TWO RATIONAL NODAL QUARTIC THREEFOLDS

IVAN CHELTsov AND CONSTANTIN SHRAMOV

Abstract. We prove that the quartic threefolds defined by
\[ \sum_{i=0}^{5} x_i = \sum_{i=0}^{5} x_i^4 - t \left( \sum_{i=0}^{5} x_i^2 \right)^2 = 0 \]
in \( \mathbb{P}^5 \) are rational for \( t = \frac{1}{6} \) and \( t = \frac{7}{10} \).

1. Introduction

Consider the six-dimensional permutation representation \( W \) of the group \( S_6 \). Choose coordinates \( x_0, \ldots, x_5 \) in \( W \) so that they are permuted by \( S_6 \). Then \( x_0, \ldots, x_5 \) also serve as homogeneous coordinates in the projective space \( \mathbb{P}^5 = \mathbb{P}(W) \).

Let us identify \( \mathbb{P}^4 \) with a hyperplane
\[ x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0 \]
in \( \mathbb{P}^5 \). Denote by \( X_t \) the quartic threefold in \( \mathbb{P}^4 \) that is given by the equation
\[ x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 = t \left( x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \right)^2, \]
where \( t \) is an element of the ground field, which we will always assume to be the field \( \mathbb{C} \) of complex numbers. Then \( X_t \) is singular for every \( t \in \mathbb{C} \). Indeed, denote by \( \Sigma_{30} \) the \( S_6 \)-orbit of the point \( [1 : 1 : \omega : \omega^2 : \omega^3] \), where \( \omega = e^{\frac{2 \pi i}{3}} \). Then \( |\Sigma_{30}| = 30 \), and \( X_t \) is singular at every point of \( \Sigma_{30} \) for every \( t \in \mathbb{C} \) (see, for example, [12, Theorem 4.1]).

The possible singularities of the quartic threefold \( X_t \) have been described by van der Geer in [12, Theorem 4.1]. To recall his description, denote by \( L_{15} \) the \( S_6 \)-orbit of the line that passes through the points \( [1 : 0 : -1 : 1 : 0 : -1] \) and \( [0 : 1 : -1 : 0 : 1 : -1] \), and denote by \( \Sigma_6 \), \( \Sigma_{10} \), and \( \Sigma_{15} \) the \( S_6 \)-orbits of the points \( [-5 : 1 : 1 : 1 : 1 : 1], [-1 : -1 : -1 : 1 : 1 : 1], \) and \( [1 : -1 : 0 : 0 : 0 : 0] \), respectively. Then the curve \( L_{15} \) is a union of fifteen lines, while \( |\Sigma_6| = 6 \), \( |\Sigma_{10}| = 10 \), and \( |\Sigma_{15}| = 15 \). Moreover, one has
\[
\text{Sing}(X_t) = \begin{cases} 
L_{15} \text{ if } t = \frac{1}{4}, \\
\Sigma_{30} \cup \Sigma_{15} \text{ if } t = \frac{1}{2}, \\
\Sigma_{30} \cup \Sigma_{10} \text{ if } t = \frac{1}{6}, \\
\Sigma_{30} \cup \Sigma_6 \text{ if } t = \frac{7}{10}, \\
\Sigma_{30} \text{ otherwise.} 
\end{cases}
\]
Furthermore, if \( t \neq \frac{1}{4} \), then all singular points of the quartic threefold \( X_t \) are isolated ordinary double points (nodes).
The threefold $X_{\frac{7}{2}}$ is classical. It is the so-called Burkhardt quartic. In [3], Burkhardt discovered that the subset $\Sigma_{30} \cup \Sigma_{15}$ is invariant under the action of the simple group $\text{PSp}_4(F_3)$ of order 25920. In [7], Coble proved that $\Sigma_{30} \cup \Sigma_{15}$ is the singular locus of the threefold $X_{\frac{7}{2}}$, and proved that $X_{\frac{7}{2}}$ is also $\text{PSp}_4(F_3)$-invariant. Later Todd proved in [22] that $X_{\frac{7}{2}}$ is rational. In [15], de Jong, Shepherd-Barron, and Van de Ven proved that $X_{\frac{7}{2}}$ is the unique quartic threefold in $\mathbb{P}^4$ with 45 singular points.

The quartic threefold $X_{\frac{1}{4}}$ is also classical. It is known as the Igusa quartic from its modular interpretation as the Satake compactification of the moduli space of Abelian surfaces with level 2 structure (see [12]). The projectively dual variety of the quartic threefold $X_{\frac{1}{4}}$ is the so-called Segre cubic. Since the Segre cubic is rational, $X_{\frac{1}{4}}$ is rational as well.

During Kul’fest conference dedicated to the 60th anniversary of Viktor Kulikov that was held in Moscow in December 2012, Alexei Bondal and Yuri Prokhorov posed

**Problem 1.2.** Determine all $t \in \mathbb{C}$ such that $X_t$ is rational.

Since $X_t$ is singular, we cannot apply Iskovskikh and Manin’s theorem from [14] to $X_t$. Similarly, we cannot apply Mella’s [18, Theorem 2] to $X_t$ either, because the quartic threefold $X_t$ is not $\mathbb{Q}$-factorial by [1, Lemma 2]. Nevertheless, Beauville proved

**Theorem 1.3** ([1]). If $t \notin \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{7}{10}\}$, then $X_t$ is non-rational.

Both $X_{\frac{7}{2}}$ and $X_{\frac{1}{4}}$ are rational. The goal of this paper is to prove

**Theorem 1.4.** The quartic threefolds $X_{\frac{7}{2}}$ and $X_{\frac{1}{4}}$ are also rational.

Surprisingly, the proof of Theorem 1.4 goes back to two classical papers of Todd. Namely, we will construct an explicit $\mathfrak{A}_6$-birational map $\mathbb{P}^3 \dashrightarrow X_{\frac{7}{10}}$ that is a special case of Todd’s construction from [20]. Similarly, we will construct an explicit $\mathfrak{S}_5$-birational map $\mathbb{P}^3 \dashrightarrow X_{\frac{1}{6}}$ that is a degeneration of Todd’s construction from [21]. We emphasize that our proof is self-contained, i.e. it does not rely on the results proved in [20] and [21], but recovers the necessary facts in our particular situation using additional symmetries arising from group actions.

**Remark 1.5.** Todd proved in [22] that the Burkhardt quartic $X_{\frac{7}{2}}$ is determinantal (see also [19, §5.1]). The constructions of our birational maps $\mathbb{P}^3 \dashrightarrow X_{\frac{7}{10}}$ and $\mathbb{P}^3 \dashrightarrow X_{\frac{1}{6}}$ imply that both $X_{\frac{7}{10}}$ and $X_{\frac{1}{6}}$ are determinantal (see [19, Example 6.4.2] and [19, Example 6.2.1]). Yuri Prokhorov pointed out that the quartic threefold

$$\det \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_4 & y_0 & y_3 & y_4 \\ y_2 & y_1 & y_1 & y_0 \\ y_0 & y_3 & y_2 & y_4 \end{pmatrix} = 0$$

in $\mathbb{P}^4$ with homogeneous coordinates $y_0, \ldots, y_4$ has exactly 45 singular points. Thus, it is isomorphic to the Burkhardt quartic $X_{\frac{7}{2}}$ by [15]. It would be interesting to find similar determinantal equations of the threefolds $X_{\frac{7}{10}}$ and $X_{\frac{1}{6}}$.

The plan of the paper is as follows. In Section 2 we recall some preliminary results on representations of a central extension of the group $\mathfrak{S}_6$, and some of its subgroups.
In Section 3 we collect results concerning a certain action of the group \( \mathfrak{A}_5 \) on \( \mathbb{P}^3 \), and study \( \mathfrak{A}_5 \)-invariant quartic surfaces; the reason we pay so much attention to this group is that it is contained both in \( \mathfrak{A}_6 \) and in \( \mathfrak{S}_5 \), and thus the information about its properties simplifies the study of the latter two groups. In Section 4 we collect auxiliary results about the groups \( \mathfrak{S}_6, \mathfrak{A}_6 \) and \( \mathfrak{S}_5 \), in particular about their actions on curves and their five-dimensional irreducible representations. In Section 5 we construct an \( \mathfrak{A}_6 \)-equivariant birational map \( \mathbb{P}^3 \rightarrow X_{10} \). Finally, in Section 6 we construct an \( \mathfrak{S}_5 \)-equivariant birational map \( \mathbb{P}^3 \rightarrow X_{16} \) and make some concluding remarks.

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2. Representation theory

Recall that the permutation group \( \mathfrak{S}_6 \) has two central extensions \( 2^+ \mathfrak{S}_6 \) and \( 2^- \mathfrak{S}_6 \) by the group \( \mu_2 \) with the central subgroup contained in the commutator subgroup (see [8, p. xxiii] for details). We denote the first of them (i.e. the one where the preimages of a transposition in \( \mathfrak{S}_6 \) under the natural projection have order two) by \( 2 \mathfrak{S}_6 \) to simplify notation. Similarly, for any group \( \Gamma \) we denote by \( 2 \Gamma \) a non-split central extension of \( \Gamma \) by the group \( \mu_2 \).

We start with recalling some facts about four- and five-dimensional representations of the group \( 2 \mathfrak{S}_6 \) we will be working with. A reader who is not interested in details here can skip to Corollary 2.1 or even to Section 4 where we reformulate everything in geometric language. Also, we will see in Section 4 that our further constructions do not depend much on the choice of representations, and all computations one makes for one of them actually apply to all others.

Let \( I \) and \( J \) be the trivial and the non-trivial one-dimensional representations of the group \( \mathfrak{S}_6 \), respectively. Consider the six-dimensional permutation representation \( W \) of \( \mathfrak{S}_6 \). One has

\[
W \cong I \oplus W_5 \otimes J
\]

for some irreducible representation \( W_5 \) of \( \mathfrak{S}_6 \). We can regard \( I, J \) and \( W_5 \) as representations of the group \( 2 \mathfrak{S}_6 \). Recall that there is a double cover \( \text{SL}_4(\mathbb{C}) \rightarrow \text{SO}_6(\mathbb{C}) \), see e.g. [10, Exercise 20.39]. Using it, we conclude that there is an embedding of the group \( 2 \mathfrak{S}_6 \) into \( \text{SL}_4(\mathbb{C}) \). This embedding gives rise to two four-dimensional representations of \( 2 \mathfrak{S}_6 \) that differ by a tensor product with \( J \). We fix one of these two representations \( U_4 \). Note that

\[
I \oplus W_5 \cong \Lambda^2(U_4).
\]
Recall that there are coordinates $x_0, \ldots, x_5$ in $W$ that are permuted by the group $S_6$. We will refer to a subgroup of $2.S_6$ fixing one of the corresponding points as a \textit{standard} subgroup $2.S_5$; we denote any such subgroup by $2.S_5^{st}$. A subgroup of $2.S_6$ that is isomorphic to $2.S_5$ but is not conjugate to a standard $2.S_5$ will be called a \textit{non-standard} subgroup of $2.S_5$; we denote any such subgroup by $2.S_5^{nst}$. These agree with standard and non-standard subgroups of $S_6$ isomorphic to $S_5$, although outer automorphisms of $S_6$ do not lift to $2.S_5$. Any subgroup of $2.S_6$ that is isomorphic to $2.A_5$, $2.S_4$ or $2.A_4$ and is contained in $2.S_5^{st}$ is denoted by $2.A_5^{st}$, $2.S_4^{st}$ or $2.A_4^{st}$, respectively. Similarly, any subgroup of $2.S_6$ that is isomorphic to $2.A_5$, $2.S_4$ or $2.A_4$ and is contained in $2.S_5^{nst}$ is denoted by $2.A_5^{nst}$, $2.S_4^{nst}$ or $2.A_4^{nst}$, respectively.

The values of characters of important representations of the group $2.S_6$, and the information about some of its subgroups are presented in Table 1, cf. [8, p. 5]. The first two columns of Table 1 describe conjugacy classes of elements of the group $2.S_6$. The first column lists the orders of the elements in the corresponding conjugacy class, and the second column, except for the entries in the second and the third row, gives a cycle type of the image of an element under projection to $S_6$ (for example, $[3,2]$ denotes a product of two disjoint cycles of lengths 3 and 2). By id we denote the identity element of $2.S_6$, and $z$ denotes the unique non-trivial central element of $2.S_6$. Note that the preimages of some of conjugacy classes in $S_6$ split into a union of two conjugacy classes in $2.S_6$. The next three columns list the values of the characters of the representations $W, W_5$ and $U_4$ of $2.S_6$. Note that there is no real ambiguity in the choice of $\sqrt{-3}$ since we did not specify any way to distinguish the two conjugacy classes in $2.S_6$ whose elements are projected to cycles of length 6 in $S_6$ up to this point (note that the two ways to choose a sign here is exactly a tensor multiplication of the representation with $J$, i.e. the choice between two homomorphisms of $2.S_6$ to $SL_4(\mathbb{C})$ having the same image). The remaining columns list the numbers of elements from each of the conjugacy classes of $2.S_6$ in subgroups of certain types. By $2.F_{36}$ (respectively, by $2.F_{20}$, by $2.D_{12}^{nst}$) we denote a subgroup of $2.S_6$ (respectively, of $2.S_6$, or of $2.S_5^{nst}$) isomorphic to a central extension of $F_{36}$ (respectively, of $F_{20}$, or of $D_{12}$) by $\mu_2$. A subgroup $2.F_{20}$ is actually contained in a subgroup $2.S_5^{st}$ and in a subgroup $2.S_5^{nst}$. Note that the intersection of a conjugacy class in a group with a subgroup may (and often does) split into several conjugacy classes in this subgroup.

It is immediate to see from Table 1 that $U_4$ is an irreducible representation of the group $2.S_6$. Using the information provided by Table 1 we immediately obtain the following results.

