Bohrification

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Abstract

The aim of this chapter is to construct new foundations for quantum logic and quantum spaces. This is accomplished by merging algebraic quantum theory and topos theory (encompassing the theory of locales or frames, of which toposes in a sense form the ultimate generalization). In a nutshell, the relation between these fields is as follows.

First, our mathematical interpretation of Bohr's 'doctrine of classical concepts' is that the empirical content of a quantum theory described by a noncommutative (unital) C*-algebra A is contained in the family of its commutative (unital) C*-algebras, partially ordered by inclusion. Seen as a category, the ensuing poset $\mathcal{C}(A)$ canonically defines the topos $[\mathcal{C}(A), \mathbf{Set}]$ of covariant functors from $\mathcal{C}(A)$ to the category **Set** of sets and functions. This topos contains the 'Bohrification' $\underline{A}$ of A, defined as the tautological functor $C \mapsto C$, as an internal commutative C*-algebra.

Second, according to the topos-valid Gelfand duality theorem of Banaschewski and Mulvey, $\underline{A}$ has a Gelfand spectrum $\underline{\Sigma}(\underline{A})$, which is a locale internal to the topos $[\mathcal{C}(A), \mathbf{Set}]$. We interpret its external description Σ_A (in the sense of Joyal and Tierney), as the 'Bohrified' phase space of the physical system described by A. As in classical physics, the open subsets of Σ_A correspond to (atomic) propositions, so that the 'Bohrified' quantum logic of A is given by the Heyting algebra structure of Σ_A.

The key difference between this logic and its classical counterpart is that the former does not satisfy the law of the excluded middle, and hence is intuitionistic. When A contains sufficiently many projections (as in the case where A is a von Neumann algebra, or, more generally, a Rickart C*-algebra), the intuitionistic quantum logic Σ_A of A may also be compared with the traditional quantum logic $\mathrm{Proj}(A)$, i.e. the orthomodular lattice of projections in A. This time, the main difference is that Σ_A is distributive (even when A is noncommutative), while $\mathrm{Proj}(A)$ is not.

This chapter is a streamlined synthesis of our earlier papers in Comm. Math. Phys. (arXiv:0709.4364), Found Phys. (arXiv:0902.3201) and Synthese (arXiv:0905.2275). See also [51].

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1 Introduction

More than a decade ago, Chris Isham proposed a topos-theoretic approach to quantum mechanics, initially in the context of the Consistent Histories approach [55], and subsequently (in collaboration with Jeremy Butterfield) in relationship with the Kochen–Specker Theorem [19–21] (see also [22] with John Hamilton). More recently, jointly with Andreas Döring, Isham expanded the topos approach so as to provide a new mathematical foundation for all of physics [37, 38]. One of the most interesting features of their approach is, in our opinion, the so-called *Daseinisation* map, which should play an important role in determining the empirical content of the formalism.

Over roughly the same period, in an independent development, Bernhard Banaschewski and Chris Mulvey published a series of papers on the extension of Gelfand duality (which in its usual form establishes a categorical duality between unital commutative C*-algebras and compact Hausdorff spaces, see e.g. [56, 67]) to arbitrary toposes (with natural numbers object) [6–8]. One of the main features of this extension is that the Gelfand spectrum of a commutative C*-algebra is no longer defined as a space, but as a locale (i.e. a lattice satisfying an infinite distributive law [56], see also Section 2 below). Briefly, locales describe spaces through their topologies instead of through their points, and the notion of a locale continues to make sense even in the absence of points (whence the alternative name of “pointfree topology” for the theory of locales). It then becomes apparent that Gelfand duality in the category **Set** of sets and functions is exceptional (compared to the situation in arbitrary toposes), in that the localic Gelfand spectrum of a commutative C* -algebra is spatial (i.e., it is fully described by its points). In the context of constructive mathematics (which differs from topos theory in a number of ways, notably in the latter being impredicative), the work of Banaschewski and Mulvey was taken up by Thierry Coquand [28]. He provided a direct lattice-theoretic description of the localic Gelfand spectrum, which will form the basis of its explicit computation in Section 4 below. This, finally, led to a completely constructive version of Gelfand duality [29, 31].

The third development that fed the research reported here was the program of relating Niels Bohr’s ideas on the foundations of quantum mechanics [13] (and, more generally, the problem of explaining the appearance of the classical world [65]) to the formalism of algebraic quantum theory [64, 66]. Note that this formalism was initially developed in response to the mathematical difficulties posed by quantum field theory [47], but it subsequently turned out to be relevant to a large number of issues in quantum theory, including its axiomatization and its relationship with classical physics [25, 48, 63].

The present work merges these three tracks, which (to the best of our knowledge) so far have been pursued independently. It is based on an *ab initio* redevelopment of quantum physics in the setting of topos theory, published in a series of papers [23, 52–54] (see also [51]), of which the present chapter forms a streamlined and self-contained synthesis, written with the benefit of hindsight.

Our approach is based on a specific mathematical interpretation of Bohr’s ‘doctrine of classical concepts’ [78], which in its original form states, roughly speaking, that the empirical content of a quantum theory is entirely contained in its effects on classical physics. In other words, the quantum world can only be seen through classical glasses. In view of the obscure and wholly unmathematical way of Bohr’s writings, it is not a priori clear what this means mathematically, but we interpret this doctrine as follows: all physically relevant information contained in a noncommutative (unital) C*-algebra A (in its role of the algebra of observables of some quantum system) is contained in the family of its commutative unital C*-algebras.

The role of topos theory, then, is to describe this family as a single commutative unital C*-algebra, as follows. Let $\mathcal{C}(A)$ be the poset of all commutative unital C*-algebras of A, partially ordered by inclusion. This poset canonically defines the topos $[\mathcal{C}(A), \mathbf{Set}]$ of covariant functors from $\mathcal{C}(A)$ (seen as a category, with a unique arrow from C to D if $C \subseteq D$ and no arrow otherwise) to the category **Set** of sets and functions. Perhaps the simplest such functor is the tautological one, mapping $C \in \mathcal{C}(A)$ to $C \in \mathbf{Set}$ (with slight abuse of notation), and mapping an arrow $C \subseteq D$ to the inclusion $C \hookrightarrow D$. We denote this functor by $\underline{A}$ and call it the 'Bohrification' of A. The point is that $\underline{A}$ is a (unital) commutative C*-algebra internal to the topos $[\mathcal{C}(A), \mathbf{Set}]$ under natural operations, and as such it has a localic Gelfand spectrum $\underline{\Sigma}(\underline{A})$ by the Gelfand duality theorem of Banaschewski and Mulvey mentioned above.

The easiest way to study this locale is by means of its external description [58], which is a locale map $f : \Sigma_A \to \mathcal{C}(A)$ (where the poset $\mathcal{C}(A)$ is seen as a topological space in its Alexandrov topology). Denoting the frame or Heyting algebra associated to Σ_A by $\mathcal{O}(\Sigma_A)$, we now identify the (formal) open subsets of Σ_A, defined as the elements of $\mathcal{O}(\Sigma_A)$, with the atomic propositions about the quantum system A. The logical structure of these propositions is then controlled by the Heyting algebra structure of $\mathcal{O}(\Sigma_A)$, so that we have found a quantum analogue of the logical structure of classical physics, the locale Σ_A playing the role of a quantum phase space. As in the classical case, this object carries both spatial and logical aspects, corresponding to the locale Σ_A and the Heyting algebra (or frame) $\mathcal{O}(\Sigma_A)$, respectively.

The key difference between the classical and the quantum case lies in the fact that $\mathcal{O}(\Sigma_A)$ is non-Boolean whenever A is noncommutative. It has to be emphasised, though, that the lattice $\mathcal{O}(\Sigma_A)$ is always distributive; this makes our intuitionistic approach to quantum logic fundamentally different from the traditional one initiated by Birkhoff and von Neumann [11]. Indeed, if A contains sufficiently many projections (as in the case where A is a von Neumann algebra, or, more generally, a Rickart C*-algebra), then the orthomodular lattice $\mathrm{Proj}(A)$ of projections in A (which is the starting point for quantum logic in the context of algebraic quantum theory [72]) is nondistributive whenever A is noncommutative. This feature of quantum logic leads to a number of problems with its interpretation as well as with its structure as a deductive theory, which are circumvented in our approach (see [53] for a detailed discussion of the conceptual points involved).

The plan of this chapter is as follows. Section 2 is a brief introduction to locales and toposes. In Section 3 we give a constructive definition of C*-algebras that can be interpreted in any topos, and review the topos-valid Gelfand duality theory mentioned above. In Section 4 we construct the internal C*-algebra $\underline{A}$ and its localic Gelfand spectrum $\underline{\Sigma}(\underline{A})$, computing the external description Σ_A of the latter explicitly. Section 5 gives a detailed mathematical comparison of the intuitionistic quantum logic $\mathcal{O}(\Sigma_A)$ with its traditional counterpart $\mathrm{Proj}(A)$. Finally, in Section 6 we discuss how a state on A gives rise to a probability integral on $\underline{A}_{\mathrm{sa}}$ within the topos $[\mathcal{C}(A), \mathbf{Set}]$, give our analogue of the *Daseinisation* map of Döring and Isham, and formulate and compute the associated state-proposition pairing.

2 Locales and toposes

This section introduces locales and toposes by summarising well-known results. Both are generalisations of the concept of topological space, and both also carry logical structures. We start with complete Heyting algebras. These can be made into categories in several ways. We consider a logical, an order theoretical, and a spatial perspective.

2.1 Definition A partially ordered set X is called a *lattice* when it has binary joins (least upper bounds, suprema) and meets (greatest lower bounds, infima). It is called a *bounded lattice* when it moreover has a least element 0 and a greatest element 1. It is called a *complete lattice* when it has joins and meets of arbitrary subsets of X. A bounded lattice X is called a *Heyting algebra* when, regarding X as a category, $(\_) \wedge x$ has a right adjoint $x \Rightarrow (\_)$ for every $x \in X$. Explicitly, a Heyting algebra X comes with a monotone function $\Rightarrow \colon X^{\mathrm{op}} \times X \to X$ satisfying $x \leq (y \Rightarrow z)$ if and only if $x \wedge y \leq z$.

2.2 A *Boolean algebra* is a Heyting algebra in which $\neg\neg x = x$ for all x, where $\neg x$ is defined to be $(x \Rightarrow 0)$.

2.3 Definition A *morphism of complete Heyting algebras* is a function that preserves the operations $\wedge$, $\bigvee$ and $\Rightarrow$, as well as the constants 0 and 1. We denote the category of complete Heyting algebras and their morphisms by **CHey**. This gives a logical perspective on complete Heyting algebras.

2.4 Definition Heyting algebras are necessarily distributive, *i.e.* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, since $(\_) \wedge x$ has a right adjoint and hence preserves colimits. When a Heyting algebra is complete, arbitrary joins exist, whence the following infinitary distributive law holds:

$$\Big(\bigvee_{i \in I} y_i\Big) \wedge x = \bigvee_{i \in I} (y_i \wedge x). \tag{1}$$

Conversely, a complete lattice that satisfies this infinitary distributive law is a Heyting algebra by defining $y \Rightarrow z = \bigvee\{x \mid x \wedge y \leq z\}$. This gives an order-theoretical perspective on complete Heyting algebras. The category **Frm** of *frames* has complete Heyting algebras as objects; morphisms are functions that preserve finite meets and arbitrary joins. The categories **Frm** and **CHey** are not identical, because a morphism of frames does not necessarily have to preserve the Heyting implication.

2.5 Definition The category **Loc** of *locales* is the opposite of the category of frames. This gives a spatial perspective on complete Heyting algebras.

2.6 Example To see why locales provide a spatial perspective, let X be a topological space. Denote its topology, *i.e.* the collection of open sets in X, by $\mathcal{O}(X)$. Ordered by inclusion, $\mathcal{O}(X)$ satisfies (1), and is therefore a frame. If $f \colon X \to Y$ is a continuous function between topological spaces, then its inverse image $f^{-1} \colon \mathcal{O}(Y) \to \mathcal{O}(X)$ is a morphism of frames. We can also consider $\mathcal{O}(f) = f^{-1}$ as a morphism $\mathcal{O}(X) \to \mathcal{O}(Y)$ of locales, in the same direction as the original function f. Thus, $\mathcal{O}(\_)$ is a covariant functor from the category **Top** of topological spaces and continuous maps to the category **Loc** of locales.

2.7 Convention To emphasise the spatial aspect of locales, we will follow the convention that a locale is denoted by X, and the corresponding frame by $\mathcal{O}(X)$ (whether or not the the frame comes from a topological space) [69, 85]. Also, we will denote a morphism of locales by $f\colon X \to Y$, and the corresponding frame morphism by $f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ (whether or not f^{-1} is indeed the pullback of a function between topological spaces). A fortiori, we will write $C(X,Y)$ for $\mathbf{Loc}(X,Y) = \mathbf{Frm}(\mathcal{O}(Y), \mathcal{O}(X))$.

2.8 A point x of a topological space X may be identified with a continuous function $1 \to X$, where 1 is a singleton set with its unique topology. Extending this to locales, a *point* of a locale X is a locale map $1 \to X$, or equivalently, a frame map $\mathcal{O}(X) \to \mathcal{O}(1)$. Here, $\mathcal{O}(1) = \{0,1\} = \Omega$ is the subobject classifier of **Set**, as we will see in Example 2.18 below.

Likewise, an *open* of a locale X is defined as a locale morphism $X \to S$, where S is the locale defined by the *Sierpinski space*, *i.e.* $\{0,1\}$ with $\{1\}$ as the only nontrivial open. The corresponding frame morphism $\mathcal{O}(S) \to \mathcal{O}(X)$ is determined by its value at 1, so that we may consider opens in X as morphisms $1 \to \mathcal{O}(X)$ in **Set**. If X is a genuine topological space and $\mathcal{O}(X)$ its collection of opens, then each such morphism $1 \to \mathcal{O}(X)$ corresponds to an open subset of X in the usual sense.

The set $\mathrm{Pt}(X)$ of points of a locale X may be topologised in a natural way, by declaring its opens to be the sets of the form $\mathrm{Pt}(U) = \{p \in \mathrm{Pt}(X) \mid p^{-1}(U) = 1\}$ for some open $U \in \mathcal{O}(X)$. This defines a functor $\mathrm{Pt}\colon \mathbf{Loc} \to \mathbf{Top}$ [56, Theorem II.1.4]. In fact, there is an adjunction

$$\begin{tikzcd} \mathbf{Top} \arrow[r, shift left, "\mathcal{O}(\_)"] \arrow[r, phantom, "\perp"] & \mathbf{Loc}. \arrow[l, shift left, "\mathrm{Pt}"] \end{tikzcd}$$

It restricts to an equivalence between so-called *spatial* locales and *sober* topological spaces. Any Hausdorff topological space is sober [56, Lemma I.1.6].

2.9 Example Let $(P, \leq)$ be a partially ordered set. This can be turned into a topological space by endowing it with the *Alexandrov topology*, in which open subsets are upper sets in P; principal upper sets form a basis for the topology. The associated locale $\mathrm{Alx}(P) = \mathcal{O}(P)$ thus consists of the upper sets UP in P.

If we give a set P the discrete order, then the Alexandrov topology on it is the discrete topology (in which every subset is open), and so $\mathcal{O}(P)$ is just the power set $\mathcal{P}(P)$.

As another example, we now study a way to construct frames (locales) by generators and relations. The generators form a meet-semilattice, and the relations are combined into one suitable so-called covering relation. This technique has been developed in the context of formal topology [76, 77].

2.10 Definition Let L be a meet-semilattice. A *covering relation* on L is a relation $\lhd \subseteq L \times \mathcal{P}(L)$, written as $x \lhd U$ when $(x,U) \in \lhd$, satisfying:

(a) if $x \in U$ then $x \lhd U$;

(b) if $x \lhd U$ and $U \lhd V$ (*i.e.* $y \lhd V$ for all $y \in U$) then $x \lhd V$;

(c) if $x \lhd U$ then $x \wedge y \lhd U$;

(d) if $x \in U$ and $x \in V$, then $x \lhd U \wedge V$ (where $U \wedge V = \{x \wedge y \mid x \in U, y \in V\}$).

2.11 Example If $X \in \mathbf{Top}$, then $\mathcal{O}(X)$ has a covering relation defined by $U \lhd \mathcal{U}$ iff $U \subseteq \bigcup \mathcal{U}$, *i.e.* iff $\mathcal{U}$ covers U.

2.12 Definition Let DL be the partially ordered set of all lower sets in a meet-semilattice L, ordered by inclusion. A covering relation $\lhd$ on L induces a closure operation $\overline{(\_)}\colon DL \to DL$, namely $\overline{U} = \{x \in L \mid x \lhd U\}$. We define

$$\mathcal{F}(L, \lhd) = \{U \in DL \mid \overline{U} = U\} = \{U \in \mathcal{P}(L) \mid x \lhd U \Rightarrow x \in U\}. \tag{2}$$

As $\overline{(\_)}$ is a closure operation, and DL is a frame [56, Section 1.2], so is $\mathcal{F}(L, \lhd)$.

2.13 Proposition *The frame $\mathcal{F}(L, \lhd)$ is the free frame on a meet-semilattice L satisfying $x \leq \bigvee U$ whenever $x \lhd U$ for the covering relation $\lhd$. The canonical inclusion $i\colon L \to \mathcal{F}(L, \lhd)$, defined by $i(x) = \overline{(\downarrow x)}$, is the universal map satisfying $i(x) \leq \bigvee U$ whenever $x \lhd U$. That is, if $f\colon L \to F$ is a morphism of meet-semilattices into a frame F satisfying $f(x) \leq \bigvee f(U)$ if $x \lhd U$, then f factors uniquely through i.*

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{F}(L,\lhd) \arrow[d, dotted] \\ & F \end{tikzcd}$$

If f generates F, in the sense that $V = \bigvee\{f(x) \mid x \in L, f(x) \leq V\}$ for each $V \in F$, there is an isomorphism of frames $F \cong \mathcal{F}(L, \lhd)$ where $x \lhd U$ iff $f(x) \leq \bigvee f(U)$.

Proof Given f, define $g\colon \mathcal{F}(L, \lhd) \to F$ by $g(U) = f(\bigvee U)$. For $x, y \in L$ satisfying $x \lhd \downarrow y$, one then has $f(x) \leq g(\bigvee \downarrow y) = f(y)$, whence $g \circ i(y) = \bigvee\{f(x) \mid x \lhd \downarrow y\} \leq f(y)$. Conversely, $y \lhd \downarrow y$ because $y \in \downarrow y$, so that $f(y) \leq \bigvee\{f(x) \mid x \lhd \downarrow y\} = g \circ i(y)$. Therefore $g \circ i = f$. Moreover, g is the unique such frame morphism. The second claim is proven in [4, Theorem 12]. □

2.14 Definition Let $(L, \lhd)$ and $(M, \blacktriangleleft)$ be meet-semilattices with covering relations. A *continuous map* $f\colon (M, \blacktriangleleft) \to (L, \lhd)$ is a function $f^*\colon L \to \mathcal{P}(M)$ with:

(a) $f^*(L) = M$;

(b) $f^*(x) \wedge f^*(y) \blacktriangleleft f^*(x \wedge y)$;

(c) if $x \lhd U$ then $f^*(x) \blacktriangleleft f^*(U)$ (where $f^*(U) = \bigcup_{u \in U} f^*(U)$).

We identify two such functions if $f_1^*(x) \blacktriangleleft f_2^*(x)$ and $f_2^*(x) \blacktriangleleft f_1^*(x)$ for all $x \in L$.

2.15 Proposition *Each continuous map $f\colon (M, \blacktriangleleft) \to (L, \lhd)$ is equivalent to a frame morphism $\mathcal{F}(f)\colon \mathcal{F}(L, \lhd) \to \mathcal{F}(M, \blacktriangleleft)$ given by $\mathcal{F}(f)(U) = \overline{f^*(U)}$.*

2.16 In fact, the previous proposition extends to an equivalence $\mathcal{F}$ between the category of frames and that of formal topologies, which a generalisation of the above triples $(L, \leq, \lhd)$, where $\leq$ is merely required to be a preorder. In this more general case, the axioms on the covering relation $\lhd$ take a slightly different form. For this, including the proof of the previous proposition, we refer to [4, 9, 71].

