REMARKS ON THE THIN OBSTACLE PROBLEM AND CONSTRAINED GINIBRE ENSEMBLES

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Abstract. We consider the problem of constrained Ginibre ensemble with prescribed portion of eigenvalues on a given curve \( \Gamma \subset \mathbb{R}^2 \) and relate it to a thin obstacle problem. The key step in the proof is the \( H^1 \) estimate for the logarithmic potential of the equilibrium measure. The coincidence set has two components: one in \( \Gamma \) and another one in \( \mathbb{R}^2 \setminus \Gamma \) which are well separated. Our main result here asserts that this obstacle problem is well posed in \( H^1(\mathbb{R}^2) \) which improves previous results in \( H^1_{loc}(\mathbb{R}^2) \).

1. Introduction

Let \( \Gamma \) be a regular curve in \( \mathbb{R}^2 \) with locally finite length and \( \mathcal{M}_a \) the set of all probability measures such that

\[
\mu(\Gamma) \geq a, \quad a \in (0, 1).
\]

By an abuse of notation we let \( \Gamma : \mathbb{R} \to \mathbb{R}^2 \) be the arc-length parameterization of the curve such that

\[
|\dot{\Gamma}(t)| = 1, \quad t \in \mathbb{R}.
\]

In this paper we consider the minimizers of the energy

\[
I[\mu] = \int \int \log \frac{1}{|x - y|} d\mu(x) d\mu(y) + \int Q d\mu
\]

where \( Q(x) \) is a given function such that the weight function \( w = e^{-Q} \) on \( \mathbb{R}^2 \) is admissible (see Definition 1.1 p.26 [8]). This means that \( w \) satisfies the following three conditions:

(H1) \( w \) is upper semi-continuous;

(H2) \( \{ w \in \mathbb{R}^2 \ s.t. \ w(z) > 0 \} \) has positive capacity;

(H3) \( |z| w(z) \to 0 \) as \( |z| \to \infty \).

In higher dimensions \( \mathbb{R}^d, d \geq 3 \) one can consider more general kernels

\[
K(x - y) = \begin{cases} 
\log \frac{1}{|x - y|}, & d = 2, \\
\frac{1}{|x - y|^{d-2}}, & d \geq 3,
\end{cases}
\]

with \( \Gamma \) being a Lyapunov surface in \( \mathbb{R}^d \) and define the energy as follows

\[
I[\mu] = \int \int K(x - y) d\mu(x) d\mu(y) + \int Q d\mu.
\]

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In this note we mostly confine ourselves with quadratic potentials \( Q(x) = |x|^2 \) in \( \mathbb{R}^2 \), although all our results remain valid for more general \( Q \) satisfying (H1) – (H3). Furthermore, our main result on global \( L^2 \) estimate of the gradient of the equilibrium potential with kernel \( K(x-y) = |x-y|^{-d} \) remains valid in \( \mathbb{R}^d, d \geq 3 \), see Theorem 4.1.

The functional \( I[\mu] \), with \( Q = |x|^2, d = 2 \), arises in the description of the convergence of the spectral measure of square \( N \times N \) matrices with complex independent, standard Gaussian entries (i.e., the Ginibre ensemble) as \( N \to \infty \). In case when there are no constraints imposed on the eigenvalues, it is well known that the eigenvalues spread evenly in the ball of radius \( \sqrt{N} \), and after renormalization by a factor \( \frac{1}{\sqrt{N}} \) the normalized spectral measure converges to the characteristic function of the unit disc. This is known as the circular law [4], [2]. In this context the functional \( I \) is used to prove large deviation principles for the spectral measure.