\textbf{Corollary 2.1.} Let $\Gamma$ be a subgroup of $2.S_6$. After restriction to the subgroup $\Gamma$ the $2.S_6$-representation $U_4$

(i) remains irreducible, if $\Gamma$ is one of the subgroups $2.A_6$, $2.S_5^{nst}$, $2.A_5^{nst}$, $2.S_4^{nst}$, $2.F_{36}$, or $2.F_{20}$;

(ii) splits into a sum of two non-isomorphic irreducible two-dimensional representations, if $\Gamma$ is one of the subgroups $2.A_5^{st}$, $2.A_4^{nst}$, or $2.D_{12}^{nst}$.

\textbf{Proof.} Compute inner products of the corresponding characters with themselves, and keep in mind that neither of the groups $2.A_5^{st}$, $2.A_4^{nst}$, and $2.D_{12}^{nst}$ has an irreducible three-dimensional representation with a non-trivial action of the central subgroup. \hfill $\square$

\textbf{Remark 2.2.} By Corollary 2.1(i), the $2.S_5^{nst}$-representation $U_4$ is irreducible. One can check that it is not induced from any proper subgroup of $2.S_5^{nst}$, i.e. it defines a primitive
Table 1. Characters and subgroups of the group $2S_6$

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subgroup isomorphic to $2.\mathfrak{S}_5$ in $\text{GL}_4(\mathbb{C})$. Note that this subgroup is not present in the list given in [11 §8.5]. It is still listed by some other classical surveys, see e.g. [2, §119].

**Corollary 2.3.** Let $\Gamma$ be a subgroup of $2.\mathfrak{S}_6$. After restriction to the subgroup $\Gamma$ the $2.\mathfrak{S}_6$-representation $\mathbb{W}_5$

(i) remains irreducible, if $\Gamma$ is one of the subgroups $2.\mathfrak{A}_6$, $2.\mathfrak{S}_5^{\text{nst}}$, or $2.\mathfrak{A}_5^{\text{nst}}$;

(ii) splits into a sum of the trivial and an irreducible four-dimensional representation if $\Gamma$ is a subgroup $2.\mathfrak{A}_5^{\text{nst}}$;

(iii) splits into a sum of the trivial and two different irreducible two-dimensional representations if $\Gamma$ is a subgroup $2.\text{D}_{12}^{\text{nst}}$.

In the sequel we will denote the restrictions of the $2.\mathfrak{S}_6$-representations $\mathbb{U}_4$ and $\mathbb{W}_5$ to various subgroups by the same symbols for simplicity. The next two corollaries are implied by direct computations (we used GAP software [11] to perform them).

**Corollary 2.4.** The following assertions hold:

(i) the $\mathfrak{A}_6$-representation $\text{Sym}^2(\mathbb{U}_4')$ does not contain one-dimensional subrepresentations;

(ii) the $\mathfrak{A}_6$-representation $\text{Sym}^4(\mathbb{U}_4')$ does not contain one-dimensional subrepresentations;

(iii) the $\mathfrak{A}_5^{\text{nst}}$-representation $\text{Sym}^2(\mathbb{U}_4')$ splits into a sum of two different irreducible three-dimensional representations and one irreducible four-dimensional representation;

(iv) the $2.\mathfrak{A}_5^{\text{nst}}$-representation $\text{Sym}^3(\mathbb{U}_4')$ does not contain one-dimensional subrepresentations;

(v) the $\mathfrak{A}_5^{\text{nst}}$-representation $\text{Sym}^4(\mathbb{U}_4')$ has a unique two-dimensional subrepresentation, and this subrepresentation splits into a sum of two trivial representations of $\mathfrak{A}_5^{\text{nst}}$.

Recall that all representations of a symmetric group are self-dual. Therefore, to study invariant hypersurfaces in $\mathbb{P}(\mathbb{W}_5)$ we will use the following result.

**Corollary 2.5.** Let $\Gamma$ be one of the groups $\mathfrak{S}_6$, $\mathfrak{A}_6$ or $\mathfrak{S}_5^{\text{nst}}$. Then

(i) the $\Gamma$-representation $\text{Sym}^2(\mathbb{W}_5)$ has a unique one-dimensional subrepresentation;

(ii) the $\Gamma$-representation $\text{Sym}^4(\mathbb{W}_5)$ has a unique two-dimensional subrepresentation, and this subrepresentation splits into a sum of two trivial representations of $\Gamma$.

We conclude this section by recalling some information about several subgroups of $2.\mathfrak{S}_6$ that are smaller than those listed in Table[1]. Namely, we list in Table[2] orders, types and numbers of elements in certain subgroups of $2.\mathfrak{A}_5^{\text{nst}}$. We keep the notation used in Table[1] By $2.\mathfrak{S}_3'$ we denote a subgroup of $2.\mathfrak{A}_5^{\text{nst}}$ isomorphic to $2.\mathfrak{S}_3$. Note that the preimage in $2.\mathfrak{S}_6$ of any subgroup $\mu_6 \subset \mathfrak{S}_6$ is isomorphic to $\mu_{10}$.

Looking at Table[2] (and keeping in mind character values provided by Table[1]) we immediately obtain the following.

**Corollary 2.6.** Let $\Gamma$ be a subgroup of $2.\mathfrak{A}_5^{\text{nst}} \subset 2.\mathfrak{S}_6$. After restriction to $\Gamma$ the $2.\mathfrak{S}_6$-representation $\mathbb{U}_4$

(i) splits into a sum of two non-isomorphic irreducible two-dimensional representations if $\Gamma$ is a subgroup $2.\text{D}_{10}$;

(ii) splits into a sum of an irreducible two-dimensional representation and two non-isomorphic one-dimensional representations if $\Gamma$ is a subgroup $2.\mathfrak{S}_3'$. 
Table 2. Subgroups of $2\mathfrak{A}_5^{nst}$

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<th>type</th>
<th>2.D$_{10}$</th>
<th>2.$\mathfrak{S}_3'$</th>
<th>2.$(\mu_2 \times \mu_2)$</th>
<th>$\mu_{10}$</th>
</tr>
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</table>

(iii) splits into a sum of two isomorphic irreducible two-dimensional representations if $\Gamma$ is a subgroup $2.(\mu_2 \times \mu_2)$;

(iv) splits into a sum of four pairwise non-isomorphic one-dimensional representations if $\Gamma$ is a subgroup $\mu_{10}$.

3. Icosahedral group in three dimensions

In this section, we consider the action of the group $\mathfrak{A}_5$ on the projective space $\mathbb{P}^3$ arising from a non-standard embedding of $\mathfrak{A}_5 \hookrightarrow \mathfrak{S}_6$. Namely, we identify $\mathbb{P}^3$ with the projectivization $\mathbb{P}(U_4)$, where $U_4$ is the restriction of the four-dimensional irreducible representation of the group $2\mathfrak{S}_6$ introduced in Section 2 to a subgroup $2\mathfrak{A}_5^{nst}$ (which we will refer to as just $2\mathfrak{A}_5$ in this section). Recall from Corollary 2.1(i) that $U_4$ is an irreducible representation of $2\mathfrak{A}_5$.

Remark 3.1 (see e. g. [8, p. 2]). Let $\Gamma$ be a proper subgroup of $\mathfrak{A}_5$ such that the index of $\Gamma$ is at most 15. Then $\Gamma$ is isomorphic either to $\mathfrak{A}_4$, or to $D_{10}$, or to $\mathfrak{S}_3$, or to $\mu_5$, or to $\mu_1 \times \mu_2$. In particular, if $\mathfrak{A}_5$ acts transitively on the set of $r < 15$ elements, then $r \in \{5, 6, 10, 12\}$.

Lemma 3.2. Let $\Omega$ be an $\mathfrak{A}_5$-orbit of length $r \leq 15$ in $\mathbb{P}^3$. Then either $r = 10$, or $r = 12$. Moreover, $\mathbb{P}^3$ contains exactly two $\mathfrak{A}_5$-orbits of length 10 and exactly two $\mathfrak{A}_5$-orbits of length 12.

Proof. By Remark 3.1 one has $r \in \{1, 5, 6, 10, 12, 15\}$. The case $r = 1$ is impossible since $U_4$ is an irreducible $2\mathfrak{A}_5$-representation. Restricting $U_4$ to subgroups of $2\mathfrak{A}_5$ isomorphic to $2\mathfrak{A}_4$, $2.D_{10}$, and $2.(\mu_2 \times \mu_2)$, and applying Corollaries 2.1(ii) and 2.6(i),(iii), we see that $r \not\in \{5, 6, 15\}$.

Restricting $U_4$ to a subgroup of $2\mathfrak{A}_5$ isomorphic to $2\mathfrak{S}_3$, applying Corollary 2.6(ii) and keeping in mind that there are ten subgroups isomorphic to $\mathfrak{S}_3$ in $\mathfrak{A}_5$, we see that $\mathbb{P}^3$ contains exactly two $\mathfrak{A}_5$-orbits of length 10.

Finally, restricting $U_4$ to a subgroup of $2\mathfrak{A}_5$ isomorphic to $\mu_{10}$, applying Corollary 2.6(iv) and keeping in mind that there are six subgroups isomorphic to $\mu_5$ in $\mathfrak{A}_5$, we see that $\mathbb{P}^3$ contains exactly two $\mathfrak{A}_5$-orbits of length 12. □

Lemma 3.3. There are no $\mathfrak{A}_5$-invariant surfaces of degree at most three in $\mathbb{P}^3$.

Proof. Apply Corollary 2.4(iii),(iv). □
By Corollary 2.1(ii), the subgroup $\mathfrak{A}_4 \subset \mathfrak{A}_5$ leaves invariant two disjoint lines in $\mathbb{P}^3$, say $L_1$ and $L'_1$. Let $L_1, \ldots, L_5$ be the $\mathfrak{A}_5$-orbit of the line $L_1$, and let $L'_1, \ldots, L'_5$ be the $\mathfrak{A}_5$-orbit of the line $L'_1$.

**Lemma 3.4.** The lines $L_1, \ldots, L_5$ (respectively, the lines $L'_1, \ldots, L'_5$) are pairwise disjoint.

*Proof.* Suppose that some of the lines $L_1, \ldots, L_5$ have a common point. Since the action of $\mathfrak{A}_5$ on the set $\{L_1, \ldots, L_5\}$ is doubly transitive, this implies that every two of the lines $L_1, \ldots, L_5$ have a common point. Therefore, either all lines $L_1, \ldots, L_5$ are coplanar, or all of them pass through one point. Both of these cases are impossible since the $2\mathfrak{A}_5$-representation $\mathbb{U}_4$ is irreducible by Corollary 2.1(i). Therefore, the lines $L_1, \ldots, L_5$ are pairwise disjoint. A similar argument applies to the lines $L'_1, \ldots, L'_5$. □

**Corollary 3.5.** Any $\mathfrak{A}_5$-orbit contained in the union $L_1 \cup \ldots \cup L_5$ has length at least 20.

*Proof.* Corollary 2.1(ii) implies that the stabilizer $\Gamma \cong \mathfrak{A}_4$ of the line $L_1$ acts on $L_1$ faithfully. Therefore, the length of any $\Gamma$-orbit contained in $L_1$ is at least four. Thus the required assertion follows from Lemma 3.4. □

We are going to describe the configuration formed by the lines $L_1, \ldots, L_5, L'_1, \ldots, L'_5$.

**Definition 3.6.** Let $T_1, \ldots, T_5, T'_1, \ldots, T'_5$ be different lines in a projective space. We say that they form a **double five configuration** if the following conditions hold:

- the lines $T_1, \ldots, T_5$ (respectively, the lines $T'_1, \ldots, T'_5$) are pairwise disjoint;
- for every $i$ the lines $T_i$ and $T'_i$ are disjoint;
- for every $i \neq j$ the line $T_i$ meets the line $T'_j$.

**Lemma 3.7.** The lines $L_1, \ldots, L_5, L'_1, \ldots, L'_5$ form a double five configuration. Moreover, the only line in $\mathbb{P}^3$ that intersects all lines of $L_1, \ldots, L_5$ but $L_i$ is the line $L'_i$, and the only line in $\mathbb{P}^3$ that intersects all lines of $L'_1, \ldots, L'_5$ but $L'_i$ is the line $L_i$.

*Proof.* For any $i$ the lines $L_i$ and $L'_i$ are disjoint by construction. The lines $L_1, \ldots, L_5$ (respectively, the lines $L'_1, \ldots, L'_5$) are pairwise disjoint by Lemma 3.4.

Since any three pairwise skew lines in $\mathbb{P}^3$ are contained in a smooth quadric surface, and an intersection of two different quadric surfaces in $\mathbb{P}^3$ cannot contain three pairwise skew lines, we see that for any three indices $1 \leq i < j < k \leq 5$ there is a unique quadric surface $Q_{ijk}$ in $\mathbb{P}^3$ passing through the lines $L_i, L_j$ and $L_k$. Moreover, the quadric $Q_{ijk}$ is smooth. Note also that the quadric $Q_{ijk}$ is not $\mathfrak{A}_5$-invariant by Lemma 3.3. This implies that all five lines $L_1, \ldots, L_5$ are not contained in a quadric.

Therefore, we may assume that the quadric $Q_{123}$ does not contain the line $L_4$. It is well-known that in this case either there is a unique line $L$ meeting all four lines $L_1, \ldots, L_4$, or there are exactly two lines $L$ and $L'$ meeting $L_1, \ldots, L_4$. In the latter case the stabilizer $\Gamma \subset \mathfrak{A}_5$ of the quadruple $L_1, \ldots, L_4$ (i.e. the stabilizer of the line $L_5$) preserves the lines $L_5$, $L$ and $L'$. On the other hand, the lines $L$ and $L'$ are different from $L_5$ since $L_5$ meets neither of the lines $L_1, \ldots, L_4$; moreover, the group $\Gamma \cong \mathfrak{A}_4$ fixes both $L$ and $L'$. But $\Gamma$ cannot fix three different lines in $\mathbb{P}^3$ by Corollary 2.1(ii). The contradiction shows that there is a unique line $L$ meeting $L_1, \ldots, L_4$. Again we see that $L \neq L_5$, so that $L = L'_5$ by Corollary 2.1(ii).