We now generalise the concept of locales by introducing toposes.

2.17 A *subobject classifier* in a category $\mathbf{C}$ with a terminal object 1 is a monomorphism $\top\colon 1 \to \Omega$ such that for any mono $m\colon M \to X$ there is a unique $\chi_m\colon X \to \Omega$ such that the following diagram is a pullback:

$$\begin{tikzcd} M \arrow[r] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, tail, "\top"] \\ X \arrow[r, "\chi_m"'] & \Omega. \end{tikzcd}$$

Sometimes the object Ω alone is referred to as the subobject classifier [57, A1.6]. Hence a subobject classifier Ω induces a natural isomorphism $\mathrm{Sub}(X) \cong \mathbf{C}(X, \Omega)$, where the former functor acts on morphisms by pullback, the latter acts by precomposition, and the correspondence is the specific pullback $[m] \mapsto \chi_m$ above.

2.18 Example The category **Set** has a subobject classifier $\Omega = \{0, 1\}$, with the morphism $\top\colon 1 \to \Omega$ determined by $\top(*) = 1$.

For any small category $\mathbf{C}$, the functor category $[\mathbf{C}, \mathbf{Set}]$ has a subobject classifier, which we now describe. A *cosieve* S on an object $X \in \mathbf{C}$ is a collection of morphisms with domain X such that $f \in S$ implies $g \circ f \in S$ for any morphism g that is composable with f. For $X \in \mathbf{C}$, elements of $\Omega(X)$ are the cosieves on X [57, A1.6.6]. On a morphism $f\colon X \to Y$, the action $\Omega(f)\colon \Omega(X) \to \Omega(Y)$ is given by

$$\Omega(f)(S) = \{g\colon Y \to Z \mid Z \in \mathbf{C}, g \circ f \in S\}.$$

Moreover, one has $F \in \mathrm{Sub}(G)$ for functors $F, G\colon \mathbf{C} \rightrightarrows \mathbf{Set}$ if and only F is a *subfunctor* of G, in that $F(X) \subseteq G(X)$ for all $X \in \mathbf{C}$.

In the especially easy case that $\mathbf{C}$ is a partially ordered set, seen as a category, a cosieve S on X is just an *upper set* above X, in the sense that $Y \in S$ and $Y \leq Z$ imply $Z \in S$ and $X \leq Y$.

2.19 Definition A *topos* is a category that has finite limits, exponentials (*i.e.* right adjoints $(\_)^X$ to $(\_) \times X$), and a subobject classifier (see 2.17).

2.20 Example The category **Set** of sets and functions is a topos: the exponential Y^X is the set of functions $X \to Y$, and the set $\Omega = \{0, 1\}$ is a subobject classifier (see Example 2.18).

For any small category $\mathbf{C}$, the functor category $[\mathbf{C}, \mathbf{Set}]$ is a topos. Limits are computed pointwise [14, Theorem 2.15.2], exponentials are defined via the Yoneda embedding [69, Proposition I.6.1], and the cosieve functor Ω of Example 2.18 is a subobject classifier.

2.21 Example Without further explanation, let us mention that a *sheaf* over a locale X is a functor from X^{op} (where the locale X is regarded as a category via its order structure) to **Set** that satisfies a certain continuity condition. The category $\mathrm{Sh}(X)$ of sheaves over a locale X is a topos. Its subobject classifier is $\Omega(x) = {\downarrow}x$ [15, Example 5.2.3].

The categories $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ are equivalent if and only if the locales X and Y are isomorphic. Thus, toposes are generalisations of locales and hence of topological spaces. Moreover, a morphism $X \to Y$ of locales induces morphisms $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ of a specific form: a so-called *geometric morphism* $\mathbf{S} \to \mathbf{T}$ between toposes is a pair of functors $f^*\colon \mathbf{T} \to \mathbf{S}$ and $f_*\colon \mathbf{S} \to \mathbf{T}$, of which f^* preserves finite limits, with $f^* \dashv f_*$. We denote the category of toposes and geometric morphisms by **Topos**.

2.22 If X is the locale resulting from putting the Alexandrov topology on a poset P, then $[P, \mathbf{Set}] \cong \mathrm{Sh}(X)$. In this sense Example 2.20 is a special case of Example 2.21. We call the category $[P, \mathbf{Set}]$ for a poset P a *Kripke topos*.

One could say that sheaves are the prime example of a topos in that they exhibit its spatial character as a generalisation of topology. However, this chapter is primarily concerned with functor toposes, and will therefore not mention sheaves again. We now switch to the logical aspect inherent in toposes, by sketching their internal language and its semantics. For a precise description, we refer to [57, Part D], [69, Chapter VI], or [15, Chapter 6].

2.23 In a (cocomplete) topos $\mathbf{T}$, each subobject lattice $\mathrm{Sub}(X)$ is a (complete) Heyting algebra. Moreover, pullback $f^{-1}\colon \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ along $f\colon X \to Y$ is a morphism of (complete) Heyting algebras. Finally, there are always both left and right adjoints $\exists_f$ and $\forall_f$ to f^{-1}. This means that we can write down properties about objects and morphisms in $\mathbf{T}$ using familiar first order logic. For example, the formula $\forall_{x \in M} \forall_{y \in M} . x \cdot y = y \cdot x$ makes sense for any object M and morphism $\cdot\colon M \times M \to M$ in $\mathbf{T}$, and is interpreted as follows. First, the subformula $x \cdot y = y \cdot x$ is interpreted as the subobject $a\colon A \rightarrowtail M \times M$ given by the equaliser of $M \times M \xrightarrow{\cdot} M$ and $M \times M \xrightarrow{\gamma} M \times M \xrightarrow{\cdot} M$. Next, the subformula $\forall_{y \in M} . x \cdot y = y \cdot x$ is interpreted as the subobject $b = \forall_{\pi_1}(a) \in \mathrm{Sub}(M)$, where $\pi_1\colon M \times M \to M$. Finally, the whole formula $\forall_{x \in M} \forall_{y \in M} . x \cdot y = y \cdot x$ is interpreted as the subobject $c = \forall_{\pi}(b) \in \mathrm{Sub}(1)$, where $\pi\colon M \to 1$. The subobject $c \in \mathrm{Sub}(1)$ is classified by a unique $\chi_c\colon 1 \to \Omega$. This, then, is the *truth value* of the formula. In general, a formula φ is said to *hold* in the topos $\mathbf{T}$, denoted by $\Vdash \varphi$, when its truth value factors through the subobject classifier $\top\colon 1 \to \Omega$.

If $\mathbf{T} = \mathbf{Set}$, the subobject a is simply the set $\{(x, y) \in M \times M \mid x \cdot y = y \cdot x\}$, and therefore the truth value of the formula is $1 \in \Omega$ if for all $x, y \in M$ we have $x \cdot y = y \cdot x$, and $0 \in \Omega$ otherwise. But the above interpretation can be given in any topos $\mathbf{T}$, even if there are few or no 'elements' $1 \to M$. Thus we can often reason about objects in a topos $\mathbf{T}$ as if they were sets. Indeed, the fact that a topos has exponentials and a subobject classifier means that we can use higher order logic to describe properties of its objects, by interpreting a power set $\mathcal{P}(X)$ as the exponential Ω^X, and the inhabitation relation $\in$ as the subobject of $X \times \Omega^X$ that is classified by the transpose $X \times \Omega^X \to \Omega$ of $\mathrm{id}\colon \Omega^X \to \Omega^X$. All this can be made precise by defining the *internal* or *Mitchell-Bénabou language* of a topos, which prescribes in detail which logical formulae about the objects and morphisms of a topos are "grammatically correct" and which ones hold.

2.24 The interpretation of the internal language takes an especially easy form in Kripke toposes. We now give this special case of the so-called *Kripke-Joyal semantics*. First, let us write $[\![t]\!]$ for the interpretation of a term t as in 2.23. For example, in the notation of 2.23, $[\![x]\!]$ is the morphism $\mathrm{id}\colon M \to M$, and $[\![x \cdot y]\!]$ is the morphism $\cdot\colon M \times M \to M$. We now inductively define $p \Vdash \varphi(\vec{a})$ for $p \in P$, a formula φ in the language of $[P, \mathbf{Set}]$ with free variables x_i of type X_i, and $\vec{a} = (a_1, \ldots, a_n)$ with $a_i \in X_i(p)$:

- $p \Vdash (t = t')(\vec{a})$ if and only if $[\![t]\!]_p(\vec{a}) = [\![t']\!]_p(\vec{a})$;
- $p \Vdash R(t_1, \ldots, t_k)(\vec{a})$ if and only if $([\![t_1]\!]_p(\vec{a}), \ldots, [\![t_k]\!](\vec{a})) \in R(p)$, where R is a relation on $X_1 \times \cdots \times X_n$ interpreted as a subobject of $X_1 \times \cdots \times X_n$;
- $p \Vdash (\varphi \wedge \psi)(\vec{a})$ if and only if $p \Vdash \varphi(\vec{a})$ and $p \Vdash \varphi(\vec{a})$;
- $p \Vdash (\varphi \vee \psi)(\vec{a})$ if and only if $p \Vdash \varphi(\vec{a})$ or $p \Vdash \varphi(\vec{a})$;
- $p \Vdash (\varphi \Rightarrow \psi)(\vec{a})$ if and only if $q \Vdash \varphi(\vec{a})$ implies $q \Vdash \psi(\vec{a})$ for all $q \geq p$;
- $p \Vdash \neg\varphi(\vec{a})$ if and only if $q \Vdash \varphi(\vec{a})$ for no $q \geq p$;
- $p \Vdash \exists_{x \in X}.\varphi(\vec{a})$ if and only if $p \Vdash \varphi(a, \vec{a})$ for some $a \in X(p)$;
- $p \Vdash \forall_{x \in X}.\varphi(\vec{a})$ if and only if $q \Vdash \varphi(a, \vec{a})$ for all $q \geq p$ and $a \in X(q)$.

It turns out that φ holds in $[P, \mathbf{Set}]$, *i.e.* $\Vdash \varphi$, precisely when $p \Vdash \varphi(\vec{a})$ for all $p \in P$ and all $\vec{a} \in X_1(p) \times \cdots \times X_n(p)$.

2.25 The axioms of intuitionistic logic hold when interpreted in any topos, and there are toposes in whose internal language formulae that are not derivable from the axioms of intuitionistic logic do not hold. For example, the principle of excluded middle $\varphi \vee \neg\varphi$ does not hold in the topos $\mathrm{Sh}(\mathbb{R})$ [15, 6.7.2]. Thus, we can derive properties of objects of a topos as if they were sets, using the usual higher-order logic, as long as our reasoning is *constructive*, in the sense that we use neither the axiom of choice, nor the principle of excluded middle.

The astute reader will have noticed that the account of this chapter up to now has been constructive in this sense (including the material around Proposition 2.13). In particular, we can speak of objects in a topos $\mathbf{T}$ that satisfy the defining properties of locales as *locales within that topos*. Explicitly, these are objects L that come with morphisms $0, 1 \colon 1 \rightrightarrows L$ and $\bigwedge, \bigvee \colon \Omega^L \rightrightarrows L$ for which the defining formulae of locales, such as (1), hold in $\mathbf{T}$ [15, Section 6.11]. The category of such objects is denoted by $\mathrm{Loc}(\mathbf{T})$, so that $\mathrm{Loc}(\mathbf{Set}) \cong \mathbf{Loc}$. For the rest of this chapter we will also take care to use constructive reasoning whenever we reason in the internal language of a topos.

2.26 We have two ways of proving properties of objects and morphisms in toposes. First, we can take an *external* point of view. This occurs, for example, when we use the structure of objects in $[P, \mathbf{Set}]$ as $\mathbf{Set}$-valued functors. Secondly, we can adopt the *internal* logic of the topos, as above. In this viewpoint, we regard the topos as a 'universe of discourse'. At least intuitionistic reasoning is valid, but more logical laws might hold, depending on the topos one is studying. To end this section, we consider the internal and external points of view in several examples.

2.27 Example Let $\mathbf{T}$ be a topos, and X an object in it. Externally, one simply looks at $\mathrm{Sub}(X)$ as a set, equipped with the structure of a Heyting algebra *in the category* $\mathbf{Set}$. Internally, $\mathrm{Sub}(X)$ is described as the exponential Ω^X, or $\mathcal{P}(X)$, which is a Heyting algebra object *in the topos* $\mathbf{T}$ [69, p. 201].

2.28 Example For any poset P, the category $\mathrm{Loc}([P,\mathbf{Set}])$ is equivalent to the slice category $\mathbf{Loc}/\mathrm{Alx}(P)$ of locale morphisms $L \to \mathrm{Alx}(P)$ from some locale L to the Alexandrov topology on P (by 2.22 and [58]). Therefore, an internal locale object $\underline{L}$ in $[P,\mathbf{Set}]$ is described externally as a locale morphism $f\colon L \to \mathrm{Alx}(P)$, determined as follows. First, $\mathcal{O}(\underline{L})(P)$ is a frame in $\mathbf{Set}$, and for U in $\mathrm{Alx}(P)$, the action $\mathcal{O}(\underline{L})(P) \to \mathcal{O}(\underline{L})(U)$ on morphisms is a frame morphism. Since $\mathcal{O}(\underline{L})$ is complete, there is a left adjoint $l_U^{-1}\colon \mathcal{O}(\underline{L})(U) \to \mathcal{O}(\underline{L})(P)$, which in turn defines a frame morphism $f^{-1}\colon \mathcal{O}(\mathrm{Alx}(P)) \to \mathcal{O}(\underline{L})(P)$ by $f^{-1}(U) = l_U^{-1}(1)$. Taking $L = \mathcal{O}(\underline{L})(P)$ then yields the desired locale morphism.

2.29 Example Let L be a locale object in the Kripke topos over a poset P. Internally, a point of L is a locale morphism $1 \to L$, which is the same thing as an internal frame morphism $\mathcal{O}(L) \to \Omega$. Externally, one looks at Ω as the frame $\mathrm{Sub}(1)$ in $\mathbf{Set}$. Since $\mathrm{Sub}(1) \cong \mathcal{O}(\mathrm{Alx}(P))$ in $[P,\mathbf{Set}]$, one finds $\mathrm{Loc}([P,\mathbf{Set}]) \cong \mathbf{Loc}/\mathrm{Alx}(P)$. By Example 2.28, L has an external description as a locale morphism $f\colon K \to L$, so that points in L are described externally by sections of f, *i.e.* locale morphisms $g\colon L \to K$ satisfying $f \circ g = \mathrm{id}$.

2.30 Locales already possess a logical aspect as well as a spatial one, as the logical perspective on complete Heyting algebras translates to the spatial perspective on locales. Elements $1 \to \mathcal{O}(L)$ of the Heyting algebra $\mathcal{O}(L)$ are the opens of the associated locale L, to be thought of as propositions, whereas points of the locale correspond to models of the logical theory defined by these propositions [85].

More precisely, recall that a formula is *positive* when it is built from atomic propositions by the connectives $\wedge$ and $\vee$ only, where $\vee$ but not $\wedge$ is allowed to be indexed by an infinite set. This can be motivated observationally: to verify a proposition $\bigvee_{i\in I} p_i$, one only needs to find a single p_i, whereas to verify $\bigwedge_{i\in I} p_i$ the validity of each p_i needs to be established [3], an impossible task in practice when I is infinite. A *geometric formula* then is one of the form $\varphi \Rightarrow \psi$, where φ and ψ are positive formulae.

Thus a frame $\mathcal{O}(L)$ defines a geometric propositional theory whose propositions correspond to opens in L, combined by logical connectives given by the lattice structure of $\mathcal{O}(L)$. Conversely, a propositional geometric theory $\mathfrak{T}$ has an associated *Lindenbaum algebra* $\mathcal{O}([\mathfrak{T}])$, defined as the poset of formulae of $\mathfrak{T}$ modulo provable equivalence, ordered by entailment. This poset turns out to be a frame, and the set-theoretical models of $\mathfrak{T}$ bijectively correspond to frame morphisms $\mathcal{O}([\mathfrak{T}]) \to \{0,1\}$. Identifying $\{0,1\}$ in $\mathbf{Set}$ with $\Omega = \mathcal{O}(1)$, one finds that a model of the theory $\mathfrak{T}$ is a point $1 \to [\mathfrak{T}]$ of the locale $[\mathfrak{T}]$. More generally, by Example 2.28 one may consider a model of $\mathfrak{T}$ in a frame $\mathcal{O}(L)$ to be a locale morphism $L \to [\mathfrak{T}]$.

2.31 Example Consider models of a geometric theory $\mathfrak{T}$ in a topos $\mathbf{T}$. Externally, these are given by locale morphisms $\mathrm{Loc}(\mathbf{T}) \to [\mathfrak{T}]$ [69, Theorem X.6.1 and Section IX.5]. One may also interpret $\mathfrak{T}$ in $\mathbf{T}$ and thus define a locale $[\mathfrak{T}]_{\mathbf{T}}$ internal to $\mathbf{T}$. The points of this locale, *i.e.* the locale morphisms $1 \to [\mathfrak{T}]_{\mathbf{T}}$ or frame morphisms $\mathcal{O}([\mathfrak{T}]_{\mathbf{T}}) \to \Omega$, describe the models of $\mathfrak{T}$ in $\mathbf{T}$ internally.

2.32 Example Several important internal number systems in Kripke toposes are defined by geometric propositional theories $\mathfrak{T}$, and can be computed via Example 2.21 and 2.22. Externally, the frame $\mathcal{O}([\mathfrak{T}])$ corresponding to the interpretation of $\mathfrak{T}$ in $[P,\mathbf{Set}]$ is given by the functor $\mathcal{O}([\mathfrak{T}])\colon p \mapsto \mathcal{O}(\uparrow p \times [\mathfrak{T}])$ [23, Appendix A].

2.33 Example As an application of the previous example, we recall an explicit construction of the *Dedekind real numbers* (see [40] or [57, D4.7.4]. Define the propositional geometric theory $\mathfrak{T}_{\mathbb{R}}$ generated by formal symbols $(q,r) \in \mathbb{Q} \times \mathbb{Q}$ with $q < r$, ordered as $(q,r) \leq (q',r')$ iff $q' \leq q$ and $r \leq r'$, subject to the following relations:

$$(q_1,r_1) \wedge (q_2,r_2) = \begin{cases} (\max(q_1,q_2), \min(r_1,r_2)) & \text{if } \max(q_1,q_2) < \min(r_1,r_2) \\ 0 & \text{otherwise} \end{cases}$$

$$(q,r) = \bigvee\{(q',r') \mid q < q' < r' < r\}$$

$$1 = \bigvee\{(q,r) \mid q < r\}$$

$$(q,r) = (q,r_1) \vee (q_1,r) \qquad \text{if } q \leq q_1 \leq r_1 \leq r.$$

This theory may be interpreted in any topos $\mathbf{T}$ with a natural numbers object, defining an internal locale $\mathbb{R}_{\mathbf{T}}$. Points p of $\mathbb{R}_{\mathbf{T}}$, *i.e.* frame morphisms $p^{-1} \colon \mathcal{O}(\mathbb{R}_{\mathbf{T}}) \to \Omega$, correspond to Dedekind cuts (L,U) by [69, p. 321]:

$$L = \{q \in \mathbb{Q} \mid p \models (q,\infty)\};$$
$$U = \{r \in \mathbb{Q} \mid p \models (-\infty,r)\},$$

where (q,∞) and $(-\infty,r)$ are defined in terms of the formal generators of the frame $\mathcal{O}(\mathbb{Q})$ by $(q,\infty) = \bigvee\{(q,r) \mid q < r\}$ and $(-\infty,r) = \bigvee\{(q,r) \mid q < r\}$. The notation $p \models (q,r)$ means that $m^{-1}(q,r)$ is the subobject classifier $\top \colon 1 \to \Omega$, where (q,r) is seen as a morphism $1 \to \mathbb{Q} \times \mathbb{Q} \to \mathcal{O}(\mathbb{R}_{\mathbf{T}})$. Conversely, a Dedekind cut (L,U) uniquely determines a point p by $(q,r) \mapsto \top$ iff $(q,r) \cap U \neq \emptyset$ and $(q,r) \cap L \neq \emptyset$. The Dedekind real numbers are therefore defined in any topos $\mathbf{T}$ as the subobject of $\mathcal{P}(\mathbb{Q}_{\mathbf{T}}) \times \mathcal{P}(\mathbb{Q}_{\mathbf{T}})$ consisting of those (L,U) that are points of $\mathbb{R}_{\mathbf{T}}$.