If one demands that the eigenvalues are real (i.e. when \( a = 1, \Gamma = \mathbb{R} \)) we get the so called semicircle law. More generally, one can demand that a portion of eigenvalues is contained in a prescribed set \( \Gamma \). This is considered in [2] when a portion of eigenvalues are contained in an open bounded subset of \( \mathbb{R}^2 \) and in [4] when \( \Gamma \) is a line. These problems can be related to the thin obstacle and obstacle problems respectively. The key step in proving this is to establish \( H^1_{\text{loc}}(\mathbb{R}^2) \) estimates for the logarithmic potential

\[
U^{\mu_a} = K \ast \mu_a
\]

of the corresponding equilibrium measure. The aim of this note is to show that the thin obstacle problem is well-posed in \( H^1(\mathbb{R}^2) \) by showing that in fact \( U^{\mu_a} \in H^1(\mathbb{R}^2) \), see Theorem 4.1. This improves the previous results in [2] and [4].

The paper is organized as follows: In the next section we prove the existence and uniqueness of the equilibrium measure \( \mu_a \) minimizing the energy \( I[\mu] \). In section 3 we discuss some basic properties of \( \mu_a \). In particular we show that there are two positive constants \( A_\Gamma \) and \( A_0 \) such that \( 2U^{\mu_a} + Q = A_\Gamma \) on \( \text{supp} \mu_a \cap \Gamma \) and \( 2U^{\mu_a} + Q = A_0 \) on \( \text{supp} \mu_a \setminus \Gamma \). Furthermore, \( A_\Gamma > A_0 \). This fact will be used later to show that \( \text{supp} \mu_a \setminus \Gamma \) and \( \text{supp} \mu_a \cap \Gamma \) are disjoint.

Our main result Theorem 4.1 is contained in section 4. To prove it we study the Fourier transformations of \( U^{\mu_a} \) and \( \mu_a \). It leads to some integral identity involving Bessel functions. This approach is based on a method of L. Carleson [3]. Finally, combining the results obtained, in section 5 we show that \( U^{\mu_a} \) solves the obstacle problem where the obstacle is given by

\[
\psi(x) = \begin{cases} 
\frac{1}{2}(A_\Gamma - |x|^2) & \text{if } x \in \Gamma, \\
\frac{1}{2}(A_0 - |x|^2) & \text{if } x \in \mathbb{R}^2 \setminus \Gamma.
\end{cases}
\]

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\end{cases}
\]

2. Existence of minimizers

In this section we show the existence of a unique equilibrium measure.

\textbf{Theorem 2.1.} Suppose \( d = 2, \Gamma \subset \mathbb{R}^2 \) is a regular \( C^{1,\alpha} \) smooth planar curve without self-intersections. There is a unique minimizer \( \mu_a \in \mathcal{M}_a \) of \( I[\mu] \) such that

\[
I[\mu_a] = \inf_{\mu \in \mathcal{M}_a} I[\mu].
\]

\textbf{Proof.} Observe that the uniqueness follows from the convexity of \( \mathcal{M}_a \) and can be proved as in [4]. Moreover, \( I[\mu] \) is also semicontinuous. Thus, we have to show that \( I[\mu] \) is bounded by below for all \( \mu \in \mathcal{M}_a \).
and there is at least one \( \mu_0 \) such that \( I[\mu] \) is finite. The lower bound follows as in the proof of Theorem 1.3 (a) p. 27 [8].

It remains show that the \( \inf_{\mu \in \mathcal{M}_+} I[\mu] < \infty \). Let \( \chi_D \) denote the characteristic function of the set \( D \) and take

\[
\mu = a \frac{1}{L} \mathcal{H}^1(\Gamma \cap \Omega) + (1 - a) \frac{1}{|B|} \chi_B
\]

where \( B = B_\rho(z) = \{ x \in \mathbb{R}^2 : |x - z| < \rho \} \) with small \( \rho \) such that \( B \subset \Omega, \Omega \subset \mathbb{R}^2 \) is a compact, \( L = \mathcal{H}^1(\Gamma \cap \Omega) > 0, \) and \( \text{dist}(\Gamma, B) > 0 \). Observe that for this choice of \( \mu \) we have

\[
\int_\Omega \log \frac{1}{|x - y|} d\mu(x) = \frac{1}{L} \int_0^L \log \frac{1}{|\Gamma(t) - y|} dt + \frac{1}{|B|} \int_B \log \frac{1}{|x - y|} d\mu(x).
\]