Since the group $\mathfrak{A}_5$ permutes the lines $L_1, \ldots, L_5$ transitively, we conclude that the only line in $\mathbb{P}^3$ that intersects all lines of $L_1, \ldots, L_5$ except $L_i$ is the line $L'_i$. Similarly, we see that the only line in $\mathbb{P}^3$ that intersects all lines of $L'_1, \ldots, L'_5$ except $L'_i$ is the line $L_i$. In particular, the lines $L_1, \ldots, L_5, L'_1, \ldots, L'_5$ form a double five configuration. □
Lemma 3.8. Every $\mathfrak{A}_5$-invariant curve of degree at most three in $\mathbb{P}^3$ is a twisted cubic. Moreover, there are exactly two $\mathfrak{A}_5$-invariant twisted cubic curves in $\mathbb{P}^3$.

Proof. Let $C$ be an $\mathfrak{A}_5$-invariant curve of degree at most three in $\mathbb{P}^3$. Since the $2,\mathfrak{A}_5$-representation $U_4$ is irreducible, we conclude that $C$ is a twisted cubic.

By Corollary 2.3(iii), one has

$$(3.9) \quad \text{Sym}^2(U_4) \cong V_3 \oplus V'_3 \oplus V_4,$$

where $V_3$, $V'_3$, and $V_4$, are irreducible representations of the group $\mathfrak{A}_5$ of dimensions 3, 3, and 4, respectively. Note that $V_3$ and $V'_3$ are not isomorphic.

Denote by $Q$ and $Q'$ the linear systems of quadrics in $\mathbb{P}^3$ that correspond to $V_3$ and $V'_3$, respectively. Since $\mathbb{P}^3$ does not contain $\mathfrak{A}_5$-orbits of lengths less or equal to eight by Lemma 3.2, we see that the base loci of $Q$ and $Q'$ contain $\mathfrak{A}_5$-invariant curves $C^1$ and $C^2$, respectively. The degrees of these curves must be less than four, so that they are twisted cubic curves. This also implies that the base loci of $Q$ and $Q'$ are exactly the curves $C^1$ and $C^2$, respectively.

Now take an arbitrary $\mathfrak{A}_5$-invariant twisted cubic curve $C$ in $\mathbb{P}^3$. The quadrics in $\mathbb{P}^3$ passing through $C$ define a three-dimensional $\mathfrak{A}_5$-subrepresentation in $\text{Sym}^2(U_4)$. Moreover, different $\mathfrak{A}_5$-invariant twisted cubics give different $\mathfrak{A}_5$-subrepresentations of $\text{Sym}^2(U_4)$. Thus, $(3.9)$ implies that $C$ coincides either with $C^1$ or with $C^2$. \hfill $\Box$

Keeping in mind Lemma 3.8 we will denote the two $\mathfrak{A}_5$-invariant twisted cubic curves in $\mathbb{P}^3$ by $C^1$ and $C^2$ throughout this section.

Remark 3.10. The curves $C^1$ and $C^2$ are disjoint. Indeed, otherwise, their intersection would contain at least 12, which is impossible, since a twisted cubic curve is an intersection of quadrics.

The lines in $\mathbb{P}^3$ that are tangent to the curves $C^1$ and $C^2$ sweep out quartic surfaces $S^1$ and $S^2$, respectively. These surfaces are $\mathfrak{A}_5$-invariant. The singular loci of $S^1$ and $S^2$ are the curves $C^1$ and $C^2$, respectively. In particular, the surfaces $S^1$ and $S^2$ are different. Their singularities along these curves are locally isomorphic to a product of $\mathbb{A}^1$ and an ordinary cusp.

Denote by $P$ the pencil of quartics in $\mathbb{P}^3$ generated by $S^1$ and $S^2$.

Lemma 3.11. All $\mathfrak{A}_5$-invariant quartic surfaces in $\mathbb{P}^3$ are contained in the pencil $P$.

Proof. This follows from Corollary 2.3(v). \hfill $\Box$

We proceed by describing the base locus of the pencil $P$. This was done in [1, Remark 2.6], but we reproduce the details here for the convenience of the reader.

Lemma 3.12. The base locus of the pencil $P$ is an irreducible curve $B$ of degree 16. It has 24 singular points, these points are in a union of two $\mathfrak{A}_5$-orbits of length 12, and each of them is an ordinary cusp of the curve $B$. The curve $B$ contains a unique $\mathfrak{A}_5$-orbit of length 20.

Proof. Denote by $B$ the base curve of the pencil $P$. Let us show that the curves $C^1$ and $C^2$ are not contained in $B$. Since $C^1$ is projectively normal, there is an exact sequence of $2,\mathfrak{A}_5$-representations

$$0 \to H^0(\mathcal{I}_{C^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \to H^0(\mathcal{O}_{C^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4)) \to 0,$$
Lemma 3.13. The pencil $\mathcal{P}$ contains two surfaces $\mathcal{R}^1$ and $\mathcal{R}^2$ with ordinary double singularities, such that the singular loci of $\mathcal{R}^1$ and $\mathcal{R}^2$ are $\mathsf{A}_5$-orbits of length 10. Every surface in $\mathbb{P}^3$ different from $\mathcal{S}^1$, $\mathcal{S}^2$, $\mathcal{R}^1$ and $\mathcal{R}^2$ is smooth.

Proof. Let $S$ be a surface in $\mathcal{P}$ that is different from $\mathcal{S}^1$ and $\mathcal{S}^2$. It follows from Lemma 3.3 that $S$ is irreducible. Assume that $S$ is singular.

We claim that $S$ has isolated singularities. Indeed, suppose that $S$ is singular along some $\mathsf{A}_5$-invariant curve $Z$. Taking a general plane section of $S$, we see that the degree of $Z$ is at most three. Thus, one has either $Z = \mathcal{C}^1$ or $Z = \mathcal{C}^2$ by Lemma 3.3. Since neither of these curves is contained in the base locus of $\mathcal{P}$ by Lemma 3.12, this would imply that either $S = \mathcal{S}^1$ or $S = \mathcal{S}^2$. The latter is not the case by assumption.

We see that the singularities of $S$ are isolated. Hence, $S$ contains at most two non-Du Val singular points by [23] Theorem 1] applied to the minimal resolution of singularities of the surface $S$. Since the set of all non-Du Val singular points of the surface $S$ must be $\mathsf{A}_5$-invariant, we see that $S$ has none of them by Lemma 3.2. Thus, all singularities of $S$ are Du Val.

where $\mathcal{L}_{\mathcal{C}^1}$ is the ideal sheaf of $\mathcal{C}^1$. The $2\mathsf{A}_5$-representation $H^0(\mathcal{O}_{\mathcal{C}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4))$ contains a one-dimensional subrepresentation corresponding to the unique $\mathsf{A}_5$-orbit of length 12 in $\mathcal{C}^1 \cong \mathbb{P}^1$. This shows that $\mathcal{P}$ contains a surface that does not pass through $\mathcal{C}^1$, so that $\mathcal{C}^1$ is not contained in $B$. Similarly, we see that $\mathcal{C}^2$ is not contained in $B$.

Let $\rho : \hat{\mathcal{S}}^1 \rightarrow \mathcal{S}^1$ be the normalization of the surface $\mathcal{S}^1$, and let $\hat{\mathcal{C}}^1$ be the preimage of the curve $\mathcal{C}^1$ via $\rho$. Then the action of the group $\mathsf{S}_5$, and in particular of its subgroup $\mathsf{A}_5$, lifts to $\hat{\mathcal{S}}^1$. One has $\hat{\mathcal{S}}^1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $\rho^*(\mathcal{O}_{\hat{\mathcal{S}}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(1))$ is a divisor of bi-degree $(1, 2)$. This shows that $\hat{\mathcal{C}}^1$ is of bi-degree $(1, 1)$. Thus, the action of $\mathsf{A}_5$ on $\hat{\mathcal{S}}$ is diagonal by [6, Lemma 6.4.3(i)].

Denote by $\hat{B}$ be the preimage of the curve $B$ via $\rho$. Then $\hat{B}$ is a divisor of bi-degree $(4, 8)$. Hence, the curve $\hat{B}$ is irreducible by [6, Lemma 6.4.4(i)], so that the curve $B$ is irreducible as well.

Note that the curve $\hat{B}$ is singular. Indeed, the intersection $\mathcal{S}^1 \cap \mathcal{C}^2$ is an $\mathsf{A}_5$-orbit $\Sigma_{12}$ of length 12, because $\mathcal{C}^2$ is not contained in $\mathcal{S}^1$. Similarly, we see that the intersection $\mathcal{S}^1 \cap \mathcal{C}^1$ is also an $\mathsf{A}_5$-orbit $\Sigma_{12}$ of length 12. These $\mathsf{A}_5$-orbits $\Sigma_{12}$ and $\Sigma_{12}'$ are different by Remark 3.10. Since $B$ is the scheme theoretic intersection of the surfaces $\mathcal{S}^1$ and $\mathcal{S}^2$, it must be singular at every point of $\Sigma_{12} \cup \Sigma_{12}'$. Denote by $\hat{\Sigma}_{12}$ and $\hat{\Sigma}_{12}'$ the preimages via $\rho$ of the $\mathsf{A}_5$-orbits $\Sigma_{12}$ and $\Sigma_{12}'$, respectively. Then $\hat{B}$ is singular in every point of $\hat{\Sigma}_{12}'$.

The curve $\hat{B}$ is smooth away of $\hat{\Sigma}_{12}$, because its arithmetic genus is 21, and the surface $\mathcal{S}^1$ does not contain $\mathsf{A}_5$-orbits of length less than 12. On the other hand, we have

$$\hat{B} \cap \mathcal{C}^1 = \hat{\Sigma}_{12},$$

because $\hat{B} \cdot \mathcal{C}^1 = 12$ and $\hat{\Sigma}_{12} \subset \hat{B}$. This shows that $B$ is an irreducible curve whose only singularities are the points of $\Sigma_{12} \cup \Sigma_{12}'$, and each such point is an ordinary cusp of the curve $B$. In particular, the genus of the normalization of the curve $B$ is 9. By [6] Lemma 5.1.5], this implies that $B$ contains a unique $\mathsf{A}_5$-orbit of length 20. \hfill $\square$

The following classification of $\mathsf{A}_5$-invariant quartic surfaces in $\mathbb{P}^3$ was obtained in [4] Theorem 2.4].
By [6, Lemma 6.7.3(iii)], the surface $S$ has only ordinary double singularities, the set $\text{Sing}(S)$ consists of one $A_5$-orbit, and

$|\text{Sing}(S)| \in \{5, 6, 10, 12, 15\}$.

Since $\mathbb{P}^3$ does not contain $A_5$-orbits of lengths 5, 6, and 15 by Lemma 3.2, we see that $\text{Sing}(S)$ is either an $A_5$-orbit of length 10 or an $A_5$-orbit of length 12.

Suppose that the singular locus of $S$ is an $A_5$-orbit $\Sigma_{12}$ of length 12. Then $S$ does not contain other $A_5$-orbits of length 12 by [6, Lemma 6.7.3(iv)]. Since $C^1$ is not contained in the base locus of $\mathcal{P}$ by Lemma 3.12 and $C^1$ is contained in $S^1$, we see that $C^1 \not\subset S$. Since

$$S \cdot C^1 = 12$$

and $\Sigma_{12}$ is the only $A_5$-orbit of length at most 12 in $C^1 \cong \mathbb{P}^1$, we have $S \cap C^1 = \Sigma_{12}$. Thus,

$$12 = S \cdot C^1 \geq \sum_{P \in \Sigma_{12}} \text{mult}_P(S) = 2|\Sigma_{12}| = 24,$$

which is absurd.

Therefore, we see that the singular locus of $S$ is an $A_5$-orbit $\Sigma_{12}$ of length 12. Vice versa, if an $A_5$-invariant quartic surface passes through an $A_5$-orbit of length 10, then it is singular by [6, Lemma 6.7.1(ii)]. We know from Lemma 3.2 that there are exactly two $A_5$-orbits of length 10 in $\mathbb{P}^3$, and it follows from Lemma 3.12 that they are not contained in the base locus of $\mathcal{P}$. Thus there are two surfaces $\mathcal{R}^1$ and $\mathcal{R}^2$ that are singular exactly at the points of these two $A_5$-orbits, respectively. The above argument shows that every surface in $\mathcal{P}$ except $S^1$, $S^2$, $\mathcal{R}^1$ and $\mathcal{R}^2$ is smooth. \hfill $\square$

Keeping in mind Lemma 3.13 we will denote by $\mathcal{R}^1$ and $\mathcal{R}^2$ the two nodal surfaces contained in the pencil $\mathcal{P}$ until the end of this section.

**Lemma 3.14.** There is a unique $A_5$-invariant quartic surface in $\mathbb{P}^3$ that contains the lines $L_1, \ldots, L_5$ (respectively, the lines $L'_1, \ldots, L'_5$). Moreover, this surface is smooth, and it does not contain the lines $L'_1, \ldots, L'_5$ (respectively, $L_1, \ldots, L_5$).

**Proof.** Put $\mathcal{L} = \sum_{i=1}^5 L_i$ and $\mathcal{L}' = \sum_{i=1}^5 L'_i$. Corollary 2.1(ii) implies that the stabilizer in $A_5$ of a general point of $L_1$ is trivial. Therefore, there exists a surface $S \in \mathcal{P}$ that contains all lines $L_1, \ldots, L_5$. By Lemma 3.12 such surface $S$ is unique.

We claim that $S \neq S^1$. Indeed, all lines contained in $S^1$ are tangent to the curve $C^1$, and there are no $A_5$-orbits of length five in $C^1 \cong \mathbb{P}^1$. Similarly, one has $S \neq S^2$.

We claim that $S$ is not one of the two nodal surfaces $\mathcal{R}^1$ and $\mathcal{R}^2$ contained in the pencil $\mathcal{P}$. Indeed, suppose that $S = \mathcal{R}^1$. Since the singular locus of $\mathcal{R}^1$ is an $A_5$-orbit of length 10 by Lemma 3.13, we see that the lines $L_1, \ldots, L_5$ are contained in the smooth locus of $\mathcal{R}^1$ by Corollary 3.5. On the other hand, one has $\mathcal{L}' = -10$ by Lemma 3.4. This means that $\text{rk Pic}(S)^{A_5} \geq 2$, which is impossible by [6, Lemma 6.7.3(i),(ii)].