One may identify $\mathrm{Pt}(\mathbb{R}_{\mathbf{Set}})$ with the field $\mathbb{R}$ in the usual sense, and $\mathcal{O}(\mathbb{R}_{\mathbf{Set}})$ with the usual Euclidean topology on $\mathbb{R}$.

In case $\mathbf{T} = [P, \mathbf{Set}]$ for a poset P, one finds that $\mathcal{O}(\mathbb{R}_{\mathbf{T}})$ is the functor $p \mapsto \mathcal{O}(\uparrow p \times \mathbb{R}_{\mathbf{Set}})$; *cf.* Example 2.32. The latter set may be identified with the set of monotone functions $\uparrow p \to \mathcal{O}(\mathbb{R}_{\mathbf{Set}})$. When P has a least element, the functor $\mathrm{Pt}(\mathbb{R}_{\mathbf{T}})$ may be identified with the constant functor $p \mapsto \mathbb{R}_{\mathbf{Set}}$.

3 C*-algebras

This section considers a generalisation of the concept of topological space different from locales and toposes, namely so-called C*-algebras [35, 59, 84]. These operator algebras also play a large role in quantum theory [47, 63, 80]. We first give a constructive definition of C*-algebras that can be interpreted in any topos (with a natural numbers object), after [6–8].

3.1 In any topos (with a natural numbers object) the rationals $\mathbb{Q}$ can be interpreted [69, Section VI.8], as can the *Gaussian rationals* $\mathbb{C}_{\mathbb{Q}} = \{q + ri \mid q,r \in \mathbb{Q}\}$. For example, the interpretation of $\mathbb{C}_{\mathbb{Q}}$ in the Kripke topos over a poset P is the constant functor that assigns the set $\mathbb{C}_{\mathbb{Q}}$ to each $p \in P$.

3.2 A monoid in $\mathbf{Vect}_K$ for some $K \in \mathbf{Fld}$ is called a (unital) K-*algebra*—not to be confused with Eilenberg-Moore algebras of a monad. It is called *commutative* when the multiplication of its monoid structure is. A **-algebra* is an algebra A over an involutive field, together with an antilinear involution $(\_)^* \colon A \to A$.

3.3 Definition A *seminorm* on a *-algebra A over $\mathbb{C}_\mathbb{Q}$ is a relation $N \subseteq A \times \mathbb{Q}^+$ satisfying

$$\begin{aligned}
(0,p) &\in N, \\
&\exists_{q \in \mathbb{Q}^+}.(a,q) \in N, \\
(a,q) \in N &\Rightarrow (a^*,q) \in N, \\
(a,r) \in N &\iff \exists_{q<r}.(a,q) \in N, \\
(a,q) \in N \ \wedge \ (b,r) \in N &\Rightarrow (a+q, p+r) \in N, \\
(a,q) \in N \ \wedge \ (b,r) \in N &\Rightarrow (ab, qr) \in N, \\
(a,q) \in N &\Rightarrow (za, qr) \in N \qquad\qquad (|z| < r), \\
(1,q) &\in N \qquad\qquad\qquad\qquad (q > 1),
\end{aligned}$$

for all $a, b \in A$, $q, r \in \mathbb{Q}^+$, and $z \in \mathbb{C}_\mathbb{Q}$. If this relation furthermore satisfies

$$(a^*a, q^2) \in N \iff (a,q) \in N$$

for all $a \in A$ and $q \in \mathbb{Q}^+$, then A is said to be a *pre-semi-C*-algebra*.

A seminorm N is called a *norm* if $a = 0$ whenever $(a,q) \in N$ for all $q \in \mathbb{Q}^+$. One can then formulate a suitable notion of completeness in this norm that does not rely on the axiom of choice, namely by considering Cauchy sequences of sets instead of Cauchy sequences [8]. A *C*-algebra* is a pre-semi-C*-algebra A whose seminorm is a norm in which A is complete. Notice that a C*-algebra by definition has a unit; what we defined as a C*-algebra is sometimes called a unital C*-algebra in the literature.

A morphism between C*-algebras A and B is a linear function $f \colon A \to B$ satisfying $f(ab) = f(a)f(b)$, $f(a^*) = f(a)^*$ and $f(1) = 1$. C*-algebras and their morphisms form a category **CStar**. We denote its full subcategory of commutative C*-algebras by **cCStar**.

3.4 Classically, a seminorm induces a norm, and vice versa, by $(a,q) \in N$ if and only if $\|a\| < q$.

3.5 The geometric theory $\mathfrak{T}_\mathbb{R}$ of Example 2.33 can be extended to a geometric theory $\mathfrak{T}_\mathbb{C}$ describing the complexified locale $\mathbb{C} = \mathbb{R} + i\mathbb{R}$. There are also direct descriptions that avoid a defining role of $\mathbb{R}$ [8]. In **Set**, the frame $\mathcal{O}(\mathbb{C})$ defined by $\mathfrak{T}_\mathbb{C}$ is the usual topology on the usual complex field $\mathbb{C}$. As a consequence of its completeness, a C*-algebra is automatically an algebra over $\mathbb{C}$ (and not just over $\mathbb{C}_\mathbb{Q}$, as is inherent in the definition).

3.6 Example The continuous linear operators $\mathbf{Hilb}(H,H)$ on a Hilbert space H form a C*-algebra. In fact, by the classical Gelfand-Naimark theorem, any C*-algebra can be embedded into one of this form [44].

3.7 Example A locale X is *compact* if every subset $S \subseteq X$ with $\bigvee S = 1$ has a finite subset $F \subseteq S$ with $\bigvee F = 1$. It is *regular* if $y = \bigvee(\downarrow\!\!\!\!\downarrow y)$ for all $y \in X$, where $\downarrow\!\!\!\!\downarrow y = \{x \in$

$X \mid x \ll y\}$ and $x \ll y$ iff there is a $z \in X$ with $z \wedge x = 0$ and $z \vee y = 1$. If the axiom of dependent choice is available—as in Kripke toposes [41]—then regular locales are automatically completely regular. Assuming the full axiom of choice, the category **KRegLoc** of compact regular locales in **Set** is equivalent to the category **KHausTop** of compact Hausdorff topological spaces. In general, if X is a completely regular compact locale, then $C(X, \mathbb{C})$ is a commutative C*-algebra. In fact, the following theorem shows that all commutative C*-algebras are of this form. This so-called *Gelfand duality* justifies regarding C*-algebras as "noncommutative" generalisations of topological spaces [27].

3.8 Theorem *[6–8] There is an equivalence*

$$\begin{tikzcd} \mathbf{cCStar} \arrow[r, shift left, "\Sigma"] \arrow[r, phantom, "\perp"] & \mathbf{KRegLoc}^{\mathrm{op}}. \arrow[l, shift left, "C(\_,\mathbb{C})"] \end{tikzcd}$$

The locale $\Sigma(A)$ *is called the* Gelfand spectrum *of* A. □

The previous theorem is proved in such a way that it applies in any topos. This means that we can give an explicit description of the Gelfand spectrum. The rest of this section is devoted to just that, following the reformulation which is fully constructive [28, 29].

3.9 To motivate the following description, we mention that the classical proof [43, 44] defines $\Sigma(A)$ to be the set of *characters* of A, *i.e.* nonzero multiplicative functionals $\rho \colon A \to \mathbb{C}$. This set becomes a compact Hausdorff topological space by the sub-base consisting of $\{\rho \in \Sigma(A) \mid |\rho(a) - \rho_0(a)| < \varepsilon\}$ for $a \in A$, $\rho_0 \in \Sigma$ and $\varepsilon > 0$. A much simpler choice of sub-base would be $\mathcal{D}_a = \{\rho \in \Sigma \mid \rho(a) > 0\}$ for $a \in A_{\mathrm{sa}} = \{a \in A \mid a^* = a\}$. Both the property that the ρ are multiplicative and the fact that the $\mathcal{D}_a$ form a sub-base may then be expressed lattice-theoretically by letting $\mathcal{O}(\Sigma(A))$ be the frame freely generated by the formal symbols $\mathcal{D}_a$ for $a \in A_{\mathrm{sa}}$, subject to the relations

$$\mathcal{D}_1 = 1, \tag{3}$$
$$\mathcal{D}_a \wedge \mathcal{D}_{-a} = 0, \tag{4}$$
$$\mathcal{D}_{-b^2} = 0, \tag{5}$$
$$\mathcal{D}_{a+b} \leq \mathcal{D}_a \vee \mathcal{D}_b, \tag{6}$$
$$\mathcal{D}_{ab} = (\mathcal{D}_a \wedge \mathcal{D}_b) \vee (\mathcal{D}_{-a} \wedge \mathcal{D}_{-b}), \tag{7}$$

supplemented with the 'regularity rule'

$$\mathcal{D}_a \leq \bigvee_{r \in \mathbb{Q}^+} \mathcal{D}_{a-r}. \tag{8}$$

3.10 Classically, the *Gelfand transform* $A \overset{\cong}{\to} C(\Sigma(A), \mathbb{C})$ is given by $a \mapsto \hat{a}$ with $\hat{a}(\rho) = \rho(a)$, and restricting to A_{sa} yields an isomorphism $A_{\mathrm{sa}} \cong C(\Sigma(A), \mathbb{R})$. Hence classically $\mathcal{D}_a = \{\rho \in \Sigma(A) \mid \hat{a}(\rho) > 0\}$. In a constructive setting, we must associate a locale morphism $\hat{a} \colon \Sigma(A) \to \mathbb{R}$ to each $a \in A_{\mathrm{sa}}$, which is, by definition, a frame morphism $\hat{a}^{-1} \colon \mathcal{O}(\mathbb{R}) \to \mathcal{O}(\Sigma(A))$. Aided by the intuition of 3.9, one finds that $\hat{a}^{-1}(-\infty, s) = \mathcal{D}_{s-a}$ and $\hat{a}^{-1}(r, \infty) = \mathcal{D}_{a-r}$ for basic opens. Hence $\hat{a}^{-1}(r, s) = \mathcal{D}_{s-a} \wedge \mathcal{D}_{a-r}$ for rationals $r < s$.

By Example 2.33, we have $A_{\mathrm{sa}} \cong C(\Sigma(A)), \mathbb{R}) = \Gamma(\mathrm{Pt}(\mathbb{R})_{\mathrm{Sh}(\Sigma(A))})$, where Γ is the global sections functor. Hence, A_{sa} is isomorphic (through the Gelfand transform) to the global sections of the real numbers in the topos of sheaves on its spectrum (and A itself "is" the complex numbers in the same sense).

3.11 To describe the Gelfand spectrum more explicitly, we start with the distributive lattice L_A freely generated by the formal symbols $\mathtt{D}_a$ for $a \in A_{\mathrm{sa}}$, subject to the relations (3)–(7). Being an involutive ring, A_{sa} has a positive cone $A^+ = \{a \in A_{\mathrm{sa}} \mid a \geq 0\} = \{a^2 \mid a \in A_{\mathrm{sa}}\}$. (For $A = \mathbf{Hilb}(H, H)$, one has $a \in A^+$ iff $\langle x \,|\, a(x) \rangle \geq 0$ for all $x \in H$.) The given definition of A^+ induces a partial order $\leq$ on A^+ by $a \leq b$ iff $0 \leq a - b$, with respect to which A^+ is a distributive lattice. Now we define a partial order $\preccurlyeq$ on A^+ by $a \preccurlyeq b$ iff $a \leq nb$ for some $n \in \mathbb{N}$. Define an equivalence relation on A^+ by $a \approx b$ iff $a \preccurlyeq b$ and $b \preccurlyeq a$. The lattice operations on A^+ respect $\approx$ and hence $A^+/\approx$ is a lattice. We have

$$L_A \cong A^+/\approx .$$

The image of the generator $\mathtt{D}_a$ in L_A corresponds to the equivalence class $[a^+]$ in $A^+/\approx$, where $a = a^+ - a^-$ with $a^\pm \in A^+$ in the usual way. Theorem 4.12 will show that the lattice L_A can be computed locally in certain Kripke toposes. In preparation, we now work towards Lemma 3.16 below.

3.12 Extending the geometric *propositional* logic of 2.30, the positive formulae of a geometric *predicate* logic may furthermore involve finitely many free variables and the existential quantifier $\exists$, and its axioms take the form $\forall_{x \in X}.\varphi(x) \Rightarrow \psi(x)$ for positive formulae φ, ψ. Geometric formulae form an important class of logical formulae, because they are precisely the ones whose truth value is preserved by inverse images of geometric morphisms between toposes. From their syntactic form alone, it follows that their external interpretation is determined locally in Kripke toposes, as the following lemma shows.

3.13 Lemma *[57, Corollary D1.2.14] Let $\mathfrak{T}$ be a geometric theory, and denote the category of its models in a topos $\mathbf{T}$ by $\mathbf{Model}(\mathfrak{T}, \mathbf{T})$. For any category $\mathbf{C}$, there is a canonical isomorphism of categories $\mathbf{Model}(\mathfrak{T}, [\mathbf{C}, \mathbf{Set}]) \cong [\mathbf{C}, \mathbf{Model}(\mathfrak{T}, \mathbf{Set})]$.* □

3.14 Definition A *Riesz space* is a vector space R over $\mathbb{R}$ that is simultaneously a distributive lattice, such that $f \leq g$ implies $f + h \leq g + h$ for all h, and $f \geq 0$ implies $rf \geq 0$ for all $r \in \mathbb{R}^+$ [68, Definition 11.1].

An *f-algebra* is a commutative $\mathbb{R}$-algebra R whose underlying vector space is a Riesz space in which $f, g \geq 0$ implies $fg \geq 0$, and $f \wedge g = 0$ implies $hf \wedge g = 0$ for all $h \geq 0$. Moreover, the multiplicative unit 1 has to be *strong* in the sense that for each $f \in R$ one has $-n1 \leq f \leq n1$ for some $n \in \mathbb{N}$ [86, Definition 140.8].

3.15 Example If A is a commutative C*-algebra, then A_{sa} becomes an f-algebra over $\mathbb{R}$ under the order defined in 3.11. Conversely, by the Stone-Yosida representation theorem every f-algebra over $\mathbb{R}$ can be densely embedded in $C(X, \mathbb{R})$ for some compact locale X [31]. Like commutative C*-algebras, f-algebras have a spectrum, for the definition of which we refer to [29].

3.16 Lemma *Let A be a commutative C*-algebra.*

(a) The Gelfand spectrum of A coincides with the spectrum of the f-algebra A_{sa}.

(b) The theory of f-algebras is geometric.

Proof Part (a) is proven in [29]. For (b), notice that an f-algebra over $\mathbb{Q}$ is precisely a uniquely divisible lattice-ordered ring [28, p. 151], since unique divisibility turns a ring into a $\mathbb{Q}$-algebra. The definition of a lattice-ordered ring can be written using equations only. The theory of torsion-free rings, *i.e.* if $n > 0$ and $nx = 0$ then $x = 0$, is also algebraic. The theory of divisible rings is obtained by adding infinitely many geometric axioms $\exists_y . ny = x$, one for each $n > 0$, to the algebraic theory of rings. Finally, a torsion-free divisible ring is the same as a uniquely divisible ring: if $ny = x$ and $nz = x$, then $n(y - z) = 0$, so that $y - z = 0$. We conclude that the theory of uniquely divisible lattice-ordered rings, *i.e.* f-algebras, is geometric, establishing (b). □

3.17 Proposition *The lattice L_A generating the spectrum of a commutative C*-algebra A is preserved under inverse images of geometric morphisms.*

Proof By the previous lemma, A_{sa} and hence A^+ are definable by a geometric theory. Since the relation $\approx$ of 3.11 is defined by an existential quantification, $L_A \cong A^+/\approx$ is preserved under inverse images of geometric morphisms. □

We now turn to the regularity condition (8), which is to be imposed on L_A. This condition turns out to be a special case of the relation $\ll$ (see Example 3.7).

3.18 Lemma *For all $\mathtt{D}_a, \mathtt{D}_b \in L_A$ the following are equivalent:*

(a) There exists $\mathtt{D}_c$ with $\mathtt{D}_c \vee \mathtt{D}_a = 1$ and $\mathtt{D}_c \wedge \mathtt{D}_b = 0$;

(b) There exists a rational $q > 0$ with $\mathtt{D}_b \leq \mathtt{D}_{a-q}$.

Proof Assuming (a), there exists a rational $q > 0$ with $\mathtt{D}_{c-q} \vee \mathtt{D}_{a-q} = 1$ by [28, Corollary 1.7]. Hence $\mathtt{D}_c \vee \mathtt{D}_{a-q} = 1$, so $\mathtt{D}_b = \mathtt{D}_b \wedge (\mathtt{D}_c \vee \mathtt{D}_{a-q}) = \mathtt{D}_b \wedge \mathtt{D}_{a-q} \leq \mathtt{D}_{a-q}$, establishing (b). For the converse, choose $\mathtt{D}_c = D_{q-a}$. □

3.19 In view of the above lemma, we henceforth write $\mathtt{D}_b \ll \mathtt{D}_a$ if there exists a rational $q > 0$ such that $\mathtt{D}_b \leq \mathtt{D}_{a-q}$, and note that the regularity condition (8) just states that the frame $\mathcal{O}(\Sigma(A))$ is regular [28].

We recall that an *ideal* of a lattice L is a lower set $U \subseteq L$ that is closed under finite joins; the collection of all ideals in L is denoted by $\mathrm{Idl}(L)$. An ideal U of a distributive lattice L is *regular* when $\twoheaddownarrow x \subseteq U$ implies $x \in U$. Any ideal U can be turned into a regular ideal $\overline{U}$ by means of the closure operator $\overline{(\_)} \colon DL \to DL$ defined by $\overline{U} = \{x \in L \mid \forall_{y \in L} . y \ll x \Rightarrow y \in U\}$ [24], with a canonical inclusion as in Proposition 2.13.

3.20 Theorem *The Gelfand spectrum $\mathcal{O}(\Sigma(A))$ of a commutative C*-algebra A is isomorphic to the frame $\mathrm{RIdl}(L_A)$ of all regular ideals of L_A, i.e.*

$$\mathcal{O}(\Sigma(A)) \cong \{U \in \mathrm{Idl}(L_A) \mid (\forall_{\mathtt{D}_b \in L_A} . \mathtt{D}_b \ll \mathtt{D}_a \Rightarrow \mathtt{D}_b \in U) \Rightarrow \mathtt{D}_a \in U\}.$$

In this realisation, the canonical map $f \colon L_A \to \mathcal{O}(\Sigma(A))$ is given by

$$f(\mathtt{D}_a) = \{\mathtt{D}_c \in L_A \mid \forall_{\mathtt{D}_b \in L_A} . \mathtt{D}_b \ll \mathtt{D}_c \Rightarrow \mathtt{D}_b \leq \mathtt{D}_a\}.$$

PROOF For a commutative C*-algebra A, the lattice L_A is *strongly normal* [28, Theorem 1.11], and hence *normal*. (A distributive lattice is *normal* if for all b_1, b_2 with $b_1 \vee b_2 = 1$ there are c_1, c_2 such that $c_1 \wedge c_2 = 0$ and $c_1 \vee b_1 = 1$ and $c_2 \vee b_2 = 1$.) By [24, Theorem 27], regular ideals in a normal distributive lattice form a compact regular frame. The result now follows from [28, Theorem 1.11]. $\square$

3.21 Corollary *The Gelfand spectrum of a commutative C*-algebra A is given by*

$$\mathcal{O}(\Sigma(A)) \cong \{U \in \mathrm{Idl}(L_A) \mid \forall_{a \in A_{\mathrm{sa}}} \forall_{q>0}. \mathtt{D}_{a-q} \in U \Rightarrow \mathtt{D}_a \in U\}.$$

PROOF By combining Lemma 3.18 with Theorem 3.20. $\square$

The following theorem is the key to explicitly determining the external description of the Gelfand spectrum $\mathcal{O}(\Sigma(A))$ of a C*-algebra A in a topos.