Assuming that \( \Gamma \) is given by arc-length parametrization we have for the logarithmic energy (2.1)

\[
\mathcal{L}[\mu] = \frac{a^2}{L^2} \int_0^L \int_0^L \log \left| \frac{1}{|\Gamma(t) - \Gamma(s)|} \right| dt ds + \frac{2a(1 - a)}{L|B|} \int_0^L \int_B \log \left| \frac{1}{|\Gamma(t) - y|} \right| dtdy + \frac{(1 - a)^2}{|B|^2} \int_B \int_B \log \frac{1}{|x - y|} dx dy.
\]

Since \( \text{dist}(\Gamma, B) > 0 \) then the second integral is bounded. As for the last integral then after change of variables \( x - y = \xi \) we have

\[
\int_{B_\rho(z)} \log \left| \frac{1}{|x - y|} \right| dx = \int_{B_\rho(z - y)} \log \left| \frac{1}{|\xi|} \right| d\xi \leq \int_{B_\rho(0)} \log \left| \frac{1}{|\xi|} \right| dx < \infty
\]

where we used \( |x - y| \leq \rho \) and the fact that \( \rho \) is small by construction.

It remains to check that the first integral is finite. Let us fix \( s \in [0, L] \). Then we have that

\[
\int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} dt = \int_{-s}^{L-s} \log \frac{1}{|\Gamma(t + s) - \Gamma(s)|} d\tau =
\]

\[
= \tau \log \left| \frac{1}{|\Gamma(t + s) - \Gamma(s)|} \right| - \int_{-s}^{L-s} \frac{1}{|\Gamma(t + s) - \Gamma(s)|^2} (\Gamma(t + s) - \Gamma(s)) d\tau =
\]

\[
= (L - s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} - I_0
\]

where \( I_0 \) is the last integral. Using the crude estimate

\[
|I_0| \leq \int_{-s}^{L-s} \tau \left| \frac{1}{|\Gamma(t + s) - \Gamma(s)|} \right| d\tau = \int_{-s}^{L-s} \left| \frac{1}{|\Gamma(t + s) - \Gamma(s)|} \right| d\tau =
\]

\[
= \int_{[-s, L-s] \setminus (-\delta, \delta)} \left| \frac{1}{|\Gamma(t + s) - \Gamma(s)|} \right| d\tau + \int_{-\delta}^{\delta} \left| \frac{1}{|\Gamma(t + s) - \Gamma(s)|} \right| d\tau 
\]

\[
\leq \frac{4L^2}{C_\delta} + \int_{-\delta}^{\delta} \left| \frac{1}{|\Gamma(t + s) - \Gamma(s)|} \right| d\tau
\]

because \( |\Gamma(t + s) - \Gamma(s)| \geq C_\delta \) if \( |\tau| \geq \delta \). Finally, from \( C^{1, \alpha} \) regularity of \( \Gamma \) we get

\[
|\Gamma(t + s) - \Gamma(s)| = |\tau| \left| \int_0^1 \frac{\dot{\Gamma}(\tau + s)}{d\sigma} - \int_0^1 \frac{\dot{\Gamma}(\tau + s) - \dot{\Gamma}(\sigma)}{d\sigma} \right| 
\]

\[
\geq |\tau| \left( \left| \int_0^1 \frac{\dot{\Gamma}(\sigma + s)}{d\sigma} - \int_0^1 \frac{\dot{\Gamma}(\sigma + s) - \dot{\Gamma}(s)}{d\sigma} \right| \right) 
\]

\[
\geq |\tau| (1 - \delta^\alpha).
\]
Combining (2.3) with (2.2) we get
\[ |I_0| \leq \frac{4L^2}{C_\delta} + 2\delta (1 - \delta^\alpha) < \infty. \]

Returning to the first integral in (2.1) we infer
\[
\begin{align*}
\int_0^L \int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} dt ds & \leq \int_0^L \left\{ (L - s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} + \frac{4L^2}{C_\delta} + 2\delta (1 - \delta^\alpha) \right\} ds \\
& \leq L \left[ \frac{4L^2}{C_\delta} + 2\delta (1 - \delta^\alpha) \right] + L \log \frac{1}{C_\delta} + \\
& \quad + \int_\delta^{L-\delta} \left\{ (L - s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} \right\} ds \\
& \leq C(\delta, L)
\end{align*}
\]
if we choose \( \delta > 0 \) suitably small. This finishes the proof for \( d = 2 \).