We see that the surface $S$ is different from $\mathcal{R}^1$. A similar argument shows that $S$ is different from $\mathcal{R}^2$. Hence, $S$ is smooth by Lemma 3.13.

Let us show that $S$ does not contain the lines $L'_1, \ldots, L'_5$. Suppose that it does. By Lemma 3.4 one has

$$\mathcal{L} \cdot \mathcal{L} = \mathcal{L}' \cdot \mathcal{L}' = -10.$$
By [6, Lemma 6.7.1(i)], we have \( \text{rk} \text{Pic}(S)_{\mathbb{Q}} = 2 \). Let \( \Pi_S \) be the class of a plane section of \( S \). Then the determinant of the matrix

\[
\begin{pmatrix}
\mathcal{L} \cdot \mathcal{L} & \mathcal{L} \cdot \mathcal{L}' & \Pi_S \cdot \mathcal{L} \\
\mathcal{L} \cdot \mathcal{L}' & \mathcal{L}' \cdot \mathcal{L}' & \Pi_S \cdot \mathcal{L}' \\
\Pi_S \cdot \mathcal{L} & \Pi_S \cdot \mathcal{L}' & \Pi_S \cdot \Pi_S
\end{pmatrix} = \begin{pmatrix}
-10 & 20 & 5 \\
20 & -10 & 5 \\
5 & 5 & 4
\end{pmatrix}
\]

must vanish. This is a contradiction, because it equals 12.

Applying similar arguments, we see that the lines \( L_1', \ldots, L_5' \) are contained in a unique \( \mathfrak{A}_5 \)-invariant quartic surface, this surface is smooth and does not contain the lines \( L_1, \ldots, L_5 \).

**Remark 3.15.** One can use the properties of the pencil \( \mathcal{P} \) to give an alternative proof of Lemma 3.14. Namely, we know from Lemma 3.14 that there are two (different) smooth \( \mathfrak{A}_5 \)-invariant quartic surfaces \( S \) and \( S' \) containing the lines \( L_1, \ldots, L_5 \) and \( L_1', \ldots, L_5' \), respectively. By Lemma 3.12, the base locus of the pencil \( \mathcal{P} \) is an irreducible curve \( B \) that contains a unique \( \mathfrak{A}_5 \)-orbit \( \Sigma \) of length 20. By Corollary 3.5, this implies that \( \Sigma \) is contained in the union \( L_1 \cup \ldots \cup L_5 \), because

\[
B \cdot (L_1 + \ldots + L_5) = 20
\]
on the surface \( S \). Similarly, we see that \( \Sigma \) is contained in \( L_1' \cup \ldots \cup L_5' \). These facts together with Lemma 3.14 easily imply that the lines \( L_1, \ldots, L_5 \) and \( L_1', \ldots, L_5' \) form a double five configuration.

Now we will obtain some restrictions on low degree \( \mathfrak{A}_5 \)-invariant curves in \( \mathbb{P}^3 \).

**Lemma 3.16.** Let \( C \) be an irreducible \( \mathfrak{A}_5 \)-invariant curve in \( \mathbb{P}^3 \) of degree \( d \leq 10 \). Denote by \( g \) the genus of the normalization of the curve \( C \). Then

\[
g \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|.
\]

**Proof.** Since \( \mathbb{U}_4 \) is an irreducible \( 2 \mathfrak{A}_5 \)-representation, the curve \( C \) is not contained in a plane in \( \mathbb{P}^3 \). This implies that a stabilizer in \( \mathfrak{A}_5 \) of a general point of the curve \( C \) is trivial. In particular, the \( \mathfrak{A}_5 \)-orbit of a general point of \( C \) has length \( |\mathfrak{A}_5| = 60 \).

Let \( S \) be a surface in the pencil \( \mathcal{P} \) that passes through a general point of \( C \). Then the curve \( C \) is contained in \( S \), because otherwise one would have

\[
60 \leq |S \cap C| \leq S \cdot C = 4d \leq 40,
\]

which is absurd. Since the assertion of the lemma clearly holds for the twisted cubic curves \( \mathcal{C}^1 \) and \( \mathcal{C}^2 \), we may assume that \( C \) is different from these two curves.

Suppose that \( S = \mathcal{S}^1 \). Let us use the notation of the proof of Lemma 3.12. Denote by \( \hat{C} \) the preimage of the curve \( C \) via \( \rho \). Then \( \hat{C} \) is a divisor of bi-degree \( (a, b) \) for some non-negative integers \( a \) and \( b \) such that \( d = 2a + b \). On the other hand, one has

\[
|\hat{C} \cap \mathcal{C}^1| \leq \hat{C} \cdot \mathcal{C}^1 = a + b \leq 2a + b = d \leq 10,
\]

which is impossible, since the curve \( \mathcal{C}^1 \cong \mathcal{C}^1 \cong \mathbb{P}^1 \) does not contain \( \mathfrak{A}_5 \)-orbits of length less than 12.

We see that \( S \neq \mathcal{S}^1 \). Similarly, we see that \( S \neq \mathcal{S}^2 \). By Lemma 3.13, either \( S \) is a smooth quartic K3 surface, or \( S \) is one of the surfaces \( R^1 \) and \( R^2 \). Denote by \( \Pi_S \) a plane...
section of $S$. Then
\[
\det \begin{pmatrix}
\Pi_S^2 & \Pi_S \cdot C \\
\Pi_S \cdot C & C^2
\end{pmatrix} = \det \begin{pmatrix}
4 & d \\
d & C^2
\end{pmatrix} = 4C^2 - d^2 \leq 0
\]
by the Hodge index theorem.

Suppose that $C$ is contained in the smooth locus of the surface $S$. Denote by $p_a(C)$ the arithmetic genus of the curve $C$. Then
\[
C^2 = 2p_a(C) - 2.
\]
by the adjunction formula. Thus, we get
\[
p_a(C) \leq \frac{d^2}{8} + 1.
\]
Since $g \leq p_a(C) - |\text{Sing}(C)|$, this implies the assertion of the lemma.

To complete the proof, we may assume that $C$ contains a singular point of the surface $S$. By Lemma 3.13, this means that either $S = R_1$ or $S = R_2$. The singularities of the surface $S$ are ordinary double points, and its singular locus is an $A_5$-orbit of length 10. In particular, the curve $C$ contains the whole singular locus of $S$. By [6, Theorem 6.7.1], one has $\text{Pic}(S)^{A_5} \cong \mathbb{Z}$. Since $\Pi_S^2 = 4$ and the self-intersection of any Cartier divisor on the surface $S$ is even, we see that the group $\text{Pic}(S)^{A_5}$ is generated by $\Pi_S$.

Suppose that $C$ is a Cartier divisor on $S$. Then either $C \sim \Pi_S$ or $C \sim 2\Pi_S$, because $d \leq 10$. Since the restriction map
\[
H^0(\mathcal{O}_{P^3}(n)) \to H^0(\mathcal{O}_S(n\Pi_S))
\]
is an isomorphism for $n \leq 3$, we conclude that there is an $A_5$-invariant quadric in $P^3$. This is not the case by Lemma 3.3.

Therefore, we see that $C$ is not a Cartier divisor on $S$. Since $S$ has only ordinary double points, the divisor $2C$ is Cartier. Thus
\[
2C \sim l\Pi_S,
\]
for some odd positive integer $l$. Since
\[
2d = 2C \cdot \Pi_S = l\Pi_S \cdot \Pi_S = 4l,
\]
we see that $l = \frac{d}{2}$. In particular, $d$ is even and $l \leq 5$.

Let $\theta: \tilde{S} \to S$ be the minimal resolution of singularities of the surface $S$. Denote by $\tilde{C}$ the proper transform of the curve $C$ on the surface $\tilde{S}$, and denote by $\Theta_1, \ldots, \Theta_{10}$ the exceptional curves of $\theta$. Then
\[
2\tilde{C} \sim \theta^*(\Pi_S) - m \sum_{i=1}^{10} \Theta_i,
\]
for some positive integer $m$. Moreover, $m$ is odd, because $C$ is not a Cartier divisor. We have
\[
4\tilde{C}^2 = \Pi_S^2 l^2 - 20m^2 = 4l^2 - 20m^2,
\]
which implies that $\tilde{C}^2 = l^2 - 5m^2$. Since $\tilde{C}^2$ is even, $m$ is odd and $l \leq 5$, we see that either $l = 3$ or $l = 5$.

Denote by $p_a(\tilde{C})$ the arithmetic genus of the curve $\tilde{C}$. Then
\[
l^2 - 5m^2 = \tilde{C}^2 = 2p_a(\tilde{C}) - 2.
\]
by the adjunction formula. In particular, we have

\[ 25 - 5m^2 \geq l^2 - 5m^2 \geq -2, \]

so that \( l \in \{3, 5\} \) and \( m = 1 \). The latter means that \( C \) is smooth at every point of \( \text{Sing}(S) \), so that

\[ |\text{Sing}(\tilde{C})| = |\text{Sing}(C)|. \]

If \( l = \frac{d}{2} = 3 \), then \( p_a(\tilde{C}) = 3 \). This gives

\[ g \leq p_a(\tilde{C}) - |\text{Sing}(\tilde{C})| = 3 - |\text{Sing}(C)| \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|. \]

Similarly, if \( l = \frac{d}{2} = 5 \), then \( p_a(\tilde{C}) = 11 \). This gives

\[ g \leq p_a(\tilde{C}) - |\text{Sing}(\tilde{C})| = 11 - |\text{Sing}(C)| \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|. \]

Recall from [6, Lemma 5.4.1] that there exists a unique smooth irreducible curve of genus 4 with a faithful action of the group \( A_5 \). This curve is known as the Bring’s curve. Its canonical model is a complete intersection of a quadric and a cubic in a three-dimensional projective space. However, this sextic curve does not appear in our \( P^3 = P(U_4) \) by Lemma 3.17.

**Lemma 3.17.** Let \( C \) be a smooth irreducible \( A_5 \)-invariant curve in \( P^3 \) of degree \( d \leq 6 \) and genus \( g \). Then \( g \neq 4 \).

**Proof.** Suppose that \( g = 4 \). Denote by \( \Pi_C \) the plane section of the curve \( C \). Then

\[ h^0(\mathcal{O}_C(\Pi_C)) = d - 3 + h^0(\mathcal{O}_C(K_C - \Pi_C)) \]

by the Riemann–Roch theorem. Since \( C \) is not contained in a plane, this implies that \( \Pi_C \sim K_C \). Therefore, the projective space \( P^3 \) is identified with a projectivization of an \( A_5 \)-representation \( H^0(\mathcal{O}_C(K_C))^\vee \), i.e. of a representation of the group \( 2A_5 \) where the center of \( 2A_5 \) acts trivially. The latter is not the case by construction of \( U_4 \).

**Lemma 3.18.** Let \( C \) be an irreducible smooth \( A_5 \)-invariant curve in \( P^3 \) of degree \( d = 10 \) and genus \( g \). Then \( g \neq 10 \).

**Proof.** Suppose that \( g = 10 \). By Lemma 3.12, the base locus of the pencil \( \mathcal{P} \) is an irreducible curve \( B \) of degree 16. In particular, there exists a surface \( S \in \mathcal{P} \) that does not contain \( C \). Thus, the intersection \( S \cap C \) is an \( A_5 \)-invariant set that consists of

\[ C \cdot S = 4d = 40 \]

points (counted with multiplicities). On the other hand, by [6] Lemma 5.1.5, any \( A_5 \)-orbit in \( C \) has length 12, 30, or 60.

4. LARGE SUBGROUPS OF \( \mathfrak{S}_6 \)

In this section we collect some auxiliary results about the groups \( \mathfrak{S}_6, A_6 \) and \( \mathfrak{S}_5 \). We start with recalling some general properties of the group \( A_6 \).

**Remark 4.1** (see e.g. [8] p. 4). Let \( \Gamma \) be a proper subgroup of \( A_6 \) such that the index of \( \Gamma \) is at most 15. Then \( \Gamma \) is isomorphic either to \( A_5 \), or to \( F_{36} \), or to \( \mathfrak{S}_4 \). In particular, if \( A_6 \) acts transitively on the set of \( r < 15 \) elements, then either \( r = 6 \) or \( r = 10 \).
We will need the following result about possible actions of the group \( A_6 \) on curves of small genera (cf. [5, Theorem 2.18] and [6, Lemma 5.1.5]).

**Lemma 4.2.** Suppose that \( C \) is a smooth irreducible curve of genus \( g \leq 15 \) with a non-trivial action of the group \( A_6 \). Then \( g = 10 \).

**Proof.** Let \( \Omega \subset C \) be an \( A_6 \)-orbit. Then a stabilizer of a point in \( \Omega \) is a cyclic subgroup of \( A_6 \), which implies that \( |\Omega| \in \{72, 90, 120, 180, 360\} \).

From the classification of finite subgroups of \( \text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(\mathbb{C}) \) we know that \( g \neq 0 \).

Put \( \bar{C} = C/A_6 \). Then \( \bar{C} \) is a smooth curve. Let \( \bar{g} \) be the genus of the curve \( \bar{C} \). The Riemann–Hurwitz formula gives

\[
2\bar{g} - 2 = 360(2\bar{g} - 2) + 180a_{180} + 240a_{120} + 270a_{90} + 288a_{72},
\]

where \( a_k \) is the number of \( A_6 \)-orbits in \( C \) of length \( k \).

Since \( a_k \geq 0 \) and \( 2 \leq g \leq 15 \), one has \( \bar{g} = 0 \). Thus, we obtain

\[
2\bar{g} - 2 = -720 + 180a_{180} + 240a_{120} + 270a_{90} + 288a_{72}.
\]

Going through the values \( 2 \leq g \leq 15 \), and solving this equation case by case we see that the only possibility is \( g = 10 \). \( \square \)

We proceed by recalling some general properties of the group \( \mathfrak{S}_5 \).