3.22 Theorem *For a commutative C*-algebra A, define a covering relation $\lhd$ on L_A by $x \lhd U$ iff $f(x) \leq \bigvee f(U)$, in the notation of Theorem 3.20.*

(a) One has $\mathcal{O}(\Sigma(A)) \cong \mathcal{F}(L_A, \lhd)$, under which $\mathcal{D}_a \mapsto \downarrow\mathtt{D}_a$.

(b) Then $\mathtt{D}_a \lhd U$ iff for all rational $q > 0$ there is a (Kuratowski) finite $U_0 \subseteq U$ such that $\mathtt{D}_{a-q} \leq \bigvee U_0$.

PROOF Part (a) follows from Proposition 2.13. For (b), first assume $\mathtt{D}_a \lhd U$, and let $q \in \mathbb{Q}$ satisfy $q > 0$. From (the proof of) Lemma 3.16 we have $\mathtt{D}_a \vee \mathtt{D}_{q-a} = 1$, whence $\bigvee f(U) \vee f(\mathtt{D}_{q-a}) = 1$. Because $\mathcal{O}(\Sigma(A))$ is compact, there is a finite $U_0 \subseteq U$ for which $\bigvee f(U_0) \vee f(\mathtt{D}_{q-a}) = 1$. Since $f(\mathtt{D}_a) = 1$ if and only if $\mathtt{D}_a = 1$ by Theorem 3.20, we have $\mathtt{D}_b \vee \mathtt{D}_{q-a} = 1$, where $\mathtt{D}_b = \bigvee U_0$. By (4), we have $\mathtt{D}_{a-q} \wedge \mathtt{D}_{q-a} = 0$, and hence

$$\mathtt{D}_{a-q} = \mathtt{D}_{a-q} \wedge 1 = \mathtt{D}_{a-q} \wedge (\mathtt{D}_b \vee \mathtt{D}_{q-a}) = \mathtt{D}_{a-q} \wedge \mathtt{D}_b \leq \mathtt{D}_b = \bigvee U_0.$$

For the converse, notice that $f(\mathtt{D}_a) \leq \bigvee\{f(\mathtt{D}_{a-q}) \mid q \in \mathbb{Q}, q > 0\}$ by construction. So from the assumption we have $f(\mathtt{D}_a) \leq \bigvee f(U)$ and hence $\mathtt{D}_a \lhd U$. $\square$

4 Bohrification

This section explains the technique of *Bohrification*. For a (generally) noncommutative C*-algebra A, Bohrification constructs a topos in which A becomes commutative. More precisely, to any C*-algebra A, we associate a particular commutative C*-algebra $\underline{A}$ in the Kripke topos $[\mathcal{C}(A), \mathbf{Set}]$, where $\mathcal{C}(A)$ is the set of commutative C*-subalgebras of A. By Gelfand duality, the commutative C*-algebra $\underline{A}$ has a spectrum $\Sigma(\underline{A})$, which is a locale in $[\mathcal{C}(A), \mathbf{Set}]$.

4.1 To introduce the idea, we outline the general method of Bohrification. We will subsequently give concrete examples.

Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be geometric theories whose variables range over only one type, apart from constructible types such as $\mathbb{N}$ and $\mathbb{Q}$. Suppose that $\mathfrak{T}_1$ is a subtheory of $\mathfrak{T}_2$. There is a functor $\mathcal{C}\colon \mathbf{Model}(\mathfrak{T}_1, \mathbf{Set}) \to \mathbf{Poset}$, defined on objects as $\mathcal{C}(A) = \{C \subseteq A \mid C \in$

$\mathbf{Model}(\mathfrak{T}_2, \mathbf{Set})\}$, ordered by inclusion. On a morphism $f\colon A \to B$ of $\mathbf{Model}(\mathfrak{T}_1, \mathbf{Set})$, the functor $\mathcal{C}$ acts as $\mathcal{C}(f)\colon \mathcal{C}(A) \to \mathcal{C}(B)$ by the direct image $C \mapsto f(C)$. Hence, there is a functor $\mathcal{T}\colon \mathbf{Model}(\mathfrak{T}_1, \mathbf{Set}) \to \mathbf{Topos}$, defined on objects by $\mathcal{T}(A) = [\mathcal{C}(A), \mathbf{Set}]$ and determined on morphisms by $\mathcal{T}(f)^* = (\_) \circ \mathcal{C}(f)$. Define the canonical object $\underline{A} \in \mathcal{T}(A)$ by $\underline{A}(C) = C$, acting on a morphism $D \subseteq C$ of $\mathcal{C}(A)$ as the inclusion $\underline{A}(D) \hookrightarrow \underline{A}(C)$. Then $\underline{A}$ is a model of $\mathfrak{T}_2$ in the Kripke topos $\mathcal{T}(A)$ by Lemma 3.13.

4.2 Example Let $\mathfrak{T}_1$ be the theory of groups, and $\mathfrak{T}_2$ the theory of Abelian groups. Both are geometric theories, and $\mathfrak{T}_1$ is a subtheory of $\mathfrak{T}_2$. Then $\mathcal{C}(G)$ is the collection of Abelian subgroups C of G, ordered by inclusion, and the functor $\underline{G}\colon C \mapsto C$ is an Abelian group in $\mathcal{T}(G) = [\mathcal{C}(G), \mathbf{Set}]$.

This resembles the so-called "microcosm principle", according to which structure of an internal entity depends on similar structure of the ambient category [5, 50].

We now turn to the setting of our interest: (commutative) C*-algebras. As the theory of C*-algebras is not geometric, it does not follow from the arguments of 4.1 that $\underline{A}$ will be a commutative C*-algebra in $\mathcal{T}(A)$. Theorem 4.8 below will show that the latter is nevertheless true.

4.3 Proposition *There is a functor* $\mathcal{C}\colon \mathbf{CStar} \to \mathbf{Poset}$*, defined on objects as*

$$\mathcal{C}(A) = \{C \in \mathbf{cCStar} \mid C \text{ is a C*-subalgebra of } A\},$$

ordered by inclusion. Its action $\mathcal{C}(f)\colon \mathcal{C}(A) \to \mathcal{C}(B)$ *on a morphism* $f\colon A \to B$ *of* $\mathbf{CStar}$ *is the direct image* $C \mapsto f(C)$*. Hence, there is a functor* $\mathcal{T}\colon \mathbf{CStar} \to \mathbf{Topos}$*, defined by* $\mathcal{T}(A) = [\mathcal{C}(A), \mathbf{Set}]$ *on objects and* $\mathcal{T}(f)^* = (\_) \circ \mathcal{C}(f)$ *on morphisms.*

Proof It suffices to show that $\mathcal{T}(f)^*$ is part of a geometric morphism, which follows from [69, Theorem VII.2.2]. □

4.4 Example The following example determines $\mathcal{C}(A)$ for $A = \mathbf{Hilb}(\mathbb{C}^2, \mathbb{C}^2)$, the C*-algebra of complex 2 by 2 matrices. Any C*-algebra has a single one-dimensional commutative C*-subalgebra, namely $\mathbb{C}$, the scalar multiples of the unit. Furthermore, any two-dimensional C*-subalgebra is generated by a pair of orthogonal one-dimensional projections. The one-dimensional projections in A are of the form

$$p(x, y, z) = \frac{1}{2}\begin{pmatrix} 1+x & y+iz \\ y-iz & 1-x \end{pmatrix}, \tag{9}$$

where $(x, y, z) \in \mathbb{R}^3$ satisfies $x^2 + y^2 + z^2 = 1$. Thus the one-dimensional projections in A are precisely parametrised by S^2. Since $1 - p(x, y, z) = p(-x, -y, -z)$, and pairs $(p, 1-p)$ and $(1-p, p)$ define the same C*-subalgebra, the two-dimensional elements of $\mathcal{C}(A)$ are parametrised by $S^2/\!\sim$, where $(x, y, z) \sim (-x, -y, -z)$. This space, in turn, is homeomorphic with the real projective plane $\mathbb{RP}^2$, *i.e.* the set of lines in $\mathbb{R}^3$ passing through the origin.[1] Parametrising $\mathcal{C}(A) \cong \{\mathbb{C}\} + \mathbb{RP}^2$, a point $[x, y, z] \in S^2/\!\sim$ then corresponds to the C*-algebra $C_{[x,y,z]}$ generated by the projections $\{p(x, y, z), p(-x, -y, -z)\}$. The order of $\mathcal{C}(A)$ is flat: $C < D$ iff $C = \mathbb{C}$.

[1] This space has an interesting topology that is quite different from the Alexandrov topology on $\mathcal{C}(A)$, but that we nevertheless ignore.

4.5 Example We now generalise the previous example to $A = \mathbf{Hilb}(\mathbb{C}^n, \mathbb{C}^n)$ for any $n \in \mathbb{N}$. In general, one has $\mathcal{C}(A) = \coprod_{k=1}^n \mathcal{C}(k,n)$, where $\mathcal{C}(k,n)$ denotes the collection of all k-dimensional commutative unital C*-subalgebras of A. To parametrise $\mathcal{C}(k,n)$, we first show that each of its elements C is a unitary rotation $C = UDU^*$, where $U \in SU(n)$ and D is some subalgebra contained in the algebra of all diagonal matrices. This follows from the case $k = n$, since each element of $\mathcal{C}(k,n)$ with $k < n$ is contained in some maximal commutative subalgebra. For $k = n$, note that $C \in \mathcal{C}(n,n)$ is generated by n mutually orthogonal projections $p_1, \ldots, p_n$ of rank 1. Each p_i has a single unit eigenvector u_i with eigenvalue 1; its other eigenvalues are 0. Put these u_i as columns in a matrix, called U. Then $U^* p_i U$ is diagonal for all i, for if (e_i) is the standard basis of $\mathbb{C}^n$, then $Ue_i = u_i$ for all i and hence $U^* p_i U e_i = U^* p_i u_i = U^* u_i = e_i$, while for $i \neq j$ one finds $U^* p_i U e_j = 0$. Hence the matrix $U^* p_i U$ has a one at location ii and zero's everywhere else. All other elements $a \in C$ are functions of the p_i, so that $U^* a U$ is equally well diagonal. Hence $C = UD_nU^*$, with D_n the algebra of all diagonal matrices. Thus

$$\mathcal{C}(n,n) = \{UD_nU^* \mid U \in SU(n)\},$$

with $D_n = \{\mathrm{diag}(a_1, \ldots, a_n) \mid a_i \in \mathbb{C}\}$, and $\mathcal{C}(k,n)$ for $k < n$ is obtained by partitioning $\{1, \ldots, n\}$ into k nonempty parts and demanding $a_i = a_j$ for i, j in the same part. However, because of the conjugation with arbitrary $U \in SU(n)$, two such partitions induce the same subalgebra precisely when they permute parts of equal size. Such permutations may be handled using Young tableaux [42]. As the size of a part is of more interest than the part itself, we define

$$Y(k,n) = \{(i_1, \ldots, i_k) \mid 0 < i_1 < i_2 < \cdots < i_k = n, \;\; i_{j+1} - i_j \leq i_j - i_{j-1}\}$$

(where $i_0 = 0$) as the set of partitions inducing different subalgebras. Hence

$$\begin{aligned}\mathcal{C}(k,n) \cong \big\{(p_1, \ldots, p_k) : p_j \in \mathrm{Proj}(A), \;\; &(i_1, \ldots, i_k) \in Y(k,n) \\ &\mid \dim(\mathrm{Im}(p_j)) = i_j - i_{j-1}, \;\; p_j \wedge p_{j'} = 0 \text{ for } j \neq j'\big\}.\end{aligned}$$

Now, since d-dimensional orthogonal projections in $\mathbb{C}^n$ bijectively correspond to the d-dimensional (closed) subspaces of $\mathbb{C}^n$ they project onto, we can write

$$\begin{aligned}\mathcal{C}(k,n) \cong \big\{(V_1, \ldots, V_k) : (i_1, \ldots, i_k) \in Y(k,n), &V_j \in \mathrm{Gr}(i_j - i_{j-1}, n) \\ &\mid V_j \cap V_{j'} = 0 \text{ for } j \neq j'\big\},\end{aligned}$$

where $\mathrm{Gr}(d,n) = U(n)/(U(d) \times U(n-d))$ is the well-known Grassmannian, *i.e.* the set of all d-dimensional subspaces of $\mathbb{C}^n$ [46]. In terms of the partial flag manifold

$$\mathrm{G}(i_1, \ldots, i_k; n) = \prod_{j=1}^k \mathrm{Gr}(i_j - i_{j-1}, n - i_{j-1}),$$

for $(i_1, \ldots, i_k) \in Y(k,n)$ (see [42]), we finally obtain

$$\mathcal{C}(k,n) \cong \{V \in \mathrm{G}(i;n) : i \in Y(k,n)\}/\sim,$$

where $i \sim i'$ if one arises from the other by permutations of equal-sized parts.

This is indeed generalises the previous example $n = 2$. First, for any n the set $\mathcal{C}(1, n)$ has a single element, as there is only one Young tableau for $k = 1$. Second, we have $Y(2, 2) = \{(1, 2)\}$, so that

$$\mathcal{C}(2,2) \cong (\mathrm{Gr}(1,2) \times \mathrm{Gr}(1,1))/S(2) \cong \mathrm{Gr}(1,2)/S(2) \cong \mathbb{CP}^1/S(2) \cong \mathbb{RP}^2.$$

4.6 Definition Let A be a C*-algebra. Define the functor $\underline{A}\colon \mathcal{C}(A) \to \mathbf{Set}$ by acting on objects as $\underline{A}(C) = C$, and acting on morphisms $C \subseteq D$ of $\mathcal{C}(A)$ as the inclusion $\underline{A}(C) \hookrightarrow \underline{A}(D)$. We call $\underline{A}$, or the process of obtaining it, the *Bohrification* of A.

4.7 Convention We will underline entities internal to $\mathcal{T}(A)$ to distinguish between the internal and external points of view.

The particular object $\underline{A}$ turns out to be a commutative C*-algebra in the topos $\mathcal{T}(A)$, even though the theory of C*-algebras is not geometric.

4.8 Theorem *Operations inherited from A make $\underline{A}$ a commutative C*-algebra in $\mathcal{T}(A)$. More precisely, $\underline{A}$ is a vector space over the complex field* $\mathrm{Pt}(\underline{\mathbb{C}})\colon C \mapsto \mathbb{C}$ *by*

$$\begin{array}{lll} 0\colon \underline{1} \to \underline{A}, & +\colon \underline{A} \times \underline{A} \to \underline{A}, & \cdot : \mathrm{Pt}(\underline{\mathbb{C}}) \times \underline{A} \to \underline{A}, \\ 0_C(*) = 0, & a +_C b = a + b, & z \cdot_C a = z \cdot a, \end{array}$$

and an involutive algebra through

$$\begin{array}{ll} \cdot : \underline{A} \times \underline{A} \to \underline{A}, & (\_)^*\colon \underline{A} \to \underline{A} \\ a \cdot_C b = a \cdot b, & (a^*)_C = a^*. \end{array}$$

The norm relation is the subobject $N \in \mathrm{Sub}(\underline{A} \times \underline{\mathbb{Q}^+})$ given by

$$N_C = \{(a, q) \in C \times \mathbb{Q}^+ \mid \|a\| < q\}.$$

Proof Recall (Definition 3.3) that a pre-semi-C*-algebra is a C*-algebra that is not necessarily Cauchy complete, and whose seminorm is not necessarily a norm. Since the theory of pre-semi-C*-algebras is geometric, Lemma 3.13 shows that $\underline{A}$ is a commutative pre-semi-C*-algebra in $\mathcal{T}(A)$, as in 4.1. Let us prove that $\underline{A}$ is in fact a pre-C*-algebra, *i.e.* that the seminorm is a norm. It suffices to show that $C \Vdash \forall_{a \in \underline{A}_{\mathrm{sa}}} \forall_{q \in \underline{Q^+}}.(a, q) \in N \Rightarrow a = 0$ for all $C \in \mathcal{C}(A)$. By 2.24, this means

$$\begin{aligned} &\text{for all } C' \supseteq C \text{ and } a \in C', \text{ if } C' \Vdash \forall_{q \in \underline{\mathbb{Q}^+}}.(a,q) \in N, \text{ then } C' \Vdash a = 0, \\ i.e.\ &\text{for all } C' \supseteq C \text{ and } a \in C', \text{ if } C'' \Vdash (a,q) \in N \text{ for all } C'' \supseteq C' \text{ and } q \in \mathbb{Q}^+, \\ &\qquad\qquad \text{then } C' \Vdash a = 0, \\ i.e.\ &\text{for all } C' \supseteq C \text{ and } a \in C', \text{ if } \|a\| = 0, \text{ then } a = 0. \end{aligned}$$

But this holds, since every C' is a C*-algebra.

Finally, we prove that $\underline{A}$ is in fact a C*-algebra. Since the axiom of dependent choice holds in $\mathcal{T}(A)$ [41], it suffices to prove that every *regular* Cauchy sequence converges, where a sequence (x_n) is regular Cauchy when $\|x_n - x_m\| \leq 2^{-n} + 2^{-m}$ for all $n, m \in \mathbb{N}$. Thus we need to prove

$$\begin{aligned}
& C \Vdash \forall_{n,m\in\underline{\mathbb{N}}}. \|x_n - x_m\| \leq 2^{-n} + 2^{-m} \Rightarrow \exists_{x\in\underline{A}}. \forall_{n\in\underline{\mathbb{N}}}. \|x - x_n\| \leq 2^{-n}, \\
\textit{i.e.}\ \ & \text{for all } C' \supseteq C, \text{ if } C' \Vdash (\forall_{n,m\in\underline{\mathbb{N}}}. \|x_n - x_m\| \leq 2^{-n} + 2^{-m}), \\
& \qquad\qquad\qquad \text{then } C' \Vdash \exists_{x\in\underline{A}}. \forall_{n\in\underline{\mathbb{N}}}. \|x - x_n\| \leq 2^{-n}, \\
\textit{i.e.}\ \ & \text{for all } C' \supseteq C, \text{ if } C' \Vdash \text{“}(x)_n \text{ is regular”, then } C' \Vdash \text{“}(x)_n \text{ converges”}.
\end{aligned}$$

Once again, this holds because every C' is a C*-algebra. □

4.9 Applying 3.8 to the commutative C*-algebra $\underline{A}$ in the topos $\mathcal{T}(A)$, we obtain a locale $\underline{\Sigma}(\underline{A})$ in that topos. As argued in the Introduction, $\underline{\Sigma}(\underline{A})$ is the 'state space' carrying the logic of the physical system whose observable algebra is A.

An important property of $\underline{\Sigma}(\underline{A})$ is that it is typically highly non-spatial, as the following theorem proves. This theorem is a localic extension of a topos-theoretic reformulation of the Kochen-Specker theorem [62] due to Jeremy Butterfield and Chris Isham [19–22].