Remark 2.2. If \( d \geq 3 \), \( Q(x) = |x|^2 \) then clearly \( I[\mu] \geq 0 \). The upper estimate for \( I[\mu] \) follows from a similar argument if we assume that \( \Gamma \) is a Lyapunov surface and take
\[
\mu = a \frac{1}{L} \mathcal{H}^{d-1}(\Gamma \cap \Omega) + (1 - a) \frac{1}{|B|} \chi_B
\]
with \( L = \mathcal{H}^{d-1}(\Gamma \cap \Omega) \) and \( \text{dist}(B, \Gamma) > 0 \). Therefore Theorem 2.1 remains valid for \( d \geq 3 \).

3. Basic properties of minimizers

In this section we prove some basic properties of the equilibrium measure. The arguments are along the line of those in [2]. Therefore, we mostly focus on those aspects of the proofs which are new or differ essentially. The results to follow are valid in \( \mathbb{R}^d, d \geq 2 \) unless otherwise stated.

Lemma 3.1. Let \( \mu_a \) be as in Theorem 2.1. Then \( \mu_a(\Gamma) = a \).

Proof. If the claim fails then \( \mu_a(\Gamma) > a \). Fix \( \delta \in (0, a) \) and let \( \mu_{a-\delta} \) be the minimizer of \( I[\cdot] \) over \( \mathcal{M}_{a-\delta} \supset \mathcal{M}_a \). Form \( \mu = (1 - \varepsilon)\mu_a + \varepsilon \mu_{a-\delta}, \varepsilon \in [0, 1] \). Clearly, \( \mu \in \mathcal{M}_a \) if we choose \( \varepsilon \delta \) sufficiently small because
\[
\mu(\Gamma) > a + [\mu_a(\Gamma) - a] - \varepsilon \delta.
\]

Consequently, we have from the strict convexity of \( I \)
\[
\begin{align*}
I[(1 - \varepsilon)\mu_a + \varepsilon \mu_{a-\delta}] & < (1 - \varepsilon)I[\mu_a] + \varepsilon I[\mu_{a-\delta}] = I[\mu_a] + \varepsilon (I[\mu_a] - I[\mu_a]) \\
& \leq I[\mu_a]
\end{align*}
\]
which is in contradiction with the fact that \( \mu_a \) is a minimizer. \( \square \)

Observe that the Fréchet derivative of \( I[\mu] \) is \( 2U^{\mu_a} + Q \) where
\[
U^{\mu_a}(y) = \int K(x - y) d\mu_a(x).
\]
It is convenient to consider variations of the equilibrium measure in terms of affine combinations. More precisely, let \( \mu_\varepsilon = (1 - \varepsilon)\mu_a + \varepsilon\nu, \nu \in \mathcal{M}_a, \varepsilon \in [0, 1] \), then by direct computation we have that

\[
I[\mu_\varepsilon] = (1 - \varepsilon)^2 \int \int K(x - y)d\mu_a(x)d\mu_a(y) + 2\varepsilon(1 - \varepsilon) \int \int K(x - y)d\mu_a(x)d\nu(y) + \varepsilon^2 \int \int K(x - y)d\nu(x)d\nu(y) + (1 - \varepsilon) \int Qd\mu_a + \varepsilon \int Qd\nu
\]

\[
= I[\mu_a] + \varepsilon \left( 2 \int \int K(x - y)d\mu_a(x)d(\nu(y) - \mu_a) + \int Qd(\nu - \mu_a) \right) + O(\varepsilon^2) = I[\mu_a] + \varepsilon \int (2U^{\mu_a} + Q)d(\nu - \mu_a) + O(\varepsilon^2).
\]

