to the final decrement by 1. On the other hand, if the number of decrements is smaller than the number of increments, then the final energy level is greater than (or equal to) what it was in the beginning (outcome 3). ■

B. One-counter energy games

We show that one-counter energy games of energy dimension $n \geq 2$ are undecidable, via the undecidability of the corresponding simulation games.

Theorem 12. *Simulation preorder between OCN and VASS of dimension ≥ 2 is undecidable in both directions.*

Proof: Consider a deterministic Minsky 2-counter machine M with a set of control-states Q, counters c_1 and c_2 and initial configuration $(q, (0, 0))$. It either eventually reaches the accepting state q_{halt} or runs forever. We construct an OCN

$A = (Q_A, Act, \delta_A)$ with initial configuration $(q, 0)$ and a VASS $V = (Q_V, Act, \delta_V)$ of dimension 2 with initial configuration $(q, (0,0))$ such that M halts iff $(q, 0) \not\leq (q, (0,0))$. (The construction for the other simulation direction is very similar.)

Let $Act = \{a, z, nz, c, h\}$, $Q_A = Q \cup Q'_A$ and $Q_V = Q \cup Q'_V$ where Q'_A, Q'_V contain some auxiliary control-states (see below). In the simulation game we maintain the following invariant of game configurations $((q, z), (q', (x, y)))$. If $q = q' \in Q$ then $z = x + y$. I.e., except in some auxiliary states of Q'_A, Q'_V, the OCN counter will contain the sum of the VASS counters. The idea is that the simulation game emulates the computation of M, where the two counters are stored in the VASS counter values x and y, respectively. Via the classic forcing technique, Duplicator gets to choose the next transition of M. The only possible deviation from a faithful emulation of M is where Duplicator chooses a zero-transition of M when the respective counter contains a nonzero value, e.g., $x > 0$. In this case Spoiler can win the game by forcing a comparison of z with y. If $x > 0$ then, by the above invariant, $z > y$ and Spoiler wins. Otherwise, if $x = 0$ then $z = y$ and Duplicator wins.

Since M is deterministic, there is only one transition rule for every control-state q.

If the rule of M is of the form $(q, [c_1{}^+; q'])$ then we add a rule $(q, a, +1, q')$ to δ_A and a rule $(q, a, q', (1, 0))$ to δ_V. (The other case where c_2 is incremented is symmetric.) Thus one round of the simulation game emulates the transition of M and the invariant is maintained.

Otherwise, the rule of M is of the form $(q, [\mathbf{if}\ (c_1 = 0)\ \mathbf{then}\ c_1; q'\ \mathbf{else}\ c_1{}^-; q''])$ (the case where c_2 is tested is symmetric). We add a rule $(q, a, 0, \hat{q})$ to δ_A and rules $(q, a, q_1, (0, 0))$ and $(q, a, q_2, (-1, 0))$ to δ_V. By choosing q_1 (resp. q_2) Duplicator claims that c_1 is zero (resp. nonzero). A nonzero claim is certainly correct, since the transition to q_2 decrements the counter. However, a zero claim might be false. Now Spoiler can either accept this claim or challenge it (and win iff it is false). For the case where a nonzero claim is accepted we add a rule $(\hat{q}, nz, -1, q'')$ to δ_A and a rule $(q_2, nz, q'', (0, 0))$ to δ_V. Similarly for a case where a zero claim is accepted we add a rule $(\hat{q}, z, 0, q')$ to δ_A and a rule $(q_1, z, q', (0, 0))$ to δ_V. In either case, two rounds of the simulation game emulate the transition of M and the invariant is maintained. Since Spoiler must not spuriously accept a choice that Duplicator has not made, we add transitions $(q_2, z, U, (0, 0))$, $(q_1, nz, U, (0, 0))$ and $(U, \alpha, U, (0, 0))$ to δ_V for every $\alpha \in Act$. Here Duplicator goes to the universal state U and wins the simulation game.

As explained above, a nonzero claim by Duplicator is always correct and thus cannot be challenged. The following construction implements a challenge by Spoiler to a zero claim of Duplicator. We add a transition $(\hat{q}, c, 0, q_c)$ to δ_A and transitions $(q_1, c, q_c, (0, 0))$ and $(q_2, c, U, (0, 0))$ to δ_V. If Spoiler spuriously issues a challenge to a zero claim that Duplicator has not made (where he is in state q_2) then Duplicator goes to the universal state U and wins. Otherwise, both players are in state q_c and the simulation game is in state $((q_c, z), (q_c, (x, y)))$ for $z = x + y$ by the invariant above. The challenge is evaluated by the following rules. We add a rule $(q_c, c, -1, q_c)$ to δ_A and a rule $(q_c, c, q_c, (0, -1))$ to δ_V. If the zero claim by Duplicator was false then $x > 0$ and thus $y < z$. Therefore, Spoiler wins the simulation game from $((q_c, z), (q_c, (x, y)))$, because eventually the second counter of Duplicator reaches zero before Spoiler. However, if the zero claim by Duplicator was true then $x = 0$ and $y = z$ and both players reach zero (and get stuck) at the same time, and thus Duplicator wins.

Finally, we add a transition $(q_{halt}, h, 0, q_{halt})$ to δ_A, i.e., the state q_{halt} is winning for Spoiler by a special action h.

To summarize, if M does not halt then Duplicator wins the simulation game by a faithful emulation of the infinite computation of M, because every challenge by Spoiler will also lead to a win by Duplicator. Conversely, if M halts then every faithful emulation would lead to state q_{halt} and a win by Spoiler. The only possible deviation from a faithful emulation is a false zero claim by Duplicator. However, the challenge construction above ensures that Spoiler can also win in this case. Thus Spoiler wins the simulation game iff M halts. ■

Since OCA subsume OCN (by Def. 4), this implies that simulation preorder between OCA and VASS of dimension ≥ 2 is also undecidable. Thus, with Lemma 2, we obtain the following theorem.

Theorem 13. *The fixed initial credit problem is undecidable for one-counter energy games of energy dimension $n \geq 2$. I.e., given a one-counter energy game $G = (Q_0, Q_1, \delta, \delta_0, n)$ of dimension $n \geq 2$, it is undecidable whether a configuration (q, k, E) is winning for Player 0.*

VI. Conclusion and Future Work

Our decidability results for infinite-state energy games show a surprising distinction between pushdown automata and one-counter automata. While pushdown energy games are undecidable even for the simplest case of a 1-dimensional energy, the decidability border for one-counter energy games runs between the cases of 1-dimensional and multi-dimensional energy.

Some questions for future work concern the decidability of the *unknown initial credit problem* for infinite-state energy games. We have shown the undecidability of this problem for pushdown energy games, but it remains open for one-counter energy games. While we have shown that the winning sets of 1-dimensional one-counter energy games are semilinear, our proof does not yield an effective procedure for constructing these semilinear sets (which would immediately imply the decidability of the unknown initial credit problem).

In multidimensional one-counter energy games, the winning sets are certainly not semilinear (even though they are upward-closed w.r.t. the energy). Otherwise, one could enumerate semilinear sets and effectively check (by Presburger arithmetic) whether they are winning sets containing the initial configuration, and thus obtain a positive semi-decision procedure. Together with the obvious negative semidecidability

(by expanding the game tree) this would yield an impossible decision procedure for the fixed initial credit problem. In spite of this, the unknown initial credit problem could still be decidable.

The unknown initial credit problem for energy games is closely related to limit-average games. While even 1-dimensional limit-average games are undecidable for pushdown automata [9], the decidability of (multi-dimensional) limit-average games on one-counter automata is open.

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