4.10 Theorem *Let H be a Hilbert space with $\dim(H) > 2$, and $A = \mathbf{Hilb}(H,H)$. The locale $\underline{\Sigma}(\underline{A})$ has no points.*

Proof A point $\underline{\rho}\colon \underline{1} \to \underline{\Sigma}(\underline{A})$ of the locale $\underline{\Sigma}(\underline{A})$ (see 2.8) may be combined with $a \in \underline{A}_{\mathrm{sa}}$, with Gelfand transform $\hat{a}\colon \underline{\Sigma}(\underline{A}) \to \underline{\mathbb{R}}$ as in 3.10, so as to produce a point $\hat{a} \circ \underline{\rho}\colon \underline{1} \to \underline{\mathbb{R}}$ of the locale $\underline{\mathbb{R}}$. This yields a map $\underline{V}_\rho\colon \underline{A}_{\mathrm{sa}} \to \mathrm{Pt}(\underline{\mathbb{R}})$, which turns out to be a multiplicative functional [6, 8, 28]. Being a morphism in $\mathcal{T}(A)$, the map $\underline{V}_\rho$ is a natural transformation, with components $\underline{V}_\rho(C)\colon \underline{A}_{\mathrm{sa}}(C) \to \mathrm{Pt}(\underline{\mathbb{R}})(C)$; by Definition 4.6 and Example 2.33, this is just $\underline{V}_\rho(C)\colon C_{\mathrm{sa}} \to \mathbb{R}$. Hence one has a multiplicative functional $\underline{V}_\rho(C)$ for each $C \in \mathcal{C}(A)$ in the usual sense, with the naturality, or 'noncontextuality', property that if $C \subseteq D$, then the restriction of $\underline{V}_\rho(D)$ to C_{sa} is $\underline{V}_\rho(C)$. But that is precisely the kind of function on $\mathbf{Hilb}(H,H)$ of which the Kochen-Specker theorem proves the nonexistence [62]. □

4.11 The previous theorem holds for more general C*-algebras than $\mathbf{Hilb}(H,H)$ (for large enough Hilbert spaces H); see [36] for results on von Neumann algebras. A C*-algebra A is called *simple* when its closed two-sided ideals are trivial, and *infinite* when there is an $a \in A$ with $a^*a = 1$ but $aa^* \neq 1$ [32]. A simple infinite C*-algebra does not admit a dispersion-free quasi-state [49], whence the previous theorem holds for such C*-algebras as well.

The rest of this section is devoted to describing the structure of the Gelfand spectrum $\underline{\Sigma}(\underline{A})$ of the Bohrification $\underline{A}$ of A from the external point of view.

4.12 Theorem *For a C*-algebra A and each $C \in \mathcal{C}(A)$, one has $\underline{L}_{\underline{A}}(C) = L_C$. Moreover $\underline{L}_{\underline{A}}(C \subseteq D)\colon L_C \to L_D$ is a frame morphism that maps each generator $\mathtt{D}_c$ for $c \in C_{\mathrm{sa}}$ to the same generator for the spectrum of D.*

Proof This follows from Lemma 3.13 and Proposition 3.17. □

4.13 The next corollary interprets $\mathtt{D}_a \lhd U$ in our situation, showing that also the covering relation $\lhd$ can be computed locally. To do so, we introduce the notation $\underline{L}_{\underline{A}|\uparrow C}$ for the restriction of the functor $\underline{L}_{\underline{A}} \colon \mathcal{C}(A) \to \mathbf{Set}$ to $\uparrow C \subseteq \mathcal{C}(A)$. Then $\underline{\Omega}^{\underline{L}_{\underline{A}}}(C) \cong \mathrm{Sub}(\underline{L}_{\underline{A}|\uparrow C})$ by [69, Section II.8]. Hence, by Kripke-Joyal semantics, *cf.* 2.24, the formal variables $\mathtt{D}_a$ and U in $C \Vdash \mathtt{D}_a \lhd U$ for $C \in \mathcal{C}(A)$ are to be instantiated with actual elements $D_c \in L_C = \underline{L}_{\underline{A}}(C)$ and a subfunctor $\underline{U} \colon \uparrow C \to \mathbf{Set}$ of $\underline{L}_{\underline{A}|\uparrow C}$. Since $\lhd$ is a subfunctor of $\underline{L}_{\underline{A}} \times \underline{\mathcal{P}}(\underline{L}_{\underline{A}})$, we can speak of $\lhd_C$ for $C \in \mathcal{C}(A)$ as the relation $\underline{L}_{\underline{A}}(C) \times \underline{\mathcal{P}}(\underline{L}_{\underline{A}})$ induced by evaluation at C.

4.14 Corollary *The covering relation $\lhd$ of Theorem 3.22 is computed locally. That is, for $C \in \mathcal{C}(A)$, $D_c \in L_C$ and $\underline{U} \in \mathrm{Sub}(\underline{L}_{\underline{A}|\uparrow C})$, the following are equivalent:*

(a) $C \Vdash \mathtt{D}_a \lhd U(D_c, \underline{U})$*;*

(b) $D_c \lhd_C \underline{U}(C)$*;*

(c) for every rational $q > 0$ there is a finite $U_0 \subseteq \underline{U}(C)$ with $D_{c-q} \leq \bigvee U_0$.

Proof The equivalence of (b) and (c) follows from Theorem 3.22. We prove the equivalence of (a) and (c). Assume, without loss of generality, that $\bigvee U_0 \in U$, so that U_0 may be replaced by $\mathtt{D}_b = \bigvee U_0$. Hence the formula $\mathtt{D}_a \lhd U$ in (a) means

$$\forall_{q>0} \exists_{\mathtt{D}_b \in L_A}.(\mathtt{D}_b \in U \wedge \mathtt{D}_{a-q} \leq \mathtt{D}_b).$$

We interpret this formula step by step, as in 2.24. First, $C \Vdash (\mathtt{D}_a \in U)(D_c, \underline{U})$ iff for all $D \supseteq C$ one has $D_c \in \underline{U}(D)$. As $\underline{U}(C) \subseteq \underline{U}(D)$, this is the case iff $D_c \in \underline{U}(C)$. Also one has $C \Vdash (\mathtt{D}_b \leq \mathtt{D}_a)(D_{c'}, D_c)$ iff $D_{c'} \leq D_c$ in L_C. Hence, $C \Vdash (\exists_{\mathtt{D}_b \in L_A}.\mathtt{D}_b \in U \wedge \mathtt{D}_{a-q} \leq \mathtt{D}_b)(D_c, \underline{U})$ iff there is $D_{c'} \in \underline{U}(C)$ with $D_{c-q} \leq D_{c'}$. Finally, $C \Vdash (\forall_{q>0} \exists_{\mathtt{D}_b \in L_A}.\mathtt{D}_b \in U \wedge \mathtt{D}_{a-q} \leq \mathtt{D}_b)(D_c, \underline{U})$ iff for all $D \supseteq C$ and all rational $q > 0$ there is $D_d \in \underline{U}(D)$ such that $D_{c-q} \leq D_d$, where $D_c \in L_C \subseteq L_D$ by Theorem 4.12 and $\underline{U} \in \mathrm{Sub}(\underline{L}_{\underline{A}|\uparrow C}) \subseteq \mathrm{Sub}(\underline{L}_{\underline{A}|\uparrow D})$ by restriction. This holds at all $D \supseteq C$ iff it holds at C, because $\underline{U}(C) \subseteq \underline{U}(D)$, whence one can take $D_d = D_{c'}$. □

4.15 The following theorem explicitly determines the Gelfand spectrum $\underline{\Sigma}(\underline{A})$ from the external point of view. It turns out that the functor $\underline{\Sigma}(\underline{A})$ is completely determined by its value $\underline{\Sigma}(\underline{A})(\mathbb{C})$ at the least element $\mathbb{C}$ of $\mathcal{C}(A)$. Therefore, we abbreviate $\underline{\Sigma}(\underline{A})(\mathbb{C})$ by Σ_A, and call it the *Bohrified state space* of A.

4.16 Theorem *For a C^*-algebra A:*

(a) At $C \in \mathcal{C}(A)$, the set $\mathcal{O}(\underline{\Sigma}(\underline{A}))(C)$ consists of the subfunctors $\underline{U} \in \mathrm{Sub}(\underline{L}_{\underline{A}|\uparrow C})$ satisfying $D_d \lhd_D \underline{U}(D) \Rightarrow D_d \in \underline{U}(D)$ for all $D \supseteq C$ and $D_d \in L_D$.

(b) In particular, the set $\mathcal{O}(\underline{\Sigma}(\underline{A}))(\mathbb{C})$ consists of the subfunctors $\underline{U} \in \mathrm{Sub}(\underline{L}_{\underline{A}})$ satisfying $D_c \lhd_C \underline{U}(C) \Rightarrow D_c \in \underline{U}(C)$ for all $C \in \mathcal{C}(A)$ and $D_c \in L_C$.

(c) The action $\mathcal{O}(\underline{\Sigma}(\underline{A})) \to \mathcal{O}(\underline{\Sigma}(\underline{A}))$ of $\mathcal{O}(\underline{\Sigma}(\underline{A}))$ on a morphism $C \subseteq D$ of $\mathcal{C}(A)$ is given by truncating $\underline{U} \colon \uparrow C \to \mathbf{Set}$ to $\uparrow D$.

(d) *The external description of $\mathcal{O}(\underline{\Sigma}(\underline{A}))$ is the frame morphism*

$$f^{-1}\colon \mathcal{O}(\mathrm{Alx}(\mathcal{C}(A))) \to \mathcal{O}(\underline{\Sigma}(\underline{A}))(\mathbb{C}),$$

given on basic opens $\uparrow D \in \mathcal{O}(\mathrm{Alx}(\mathcal{C}(A)))$ by

$$f^{-1}(\uparrow D)(E) = \begin{cases} L_E & \text{if } E \supseteq D, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof By Theorem 3.22(a) and (2), $\mathcal{O}(\underline{\Sigma}(\underline{A}))$ is the subobject of $\underline{\Omega}^{\underline{L}_A}$ defined by the formula $\forall_{\mathtt{D}_a \in L_A}.\mathtt{D}_a \lhd U \Rightarrow \mathtt{D}_a \in U$. As in 4.13, elements $\underline{U} \in \mathcal{O}(\underline{\Sigma}(\underline{A}))(C)$ may be identified with subfunctors of $\underline{L}_{\underline{A}|\uparrow C}$. Hence, by Corollary 4.14, we have $\underline{U} \in \mathcal{O}(\underline{\Sigma}(\underline{A}))$ if and only if

$$\forall_{D \supseteq C} \forall_{D_d \in L_D} \forall_{E \supseteq D}. D_d \lhd_E \underline{U}(E) \Rightarrow D_d \in \underline{U}(E),$$

where D_d is regarded as an element of L_E. This is equivalent to the apparently weaker condition

$$\forall_{D \supseteq C} \forall_{D_d \in L_D}. D_d \lhd_D \underline{U}(D) \Rightarrow D_d \in \underline{U}(D),$$

because the latter applied at $D = E$ actually implies the former condition since $D_d \in L_D$ also lies in L_E. This proves (a), (b) and (c). Part (d) follows from Example 2.28. $\square$

5 Projections

This section compares the quantum state spaces $\mathcal{O}(\underline{\Sigma}(\underline{A}))$ with quantum logic in the sense of [11]. In the setting of operator algebras, this more traditional quantum logic is concerned with projections; we denote the set of projections of a C*-algebra A by

$$\mathrm{Proj}(A) = \{p \in A \mid p^* = p = p \circ p\}.$$

A generic C*-algebra may not have enough projections: for example, if A is a commutative C*-algebra whose Gelfand spectrum $\Sigma(A)$ is connected, then A has no projections except for 0 and 1. Hence we need to specialise to C*-algebras that have enough projections. The best-know such class consist of von Neumann algebras, but in fact the most general class of C*-algebras that are generated by their projections *and* can easily be Bohrified turns out to consist of so-called *Rickart C*-algebras*. To motivate this choice, we start by recalling several types of C*-algebras and known results about their spectra.

5.1 Definition Let A be a C*-algebra. Define $R(S) = \{a \in A \mid \forall_{s \in S}. sa = 0\}$ to be the *right annihilator* of some subset $S \subseteq A$. Then A is said to be:

(a) a *von Neumann algebra* if it is the dual of some Banach space [75];

(b) an *AW*-algebra* if for each nonempty $S \subseteq A$ there is a $p \in \mathrm{Proj}(A)$ satisfying $R(S) = pA$ [61];

(c) a *Rickart C*-algebra* if for each $x \in A$ there is a $p \in \mathrm{Proj}(A)$ satisfying $R(\{x\}) = pA$ [73];

(d) a *spectral C*-algebra* if for each $a \in A^+$ and each $r, s \in (0, \infty)$ with $r < s$, there is a $p \in \mathrm{Proj}(A)$ satisfying $ap \geq rp$ and $a(1-p) \leq s(1-p)$ [82].

In all cases, the projection p turns out to be unique. Each class contains the previous one(s).

5.2 To prepare for what follows, we recall the *Stone representation theorem* [56]. This theorem states that any Boolean algebra B (in the topos **Set**) is isomorphic to the lattice $\mathcal{B}(X)$ of clopen subsets of a *Stone space* X, *i.e.* a compact Hausdorff space that is *totally* disconnected, in that its only connected subsets are singletons. Equivalently, a Stone space is compact, T_0, and has a basis of clopen sets. The space X is uniquely determined by B up to homeomorphism, and hence may be written $\hat{\Sigma}(B)$; one model for it is given by the set of all maximal filters in B, topologized by declaring that for each $b \in B$, the set of all maximal filters containing b is a basic open for $\hat{\Sigma}(B)$. Another description is based on the isomorphism

$$\mathcal{O}(\hat{\Sigma}(B)) \cong \mathrm{Idl}(B) \tag{10}$$

of locales, where the left-hand side is the topology of $\hat{\Sigma}(B)$, and the right-hand side is the ideal completion of B (seen as a distributive lattice). This leads to an equivalent model of $\hat{\Sigma}(B)$, namely as $\mathrm{Pt}(\mathrm{Idl}(B))$ with its canonical topology (cf. 2.8); see Corollaries II.4.4 and II.3.3 and Proposition II.3.2 in [56]. Compare this with Theorem 3.20, which states that the Gelfand spectrum $\Sigma(A)$ of a unital commutative C*-algebra A may be given as

$$\mathcal{O}(\Sigma(A)) \cong \mathrm{RIdl}(L_A). \tag{11}$$

The analogy between (10) and (11) is more than an optical one. A Stone space X gives rise to a Boolean algebra $\mathcal{B}(X)$ as well as to a commutative C*-algebra $C(X, \mathbb{C})$. Conversely, if A is a commutative C*-algebra, then $\mathrm{Proj}(A)$ is isomorphic with the Boolean lattice $\mathcal{B}(\Sigma(A))$ of clopens in $\Sigma(A)$. If we regard $\Sigma(A)$ as consisting of characters as in 3.9, then this isomorphism is given by

$$\begin{aligned} \mathrm{Proj}(A) &\xrightarrow{\cong} \mathcal{B}(\Sigma(A)) \\ p &\mapsto \{\sigma \in \Sigma(A) \mid \sigma(p) \neq 0\}, \end{aligned}$$

where $\hat{p}$ is the Gelfand transform of p as in 3.10.

To start with a familiar case, a von Neumann algebra A is commutative if and only if $\mathrm{Proj}(A)$ is a Boolean algebra [72, Proposition 4.16]. In that case, the Gelfand spectrum $\Sigma(A)$ of A may be identified with the Stone spectrum of $\mathrm{Proj}(A)$; passing to the respective topologies, in view of (10) we therefore have

$$\mathcal{O}(\Sigma(A)) \cong \mathrm{Idl}(\mathrm{Proj}(A)). \tag{12}$$

In fact, this holds more generally in two different ways: firstly, it is true for the larger class of Rickart C*-algebras, and secondly, the proof is constructive and hence the result holds in arbitrary toposes; see Theorem 5.15 below. As to the first point, in **Set** (where the locales in question are spatial), we may conclude from this theorem that for a commutative Rickart C*-algebra A one has a homeomorphism

$$\Sigma(A) \cong \hat{\Sigma}(\mathrm{Proj}(A)), \tag{13}$$

a result that so far had only been known for von Neumann algebras.

5.3 One reason for dissatisfaction with von Neumann algebras is that the above correspondence between Boolean algebras and commutative von Neumann algebras is not bijective. Indeed, if A is a commutative von Neumann algebra, then $\mathrm{Proj}(A)$ is complete, so that $\Sigma(A)$ is not merely Stone but *Stonean*, *i.e.* compact, Hausdorff and *extremely* disconnected, in that the closure of every open set is open. (The Stone spectrum of a Boolean algebra L is Stonean if and only if L is complete.) But commutative von Neumann algebras do not correspond bijectively to complete Boolean algebras either, since the Gelfand spectrum of a commutative von Neumann algebra is not merely Stone but has the stronger property of being *hyperstonean*, in that it admits sufficiently many positive normal measures [84, Definition 1.14]. Indeed, a commutative C*-algebra A is a von Neumann algebra if and only if its Gelfand spectrum (and hence the Stone spectrum of its projection lattice) is hyperstonean.

5.4 Theorem *A commutative C*-algebra A is:*

(a) a von Neumann algebra if and only if $\Sigma(A)$ *is hyperstonean [84, Section III.1];*

(b) an AW-algebra if and only if* $\Sigma(A)$ *is Stonean, if and only if* $\Sigma(A)$ *is Stone and* $\mathcal{B}(\Sigma(A))$ *is complete [10, Theorem 1.7.1];*

(c) a Rickart C-algebra if and only if* $\Sigma(A)$ *is Stone and* $\mathcal{B}(\Sigma(A))$ *is countably complete [10, Theorem 1.8.1];*

(d) a spectral C-algebra if and only if* $\Sigma(A)$ *is Stone [82, Section 9.7].* □

5.5 Although spectral C*-algebras are the most general class in Definition 5.1, their projections may not form a lattice in the noncommutative case. A major advantage of Rickart C*-algebras is that their projections do, as in the following proposition. Rickart C*-algebras are also of interest for classification programmes, as follows. The class of so-called *real rank zero* C*-algebras has been classified using K-theory. This is a functor K from **CStar** to graded Abelian groups. In fact, it is currently believed that real rank zero C*-algebras are the widest class of C*-algebras for which $A \cong B$ if and only if $K(A) \cong K(B)$ [74, Section 3]. Rickart C*-algebras are always real rank zero [12, Theorem 6.1.2].

5.6 Proposition *Let A be a Rickart C*-algebra.*

(a) If it is ordered by $p \leq q \Leftrightarrow pA \subseteq qA$, *then* $\mathrm{Proj}(A)$ *is a countably complete lattice [10, Proposition 1.3.7 and Lemma 1.8.3].*

(b) If A is commutative, then it is the (norm-)closed linear span of $\mathrm{Proj}(A)$ *[10, Proposition 1.8.1.(3)].*

(c) If A is commutative, then it is monotone countably complete, i.e. each increasing bounded sequence in A_{sa} *has a supremum in A [82, Proposition 9.2.6.1].* □

5.7 Definition 5.1(a) requires the so-called ultraweak or σ-weak topology, which is hard to internalise to a topos. There are constructive definitions of von Neumann algebras [34, 81], but they rely on the strong operator topology, which is hard to internalise, too. Furthermore, the latter rely on the axiom of dependent choice. Although this holds in Kripke toposes, we prefer to consider Rickart C*-algebras. All one loses in this generalisation is that the projection lattice is only countably complete instead of complete—this is not a source of tremendous worry, because countable completeness of $\mathrm{Proj}(A)$ implies completeness if A has a faithful representation on a separable Hilbert space. Moreover, Rickart C*-algebras can easily be Bohrified, as Theorem 5.10 below shows.

5.8 Proposition *For a commutative C*-algebra A, the following are equivalent:*

(a) A is Rickart;

(b) for each $a \in A$ there is a (unique) $[a = 0] \in \mathrm{Proj}(A)$ such that $a[a = 0] = 0$, and $b = b[a = 0]$ when $ab = 0$;

(c) for each $a \in A_{\mathrm{sa}}$ there is a (unique) $[a > 0] \in \mathrm{Proj}(A)$ such that $[a > 0]a = a^+$ and $[a > 0][-a > 0] = 0$.