**Remark 4.3** (see e.g. [8, p. 2]). Let \( \Gamma \) be a proper subgroup of \( \mathfrak{S}_5 \) such that the index of \( \Gamma \) is less than 12. Then \( \Gamma \) is isomorphic either to \( \mathfrak{A}_5 \), or to \( \mathfrak{S}_4 \), or to \( F_{20} \), or to \( \mathfrak{A}_4 \), or to \( D_{12} \). In particular, if \( \mathfrak{S}_5 \) acts transitively on the set of \( r < 12 \) elements, then \( r \in \{2, 5, 6, 10\} \).

**Lemma 4.4.** The group \( \mathfrak{S}_5 \) cannot act faithfully on a smooth irreducible curve of genus 5.

**Proof.** Suppose that \( C \) is a curve of genus 5 with a faithful action of \( \mathfrak{S}_5 \). Considering the action of the subgroup \( \mathfrak{A}_5 \subset \mathfrak{S}_5 \) on \( C \) and applying [6, Lemma 5.4.3], we see that \( C \) is hyperelliptic. This gives a natural homomorphism

\[
\theta : \mathfrak{S}_5 \to \text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2(\mathbb{C})
\]

whose kernel is either trivial or isomorphic to \( \mu_2 \). Thus \( \theta \) is injective, which gives a contradiction. \( \square \)

Now we will prove some auxiliary facts about actions of the groups \( \mathfrak{S}_6, \mathfrak{A}_6 \) and \( \mathfrak{S}_5 \) on the four-dimensional projective space.

**Remark 4.5.** The group \( \mathfrak{S}_6 \) has exactly four irreducible five-dimensional representations (see e.g. [8, p. 5]). Starting from one of them, one more can be obtained by a twist by an outer automorphism of \( \mathfrak{S}_6 \), and two remaining ones are obtained from these two by a tensor product with the sign representation. Although these four representations are not isomorphic, the images of \( \mathfrak{S}_6 \) in \( \text{PGL}_5(\mathbb{C}) \) under them are the same. Every irreducible five-dimensional representation of \( \mathfrak{S}_6 \) restricts to an irreducible representation of the subgroup \( \mathfrak{A}_6 \subset \mathfrak{S}_6 \), and restricts to an irreducible representation of the *some* of the subgroups \( \mathfrak{S}_5 \subset \mathfrak{S}_6 \). The group \( \mathfrak{A}_6 \) has exactly two irreducible five-dimensional representations, each of them arising this way (see e.g. [8, p. 5]). Similarly, the group \( \mathfrak{S}_5 \) has exactly two irreducible five-dimensional representations, each of them arising this
way (see e.g. [8, p. 2]). Note also that every five-dimensional representation of a group \(A_6\) or \(S_5\) that does not contain one-dimensional subrepresentations is irreducible.

Let \(V_5\) be an irreducible five-dimensional representation of the group \(S_6\). Put \(P^4 = \mathbb{P}(V_5)\). Keeping in mind Remark [4.5], we see that the image of the corresponding homomorphism \(S_6\) to \(\text{PGL}_5(\mathbb{C})\) is the same for any choice of \(V_5\), and thus the \(S_6\)-orbits and \(S_6\)-invariant hypersurfaces in \(P^4\) do not depend on \(V_5\) either.

Remark [4.5] implies that there are six linear forms \(x_0, \ldots, x_5\) on \(P^4\) that are permuted by the group \(S_6\) (cf. Sections 1 and 2). Indeed, up to a twist by an outer automorphism of \(S_6\) and a tensor product with the sign representation, \(V_5\) is a subrepresentation of the six-dimensional representation \(W\) of \(S_6\), so that one can take restrictions of the natural coordinates in \(W\) to be these linear forms. Let \(Q\) be the three-dimensional quadric in \(P^4\) given by equation

\[
(4.6) \quad x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0. 
\]

The quadric \(Q\) is smooth and \(S_6\)-invariant. Note also that equation (1.1) makes sense in our \(P^4\).

We will use the notation introduced above until the end of the paper.

**Lemma 4.7.** Let \(\Gamma\) be either the group \(S_6\), or its subgroup \(A_6\), or a subgroup \(S_5\) of \(S_6\) such that \(V_5\) is an irreducible representation of \(\Gamma\). Then the only \(\Gamma\)-invariant quadric threefold in \(P^4\) is the quadric \(Q\). Similarly, every (reduced) \(\Gamma\)-invariant quartic threefold in \(P^4\) is given by equation (1.1) for some \(t \in \mathbb{C}\).

**Proof.** Apply Corollary 2.5. \(\square\)

By a small abuse of notation we will refer to the points in \(P^4\) using \(x_i\) as if they were homogeneous coordinates, i.e. a point in \(P^4\) will be encoded by a ratio of six linear forms \(x_i\). As in Section 1, let \(\Sigma_6\) and \(\Sigma_{10}\) are the \(S_6\)-orbits of the points \([-5:1:1:1:1:1]\) and \([-1:-1:-1:1:1:1]\), respectively. Looking at equation (1.6), we obtain

**Corollary 4.8.** The quadric \(Q\) does not contain the \(S_6\)-orbits \(\Sigma_6\) and \(\Sigma_{10}\).

Now we will have a look at the action of the group \(A_6\) on \(P^4\). Note that \(V_5\) is an irreducible \(A_6\)-representation by Remark [4.5].

**Lemma 4.9.** There are no \(A_6\)-orbits of length less than six in \(P^4\). Moreover, the only \(A_6\)-orbit of length six in \(P^4\) is \(\Sigma_6\).

**Proof.** The only subgroup of \(A_6\) of index less than six is \(A_6\) itself (cf. Remark [4.1]), so that the first assertion of the lemma follows from irreducibility of the \(A_6\)-representation \(V_5\). Also, the only subgroups of \(A_6\) of index six are \(A_5^{st}\) and \(A_5^{nst}\), so that the second assertion of the lemma also follows from Corollary 2.3. \(\square\)

**Lemma 4.10.** Let \(X\) be an \(A_6\)-invariant quartic threefold in \(P^4\) that contains an \(A_6\)-orbit of length at most six. Then \(X = X_{\frac{7}{10}}\).

**Proof.** By Lemma 4.7 one has \(X = X_t\) for some \(t \in \mathbb{C}\), and by Lemma 4.9 the \(A_6\)-orbit \(\Sigma_6\) is contained in \(X_t\). Since \(\Sigma_6\) is not contained in the quadric \(Q\) by Corollary 4.8 we see that there is a unique \(t \in \mathbb{C}\) such that \(\Sigma_6\) is contained in a quartic given by equation (1.1). Therefore, we conclude that \(t = \frac{7}{10}\). \(\square\)
Now we will make a couple of observations about the action of the group \( \mathcal{S}_5 \) on \( \mathbb{P}^4 \). We choose \( \mathcal{S}_5 \) to be a subgroup of \( \mathcal{S}_6 \) such that \( \mathcal{V}_5 \) is an irreducible \( \mathcal{S}_5 \)-representation (cf. Remark 1.5 and Corollary 2.3).

**Lemma 4.11.** Let \( P \in \mathbb{P}^4 \) be a point such that its stabilizer in \( \mathcal{S}_5 \) contains a subgroup isomorphic to \( D_{12} \). Then the \( \mathcal{S}_5 \)-orbit of \( P \) is \( \Sigma_{10} \).

**Proof.** By Corollary 2.3(iii), the point in \( \mathbb{P}^4 \) fixed by a subgroup \( D_{12} \subset \mathcal{S}_5 \) is unique. On the other hand, it is straightforward to check that a stabilizer in \( \mathcal{S}_5 \) of a point of \( \Sigma_{10} \) contains a subgroup isomorphic to \( D_{12} \). It remains to notice that the latter stabilizer is actually isomorphic to \( D_{12} \), since the only subgroups of \( \mathcal{S}_5 \) that contain \( D_{12} \) are \( D_{12} \) and \( \mathcal{S}_5 \) itself, while \( \mathcal{S}_5 \) has no fixed points on \( \mathbb{P}^4 \). \( \square \)

**Lemma 4.12.** Let \( X \) be an \( \mathcal{S}_5 \)-invariant quartic threefold in \( \mathbb{P}^4 \) that contains \( \Sigma_{10} \). Then \( X = X_{\frac{1}{6}} \).

**Proof.** By Lemma 4.7, one has \( X = X_t \) for some \( t \in \mathbb{C} \). Since \( \Sigma_{10} \) is not contained in the quadric \( Q \) by Corollary 4.8, we see that there is a unique \( t \in \mathbb{C} \) such that \( \Sigma_{10} \) is contained in a quartic given by equation (1.1). Therefore, we conclude that \( t = \frac{1}{6} \). \( \square \)

## 5. Rationality of the Quartic Threefold \( X_{\frac{7}{10}} \)

In this section we will construct an explicit \( \mathfrak{A}_6 \)-equivariant birational map \( \mathbb{P}^3 \dasharrow X_{\frac{7}{10}} \). Implicitly, the construction of this map first appeared in the proof of \( \mathfrak{A} \) Theorem 1.20. Here we will present a much simplified proof of its existence.

We identify \( \mathbb{P}^3 \) with the projectivization \( \mathbb{P}(U_4) \), where \( U_4 \) is the restriction of the four-dimensional irreducible representation of the group \( 2.\mathfrak{S}_6 \) introduced in Section 2 to the subgroup \( 2.\mathfrak{A}_6 \). By Corollary 2.11(i), the \( 2.\mathfrak{A}_6 \)-representation \( U_4 \) is irreducible.

**Lemma 5.1.** There are no \( \mathfrak{A}_6 \)-invariant surfaces of odd degree in \( \mathbb{P}^3 \), and no \( \mathfrak{A}_6 \)-invariant pencils of surfaces of odd degree in \( \mathbb{P}^3 \). Moreover, there are no \( \mathfrak{A}_6 \)-invariant quadric and quartic surfaces in \( \mathbb{P}^3 \).

**Proof.** Recall that the only one-dimensional representation of the group \( 2.\mathfrak{A}_6 \) is the trivial representation. Therefore, any \( \mathfrak{A}_6 \)-invariant surface of odd degree \( d \) in \( \mathbb{P}^3 \) gives rise to a trivial \( 2.\mathfrak{A}_6 \)-subrepresentation in \( R_d = \text{Sym}^d(U_4) \). On the other hand, the non-trivial central element \( z \) of \( 2.\mathfrak{A}_6 \) acts on \( R_d \) by a scalar matrix with diagonal entries equal to \(-1\), which shows that \( R_d \) does not contain trivial \( 2.\mathfrak{A}_6 \)-representations. Also, since the only two-dimensional representation of \( 2.\mathfrak{A}_6 \) is the sum of two trivial representations, this implies that there are no \( \mathfrak{A}_6 \)-invariant pencils of surfaces of odd degree in \( \mathbb{P}^3 \).

The last assertion of the lemma follows from Corollary 2.11(i),(ii). \( \square \)

**Lemma 5.2.** Let \( \Omega \) be an \( \mathfrak{A}_6 \)-orbit in \( \mathbb{P}^3 \). Then \( |\Omega| \geq 16 \).

**Proof.** Lemma 5.1 implies that there are no \( \mathfrak{A}_6 \)-orbits of odd length in \( \mathbb{P}^3 \). Thus, if \( \Omega \) is an \( \mathfrak{A}_6 \)-orbit in \( \mathbb{P}^3 \) of length at most 15, then by Remark 4.1 a stabilizer of its general point is isomorphic either to \( \mathfrak{A}_5 \) or to \( F_{36} \). Both of these cases are impossible by Corollary 2.11. \( \square \)

Actually, the minimal degree of an \( \mathfrak{A}_6 \)-invariant surface in \( \mathbb{P}^3 \) equals 8 (see [5, Lemma 3.7]), and the minimal length of an \( \mathfrak{A}_6 \)-orbit in \( \mathbb{P}^3 \) equals 36 (see [5, Lemma 3.8]), but we will not need this here.
Lemma 5.3 (cf. \[5\] Lemma 4.26)]. Let \( C \) be a smooth irreducible \( \mathfrak{A}_6 \)-invariant curve of degree 9 and genus \( g \) in \( \mathbb{P}^3 \). Then \( g \neq 10 \).

Proof. Suppose that \( g = 10 \). Then it follows from [13 Example 6.4.3] that either \( C \) is contained in a unique quadric surface in \( \mathbb{P}^3 \), or \( C \) is a complete intersection of two cubic surfaces in \( \mathbb{P}^3 \). The former case is impossible, since there are no \( \mathfrak{A}_6 \)-invariant quadrics in \( \mathbb{P}^3 \) by Lemma 5.1. The latter case is impossible, because there are no \( \mathfrak{A}_6 \)-invariant pencils of cubic surfaces in \( \mathbb{P}^3 \) by Lemma 5.1. \( \square \)

Recall that the group \( \mathfrak{A}_6 \) contains six standard subgroups isomorphic to \( \mathfrak{A}_5 \) and six non-standard subgroups isomorphic to \( \mathfrak{A}_5 \) (see the conventions made in Section 2). Denote the former ones by \( H_1', \ldots, H_6' \), and denote the latter ones by \( H_1, \ldots, H_6 \). By Corollary 2.1(ii), each group \( H'_i \) leaves invariant two lines \( L_1^i \) and \( L_2^i \) in \( \mathbb{P}^3 \). Note that each group \( H_i \) permutes transitively the lines \( L_1^i, \ldots, L_6^i \) (respectively, \( L_2^i, \ldots, L_6^i \)).

Put \( \mathcal{L}^1 = L_1^1 + \ldots + L_6^1 \) and \( \mathcal{L}^2 = L_1^2 + \ldots + L_6^2 \). Then the curves \( \mathcal{L}^1 \) and \( \mathcal{L}^2 \) are \( \mathfrak{A}_6 \)-invariant, and the curve \( \mathcal{L}^1 + \mathcal{L}^2 \) is \( \mathfrak{S}_6 \)-invariant.