Since \( \mu_a \) is the minimizer then \( I[\mu_a] \leq I[\mu] \), and after sending \( \varepsilon \to 0 \) it follows that

\[
(3.2) \quad \int (2U^{\mu_a} + Q)d(\nu - \mu_a) \geq 0, \quad \forall \nu \in \mathcal{M}_a.
\]

**Lemma 3.2.** Let \( A_\Gamma = \frac{1}{a} \int \Gamma (2U^{\mu_a} + Q)d\mu_a \) then quasi everywhere

\[
(3.3) \quad 2U^{\mu_a} + Q = A_\Gamma \quad \text{on} \quad \Gamma \cap \text{supp} \, \mu_a,
\]

\[
2U^{\mu_a} + Q \geq A_\Gamma \quad \text{on} \quad \Gamma.
\]

Similarly, let us denote \( A_0 = \frac{1}{1 - a} \int_{\mathbb{R}^2 \setminus \Gamma} (2U^{\mu_a} + Q)d\mu_a \) then

\[
(3.4) \quad 2U^{\mu_a} + Q = A_0 \quad \text{on} \quad \text{supp} \, \mu_a \setminus \Gamma,
\]

\[
2U^{\mu_a} + Q \geq A_0 \quad \text{on} \quad \mathbb{R}^2 \setminus (\text{supp} \, \mu_a \setminus \Gamma).
\]

Furthermore,

\[
(3.5) \quad A_\Gamma > A_0.
\]

**Proof.** We first prove (3.3). Suppose that there is a set capacitable \( E \) of positive capacity such that \( \Gamma \cap E \) has zero capacity and

\[
2U^{\mu_a} + Q < A_\Gamma - \delta \quad \text{q.e. on} \quad E
\]

for some positive \( \delta \). Let \( \mu_E \) be the equilibrium measure of \( E \) and form \( \nu = \mu_a \mathbb{1}_{(\mathbb{R}^2 \setminus \Gamma)} + a\mu_E \). Clearly \( \nu \in \mathcal{M}_a \). Therefore, in view of (3.1) for the measure \( \mu_\varepsilon = \varepsilon \mu_a + (1 - \varepsilon)\nu \in \mathcal{M}_a \) we get

\[
I[\mu_\varepsilon] = I[\mu_a] + \varepsilon \left( 2 \int \int K(x - y)d\mu_a(x)d(\nu(y) - \mu_a) + \int Qd(\nu - \mu_a) \right) + O(\varepsilon^2)
\]

\[
= I[\mu_a] + \varepsilon \int (2U^{\mu_a} + Q)d(a\mu_E - \mu_a) + O(\varepsilon^2)
\]

\[
= I[\mu_a] + \varepsilon \left( a \int \Gamma (2U^{\mu_a} + Q)d\mu_E - aA_\Gamma \right) + O(\varepsilon^2)
\]

\[
< I[\mu_a] - a\varepsilon\delta + O(\varepsilon^2)
\]

\[
< I[\mu_a]
\]
if $\varepsilon$ and $\delta$ are sufficiently small. This will be in contradiction with the fact that $\mu_a$ is the minimizer. Thus we have proved that $2U^{\mu_a} + Q \geq A_\Gamma$ q.e. on $\Gamma$.

Next we show that on $\text{supp } \mu_a \cap \Gamma$ we have $2U^{\mu_a} + Q = A_\Gamma$ q.e. Indeed, from the definition of $A_\Gamma$ it follows

$$aA_\Gamma = \int_\Gamma (2U^{\mu_a} + Q) d\mu_a \geq aA_\Gamma$$

where the last inequality follows from the first inequality in (3.3). The proof of (3.4) is similar. In order to prove the last claim $A_\Gamma > A_0$ we first observe that there exists a measure $\nu \in \mathcal{M}_a$ such that

- $a > \nu(\Gamma)$,
- $I[\nu] \leq I[\mu_a]$.