Proof For the equivalence of (a) and (b) we refer to [10, Proposition 1.3.3]. Assuming (b) and defining $[a > 0] = 1 - [a^+ = 0]$, we have

$$\begin{aligned}[a > 0]a &= (1 - [a^+ = 0])(a^+ - a^-)\\ &= a^+ - a^- - a^+[a^+ = 0] + a^-[a^+ = 0]\\ &= a^+, \qquad \text{(since } a^-a^+ = 0\text{, so that } a^-[a^+ = 0] = a^-)\end{aligned}$$

and similarly $a^-[a > 0] = a^- - a^-[a^+ = 0] = 0$, whence

$$\begin{aligned}[a > 0][-a > 0] &= [a > 0](1 - [(-a)^+ = 0])\\ &= [a > 0] - [a > 0][a^- = 0] = 0, \qquad \text{(since } [a^-[a > 0] = 0)\end{aligned}$$

establishing (c). For the converse, notice that it suffices to handle the case $a \in A^+$: decomposing general $a \in A$ into four positives we obtain $[a = 0]$ by multiplying the four associated projections. Assuming (c) and $a \in A^+$, define $[a = 0] = 1 - [a > 0]$. Then $a[a = 0] = (1 - [a > 0]) = a^+ - a[a > 0] = 0$. If $ab = 0$ for $b \in A$, then

$$\mathtt{D}_{b[a>0]} = \mathtt{D}_{b\wedge[a>0]} = \mathtt{D}_b \wedge \mathtt{D}_{[a>0]} = \mathtt{D}_b \wedge \mathtt{D}_a = \mathtt{D}_{ba} = \mathtt{D}_0,$$

so that $b[a < 0] \preccurlyeq 0$ by 3.11. That is, $b[a < 0] \leq n \cdot 0 = 0$ for some $n \in \mathbb{N}$. □

5.9 Parallel to Proposition 4.3, we define $\mathcal{C}_{\mathrm{R}}(A)$ to be the collection of all commutative Rickart C*-subalgebras C of A, and $\mathcal{T}_{\mathrm{R}}(A) = [\mathcal{C}_{\mathrm{R}}(A), \mathbf{Set}]$. The Bohrification $\underline{A}$ of a Rickart C*-algebra A is then defined by $\underline{A}(C) = C$, just as in Definition 4.6.

5.10 Theorem *Let A be a Rickart C*-algebra. Then $\underline{A}$ is a commutative Rickart C*-algebra in $\mathcal{T}_{\mathrm{R}}(A)$.*

Proof By Theorem 4.8, we already know that $\underline{A}$ is a commutative C*-algebra in $\mathcal{T}_{\mathrm{R}}(A)$. Proposition 5.8 captures the property of a commutative C*-algebra being Rickart in a geometric formula. Hence, by Lemma 3.13, $\underline{A}$ is Rickart since every $C \in \mathcal{C}_{\mathrm{R}}(A)$ is. □

We now work towards an explicit formula for the external description of the Gelfand spectrum of the Bohrification of a Rickart C*-algebra.

5.11 Lemma *Let A be a commutative Rickart C*-algebra, and $a, b \in A$ self-adjoint. If $ab \geq a$, then $a \preccurlyeq b$, i.e. $\mathtt{D}_a \leq \mathtt{D}_b$.*

Proof If $a \leq ab$ then certainly $a \preccurlyeq ab$. Hence $\mathtt{D}_a \leq \mathtt{D}_{ab} = \mathtt{D}_a \wedge \mathtt{D}_b$. In other words, $\mathtt{D}_a \leq \mathtt{D}_b$, whence $a \preccurlyeq b$. □

5.12 Definition Recall that a function f between posets satisfying $f(x) \geq f(y)$ when $x \leq y$ is called antitone. A *pseudocomplement* on a distributive lattice L is an antitone function $\neg \colon L \to L$ satisfying $x \wedge y = 0$ iff $x \leq \neg y$. Compare 2.2.

5.13 Proposition *For a commutative Rickart C*-algebra A, the lattice L_A has a pseudocomplement, determined by $\neg \mathtt{D}_a = \mathtt{D}_{[a=0]}$ for $a \in A^+$.*

Proof Without loss of generality, let $b \leq 1$. Then

$$\begin{aligned}
\mathtt{D}_a \wedge \mathtt{D}_b = 0 &\iff \mathtt{D}_{ab} = \mathtt{D}_0 \\
&\iff ab = 0 \\
&\iff b[a=0] = b && \text{($\Rightarrow$ by Proposition 5.8)} \\
&\iff b \preccurlyeq [a=0] && \text{($\Leftarrow$ since $b \leq 1$, $\Rightarrow$ by Lemma 5.11)} \\
&\iff \mathtt{D}_b \leq \mathtt{D}_{[a=0]} = \neg \mathtt{D}_a.
\end{aligned}$$

To see that $\neg$ is antitone, suppose that $\mathtt{D}_a \leq \mathtt{D}_b$. Then $a \preccurlyeq b$, so $a \leq nb$ for some $n \in \mathbb{N}$. Hence $[b=0]a \leq [b=0]bn = 0$, so that $\neg \mathtt{D}_b \wedge \mathtt{D}_a = \mathtt{D}_{[b=0]a} = 0$, and therefore $\neg \mathtt{D}_b \leq \neg \mathtt{D}_a$.□

5.14 Lemma *If A is a commutative Rickart C*-algebra, then the lattice L_A satisfies $\mathtt{D}_a \leq \bigvee_{r \in \mathbb{Q}^+} \mathtt{D}_{[a-r>0]}$ for all $a \in A^+$.*

Proof Since $[a>0]a = a^+ \geq a$, Lemma 5.11 gives $a \preccurlyeq [a>0]$ and therefore $\mathtt{D}_a \leq \mathtt{D}_{[a>0]}$. Also, for $r \in \mathbb{Q}^+$ and $a \in A^+$, one has $1 \leq \frac{2}{r}((r-a) \vee a)$, whence

$$[a-r>0] \leq \frac{2}{r}((r-a) \vee a)[a-r>0] = \frac{2}{r}(a[a-r>0]).$$

Lemma 5.11 then yields $\mathtt{D}_{[a-r>0]} \leq \mathtt{D}_{\frac{2}{r}a} = \mathtt{D}_a$. In total, we have $\mathtt{D}_{[a-r>0]} \leq \mathtt{D}_a \leq \mathtt{D}_{[a>0]}$ for all $r \in \mathbb{Q}^+$, from which the statement follows. □

The following simplifies Theorem 3.20 by restricting to Rickart C*-algebras.

5.15 Theorem *The Gelfand spectrum $\mathcal{O}(\Sigma(A))$ of a commutative Rickart C*-algebra A is isomorphic to the frame* $\mathrm{Idl}(\mathrm{Proj}(A))$ *of ideals of* $\mathrm{Proj}(A)$*. Hence the regularity condition may be dropped if one uses* $\mathrm{Proj}(A)$ *instead of L_A. Moreover, $\mathcal{O}(\Sigma(A))$ is generated by the sublattice $P_A = \{\mathtt{D}_a \in L_A \mid a \in A^+, \neg\neg \mathtt{D}_a = \mathtt{D}_a\}$ of 'clopens' of L_A, which is Boolean by construction.*

PROOF Since $\neg\mathtt{D}_p = \mathtt{D}_{1-p}$ for $p \in \mathrm{Proj}(A)$, we have $\neg\neg\mathtt{D}_p = \mathtt{D}_p$. Conversely, $\neg\neg\mathtt{D}_a = \mathtt{D}_{[a>0]}$, so that each element of P_A is of the form $\mathtt{D}_a = \mathtt{D}_p$ for some $p \in \mathrm{Proj}(A)$. So $P_A = \{\mathtt{D}_p \mid p \in \mathrm{Proj}(A)\} \cong \mathrm{Proj}(A)$, since each projection $p \in \mathrm{Proj}(A)$ may be selected as the unique representative of its equivalence class $\mathtt{D}_p$ in L_A. By Lemma 5.14, we may use $\mathrm{Proj}(A)$ instead of L_A as the generating lattice for $\mathcal{O}(\Sigma(A))$. So $\mathcal{O}(\Sigma(A))$ is the collection of regular ideals of $\mathrm{Proj}(A)$ by Theorem 3.20. But since $\mathrm{Proj}(A) \cong P_A$ is Boolean, all its ideals are regular, as $\mathtt{D}_p \ll \mathtt{D}_p$ for each $p \in \mathrm{Proj}(A)$ [56]. This establishes the statement, $\mathcal{O}(\Sigma(A)) \cong \mathrm{Idl}(\mathrm{Proj}(A))$. □

We can now give a concise external description of the Gelfand spectrum of the Bohrification of a Rickart C*-algebra A, simplifying Theorem 4.16.

5.16 Theorem *The Bohrified state space Σ_A of a Rickart C*-algebra A is given by*

$$\mathcal{O}(\Sigma_A) \cong \{F \colon \mathcal{C}(A) \to \mathbf{Set} \mid F(C) \in \mathcal{O}(\Sigma(C)) \text{ and } \Sigma(C \subseteq D)(F(C)) \subseteq F(D) \text{ if } C \subseteq D\}.$$

It has a basis given by

$$\mathcal{B}(\Sigma_A) = \{G \colon \mathcal{C}(A) \to \mathrm{Proj}(A) \mid G(C) \in \mathrm{Proj}(C) \text{ and } G(C) \leq G(D) \text{ if } C \subseteq D\}.$$

More precisely, there is an injection $f \colon \mathcal{B}(\Sigma_A) \to \mathcal{O}(\Sigma_A)$ given by $f(G)(C) = \mathrm{supp}(\widehat{G(C)})$, using the Gelfand transform of 3.10 in **Set**. *Each $F \in \mathcal{O}(\Sigma_A)$ can be expressed as $F = \bigvee\{f(G) \mid G \in \mathcal{B}(\Sigma_A), f(G) \leq F\}$.*

PROOF By (the proof of) Theorem 5.15, one can use $\mathrm{Proj}(C)$ instead of $\underline{L_A}(C)$ as a generating lattice for $\mathcal{O}(\Sigma(A))$. Translating Theorem 4.16(b) in these terms yields that $\mathcal{O}(\Sigma_A)$ consists of subfunctors $\underline{U}$ of $\underline{L_A}$ for which $\underline{U}(C) \in \mathrm{Idl}(\mathrm{Proj}(C))$ at each $C \in \mathcal{C}(A)$. Notice that Theorem 4.16 holds in $\mathcal{T}_R(A)$ as well as in $\mathcal{T}(A)$ (by interpreting Theorem 3.22 in the former instead of in the latter topos). Thus we obtain a frame isomorphism $\mathrm{Idl}(\mathrm{Proj}(C)) \cong \mathcal{O}(\Sigma(C))$, and the description in the statement. □

5.17 Corollary *If A is finite-dimensional, then*

$$\mathcal{O}(\Sigma_A) \cong \{G \colon \mathcal{C}(A) \to \mathrm{Proj}(A) \mid G(C) \in \mathrm{Proj}(C) \text{ and } G(C) \leq G(D) \text{ if } C \subseteq D\}.$$

This is a complete Heyting algebra under pointwise order with respect to the usual ordering of projections. As shown in [23], the lattice $\mathcal{O}(\Sigma_A)$ is not Boolean whenever A is noncommutative, so that the intrinsic logical structure carried by Σ_A is intuitionistic. This fact may conceptually be related to the fact that the passage from the initial noncommutative C*-algebra A to its Bohrification $\underline{A}$ involves some loss of information. Furthermore, compared with the standard formalism of von Neumann, in which single projections are interpreted as (atomic) propositions, it now appears that in our 'Bohrified' description each atomic proposition $G \in \mathcal{O}(\Sigma_A)$ consists of a famiy of projections, one (namely $G(C)$) for each classical context $C \in \mathcal{C}(A)$.

We now examine the connection with quantum logic in the usual sense in some more detail. To do so, we assume that A is a Rickart C*-algebra, in which case it follows from Example 3.6 that $\mathrm{Proj}(A)$ is a countably complete orthomodular lattice. This includes the situation where A is a von Neumann algebra, in which case $\mathrm{Proj}(A)$ is a complete orthomodular lattice [72]. For the sake of completeness, we recall:

5.18 Definition A (complete) lattice X is called *orthomodular* when it is equipped with a function $\perp\colon X \to X$ that satisfies:

1. $x^{\perp\perp} = x$;
2. $y^{\perp} \leq x^{\perp}$ when $x \leq y$;
3. $x \wedge x^{\perp} = 0$ and $x \vee x^{\perp} = 1$;
4. $x \vee (x^{\perp} \wedge y) = y$ when $x \leq y$.

The first three requirements are sometimes called (1) "double negation", (2) "contraposition", (3) "noncontradiction" and "excluded middle", but, as argued in the Introduction, one should refrain from names suggesting a logical interpretation. If these are satisfied, the lattice is called *orthocomplemented*. The requirement (4), called the orthomodular law, is a weakening of distributivity.

Hence, a Boolean algebra is a lattice that is at the same time a Heyting algebra and an orthomodular lattice with the same operations, *i.e.* $x = x^{\perp\perp}$ for all x, where $x^{\perp}$ is defined to be $(x \Rightarrow 0)$. It is usual to denote the latter by $\neg x$ instead of $x^{\perp}$ in case the algebra is Boolean.

Using the description of the previous theorem, we are now in a position to compare our Bohrified state space $\mathcal{O}(\Sigma_A)$ to the traditional "quantum logic" $\mathrm{Proj}(A)$. To do so, we recall an alternative characterisation of orthomodular lattices.

5.19 Definition A (complete) *partial Boolean algebra* is a family $(B_i)_{i\in I}$ of (complete) Boolean algebras whose operations coincide on overlaps:

- each B_i has the same least element 0;
- $x \Rightarrow_i y$ if and only if $x \Rightarrow_j y$, when $x, y \in B_i \cap B_j$;
- if $x \Rightarrow_i y$ and $y \Rightarrow_j z$ then there is a $k \in I$ with $x \Rightarrow_k z$;
- $\neg_i x = \neg_j x$ when $x \in B_i \cap B_j$;
- $x \vee_i y = x \vee_j y$ when $x, y \in B_i \cap B_j$;
- if $y \Rightarrow_i \neg_i x$ for some $x, y \in B_i$, and $x \Rightarrow_j z$ and $y \Rightarrow_k z$, then $x, y, z \in B_l$ for some $l \in I$.

5.20 The requirements of a partial Boolean algebra imply that the amalgamation $\mathcal{A}(B) = \bigcup_{i\in I} B_i$ carries a well-defined structure $\vee, \wedge, 0, 1, \perp$, under which it becomes an orthomodular lattice. For example, $x^{\perp} = \neg_i x$ for $x \in B_i \subseteq \mathcal{A}(B)$. Conversely, any orthomodular lattice X is a partial Boolean algebra, in which I is the collection of all orthogonal subsets of $\mathcal{A}(B)$, and B_i is the sublattice of $\mathcal{A}(B)$ generated by I. Here, a subset $E \subseteq \mathcal{A}(B)$ is called orthogonal when pairs (x, y) of different elements of E are orthogonal, *i.e.* $x \leq y^{\perp}$. The generated sublattices B_i are therefore automatically Boolean. If we order I by inclusion, then $B_i \subseteq B_j$ when $i \leq j$. Thus there is an isomorphism between the categories of orthomodular lattices and partial Boolean algebras [33, 39, 60, 62].

5.21 A similar phenomenon occurs in the Heyting algebra $\mathcal{B}(\Sigma_A)$ of Theorem 5.16, when this is complete, which is the case for AW*-algebras and in particular for von Neumann algebras (provided, of course, that we require $\mathcal{C}(A)$ to consist of commutative subalgebras in the same class). Indeed, we can think of $\mathcal{B}(\Sigma_A)$ as an amalgamation of Boolean algebras: just as every B_i in Definition 5.19 is a Boolean algebra, every $\mathrm{Proj}(C)$ in Theorem 5.16 is a Boolean algebra. Hence the fact that the set I in Definition 5.19 is replaced by the partially ordered set $\mathcal{C}(A)$ and the requirement in Theorem 5.16 that G be monotone are responsible for making the partial Boolean algebra $\mathcal{O}(\Sigma_A)$ into a Heyting algebra. Indeed, this construction works more generally, as the following theorem shows. (Compare also [45] and [87], that write an orthomodular lattice as a sheaf of Boolean and distributive ones, respectively.)

5.22 Theorem *Let $(I,\leq)$ be a partially ordered set, and B_i an I-indexed family of complete Boolean algebras such that $B_i \subseteq B_j$ if $i \leq j$. Then*

$$\mathcal{B}(B) = \{f\colon I \to \bigcup_{i\in I} B_i \mid \forall_{i\in I}.f(i) \in B_i \text{ and } f \text{ monotone}\}$$

is a complete Heyting algebra, with Heyting implication

$$(g \Rightarrow h)(i) = \bigvee\{x \in B_i \mid \forall_{j\geq i}.x \leq g(j) \Rightarrow h(j)\}.$$

Proof Defining operations pointwise makes Y into a frame. For example, $f \wedge g$, defined by $(f \wedge g)(i) = f(i) \wedge_i g(i)$, is again a well-defined monotone function whose value at i lies in B_i. Hence, as in Definition 2.4, $\mathcal{B}(B)$ is a complete Heyting algebra by $(g \Rightarrow h) = \bigvee\{f \in Y \mid f \wedge g \leq h\}$. We now rewrite this Heyting implication:

$$\begin{aligned}
(g \Rightarrow h)(i) &= \big(\bigvee\{f \in \mathcal{B}(B) \mid f \wedge g \leq h\}\big)(i)\\
&= \bigvee\{f(i) \mid f \in \mathcal{B}(B), f \wedge g \leq h\}\\
&= \bigvee\{f(i) \mid f \in \mathcal{B}(B), \forall_{j\in I}.f(j) \wedge g(j) \leq h(j)\}\\
&= \bigvee\{f(i) \mid f \in \mathcal{B}(B), \forall_{j\in I}.f(j) \leq g(j) \Rightarrow h(j)\}\\
&\overset{*}{=} \bigvee\{x \in B_i \mid \forall_{j\geq i}.x \leq g(j) \Rightarrow h(j)\}.
\end{aligned}$$

To finish the proof, we establish the marked equation. First, suppose that $f \in \mathcal{B}(B)$ satisfies $f(j) \leq g(j) \Rightarrow h(j)$ for all $j \in I$. Take $x = f(i) \in B_i$. Then for all $j \geq i$ we have $x = f(i) \leq f(j) \leq g(j) \Rightarrow h(j)$. Hence the left-hand side of the marked equation is less than or equal to the right-hand side. Conversely, suppose that $x \in B_i$ satisfies $x \leq g(j) \Rightarrow h(j)$ for all $j \geq i$. Define $f\colon I \to \bigcup_{i\in I} B_i$ by $f(j) = x$ if $j \geq i$ and $f(j) = 0$ otherwise. Then f is monotone and $f(i) \in B_i$ for all $i \in I$, whence $f \in Y$. Moreover, $f(j) \leq g(j) \Rightarrow h(j)$ for all $j \in I$. Since $f(i) \leq x$, the right-hand side is less than or equal to the left-hand side. □

5.23 Proposition *Let $(I,\leq)$ be a partially ordered set. Let $(B_i)_{i\in I}$ be complete partial Boolean algebra, and suppose that $B_i \subseteq B_j$ for $i \leq j$. Then there is an injection $D\colon \mathcal{A}(B) \to \mathcal{B}(B)$. This injection reflects the order: if $D(x) \leq D(y)$ in Y, then $x \leq y$ in X.*

PROOF Define $D(x)(i) = x$ if $x \in B_i$ and $D(x)(i) = 0$ if $x \notin B_i$. Suppose that $D(x) = D(y)$. Then for all $i \in I$ we have $x \in B_i$ iff $y \in B_i$. Since $x \in \mathcal{A}(B) = \bigcup_{i \in I} B_i$, there is some $i \in I$ with $x \in B_i$. For this particular i, we have $x = D(x)(i) = D(y)(i) = y$. Hence D is injective. If $D(x) \leq D(y)$ for $x, y \in \mathcal{A}(B)$, pick $i \in I$ such that $x \in B_i$. Unless $x = 0$, we have $x = D(x)(i) \leq D(y)(i) = y$. □

5.24 In the situation of the previous proposition, the Heyting algebra $\mathcal{B}(B)$ comes with its Heyting implication, whereas the orthomodular lattice $\mathcal{A}(B)$ has a so-called *Sasaki hook* $\Rightarrow_{\mathrm{S}}$, satisfying the adjunction $x \leq y \Rightarrow_{\mathrm{S}} z$ iff $x \wedge y \leq z$ only for y and z that are compatible. This is the case if and only if y and z generate a Boolean subalgebra, *i.e.* if and only if $y, z \in B_i$ for some $i \in I$. In that case, the Sasaki hook $\Rightarrow_{\mathrm{S}}$ coincides with the implication $\Rightarrow$ of B_i. Hence

$$\begin{aligned}(D(x) \Rightarrow D(y))(i) &= \bigvee \{z \in B_i \mid \forall_{j \geq i}. z \leq D(x)(j) \Rightarrow D(y)(j)\} \\ &= \bigvee \{z \in B_i \mid z \leq x \Rightarrow y\} \\ &= (x \Rightarrow_{\mathrm{S}} y).\end{aligned}$$

In particular, we find that $\Rightarrow$ and $\Rightarrow_{\mathrm{S}}$ coincide on $B_i \times B_i$ for $i \in I$; furthermore, this is precisely the case in which the Sasaki hook satisfies the defining adjunction for (Heyting) implications.