Lemma 5.4. The lines \( L_1^1, \ldots, L_6^1 \) (respectively, the lines \( L_1^2, \ldots, L_6^2 \)) are pairwise disjoint. Moreover, the curves \( \mathcal{L}^1 \) and \( \mathcal{L}^2 \) are disjoint.

Proof. We use an argument similar to one in the proof of Lemma 3.4. Suppose that some of the lines \( L_1, \ldots, L_6 \) have a common point. Since the action of \( \mathfrak{A}_6 \) on the set \( \{ L_1, \ldots, L_6 \} \) is doubly transitive, this implies that any two of the lines \( L_1, \ldots, L_6 \) have a common point. Therefore, either all lines \( L_1, \ldots, L_6 \) are coplanar, or all of them pass through one point. Both of these cases are impossible since \( \mathfrak{U}_4 \) is an irreducible \( 2 \cdot \mathfrak{A}_6 \)-representation (see Corollary 2.1(i)). Therefore, the lines \( L_1, \ldots, L_6 \) are pairwise disjoint. A similar argument applies to the lines \( L_1', \ldots, L_6' \).

Suppose that some of the lines \( L_1, \ldots, L_6 \), say, \( L_1 \), intersects some of the lines \( L_1', \ldots, L_6' \). Since the lines \( L_1 \) and \( L_2 \) are disjoint by construction, we may assume that \( L_1 \) intersects \( L_2 \). Since the stabilizer \( H'_1 \subset \mathfrak{A}_6 \) of \( L_1 \) acts transitively on the lines \( L_2, \ldots, L_6 \), we conclude that all five lines \( L_2, \ldots, L_6 \) intersect \( L_1 \). Therefore, the line \( L_1 \) contains a subset of at most five points that is invariant with respect to the group \( H'_1 \cong \mathfrak{A}_5 \), which is a contradiction. Thus, \( \mathcal{L}^1 \) and \( \mathcal{L}^2 \) are disjoint. \( \square \)

Lemma 5.5. Let \( C \) be an \( \mathfrak{A}_6 \)-invariant curve in \( \mathbb{P}^3 \) of degree \( d \leq 10 \). Then either \( C = \mathcal{L}^1 \) or \( C = \mathcal{L}^2 \).

Proof. Suppose first that \( C \) is reducible. We may assume that \( \mathfrak{A}_6 \) permutes the irreducible components of \( C \) transitively. Thus, \( C \) has either 6 or 10 irreducible components by Remark 4.11 and these irreducible components are lines. By Remark 4.11 and Corollary 2.1 the latter case is impossible, and in the former case one has either \( C = \mathcal{L}^1 \) or \( C = \mathcal{L}^2 \).

Therefore, we assume that the curve \( C \) is irreducible. Let \( g \) be the genus of the normalization of the curve \( C \). We have

\[
(5.6) \quad g \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)| \leq 13 - |\text{Sing}(C)|
\]

by Lemma 3.16. This implies that the curve \( C \) is smooth, because \( \mathbb{P}^3 \) does not contain \( \mathfrak{A}_6 \)-orbits of length less than 16 by Lemma 5.2.

If \( d \leq 8 \), then (5.6) gives \( g \leq 9 \). This is impossible by Lemma 4.2.

If \( d = 9 \), then (5.6) gives \( g \leq 11 \), so that \( g = 10 \) by Lemma 4.2. This is impossible by Lemma 5.3.
Therefore, we see that \( d = 10 \). Thus, (5.6) gives \( g \leq 13 \), so that \( g = 10 \) by Lemma 4.2. The latter is impossible by Lemma 3.18. \( \square \)

Denote by \( \mathcal{M} \) the linear system on \( \mathbb{P}^3 \) consisting of all quartic surfaces passing through the lines \( L_1^1, \ldots, L_6^1 \). Then \( \mathcal{M} \) is not empty. In fact, its dimension is at least four by parameter count. Moreover, the linear system \( \mathcal{M} \) does not have base components by Lemma 5.1.

**Lemma 5.7.** The base locus of \( \mathcal{M} \) does not contain curves except the lines \( L_1^1, \ldots, L_6^1 \). Moreover, a general surface in \( \mathcal{M} \) is smooth at a general point of each of the lines \( L_1^1, \ldots, L_6^1 \).

**Proof.** Denote by \( Z \) the union of the curves that are contained in the base locus of \( \mathcal{M} \) and are different from the lines \( L_1^1, \ldots, L_6^1 \). Then \( Z \) is a (possibly empty) \( \mathcal{A}_6 \)-invariant curve. Denote its degree by \( d \). Pick two general surfaces \( M_1 \) and \( M_2 \) in \( \mathcal{M} \). Then
\[
M_1 \cdot M_2 = Z + mL^1 + \Delta,
\]
where \( m \) is a positive integer, and \( \Delta \) is an effective one-cycle on \( \mathbb{P}^3 \) that contains none of the lines \( L_1^1, \ldots, L_6^1 \). Note that \( \Delta \) may contain irreducible components of the curve \( Z \). Let \( \Pi \) be a plane in \( \mathbb{P}^3 \). Then
\[
16 = \Pi \cdot M_1 \cdot M_2 = \Pi \cdot Z + m \Pi \cdot L^1 + \Pi \cdot \Delta = d + 6m + \Pi \cdot \Delta \leq d + 6m,
\]
which implies that \( m \leq 2 \) and \( d \leq 10 \). By Lemma 5.5 we have \( d = 0 \), so that \( Z \) is empty. Since
\[
2 \geq m \geq \text{mult}_{L_i^1}(M_1) \text{mult}_{L_i^1}(M_2),
\]
we see that a general surface in \( \mathcal{M} \) is smooth at a general point of \( L_i^1 \). \( \square \)

Let \( \alpha: U \to \mathbb{P}^3 \) be a blow up along the lines \( L_1^1, \ldots, L_6^1 \). Then \( -K^3_U = 4 \), and the action of \( \mathcal{A}_6 \) lifts to \( U \). Denote by \( E_1, \ldots, E_6 \) the \( \alpha \)-exceptional surfaces that are mapped to \( L_1^1, \ldots, L_6^1 \), respectively.

**Lemma 5.8.** The action of the stabilizer \( H_i^1 \cong \mathcal{A}_5 \) in \( \mathcal{A}_6 \) of the line \( L_i^1 \) on the surface \( E_i \cong \mathbb{P}^1 \times \mathbb{P}^1 \) twisted diagonal, i.e., \( E_i \) is identified with \( \mathbb{P}(U_2) \times \mathbb{P}(U'_2) \), where \( U_2 \) and \( U'_2 \) are different two-dimensional irreducible representations of the group \( 2.\mathcal{A}_5 \).

**Proof.** This follows from Corollary 2.1(ii). \( \square \)

Let us denote by \( \mathcal{M}_U \) the proper transform of the linear system \( \mathcal{M} \) on the threefold \( U \). Then \( \mathcal{M}_U \sim -K_U \) by Lemma 5.7.

**Lemma 5.9.** The linear system \( \mathcal{M}_U \) is base point free.

**Proof.** Let us first show that \( \mathcal{M}_U \) is free from base curves. Suppose that the base locus of the linear system \( \mathcal{M}_U \) contains some curves. Then each of these curves is contained in some of the \( \alpha \)-exceptional surfaces by Lemma 5.7. Denote by \( Z \) the union of all such curves that are contained in \( E_1 \). Then \( Z \) is an \( H_i^1 \)-invariant curve. For some non-negative integers \( a \) and \( b \), one has
\[
Z \sim as + bl,
\]
where \( s \) is a section of the natural projection \( E_1 \to L_1^1 \) such that \( s^2 = 0 \) on \( E_1 \), and \( l \) is a fiber of this projection. On the other hand, we have
\[
\mathcal{M}_U|_{E_1} \sim -K_{U_{E_1}} \sim s + 3l.
\]
This gives $a \leq 1$ and $b \leq 3$. Since the action of $H'_1$ on the surface $E_1$ is twisted diagonal by Lemma 5.8, the latter is impossible by [6, Lemma 6.4.2(i)] and [6, Lemma 6.4.11(o)].

We see that $\mathcal{M}_U$ is free from base curves. Since $\mathcal{M}_U \sim -K_U$, the linear system $\mathcal{M}_U$ cannot have more than $-K^3_U = 4$ base points. By Lemma 5.2, this implies that $\mathcal{M}_U$ is base point free.

\textbf{Corollary 5.10.} The base locus of the linear system $\mathcal{M}$ consists of the lines $L_1^1, \ldots, L_6^1$.

By Lemma 5.9, the divisor $-K_U$ is nef. Since $-K^3_U = 4$, it is also big. Thus, we have

$$h^1\left(O_U(-K_U)\right) = h^2\left(O_U(-K_U)\right) = 0$$

by the Kawamata–Viehweg vanishing theorem (see [17]). Hence, the Riemann–Roch formula gives

$$h^0(O_U(-K_U)) = 5.$$  \hspace{1cm} (5.11)

In particular, we see that $| - K_U | = \mathcal{M}_U$.

\textbf{Lemma 5.12.} The $\mathfrak{a}_6$-representation $H^0(O_U(-K_U))$ is irreducible.

\textit{Proof.} By Lemma 5.1, there are no $\mathfrak{a}_6$-invariant quartic surfaces in $\mathbb{P}^3$. This implies that $H^0(O_U(-K_U))$ does not contain one-dimensional subrepresentations. Hence it is irreducible by Remark 4.5. \hfill \Box

Lemma 5.9 together with (5.11) implies that there is an $\mathfrak{a}_6$-equivariant commutative diagram

$$\begin{array}{ccc}
\mathbb{P}^3 & \xrightarrow{\phi} & \mathbb{P}^4 \\
\downarrow{\alpha} & & \downarrow{\beta} \\
U & & \\
\end{array} \hspace{1cm} (5.13)$$

where $\phi$ is a rational map given by $\mathcal{M}$, and $\beta$ is a morphism given by the anticanonical linear system $| - K_U |$. By Lemma 5.12, the projective space $\mathbb{P}^4$ in (5.13) is a projectivization of an irreducible $\mathfrak{a}_6$-representation.

Recall from Lemma 3.8 that $\mathbb{P}^3$ contains exactly two $H_1$-invariant twisted cubic curves $\mathcal{C}_1^1$ and $\mathcal{C}_2^1$.

\textbf{Lemma 5.14.} The curve $\mathcal{L}^1$ intersects exactly one curve among $\mathcal{C}_1^1$ and $\mathcal{C}_2^1$. Moreover, each line among $L_1^1, \ldots, L_6^1$ contains two points of this intersection. Similarly, the curve $\mathcal{L}^2$ intersects exactly one curve among $\mathcal{C}_1^2$ and $\mathcal{C}_2^2$, and this curve is different from the one that intersects $\mathcal{L}^1$.

\textit{Proof.} By Remark 3.11, the stabilizer in $H_1$ of the curve $L_1^1$ is isomorphic to $D_{10}$, and thus it has an orbit of length 2 on $L_1^1$. Thus, the curve $\mathcal{L}^1$ contains an $H_1$-orbit $\Sigma_{12}^1$ of length 12 by Lemma 3.2. Similarly, the curve $\mathcal{L}^2$ contains an $H_1$-orbit $\Sigma_{12}^2$ of length 12. By Lemma 5.4, one has $\Sigma_{12}^1 \neq \Sigma_{12}^2$. Moreover, $\Sigma_{12}^1$ and $\Sigma_{12}^2$ are the only $H_1$-orbits in $\mathbb{P}^3$ of length 12 by Lemma 3.2. Since $\mathcal{C}_1^1$ and $\mathcal{C}_1^2$ are disjoint by Remark 3.10 and each of them contains an $H_1$-orbit of length 12, we see that either $\Sigma_{12}^1 \subset \mathcal{C}_1^1$ and $\Sigma_{12}^2 \subset \mathcal{C}_2^2$, or $\Sigma_{12}^2 \subset \mathcal{C}_1^2$ and $\Sigma_{12}^1 \subset \mathcal{C}_1^1$. Since a line cannot have more than two common points with a twisted cubic, this also implies the last assertion of the lemma. \hfill \Box
Without loss of generality, we may assume that the curve \( L^1 \) intersects \( C^1 \), and the curve \( L^2 \) intersects \( C^2 \). Let \( C_1 \), \ldots, \( C_6 \) be the \( \mathfrak{A}_6 \)-orbit of the curve \( C^1 \), and let \( C_1' \), \ldots, \( C_6' \) be the \( \mathfrak{A}_6 \)-orbit of the curve \( C^2 \). By Lemma 5.8 the curves \( C_i \) and \( C_i' \) are the only twisted cubic curves in \( \mathbb{P}^3 \) that are \( H_i \)-invariant. By Lemma 5.14 we have

**Corollary 5.15.** Every twisted cubic curve \( C^1 \) intersects each line among \( L^1_1, \ldots, L^1_6 \) by two points. Similarly, every twisted cubic curve \( C^2 \) intersects each line among \( L^2_1, \ldots, L^2_6 \) by two points.

Denote by \( \tilde{C}_1 \), \ldots, \( \tilde{C}_6 \) the proper transforms on \( U \) of the curves \( C_1 \), \ldots, \( C_6 \), respectively.

**Lemma 5.16.** One has \(-K_U \cdot \tilde{C}_1 = \ldots = -K_U \cdot \tilde{C}_6 = 0\).

**Proof.** This follows from Corollary 5.15. \( \square \)

We see that each curve \( \tilde{C}_i \) is contracted by \( \beta \) to a point. Since the \( \mathfrak{A}_6 \)-orbit of \( \tilde{C}_1 \) consists of six curves, we also obtain the following.

**Corollary 5.17.** The image of the morphism \( \beta \) contains an \( \mathfrak{A}_6 \)-orbit of length at most six.

Since \(-K^3_U = 4\), the image of \( \beta \) is either an \( \mathfrak{A}_6 \)-invariant quartic threefold or an \( \mathfrak{A}_6 \)-invariant quadric threefold. Using results of [24], one can show that the latter case is impossible. However, this immediately follows from Corollary 5.17. Indeed, an \( \mathfrak{A}_6 \)-orbit of length at most six cannot be contained in the \( \mathfrak{A}_6 \)-invariant quadric by Corollary 4.8 and Lemma 4.9.