First notice that $\mathcal{M}_a \subset \mathcal{M}_{a-\delta}$ for $\delta \in (0, a)$. Fix such $\delta > 0$ and let $\mu_{a-\delta}$ be the minimizer of $I[\cdot]$ over $\mathcal{M}_{a-\delta}$. Then by Lemma 3.1 $\mu_{a-\delta}(\Gamma) = a - \delta < a$ and $I[\mu_{a-\delta}] = \inf_{\mathcal{M}_{a-\delta}} I[\mu] \leq I[\mu_a] = \inf_{\mathcal{M}_a} I[\mu]$. Therefore one can take $\nu = \mu_{a-\delta}$.

From the strict convexity of $I$ it follows that

$$I[\nu] > I[\mu_a] + \langle DI[\mu_a], \nu - \mu_a \rangle$$

where $DI[\mu] = 2U^{\mu} + Q$ is the Fréchet derivative of $I[\mu]$. Therefore, from the properties of $\nu$ we infer

(3.7) $0 \geq I[\nu] - I[\mu_a] > \langle DI[\mu_a], \nu - \mu_a \rangle$

or equivalently

$$\langle 2U^{\mu_a} + Q, \nu - \mu_a \rangle < 0.$$  

On the other hand

(3.8) $\int_\Gamma (2U^{\mu_a} + Q) d\mu_a = aA_\Gamma + (1 - a)A_0$

while

$$\int_\Gamma (2U^{\mu_a} + Q) d\nu = \int_\Gamma (2U^{\mu_a} + Q) d\nu + \int_{\mathbb{R}^2 \setminus \Gamma} (2U^{\mu_a} + Q) d\nu \geq \nu(\Gamma) A_\Gamma + \nu(\mathbb{R}^2 \setminus \Gamma) A_0.$$  

This together with (3.8), (3.7) yields

$$aA_\Gamma + (1 - a)A_0 > \nu(\Gamma) A_\Gamma + (1 - \nu(\Gamma)) A_0 \Rightarrow A_0 (\nu(\Gamma) - a) > A_\Gamma (\nu(\Gamma) - a).$$

Finally, the property $\nu(\Gamma) < a$ implies that $A_\Gamma > A_0$. \hfill \square

**Corollary 3.3.** $\text{supp } \mu_a$ is compact.

**Proof.** If $d \geq 3$ then $K(x - y) \geq 0$, hence by Lemma 3.2 for $x \in \text{supp } \mu_a$ we have

(3.9) $\max(A_\Gamma, A_0) \geq 2U^{\mu_a}(x) + Q(x) \geq Q(x) \to \infty$ if $|x| \to \infty$

which is a contradiction. If $d = 2$ then from the triangle inequality we get that

(3.10) $K(x - y) \geq -\log |x| - \log \left(1 + \frac{|y|}{|x|}\right).$

Consequently, for $x \in \text{supp } \mu_a$

$$\max(A_\Gamma, A_0) \geq 2U^{\mu_a}(x) + Q(x) \geq Q(x) - 2 \log |x| - \int \log \left(1 + \frac{|y|}{|x|}\right) d\mu_a$$

$$= Q(x) - 2 \log |x| + O(1) \to \infty \quad \text{if } |x| \to \infty.$$
for sufficiently large $|x|$, where the last inequality follows from (4.12) and $\int Qd\mu_a < I[\mu_a] < \infty$. Since $Q = |x|^2$ (or for the general case from the hypotheses on $Q$ (H1) – (H3)) it again follows that $\text{supp } \mu_a$ is bounded. □

4. Global $L^2$ estimates for $U^{\mu_a}$ and $\nabla U^{\mu_a}$

Our main result is contained in the following

**Theorem 4.1.** Let $U^{\mu_a}(y) = \int K(x - y)d\mu_a$, if $d \geq 3$ then $\nabla U^{\mu_a} \in L^2(\mathbb{R}^d)$. If $d = 2$ then $U^{\mu_a} \in H^1(\mathbb{R}^2)$.