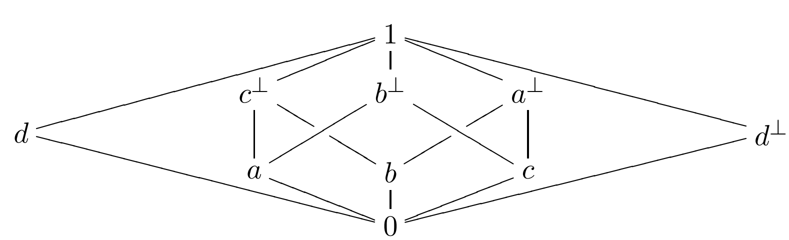
However, the canonical injection D need not turn Sasaki hooks into implications in general. One finds:

$$D(x \Rightarrow_{\mathrm{S}} y)(i) = \begin{bmatrix} x^{\perp} \vee (x \wedge y) & \text{if } x \Rightarrow_{\mathrm{S}} y \in B_i \\ 0 & \text{otherwise} \end{bmatrix},$$

$$(D(x) \Rightarrow D(b))(i) = \bigvee \left\{ z \in B_i \mid \forall_{j \geq i}. z \leq \begin{bmatrix} 1 & \text{if } x \notin B_j \\ x^{\perp} & \text{if } x \in B_j, y \notin B_j \\ x^{\perp} \vee y & \text{if } x, y \in B_j \end{bmatrix} \right\}.$$

So if $x \notin B_j$ for any $j \geq i$, we have $D(x \Rightarrow_{\mathrm{S}} y)(i) = 0 \neq 1 = (D(x) \Rightarrow D(y))(i)$.

5.25 To end this section, we consider the so-called *Bruns-Lakser completion* [16, 26, 83]. The Bruns-Lakser completion of a complete lattice is a complete Heyting algebra that contains the original lattice join-densely. It is the universal in that this inclusion preserves existing distributive joins. Explicitly, the Bruns-Lakser completion of a lattice L is the collection $\mathrm{DIdl}(L)$ of its *distributive ideals*. Here, an ideal M is called *distributive* when ($\bigvee M$ exists and) $(\bigvee M) \wedge x = \bigvee_{m \in M}(m \wedge x)$ for all $x \in L$. Now consider the orthomodular lattice X with the following Hasse diagram.



This contains precisely five Boolean algebras, namely $B_0 = \{0,1\}$ and $B_i = \{0,1,i,i^\perp\}$ for $i \in \{a,b,c,d\}$. Hence $X = \mathcal{A}(B)$ when we take $I = \{0,a,b,c,d\}$ ordered by $i < j$ iff $i = 0$. The monotony requirement in $\mathcal{B}(B)$ becomes $\forall_{i\in\{a,b,c,d\}}.f(0) \leq f(i)$. If $f(0) = 0 \in B_0$, this requirement is vacuous. But if $f(0) = 1 \in B_0$, the other values of f are already fixed. Thus one finds that $\mathcal{B}(B) \cong (B_1 \times B_2 \times B_3 \times B_4) + 1$ has 257 elements.

On the other hand, the distributive ideals of X are given by

$$\begin{aligned}\mathrm{DI}(X) = \Big\{ \big(\bigcup_{x\in A} \downarrow x\big) \cup \big(\bigcup_{y\in B} \downarrow y\big) \;\Big|\; A \subseteq \{a,b,c,d,d^\perp\}, B \subseteq \{a^\perp, b^\perp, c^\perp\}\Big\} \\ - \{\emptyset\} + \{X\}.\end{aligned}$$

In the terminology of [83],

$$\mathcal{J}_{\mathrm{dis}}(x) = \{S \subseteq \downarrow x \mid x \in S\},$$

i.e. the covering relation is the trivial one, and $\mathrm{DI}(X)$ is the Alexandrov topology (as a frame/locale). We are unaware of instances of the Bruns–Lakser completion of orthomodular lattices that occur naturally in quantum physics but lead to Heyting algebras different from ideal completions. The set $\mathrm{DI}(X)$ has 72 elements.

The canonical injection D of Proposition 5.23 need not preserve the order, and hence does not satisfy the universal requirement of which the Bruns–Lakser completion is the solution. Therefore, it is unproblemetic to conclude that the construction in Theorem 5.22 differs from the Bruns–Lakser completion.

6 States and observables

This final section considers some relationships between the external C*-algebra A and its Bohrification $\underline{A}$. For example, we discuss how a state on A in the operator algebraic sense gives rise to a probability integral on $\underline{A}_{\mathrm{sa}}$ within $\mathcal{T}(A)$. The latter corresponds to a suitably adapted version of a probability measure on $\mathcal{O}(\underline{\Sigma}(\underline{A}))$, justifying the name "Bohrified" state space. We also consider how so-called *Daseinisation* translates an external proposition about an observable $a \in A_{\mathrm{sa}}$ into a subobject of the Bohrified state space. The internalised state and observable are then combined to give a truth value.

6.1 Definition A linear functional $\rho\colon A \to \mathbb{C}$ on a C*-algebra A is called *positive* when $\rho(A^+) \subseteq \mathbb{R}^+$. It is a *state* when it is positive and satisfies $\rho(1) = 1$. A state ρ is *pure* when $\rho = t\rho' + (1-t)\rho''$ for some $t \in (0,1)$ and some states ρ', ρ'' implies $\rho' = \rho''$. Otherwise, it is called *mixed*. A state is called *faithful* when $\rho(a) = 0$ implies $a = 0$ for all $a \in A^+$.

States are automatically Hermitian, in the sense that $\rho(a^*)$ is the complex conjugate of $\rho(a)$, or equivalently, $\rho(a) \in \mathbb{R}$ for $a \in A_{\mathrm{sa}}$.

6.2 Example If $A = \mathbf{Hilb}(X,X)$ for some Hilbert space X, each unit vector $x \in X$ defines a pure state on A by $\rho_x(a) = \langle x \,|\, a(x)\rangle$. Normal mixed states ρ arise from countable sequences (r_i) of numbers satisfying $0 \leq r_i \leq 1$ and $\sum_i r_i = 1$, coupled with a family (x_i) of $x_i \in X$, through $\rho(a) = \sum_i r_i \rho_{x_i}(a)$. This state is faithful when (x_i) comprise an orthonormal basis of X and each $r_i > 0$.

Taking Bohr's doctrine of classical concepts seriously means accepting that two operators can only be added in a meaningful way when they commute, leading to the following notion [1, 17, 18, 70].

6.3 Definition A *quasi-linear* functional on a C*-algebra A is a map $\rho\colon A \to \mathbb{C}$ that is linear on all commutative subalgebras and satisfies $\rho(a+ib) = \rho(a) + i\rho(b)$ for all (possibly noncommuting) $a, b \in A_{\mathrm{sa}}$. It is called *positive* when $\rho(A^+) \subseteq A^+$, and it is called a *quasi-state* when furthermore $\rho(1) = 1$.

This kind of quasi-linearity determines when some property P of A descends to a corresponding property $\underline{P}$ for the Bohrification $\underline{A}$, as the following lemma shows. To be precise, for $P \subseteq A$, define $\underline{P} \in \mathrm{Sub}(\underline{A})$ by $\underline{P}(C) = P \cap C$. A property $P \subseteq A$ is called *quasi-linear* when $a, b \in P \cap A_{\mathrm{sa}}$ implies $ra + isb \in P$ for all $r, s \in \mathbb{R}$.

6.4 Lemma *Let A be a C*-algebra, and let $P \subseteq A$ be a quasi-linear property. Then $P = A$ if and only if $\underline{P} = \underline{A}$.*

Proof One implication is trivial; for the other, suppose that $\underline{P} = \underline{A}$. For $a \in A$, denote by $C^*(a)$ the C*-subalgebra generated by a (and 1). When a is self-adjoint, $C^*(a)$ is commutative. So $A_{\mathrm{sa}} \subseteq P$, whence by quasi-linearity of P and the unique decomposition of elements in a real and imaginary part, we have $A \subseteq P$. □

6.5 Definition An *integral* on a Riesz space R is a linear functional $I\colon R \to \mathbb{R}$ that is positive, *i.e.* if $f \geq 0$ then also $I(f) \geq 0$. If R has a strong unit 1 (see Definition 3.14), then an integral I satisfying $I(1) = 1$ is called a *probability integral*. An integral I is *faithful* when $I(f) = 0$ and $f \geq 0$ imply $f = 0$.

6.6 Example Except in the degenerate case $I(1) = 0$, any integral can obviously be normalised to a probability integral. The prime example of an integral is the Riemann or Lebesgue integral on the ordered vector space $C([0,1], \mathbb{C})$. More generally, any positive linear functional on a commutative C*-algebra provides an example, states yielding probability integrals.

6.7 Definition Let R be a Riesz space. We now define the locale $\mathcal{I}(R)$ of probability integrals on R. First, let $\mathrm{Int}(R)$ be the distributive lattice freely generated by symbols $\mathtt{P}_f$ for $f \in R$, subject to the relations

$$\begin{aligned} \mathtt{P}_1 &= 1, \\ \mathtt{P}_f \wedge \mathtt{P}_{-f} &= 0, \\ \mathtt{P}_{f+g} &\leq \mathtt{P}_f \vee \mathtt{P}_g, \\ \mathtt{P}_f &= 0 \qquad\qquad\qquad\qquad (\text{for } f \leq 0). \end{aligned}$$

This lattice generates a frame $\mathcal{O}(\mathcal{I}(R))$ by furthermore imposing the regularity condition

$$\mathtt{P}_f = \bigvee \{\mathtt{P}_{f-q} \mid q \in \mathbb{Q}, q > 0\}.$$

6.8 Classically, points p of $\mathcal{I}(R)$ correspond to probability integrals I on R, by mapping I to the point p_I given by $p_I(\mathtt{P}_f) = 1$ iff $I(f) > 0$. Conversely, a point p defines an integral $I_p = (\{q \in \mathbb{Q} \mid p \models \mathtt{P}_{f-q}\}, \{r \in \mathbb{Q} \mid p \models \mathtt{P}_{r-f}\})$, which is a Dedekind cut by the relations imposed on $\mathtt{P}_{(\_)}$, as in Example 2.33. Therefore, intuitively, $\mathtt{P}_f = \{\rho\colon R \to \mathbb{R} \mid \rho(f) > 0, \rho \text{ positive linear}\}$.

Classically, for a locally compact Hausdorff space X, the Riesz-Markov theorem provides a duality between integrals on a Riesz space $\{f \in C(X,\mathbb{R}) \mid \mathrm{supp}(f) \text{ compact}\}$ and regular measures on the Borel subsets of X. Constructively, one uses so-called valuations, which are only defined on open subsets of X, instead of measures. Theorem 6.13 below gives a constructively valid version of the Riesz-Markov theorem. In preparation we consider a suitable constructive version of measures.

6.9 Classically, points of the locale $\mathbb{R}$ of Example 2.33 are Dedekind cuts (L,U) (and $\mathcal{O}(\mathbb{R})$ is the usual Euclidean topology). We now introduce two variations on the locale $\mathbb{R}$. First, consider the locale $\mathbb{R}_l$ that is generated by formal symbols $q \in \mathbb{Q}$ subject to the following relations:

$$q \wedge r = \min(q,r), \qquad q = \bigvee\{r \mid r > q\}, \qquad 1 = \bigvee\{q \mid q \in \mathbb{Q}\}.$$

Classically, its points are *lower reals*, and locale morphisms to $\mathbb{R}_l$ correspond to lower-semicontinuous real-valued functions. Restricting generators to $0 \leq q \leq 1$ yields a locale denoted $[0,1]_l$.

6.10 Secondly, let $\mathbb{IR}$ be the locale defined by the very same generators (q,r) and relations as in Example 2.33, except that we omit the fourth relation $(q,r) = (q,r_1) \vee (q_1,r)$ for $q \leq q_1 \leq r_1 \leq r$. The effect is that, classically, points of $\mathbb{IR}$ again correspond to pairs (L,U) as in Example 2.33, except that the lower real L and the upper real U need not combine into a Dedekind cut, as the 'kissing' requirement is no longer in effect. Classically, a point (L,U) of $\mathbb{IR}$ corresponds to a compact interval $[\sup(L), \inf(U)]$ (including the singletons $[x,x] = \{x\}$). Ordered by reverse inclusion, the topology they carry is the *Scott topology* [2] whose closed sets are lower sets that are closed under directed joins. Hence, each open interval (q,r) in $\mathbb{R}$ (with $q = -\infty$ and $r = \infty$ allowed) corresponds to a Scott open $\{[a,b] \mid q < a \leq b < r\}$ in $\mathbb{IR}$, and these form the basis of the Scott topology. Therefore, $\mathbb{IR}$ is also called the *interval domain* [71, 79]. One can think of it as approximations of real numbers by rational intervals, interpreting each individual interval as finitary information about the real number under scrutiny. The ordering by reverse inclusion is then explained as a smaller interval means that more information is available about the real number.

In a Kripke topos $[P, \mathbf{Set}]$ over a poset P with a least element, one has $\mathcal{O}(\underline{\mathbb{IR}})(p) = \mathcal{O}((\uparrow p) \times \mathbb{IR})$, which may be identified with the set of monotone functions from $\uparrow p$ to $\mathcal{O}(\mathbb{IR})$. This follows by carefully adapting the proof of [69, Theorem VI.8.2].

6.11 Definition A *continuous probability valuation* on a locale X is a monotone function $\mu\colon \mathcal{O}(X) \to \mathcal{O}([0,1]_l)$ such that $\mu(1) = 1$ as well as $\mu(U) + \mu(V) = \mu(U \wedge V) + \mu(U \vee V)$ and $\mu(\bigvee_i U_i) = \bigvee_i \mu(U_i)$ for a directed family (U_i). Like integrals, continuous probability valuations organise themselves in a locale $\mathcal{V}(X)$.

6.12 Example If X is a compact Hausdorff space, a continuous probability valuation on $\mathcal{O}(X)$ is the same thing as a regular probability measure on X.

6.13 Theorem *[30] Let R be an f-algebra and Σ its spectrum. Then the locales $\mathcal{I}(R)$ and $\mathcal{V}(\Sigma)$ are isomorphic. A continuous probability valuation μ gives a probability integral by*

$$I_\mu(f) = (\sup_{(s_i)} \sum s_i \mu(\mathtt{D}_{f-s_i} \wedge \mathtt{D}_{s_{i+1}-f}), \inf_{(s_i)} \sum s_{i+1}(1 - \mu(\mathtt{D}_{s_i-f}) - \mu(\mathtt{D}_{f-s_{i+1}}))),$$

where (s_i) is a partition of $[a,b]$ such that $a \leq f \leq b$. Conversely, a probability integral I gives a continuous probability valuation

$$\mu_I(\mathtt{D}_a) = \sup\{I(na^+ \wedge 1) \mid n \in \mathbb{N}\}. \qquad \square$$

6.14 Corollary *For a C*-algebra A, the locale $\mathcal{I}(\underline{A})$ in $\mathcal{T}(A)$ of probability integrals on $\underline{A}_{\mathrm{sa}}$ is isomorphic to the locale $\mathcal{V}(\underline{\Sigma}(\underline{A}))$ in $\mathcal{T}(A)$ of continuous probability valuations on $\underline{\Sigma}(\underline{A})$.*

Proof Interpret Theorem 6.13—whose proof is constructive—in $\mathcal{T}(A)$. $\square$

6.15 Theorem *There is a bijective correspondence between (faithful) quasi-states on a C*-algebra A and (faithful) probability integrals on $\underline{A}_{\mathrm{sa}}$.*

Proof Every quasi-state ρ gives a natural transformation $I_\rho \colon \underline{A}_{\mathrm{sa}} \to \underline{\mathbb{R}}$ whose component $(I_\rho)_C \colon C_{\mathrm{sa}} \to \mathbb{R}$ is the restriction $\rho_{|C_{\mathrm{sa}}}$ of ρ to $C_{\mathrm{sa}} \subseteq A_{\mathrm{sa}}$. Conversely, let $I \colon \underline{A}_{\mathrm{sa}} \to \underline{\mathbb{R}}$ be an integral. Define $\rho \colon A_{\mathrm{sa}} \to \mathbb{R}$ by $\rho(a) = I_{C^*(a)}(a)$, where $C^*(a)$ is the sub-C*-algebra generated by a. For commuting $a, b \in A_{\mathrm{sa}}$, then

$$\begin{aligned} \rho(a+b) &= I_{C^*(a+b)}(a+b) \\ &= I_{C^*(a,b)}(a+b) \\ &= I_{C^*(a,b)}(a) + I_{C^*(a,b)}(b) \\ &= I_{C^*(a)}(a) + I_{C^*(b)}(b) \\ &= \rho(a) + \rho(b), \end{aligned}$$

because I is a natural transformation, $C^*(a) \cup C^*(b) \subseteq C^*(a,b) \supseteq C^*(a+b)$, and I is locally linear. Moreover, ρ is positive because I is locally positive, by Lemma 6.4. Hence we have defined ρ on A_{sa} and may extend it to A by complex linearity. It is clear that the two maps $I \mapsto \rho$ and $\rho \mapsto I$ are each other's inverse. $\square$

6.16 Let ρ be a (quasi-)state on a C*-algebra A. Then μ_ρ is a continuous probability valuation on $\mathcal{O}(\underline{\Sigma}(\underline{A}))$. Hence $\mu_\rho(\_) = 1$ is a term of the internal language of $\mathcal{T}(A)$ with one free variable of type $\mathcal{O}(\underline{\Sigma}(\underline{A}))$. Its interpretation $[\![\mu_\rho(\_) = 1]\!]$ defines a subobject of $\mathcal{O}(\underline{\Sigma}(\underline{A}))$, or equivalently, a morphism $[\rho] \colon \mathcal{O}(\underline{\Sigma}(\underline{A})) \to \underline{\Omega}$.

For Rickart C*-algebras, we can make Theorem 6.15 a bit more precise.

6.17 Definition (a) A *probability measure* on a countably complete orthomodular lattice X is a function $\mu \colon X \to [0,1]_l$ that on any countably complete Boolean sublattice of X restricts to a probability measure (in the traditional sense).

(b) A *probability valuation* on an orthomodular lattice X is a function $\mu \colon X \to [0,1]_l$ such that $\mu(0) = 0$, $\mu(1) = 1$, $\mu(x) + \mu(y) = \mu(x \wedge y) + \mu(x \vee y)$, and if $x \leq y$ then $\mu(x) \leq \mu(y)$.

6.18 Lemma *Let μ be a probability valuation on a Boolean algebra X. Then $\mu(x)$ is a Dedekind cut for any $x \in X$.*

PROOF Since X is Boolean, we have $\mu(\neg x) = 1 - \mu(x)$. Let $q, r \in \mathbb{Q}$, and suppose that $q < r$. We have to prove that $q < \mu(x)$ or $\mu(x) \leq r$. As the inequalities concern rationals, it suffices to prove that $q < \mu(x)$ or $1 - r < 1 - \mu(x) = \mu(\neg x)$. This follows from $1 - (r - q) < 1 = \mu(1) = \mu(x \vee \neg x)$ and $q - r < 0 = \mu(0) = \mu(x \wedge \neg x)$. □

The following theorem relates Definition 6.11 and Definition 6.17. Definition 6.11 will be applied to the Gelfand spectrum $\underline{\Sigma}(\underline{A})$ of the Bohrification of a Rickart C*-algebra A. Part (a) of Definition 6.17 will be applied to $\mathrm{Proj}(A)$ in **Set** for a Rickart C*-algebra A, and part (b) will be applied to the lattice $\underline{P}_{\underline{A}}$ of Theorem 5.15 in $\mathcal{T}(A)$.