**Corollary 5.18.** The morphism \( \beta \) is birational onto its image, and its image is a quartic threefold.

Now Lemma 4.10 implies that the image of \( \beta \) is the quartic \( X_{\mathfrak{A}_6} \). This proves

**Corollary 5.19.** The threefold \( X_{\mathfrak{A}_6} \) is rational.

Let us conclude this section by recalling two related results proved in [5, §4]. The commutative diagram (5.13) can be extended to an \( \mathfrak{A}_6 \)-equivariant commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\rho} & U \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
X_{\mathfrak{A}_6} & \xrightarrow{\sigma} & X_{\mathfrak{A}_6} \\
\downarrow{\phi} & & \downarrow{\phi} \\
\mathbb{P}^3 & \xrightarrow{\psi} & \mathbb{P}^3.
\end{array}
\]

Here \( \sigma \) is an automorphism of the quartic threefold \( X_{\mathfrak{A}_6} \) given by an odd permutation in \( \mathfrak{S}_6 \) acting on \( \mathbb{P}^4 \), cf. Remark 4.5. The birational map \( \rho \) is a composition of Atiyah flops in 36 curves contracted by \( \gamma \), and the birational map \( \psi \) is not regular.

The diagram (5.20) is a so-called \( \mathfrak{A}_6 \)-Sarkisov link. The subgroup \( \mathfrak{A}_6 \subset \text{Aut}(\mathbb{P}^3) \) together with \( \psi \in \text{Bir}^{\mathfrak{A}_6}(\mathbb{P}^3) \) generates a subgroup isomorphic to \( \mathfrak{S}_6 \). Moreover, the subgroup

\[
\text{Aut}^{\mathfrak{A}_6}(\mathbb{P}^3) \subset \text{Bir}^{\mathfrak{A}_6}(\mathbb{P}^3)
\]
6. Rationality of the quartic threefold $X_{\frac{8}{5}}$

In this section we will construct an explicit $G_5$-equivariant birational map $\mathbb{P}^3 \dashrightarrow X_{\frac{8}{5}}$. We identify $\mathbb{P}^3$ with the projectivization $\mathbb{P}(U_4)$, where $U_4$ is the restriction of the four-dimensional irreducible representation of the group $2.G_6$ introduced in Section 2 to a subgroup $2.G_5^{st}$, and denote the latter subgroup simply by $2.G_5$. By Corollary 2.1(i), the $2.G_5$-representation $U_4$ is irreducible.

**Lemma 6.1.** Let $\Omega$ be an $G_5$-orbit in $\mathbb{P}^3$. Then $|\Omega| \geq 12$.

**Proof.** Apply Remark 4.3 together with Corollary 2.1. \hfill \Box

**Lemma 6.2.** Let $C$ be an $G_5$-invariant curve in $\mathbb{P}^3$ of degree $d$. Then $d \geq 6$.

**Proof.** Suppose that $d \leq 5$. To start with, assume that $C$ is reducible and denote by $r$ the number of its irreducible components. We may assume that $G_5$ permutes the irreducible components of $C$ transitively. Thus, either $r = 2$ or $r = 5$ by Remark 4.3. If $r = 5$, the irreducible components of $C$ are lines, so that this case is impossible by Remark 4.3 and Corollary 2.1(i). Hence, we have $r = 2$, and the stabilizer of each of the two irreducible components $C_1$ and $C_2$ of $C$ is the subgroup $A_5 \subset G_5$. Moreover, in this case one has

$$\deg(C_1) = \deg(C_2) \leq 2,$$

which is impossible by Lemma 3.8.

Therefore, we assume that the curve $C$ is irreducible. Let $g$ be the genus of the normalization of the curve $C$. Then

$$g \leq \frac{d^2}{8} + 1 - |\text{Sing}(C)|$$

by Lemma 3.16, so that $g \leq 5 - |\text{Sing}(C)|$. This implies that $C$ is smooth, because there are no $G_5$-orbits of length less than 12 by Lemma 6.1.

Since $G_5$ does not act faithfully on $\mathbb{P}^1$, we see that $g \neq 0$. Thus, either $g = 4$ or $g = 5$ by [6 Lemma 5.1.5]. The former case is impossible by Lemma 3.17, while the latter case is impossible by Lemma 4.3. \hfill \Box

Recall from Section 3 that the subgroup $A_4 \subset A_5 \subset G_5$ fixes two disjoint lines $L_1$ and $L'_1$. As before, we consider the $G_5$-orbit $L_1, \ldots, L_5$ of the line $L_1$ and the $A_5$-orbit $L'_1, \ldots, L'_5$ of the line $L'_1$. By Lemma 3.7 the lines $L_1, \ldots, L_5, L'_1, \ldots, L'_5$ form a double five configuration (see Definition 3.6). Corollary 2.1(i) implies that the $G_5$-orbit of the line $L_1$ is $L_1, \ldots, L_5, L'_1, \ldots, L'_5$.

**Remark 6.3.** Any subgroup $F_{20} \subset G_5$ permutes the ten lines $L_1, \ldots, L_5, L'_1, \ldots, L'_5$ transitively. Indeed, let $c \in F_{20}$ be an element of order five. Then $c$ is not contained in a stabilizer of the line $L_1$, so that the orbit of $L_1$ with respect to the group $\Gamma \cong \mu_5$ generated by $c$ is $L_1, \ldots, L_5$. Similarly, the $\Gamma$-orbit of the line $L'_1$ is $L'_1, \ldots, L'_5$. Also, the group $F_{20}$ is not contained in $A_5$, so that the $F_{20}$-orbit of $L_1$ contains some of the lines $L'_1, \ldots, L'_5$, and thus contains all the ten lines $L_1, \ldots, L_5, L'_1, \ldots, L'_5$. 

is also isomorphic to $G_6$. By [5 Theorem 1.24], the whole group Bir$^{st}_{G_6}(\mathbb{P}^3)$ is a free product of these two copies of $G_6$ amalgamated over the original $A_6$. 


Let \( \mathcal{M} \) be the linear system on \( \mathbb{P}^3 \) consisting of all quartic surfaces passing through all lines \( L_1, \ldots, L_5 \) and \( L'_1, \ldots, L'_5 \). Then \( \mathcal{M} \) is not empty. In fact, Lemma 3.7 and parameter count imply that its dimension is at least four. Moreover, the linear system \( \mathcal{M} \) does not have base components by Lemma 3.3.

**Lemma 6.4.** The base locus of \( \mathcal{M} \) does not contain curves that are different from the lines \( L_1, \ldots, L_5, L'_1, \ldots, L'_5 \). Moreover, general surface in \( \mathcal{M} \) is smooth in a general point of each of these lines. Furthermore, two general surfaces in \( \mathcal{M} \) intersect transversally at a general point of each of these lines.

**Proof.** Denote by \( Z \) the union of all curves that are contained in the base locus of the linear system \( \mathcal{M} \) and are different from the lines \( L_1, \ldots, L_5, L'_1, \ldots, L'_5 \). Then \( Z \) is a (possibly empty) \( \mathcal{G}_5 \)-invariant curve. Denote its degree by \( d \). Pick two general surfaces \( M_1 \) and \( M_2 \) in \( \mathcal{M} \). Then

\[
M_1 \cdot M_2 = Z + m \sum_{i=1}^{5} L_i + m \sum_{i=1}^{5} L'_i + \Delta,\]

where \( m \) is a positive integer, and \( \Delta \) is an effective one-cycle on \( \mathbb{P}^3 \) that contains none of the lines \( L_1, \ldots, L_5 \) and \( L'_1, \ldots, L'_5 \). Note that \( \Delta \) may contain irreducible components of the curve \( Z \). Note also that \( \Delta \neq 0 \), because \( \mathcal{M} \) is not a pencil.

Let \( \Pi \) be a plane in \( \mathbb{P}^3 \). Then

\[
16 = \Pi \cdot Z + m \sum_{i=1}^{5} \Pi \cdot L_i + m \sum_{i=1}^{5} \Pi \cdot L'_i + \Pi \cdot \Delta = d + 10m + \Pi \cdot \Delta > d + 10m,
\]

which implies that \( m = 1 \) and \( d \leq 5 \). By Lemma 6.2 we have \( d = 0 \), so that \( Z \) is empty. Since

\[
1 \geq m \geq \text{mult}_{L_i}(M_1 \cdot M_2) \geq \text{mult}_{L_i}(M_1) \text{mult}_{L_i}(M_2)
\]

we see that a general surface in \( \mathcal{M} \) is smooth at a general point of \( L_i \), and two general surfaces in \( \mathcal{M} \) intersect transversally at a general point of \( L_i \). Similarly, we see that a general surface in \( \mathcal{M} \) is smooth at a general point of \( L'_i \), and two general surfaces in \( \mathcal{M} \) intersect transversally at a general point of \( L'_i \).

Let \( g: W \to \mathbb{P}^3 \) be a blow up along the lines \( L_1, \ldots, L_5 \), and let \( g': W' \to \mathbb{P}^3 \) be a blow up along the lines \( L'_1, \ldots, L'_5 \). Denote by \( \tilde{L}_1, \ldots, \tilde{L}_5 \) (respectively, \( \tilde{L}'_1, \ldots, \tilde{L}'_5 \)) the proper transforms of the lines \( L_1, \ldots, L_5 \) on the threefold \( W \) (respectively, on the threefold \( W' \)).

Let \( h: V \to W \) be a blow up along the curves \( \tilde{L}_1, \ldots, \tilde{L}_5 \), and let \( h': V' \to W' \) be a blow up along the curves \( \tilde{L}'_1, \ldots, \tilde{L}'_5 \). Finally, let \( \alpha: U \to \mathbb{P}^3 \) be a blow up of the (singular) curve that is a union of all lines \( L_1, \ldots, L_5, L'_1, \ldots, L'_5 \). Then \( U \) has twenty nodes by Lemma 3.7 and there exists a commutative diagram.
where \( u \) and \( u' \) are small resolutions of singularities of the threefold \( U \), and \( \tau \) is a composition of twenty Atiyah flops.

Remark 6.5. By construction, the action of group \( \mathfrak{A}_5 \) lifts to the threefolds \( W, W', V, V' \), and \( U \). Similarly, the action of the group \( \mathfrak{S}_5 \) lifts to the threefold \( U \), but this action does not lift to \( W \) and \( W' \). On the threefolds \( V \) and \( V' \), the group \( \mathfrak{S}_5 \) acts biregularly outside of the curves flopped by \( \tau \) and \( \tau^{-1} \), respectively.

Denote by \( E_1, \ldots, E_5 \) the \( g \)-exceptional surfaces that are mapped to \( L_1, \ldots, L_5 \), respectively. Similarly, denote by \( E'_1, \ldots, E'_5 \) the \( g' \)-exceptional surfaces that are mapped to \( L'_1, \ldots, L'_5 \), respectively. Then all surfaces \( E_1, \ldots, E_5, E'_1, \ldots, E'_5 \) are isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \).

Denote by \( \hat{E}_1, \ldots, \hat{E}_5 \) the \( h \)-exceptional surfaces that are mapped to the curves \( \hat{L}_{1}, \ldots, \hat{L}_{5} \), respectively. Similarly, denote by \( \hat{E}_1, \ldots, \hat{E}_5 \) the \( h' \)-exceptional surfaces that are mapped to the curves \( \hat{L}_{1}, \ldots, \hat{L}_{5} \), respectively. Denote by \( \hat{E}_1, \ldots, \hat{E}_5 \) the proper transforms on \( V \) of the surfaces \( E_1, \ldots, E_5 \), respectively. Finally, denote by \( \hat{E}'_1, \ldots, \hat{E}'_5 \) the proper transforms on \( V' \) of the surfaces \( E'_1, \ldots, E'_5 \), respectively. Then \( \tau \) maps the surfaces \( \hat{E}_1, \ldots, \hat{E}_5 \) to the surfaces \( \hat{E}_1, \ldots, \hat{E}_5 \), respectively, and it maps the surfaces \( \hat{E}'_1, \ldots, \hat{E}'_5 \) to the surfaces \( \hat{E}'_1, \ldots, \hat{E}'_5 \), respectively.

Denote by \( \mathcal{M}_W, \mathcal{M}_V, \mathcal{M}_W', \mathcal{M}_V' \), and \( \mathcal{M}_U \) the proper transforms of the linear system \( \mathcal{M} \) on the threefolds \( W, V, W', V' \), and \( U \), respectively. Then it follows from Lemma 6.3 that

\[
\mathcal{M}_W \sim -K_W, \quad \mathcal{M}_V \sim -K_V, \quad \mathcal{M}_W' \sim -K_{W'}, \quad \mathcal{M}_V' \sim -K_{V'},
\]

and \( \mathcal{M}_U \sim -K_U \).

Lemma 6.6. The base locus of the linear system \( \mathcal{M}_W \) does not contain curves that are different from the curves \( \hat{L}_1, \ldots, \hat{L}_5 \). Similarly, the base locus of \( \mathcal{M}_W' \) does not contain curves that are different from the curves \( \hat{L}_1, \ldots, \hat{L}_5 \).

Proof. It is enough to prove the first assertion of the lemma. Suppose that the base locus of the linear system \( \mathcal{M}_W \) contains an irreducible curve \( Z \) that is different from the curves \( \hat{L}_1, \ldots, \hat{L}_5 \). Then \( Z \) is contained in one of the surfaces \( E_1, \ldots, E_5 \) by Lemma 6.4.

By Lemma 6.4, the curve \( Z \) is a fiber of some of the natural projections \( E_i \to L_i \), because otherwise two general surfaces in \( \mathcal{M}_W \) would be tangent in a general point of \( L_i \). In particular, the only curves in the base locus of the linear system \( \mathcal{M}_W \) are \( \hat{L}_i \) and possibly some fibers of the projections \( E_i \to L_i \). This shows that \( -K_W \) is nef. Indeed, \( -K_W \) has positive intersections with the fibers of the projections \( E_i \to L_i \); it has trivial intersection with all curves \( \hat{L}_1, \ldots, \hat{L}_5 \), and \( -K_W \sim \mathcal{M}_W \) has non-negative intersection with any other curve.

Let \( Z_1 = Z, Z_2, \ldots, Z_r \) be the \( \mathfrak{A}_5 \)-orbit of the curve \( Z \). Then \( r \geq 20 \) by Corollary 3.5.

Pick two general surfaces \( M_1 \) and \( M_2 \) in the linear system \( \mathcal{M}_W \). By Lemma 6.4 one has

\[
M_1 \cdot M_2 = \sum_{i=1}^{5} \hat{L}_i + m \sum_{i=1}^{r} Z_i + \Delta
\]
for some positive integer \( m \) and some effective one-cycle \( \Delta \) whose support contains none of the curves \( \tilde{L}_1, \ldots, \tilde{L}_5 \) and \( Z_1, \ldots, Z_r \). Hence

\[
14 = -K_W^3 = -K_W \cdot M_1 \cdot M_2 = -K_W \cdot \left( \sum_{i=1}^{5} \tilde{L}_i' + m \sum_{i=1}^{r} Z_i + \Delta \right) = -5K_W \cdot \tilde{L}_1' - mrK_W \cdot Z - K_W \cdot \Delta = mr - K_W \cdot \Delta \geq mr \geq r \geq 20,
\]

which is absurd.

**Lemma 6.7.** The linear system \( \mathcal{M}_V \) is base point free.

*Proof.* It is enough to show that \( \mathcal{M}_V \) is free from base curves. Indeed, if the base locus of the linear system \( \mathcal{M}_V \) does not contain base curves, then \( \mathcal{M}_V \) cannot have more than \( -K_V^3 = 4 \) base points, because \( \mathcal{M}_V \sim -K_V \). On the other hand, \( V \) does not contain \( \mathcal{S}_5 \)-orbits of length less than 12, because there are no \( \mathcal{S}_5 \)-orbits of such length on \( \mathbb{P}^3 \) by Lemma 6.1.

Suppose that the base locus of the linear system \( \mathcal{M}_V \) contains an irreducible curve \( Z \). If \( Z \) is not contained in any of the surfaces \( \hat{E}_1', \ldots, \hat{E}_5' \), then the curve \( h(Z) \) is a base curve of the linear system \( \mathcal{M}_W \) and \( h(Z) \) is different from the curves \( \tilde{L}_1, \ldots, \tilde{L}_5 \). This is impossible by Lemma 6.6. Similarly, if \( Z \) is not contained in any of the surfaces \( \hat{E}_1, \ldots, \hat{E}_5 \), then the curve \( h' \circ \tau(Z) \) is a base curve of the linear system \( \mathcal{M}_W' \), that is different from the curves \( \tilde{L}_1, \ldots, \tilde{L}_5 \). This is again impossible by Lemma 6.6. Thus, \( Z \) is contained in one of the surfaces \( \hat{E}_1', \ldots, \hat{E}_5' \), and in one of the surfaces \( \hat{E}_1, \ldots, \hat{E}_5 \). In particular, the curves flopped by \( \tau \) are not contained in the base locus of \( \mathcal{M}_V \).

Without loss of generality, we may assume that \( Z = \hat{E}_1 \cap \hat{E}_2 \). Let \( C \) be the curve flopped by \( \tau \) that is contained in \( \hat{E}_1 \) and intersects \( \hat{E}_2 \). Then \( C \) intersects \( Z \) by one point. On the other hand, we have \( -K_V \cdot C = 0 \). Since \( \mathcal{M}_V \sim -K_V \), this implies that \( C \) is disjoint from a general surface in \( \mathcal{M}_V \). This is impossible, because \( C \cap Z \neq \emptyset \), while \( Z \) is contained in the base locus of the linear system \( \mathcal{M}_V \).

**Corollary 6.8.** The linear systems \( \mathcal{M}_{V'} \) and \( \mathcal{M}_U \) are also base point free.

*Proof.* Recall that \( \mathcal{M}_V \sim -K_V \). Thus, the general surface of \( \mathcal{M}_V \) is disjoint from all curves flopped by \( \tau \), because \( \mathcal{M}_V \) is base point free by Lemma 6.7.

**Corollary 6.9.** The base locus of \( \mathcal{M} \) consists of the lines \( L_1, \ldots, L_5, L'_1, \ldots, L'_5 \).

By Lemma 6.7 and Corollary 6.8, the divisors \( -K_V, -K_{V'}, \) and \( -K_U \) are nef. Since

\[
-K_V^3 = -K_{V'}^3 = -K_U^3 = 4,
\]

these divisors are also big. Thus, the Kawamata–Viehweg vanishing theorem and the Riemann–Roch formula give

\[
(6.10) \quad h^0 \left( \mathcal{O}_V(-K_V) \right) = h^0 \left( \mathcal{O}_{V'}(-K_{V'}) \right) = h^0 \left( \mathcal{O}_U(-K_U) \right) = 4.
\]

In particular, one has \( | -K_V | = \mathcal{M}_V \), \( | -K_{V'} | = \mathcal{M}_{V'} \), and \( | -K_U | = \mathcal{M}_U \).

**Lemma 6.11.** The \( \mathcal{S}_5 \)-representation \( H^0(\mathcal{O}_U(-K_U)) \) is irreducible.

*Proof.* By Lemma 3.14 there are no \( \mathcal{S}_5 \)-invariant quartic surfaces in \( \mathbb{P}^3 \) that pass through the ten lines \( L_1, \ldots, L_5, L'_1, \ldots, L'_5 \). This implies that \( H^0(\mathcal{O}_U(-K_U)) \) does not contain one-dimensional subrepresentations. Hence it is irreducible by Remark 4.5.
Lemma 6.7 together with (6.10) implies that there is an $S_5$-equivariant commutative diagram

\[
\begin{array}{ccc}
U & \alpha \ar[d] & \beta \\
\mathbb{P}^3 & \phi \ar[u] \ar[r] & \mathbb{P}^4,
\end{array}
\]

where $\phi$ is a rational map given by $M$, and $\beta$ is a morphism given by the anticanonical linear system $|{-K_U}|$. By Lemma 6.11 the projective space $\mathbb{P}^4$ in (6.12) is a projectivization of an irreducible $S_5$-representation.

For $1 \leq i < j \leq 5$, let $\Lambda_{ij}$ be the intersection line of the plane spanned by $L_i$ and $L'_j$ with the plane spanned by $L'_i$ and $L_j$. Note that the stabilizer of $\Lambda_{ij}$ in $S_5$ contains a subgroup isomorphic to $D_{12}$. Actually, this implies that the stabilizer of $\Lambda_{ij}$ in $S_5$ is isomorphic to $D_{12}$, since $D_{12}$ is a maximal proper subgroup in $S_5$ (see Remark 4.3) and there are no $S_5$-invariant lines in $\mathbb{P}^3$ by Corollary 2.1(i). Denote by $\hat{\Lambda}_{ij}$ the proper transform of the line $\Lambda_{ij}$ on the threefold $V$, and denote by $\bar{\Lambda}_{ij}$ its proper transform on $U$. Then

$$-K_V \cdot \hat{\Lambda}_{ij} = 0.$$  

Since $\nu$ is a small birational morphism, we also obtain $-K_U \cdot \bar{\Lambda}_{ij} = 0$.

We see that each curve $\bar{\Lambda}_{ij}$ is contracted by $\beta$ to a point. Note that the stabilizer of $\Lambda_{ij}$ in $S_5$ is isomorphic to $D_{12}$. Since $-K_3^3 = 4$, the image of $\beta$ is either an $S_5$-invariant quartic threefold or an $S_5$-invariant quadric threefold. Applying Corollary 4.8 together with Lemma 4.11, we obtain the following.

**Corollary 6.13.** The morphism $\beta$ is birational on its image, and its image is a quartic threefold.

Now Lemmas 4.11 and 4.12 imply that the image of $\beta$ is the quartic $X_{\frac{1}{6}}$. This proves

**Corollary 6.14.** The threefold $X_{\frac{1}{6}}$ is rational.

**Remark 6.15.** An alternative approach to the rationality of the quartic threefold $X_{\frac{1}{6}}$ was suggested in [16]. Unfortunately, its implementation seems to contradict the existence of the commutative diagram (6.12). Indeed, the paper [16] studies the action of the subgroup $F_{20} \subset S_5$ on the threefold $X_{\frac{1}{6}}$. Since all such subgroups in $S_5$ are conjugate, one may identify $F_{20}$ with a subgroup of our $S_5$. By Remark 6.3, the group $F_{20}$ permutes the ten lines $L_1, \ldots, L_5, L'_1, \ldots, L'_5$ transitively. This means that $\gamma$ is an $F_{20}$-factorialization of the quartic threefold $X_{\frac{1}{6}}$. Thus, the application of $F_{20}$-Minimal Model Program to $U$ must give the birational map $\alpha: U \to \mathbb{P}^3$. However, [16, Lemma 2.10], [16, Lemma 2.12] and [16, Lemma 2.13] exclude this possibility.

Ten curves $\bar{\Lambda}_{ij}$ are mapped by $\gamma$ to ten singular points of the threefold $X_{\frac{1}{6}}$. Twenty singular points of $U$ are mapped by $\gamma$ to another twenty singular points of $X_{\frac{1}{6}}$. Let us describe the curves in $U$ that are contracted by $\gamma$ to the remaining ten singular points of the threefold $X_{\frac{1}{6}}$. To do this, we need

**Lemma 6.16.** Let $\ell_1$, $\ell_2$, $\ell_3$ and $\ell_4$ be pairwise skew lines in $\mathbb{P}^3$. Suppose that there is a unique line $\ell \subset \mathbb{P}^3$ that intersects $\ell_1$, $\ell_2$, $\ell_3$ and $\ell_4$. Let $\pi: Y \to \mathbb{P}^3$ be a blow up of the
line $\ell$, and $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ be the exceptional divisor of $\pi$. Denote by $\ell_1, \ell_2, \ell_3$ and $\ell_4$ the proper transforms on $Y$ of the lines $\ell_1, \ell_2, \ell_3$ and $\ell_4$, respectively. Then there exists a unique curve $C \subset E$ of bi-degree $(1,1)$ that intersects the curves $\ell_1, \ell_2, \ell_3$ and $\ell_4$.

Proof. The lines $\ell_1$, $\ell_2$, and $\ell_3$ are contained in a unique quadric surface $S \subset \mathbb{P}^3$. Note that $S$ is smooth, because $\ell_1$, $\ell_2$, and $\ell_3$ are disjoint. Furthermore, the line $\ell$ is contained in $S$, because $\ell$ intersects the lines $\ell_1$, $\ell_2$, and $\ell_3$ by assumption. Moreover, the line $\ell_4$ is tangent to $S$, since otherwise there would be either two or infinitely many lines in $\mathbb{P}^3$ that intersect $\ell_1$, $\ell_2$, $\ell_3$ and $\ell_4$. Denote by $S$ the proper transform on $Y$ of the quadric surface $S$. Then $S$ contains the curves $\ell_1, \ell_2$, and $\ell_3$. Moreover, $S$ intersects the curve $\ell_4$. Thus $S|_E$ is the required curve $C$. \qed

By Lemmas 3.7 and 6.16, each surface $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ contains a unique smooth rational curve $C_i$ of bi-degree (1,1) that passes through all four points of the intersection of $E_i$ with the curves $\tilde{L}_1, \ldots, \tilde{L}_5$ (recall that $E_i \cap \tilde{L}_i = \emptyset$). Similarly, each surface $E_i' \cong \mathbb{P}^1 \times \mathbb{P}^1$ contains a unique smooth rational curve $C_i'$ of bi-degree (1,1) that passes through all four points of the intersection of $E_i'$ with the curves $\tilde{L}_1', \ldots, \tilde{L}_5'$. Denote by $\hat{C}_1, \ldots, \hat{C}_5$ the proper transforms on the threefold $V$ of the curves $C_1, \ldots, C_5$, respectively. Similarly, denote by $\hat{C}_1', \ldots, \hat{C}_5'$ the proper transforms on the threefold $V'$ of the curves $C_1', \ldots, C_5'$, respectively. Then

$$-K_V \cdot \hat{C}_i = -K_{V'} \cdot \hat{C}_i' = 0.$$  

This implies that the proper transforms of the curves $\hat{C}_1, \ldots, \hat{C}_5$ on the threefold $V$ are $(-2)$-curves on the surfaces $E_1, \ldots, E_5$, respectively. Similarly, the proper transforms of the curves $\hat{C}_1', \ldots, \hat{C}_5'$ on the threefold $V'$ are $(-2)$-curves on the surfaces $E_1', \ldots, E_5'$, respectively. Thus, all surfaces $E_1, \ldots, E_5, E_1', \ldots, E_5'$ are isomorphic to the Hirzebruch surface $\mathbb{F}_2$.

Denote by $\overline{C}_1, \ldots, \overline{C}_5, \overline{C}_1', \ldots, \overline{C}_5'$ the images of the curves $\hat{C}_1, \ldots, \hat{C}_5, \hat{C}_1', \ldots, \hat{C}_5'$ on the threefold $U$, respectively. Then

$$-K_U \cdot \overline{C}_i = -K_U \cdot \overline{C}_i' = 0,$$

because $-K_V \cdot \hat{C}_i = -K_{V'} \cdot \hat{C}_i' = 0$, and $\nu$ and $\nu'$ are small birational morphisms. Thus, the ten curves $\overline{C}_1, \ldots, \overline{C}_5, \overline{C}_1', \ldots, \overline{C}_5'$ are contracted by the morphism $\beta$ to ten singular points of $X_4$.

It would be interesting to extend the commutative diagram (6.12) to an $\mathfrak{S}_5$-Sarkisov link similar to the $\mathfrak{A}_6$-Sarkisov link (5.20).

References


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