6.19 Theorem *For a Rickart C*-algebra A, there is a bijective correspondence between:*

(a) quasi-states on A;

(b) probability measures on $\mathrm{Proj}(A)$*;*

(c) probability valuations on $\underline{P}_{\underline{A}}$;

(d) continuous probability valuations on $\underline{\Sigma}(\underline{A})$.

PROOF The correspondence between (a) and (d) is Theorem 6.15. The correspondence between (c) and (d) follows from Theorem 5.15 and the observation that valuations on a compact regular frame are determined by their behaviour on a generating lattice [30, Section 3.3]; indeed, if a frame $\mathcal{O}(X)$ is generated by L, then a probability measure μ on L yields a continuous probability valuation ν on $\mathcal{O}(X)$ by $\nu(U) = \sup\{\mu(u) \mid u \in U\}$, where $U \subseteq L$ is regarded as an element of $\mathcal{O}(X)$. Finally, we turn to the correspondence between (b) and (c). Since $\underline{\mathbb{R}}$ in $\mathcal{T}(A)$ is the constant functor $C \mapsto \mathbb{R}$ (as opposed to $\underline{\mathbb{R}_l}$), according to the previous lemma a probability valuation $\mu\colon \underline{\mathrm{Idl}}(\underline{\mathrm{Proj}}(\underline{A})) \to \underline{[0,1]_l}$ is defined by its components $\mu_C\colon \mathrm{Proj}(C) \to [0,1]$. By naturality, for $p \in \mathrm{Proj}(C)$, the real number $\mu_C(p)$ is independent of C, from which the correspondence between (b) and (c) follows immediately. □

6.20 We now turn to internalising an elementary proposition $a \in (q, r)$ concerning an observable $a \in A_{\mathrm{sa}}$ and rationals $q, r \in \mathbb{Q}$ with $q < r$. If A were commutative, then a would have a Gelfand transform $\hat{a}\colon \Sigma(A) \to \mathbb{R}$, and we could just internalise $\hat{a}^{-1}(q, r) \subseteq \Sigma(A)$ directly. For noncommutative A, there can be contexts $C \in \mathcal{C}(A)$ that do not contain a, and therefore the best we can do is approximate. Our strategy is to replace the reals $\mathbb{R}$ by the interval domain $\mathbb{IR}$ of 6.10. We will construct a locale morphism $\underline{\delta}(a)\colon \underline{\Sigma}(\underline{A}) \to \underline{\mathbb{IR}}$, called the *Daseinisation* of $a \in A_{\mathrm{sa}}$—this terminology stems from [37], but the morphism is quite different from the implementation in that article. The elementary proposition $a \in (q, r)$ is then internalised as the composite morphism

$$\underline{1} \xrightarrow{\underline{(q,r)}} \mathcal{O}(\underline{\mathbb{IR}}) \xrightarrow{\underline{\delta}(a)^{-1}} \mathcal{O}(\underline{\Sigma}(\underline{A})),$$

where $\underline{(q,r)}$ maps into the monotone function with constant value $\Downarrow(q, r)$. (As in 6.10, (q, r) is seen as an element of the generating semilattice, whereas $\Downarrow(q, r)$ is its image in the frame $\mathcal{O}(\mathbb{IR})$ under the canonical inclusion of Proposition 2.13.)

6.21 The interval domain $\mathcal{O}(\mathbb{IR})$ of 6.10 can be constructed as $\mathcal{F}(\mathbb{Q} \times_< \mathbb{Q}, \blacktriangleleft)$, as in Definition 2.12 [71]. The pertinent meet-semilattice $\mathbb{Q} \times_< \mathbb{Q}$ consists of pairs $(q,r) \in \mathbb{Q} \times \mathbb{Q}$ with $q < r$, ordered by inclusion (*i.e.* $(q,r) \leq (q',r')$ iff $q' \leq q$ and $r \leq r'$), with a least element 0 added. The covering relation $\blacktriangleleft$ is defined by $0 \blacktriangleleft U$ for all U, and $(q,r) \blacktriangleleft U$ iff for all rational q', r' with $q < q' < r' < r$ there exists $(q'', r'') \in U$ with $(q',r') \leq (q'',r'')$. In particular, we may regard $\mathcal{O}(\mathbb{IR})$ as a subobject of $\mathbb{Q} \times_< \mathbb{Q}$. As in 4.13:

$$\mathcal{O}(\underline{\mathbb{IR}})(\mathbb{C}) \cong \{\underline{F} \in \mathrm{Sub}(\underline{\mathbb{Q} \times_< \mathbb{Q}}) \mid \forall_{C \in \mathcal{C}(A)} . \underline{F}(C) \in \mathcal{O}(\mathbb{IR})\}.$$

6.22 Lemma *For a C*-algebra A and a fixed element $a \in A_{\mathrm{sa}}$, the components $\underline{d}(a)_C \colon \mathbb{Q} \times_< \mathbb{Q} \to \mathrm{Sub}(\underline{L}_{A|\uparrow C})$ given by*

$$\begin{aligned} \underline{d}(a)^*_C(q,r)(D) &= \{\mathtt{D}_{f-q} \wedge \mathtt{D}_{r-g} \mid f,g \in D_{\mathrm{sa}}, f \leq a \leq g\} \\ \underline{d}(a)^*_C(0)(D) &= \{\mathtt{D}_0\} \end{aligned}$$

form a morphism $\underline{d}(a)^* \colon \underline{\mathbb{Q} \times_< \mathbb{Q}} \to \underline{\Omega}^{\underline{L}_A}$ in $\mathcal{T}(A)$ via 4.13. This morphism is a continuous map $(\underline{L}_A, \lhd) \to (\underline{\mathbb{Q} \times_< \mathbb{Q}}, \underline{\blacktriangleleft})$ in the sense of Definition 2.14.

Notice that since $\underline{\mathbb{Q} \times_< \mathbb{Q}}(C) = \mathbb{Q} \times_< \mathbb{Q}$ for any $C \in \mathcal{C}(A)$, the natural transformation $\underline{d}(a)$ is completely determined by its component at $\mathbb{C} \in \mathcal{C}(A)$.

Proof We verify that the map defined in the statement satisfies the conditions of Definition 2.14.

(a) We have to show that $\Vdash \forall_{\mathtt{D}_a \in \underline{L}_A} \exists_{(q,r) \in \underline{\mathbb{Q} \times_< \mathbb{Q}}} . \mathtt{D}_a \in \underline{d}(a)^*(q,r)$. By interpreting via 2.24, we therefore have to prove: for all $C \in \mathcal{C}(A)$ and $\mathtt{D}_c \in L_C$ there are $(q,r) \in \mathbb{Q} \times_< \mathbb{Q}$ and $f,g \in C_{\mathrm{sa}}$ such that $f \leq a \leq g$ and $\mathtt{D}_c = \mathtt{D}_{f-q} \wedge \mathtt{D}_{r-g}$. Equivalently, we have to find $(q,r) \in \mathbb{Q} \times_< \mathbb{Q}$ and $f,g \in C_{\mathrm{sa}}$ such that $f + q \leq a \leq r + g$ and $\mathtt{D}_c = \mathtt{D}_f \wedge \mathtt{D}_{-g}$. Choosing $f = c$, $g = -c$, $q = -\|c\| - \|a\|$ and $r = \|c\| + \|a\|$ does the job, since $\mathtt{D}_c = \mathtt{D}_c \wedge \mathtt{D}_c$ and

$$f + q = c - \|c\| - \|a\| \leq -\|a\| \leq a \leq \|a\| \leq \|c\| + \|a\| - c = r + g.$$

(b) We have to show that

$$\begin{aligned} \Vdash \forall_{(q,r),(q',r') \in \underline{\mathbb{Q} \times_< \mathbb{Q}}} \forall_{u,v \in \underline{L}_A} . u \in \underline{d}(a)^*(q,r) \wedge v \in \underline{d}(a)^*(q',r') \\ \Rightarrow u \wedge v \lhd \underline{d}(a)^*((q,r) \wedge (q',r')). \end{aligned}$$

Going through the motions of 2.24, that means we have to prove: for all $(q,r),(q',r') \in \mathbb{Q} \times_< \mathbb{Q}$, $C \subseteq D \in \mathcal{C}(A)$ and $f, f', g, g' \in C_{\mathrm{sa}}$, if $(q'',r'') = (q,r) \wedge (q',r') \neq 0$, $f \leq a \leq g$ and $f' \leq a \leq g'$, then

$$\begin{aligned} &\mathtt{D}_{f-q} \wedge \mathtt{D}_{r-g} \wedge \mathtt{D}_{f'-q'} \wedge \mathtt{D}_{r'-g'} \\ &\lhd \{\mathtt{D}_{f''-q''} \wedge \mathtt{D}_{r''-g''} \mid f'', g'' \in D_{\mathrm{sa}}, f'' \leq a \leq g''\}. \end{aligned}$$

We distinguish the possible cases of (q'',r'') (which distinction is constructively valid since it concerns rationals). For example, if $(q'',r'') = (q,r')$, then $q \leq q' \leq r \leq r'$. So $\mathtt{D}_{f-q} \wedge \mathtt{D}_{r'-g'} = \mathtt{D}_{f''-q''} \wedge \mathtt{D}_{r''-g''}$ for $f'' = f$, $g'' = g'$, $q'' = q$ and $r'' = r'$, whence the statement holds by (a) and (c) of Definition 2.10. The other cases are analogous.

(c) We have to show that

$$\Vdash \forall_{(q,r)\in \underline{\mathbb{Q}\times_<\mathbb{Q}}}\forall_{U\in \underline{\mathcal{P}(\mathbb{Q}\times_<\mathbb{Q})}}.(q,r)\blacktriangleleft U \Rightarrow \underline{d}(a)^*(q,r)\lhd \bigcup_{(q',r')\in U}\underline{d}(a)^*(q',r').$$

By 2.24, we therefore have to prove: for all $(q,r)\in\mathbb{Q}\times_<\mathbb{Q}$, $U\subseteq U'\subseteq\mathbb{Q}\times_<\mathbb{Q}$, $D\in\mathcal{C}(A)$ and $f,g\in D_{\text{sa}}$, if $(q,r)\blacktriangleleft U$ and $f\leq a\leq g$, then

$$\mathtt{D}_{f-q}\wedge\mathtt{D}_{r-g}\lhd\{\mathtt{D}_{f'-q'}\wedge\mathtt{D}_{r'-g'}\mid (q',r')\in U', f',g'\in D_{\text{sa}}, f'\leq a\leq g'\}.$$

To establish this, it suffices to show $\mathtt{D}_{f-q}\wedge\mathtt{D}_{r-g}\lhd\{\mathtt{D}_{f-q'}\wedge\mathtt{D}_{r'-g}\mid (q',r')\in U\}$ when $(q,r)\blacktriangleleft U$. Let $s\in\mathbb{Q}$ satisfy $0<s$. Then one has $(q,r-s)<(q,r)$. Since $(q,r)\blacktriangleleft U$, 6.21 yields a $(q'',r'')\in U$ such that $(q,r-s)\leq(q'',r'')$, and so $r-s\leq r''$. Taking $U_0=\{(q'',r'')\}$, one has $r-g-s\leq r''-g$ and therefore $\mathtt{D}_{r-g-s}\leq\mathtt{D}_{r''-g}=\bigvee U_0$. So, by Corollary 4.14, we have $\mathtt{D}_{r-g}\lhd\{\mathtt{D}_{r'-g}\mid (q',r')\in U\}$. Similarly, one finds $\mathtt{D}_{f-q}\lhd\{\mathtt{D}_{f-q'}\mid (q',r')\in U\}$. Finally, $\mathtt{D}_{f-q}\wedge\mathtt{D}_{r-g}\lhd\{\mathtt{D}_{f-q'}\wedge\mathtt{D}_{r'-g}\mid (q',r')\in U\}$ by Definition 2.10(d). □

6.23 Definition Let A be a C*-algebra. The *Daseinisation* of $a\in A_{\text{sa}}$ is the locale morphism $\underline{\delta}(a)\colon\underline{\Sigma}(\underline{A})\to\underline{\mathbb{IR}}$, whose associated frame morphism $\underline{\delta}(a)^{-1}$ is given by $\mathcal{F}(\underline{d}(a)^*)$, where $\mathcal{F}$ is the functor of Proposition 2.15, and $\underline{d}(a)$ comes from Lemma 6.22.

6.24 Example The locale $\underline{\Sigma}(\underline{A})$ is described externally by its value at $\mathbb{C}\in\mathcal{C}(A)$, see Theorem 4.16. The component at $\mathbb{C}$ of the Daseinisation $\underline{\delta}(a)$ is given by

$$\underline{\delta}(a)^{-1}_{\mathbb{C}}(q,r)(C)=\{\mathtt{D}_{f-q}\wedge\mathtt{D}_{r-g}\mid f,g\in C_{\text{sa}}, f\leq a\leq g\}.$$

Now suppose that A is commutative. Then, classically, $\mathtt{D}_a=\{\rho\in\Sigma(A)\mid\rho(a)>0\}$ as in 3.9. Hence $\mathtt{D}_{f-r}=\{\rho\in\Sigma(A)\mid\rho(f)>r\}$, so that

$$\begin{aligned}\underline{\delta}(a)^{-1}_{\mathbb{C}}(q,r)(C)&=\bigcup_{\substack{f,g\in C_{\text{sa}}\\ f\leq a\leq g}}\{\rho\in\Sigma(A)\mid\rho(f)>q\text{ and }\rho(g)<r\}\\&=\{\rho\in\Sigma(A)\mid\exists_{f\leq a}.q<\rho(f)<r\text{ and }\exists_{g\geq a}.q<\rho(g)<r\}\\&=\{\rho\in\Sigma(A)\mid q<\rho(a)<r\}\\&=\hat{a}^{-1}(q,r).\end{aligned}$$

6.25 Proposition *The map $\underline{\delta}\colon A_{\text{sa}}\to C(\underline{\Sigma}(\underline{A}),\underline{\mathbb{IR}})$ is injective. Moreover $a\leq b$ if and only if $\underline{\delta}(a)\leq\underline{\delta}(b)$.*

Proof Suppose that $\underline{\delta}(a)=\underline{\delta}(b)$. Then for all $C\in\mathcal{C}(A)$, the sets $L_a(C)=\{f\in C_{\text{sa}}\mid f\leq a\}$ and $U_a(C)=\{g\in C_{\text{sa}}\mid a\leq g\}$ must coincide with $L_b(C)$ and $U_b(C)$, respectively. Imposing these equalities at $C=C^*(a)$ and at $C=C^*(b)$ yields $a=b$. The order in A_{sa} is clearly preserved by $\underline{\delta}$, whereas the converse implication can be shown by the same method as the first claim of the proposition. □

6.26 Given a state ρ of a C*-algebra A, an observable $a \in A_{\mathrm{sa}}$, and an interval (q,r) with rational endpoints $q, r \in \mathbb{Q}$, we can now compose the morphisms of 6.16, 6.20 and Definition 6.23 to obtain a truth value

$$\underline{1} \xrightarrow{(q,r)} \mathcal{O}(\underline{\mathbb{IR}}) \xrightarrow{\underline{\delta}(a)^{-1}} \mathcal{O}(\underline{\Sigma}(\underline{A})) \xrightarrow{[\rho]} \underline{\Omega}.$$

Unfolding definitions, we find that at $\mathbb{C} \in \mathcal{C}(A)$ this truth value is given by

$$\begin{aligned}
&([\rho] \circ \underline{\delta}(a)^{-1} \circ \underline{(q,r)})_{\mathbb{C}}(*) \\
&= [\![\mu_\rho(\underline{d}(a)^*(q,r)) = 1]\!](C) \\
&= \{C \in \mathcal{C}(A) \mid C \Vdash \mu_\rho(\underline{d}(a)^*(q,r)) = 1\} \\
&= \{C \in \mathcal{C}(A) \mid C \Vdash \mu_\rho(\bigvee_{\substack{f,g \in C_{\mathrm{sa}} \\ f \leq a \leq g}} \mathtt{D}_{f-q} \wedge \mathtt{D}_{r-g}) = 1\} \\
&= \{C \in \mathcal{C}(A) \mid C \Vdash \mu_\rho(\bigvee_{f \leq a} \mathtt{D}_{f-q}) = 1,\ C \Vdash \mu_\rho(\bigvee_{g \geq a} \mathtt{D}_{r-g}) = 1\}.
\end{aligned}$$

By Theorem 6.13 and 2.24, $C \Vdash \mu_\rho(\bigvee_{f \leq a} \mathtt{D}_{f-q}) = 1$ if and only if for all $n \in \mathbb{N}$ there are $m \in \mathbb{N}$ and $f \in C_{\mathrm{sa}}$ such that $f \leq a$ and $\rho(m(f-q)^+ \wedge 1) > 1 - \frac{1}{n}$. Hence the above truth value is given by

$$\begin{aligned}
\{C \in \mathcal{C}(A) \mid \forall_{n \in \mathbb{N}} \exists_{m \in \mathbb{N}} \exists_{f,g \in C_{\mathrm{sa}}} . f \leq a \leq g,\ \rho(m(f-q)^+ \wedge 1) > 1 - \frac{1}{n},& \\
\rho(m(r-g)^+ \wedge 1) > 1 - \frac{1}{n}\}.&
\end{aligned}$$

6.27 If A is a von Neumann algebra, the pairing formula of 6.26 simplifies further. Using the external description of the Bohrified state space in Theorem 5.16, one finds that the following are equivalent for a general open $F \in \mathcal{O}(\Sigma_A)$ and a state $\rho \colon A \to \mathbb{C}$:

$$\begin{aligned}
&C \Vdash \mu_\rho(F) = 1, \\
&C \Vdash \forall_{q \in \underline{\mathbb{Q}}, q < 1} . \mu_\rho(F) > q, \\
\text{for all } D \supseteq C \text{ and rational } q < 1:\ &D \Vdash \mu_\rho(F) > q, \\
\text{for all } D \supseteq C \text{ and rational } q < 1:\ &D \Vdash \exists_{u \in F} . \mu_\rho(u) > q, \\
\text{for all } D \supseteq C \text{ and } q < 1,\ \text{there is } p \in F(D) \text{ with } &D \Vdash \mu_\rho(p) > q, \\
\text{for all } q < 1,\ \text{there is } p \in F(C) \text{ with } &\rho(p) > q, \\
&\sup_{p \in F(C)} \rho(p) = 1.
\end{aligned}$$

By Proposition 5.8 and Theorem 5.15 one may choose basic opens $\mathcal{D}_{f-q}$ of the spectrum $\underline{\Sigma}(\underline{A}))$ corresponding to projections $[f - q > 0]$ of $\underline{\mathrm{Proj}}(\underline{A})$. Let us now return to the case $F(C) = \{\mathtt{D}_{f-q} \mid f \in C_{\mathrm{sa}}, f \leq a\}$. By Theorem 5.16, $F(C)$ is generated by projections, and by Theorem 5.4, we can take their supremum, so that $(\bigvee F)(C) = \bigvee\{[f-q>0] \mid f \in C_{\mathrm{sa}}, f \leq a\}$. Hence the above forcing condition $C \Vdash \mu_\rho(\bigvee_{f \leq a} \mathtt{D}_{f-q}) = 1$ is equivalent to $\rho(\bigvee\{[f-q>0] \mid f \in C_{\mathrm{sa}}, f \leq a\}) = 1$. Thus the pairing formula of 6.26 results in the truth value

$$\begin{aligned}
\{C \in \mathcal{C}(A) \mid &\rho(\bigvee\{[f-q>0] \mid f \in C_{\mathrm{sa}}, f \leq a\}) = 1, \\
&\rho(\bigvee\{[r-g>0] \mid g \in C_{\mathrm{sa}}, a \leq g\}) = 1\},
\end{aligned}$$

listing the "possible worlds" C in which the proposition $a \in (q, r)$ holds in state ρ in the classical sense.

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