Furthermore, there holds

\begin{equation}
\|U^{\mu_a}\|_{H^1(\mathbb{R}^2)} \leq CE[\mu_a].
\end{equation}

Here $E[\mu]$ is the energy of $\mu$ defined as $\int \int K(x - y)d\mu(x)d\mu(y)$.

**Remark 4.2.** It is shown in [3] that $E[\mu] > 0$ for any probability measure $\mu$ and $d \geq 2$. In fact, this can be seen from the proof to follow (see also Corollary 4.3).

**Proof.** The case $d \geq 3$ follows from Lemma 1.6 p. 92 [7] (see also Lemma 17 p. 95), which assert that

\[ \frac{\partial U^{\mu_a}(x)}{\partial x_i} = \int \frac{\partial K(x - y)}{\partial x_i}d\mu_a \]

almost everywhere and moreover

\[ \frac{1}{4\pi^2} \int_{\mathbb{R}^d} |\nabla U^{\mu_a}|^2 \leq \int \int K(x - y)d\mu_a(x)d\mu_a(y) = E[\mu_a]. \]

The case of the logarithmic potential follows from a modification of the argument by L. Carleson [3] Lemma 3 page 22. We begin with computing the Fourier transformation of $K$. Note that since $\text{supp } \mu_a$ is compact we can assume that $K(r) = 0$ for $r \geq r_0$ for some fixed $r_0 > 0$. We have

\[ \hat{K}(\xi) = \int K(x)e^{-2\pi i \langle x, \xi \rangle}dx = \int K(x)e^{-2\pi i \langle x, \frac{\xi}{|\xi|} \rangle}dx = \frac{1}{4\pi^2|\xi|^2} \int K\left(\frac{y}{2\pi|\xi|}\right)e^{i\langle y, \frac{\xi}{|\xi|} \rangle}dx. \]

Let us denote $K_0(y) = K\left(\frac{y}{2\pi|\xi|}\right)$ and define

\[ F(\eta) = \int K_0(y)e^{i\pi \langle y, \eta \rangle}, \quad \eta = \frac{\xi}{|\xi|}. \]

From Lemma 2 p. 21 [3] it follows that there is a universal constant $c_1$ such that

\[ F(\eta) = c_1 \int_0^\infty K_0(r)J(r)rdr, \quad |\eta| = 1 \]

where $J$ is the Bessel function

\begin{equation}
J(r) = -J''(r) - \frac{J'(r)}{r}, \quad J(0) = 1, J'(0) = 0, \quad J(r) < 1, r \neq 0.
\end{equation}
Therefore $F(\eta)$ can be further simplified as follows

\[(4.3)\quad F(\eta) = -c_1 \int_0^\infty K_0(r)(rJ(r))'dr = c_1 \int_0^{2\pi|\xi| r_0} rJ'(r)K_0'(r)dr\]

because from the definition of $K_0$ we have $\text{supp } K_0 \subset [0, 2\pi|\xi| r_0]$. Moreover, $K_0'(r) = -\frac{1}{r}$ hence

\[(4.4)\quad F(\eta) = c_1(1 - J(2\pi|\xi| r_0)).\]

Consequently,

\[(4.5)\quad \hat{K}(\xi) = \frac{c_1}{4\pi^2|\xi|^2}(1 - J(2\pi|\xi| r_0)).\]

Next we restrict $\mu_1 = \mu_a \upharpoonright \mathcal{C}$ where $\mathcal{C} \subset \text{supp } \mu_a$ is a compact such that $U^{\mu_1}$ is continuous. Observe that $\int U^{\mu_a}d\mu_a$ is finite hence $U^{\mu_a}$ is finite $\mu_a$ almost everywhere. By Theorem 1.8 p. 70 [7] for every $\varepsilon > 0$ small there is a restriction of $\mu_a$ such that

\[0 \leq \int \mu_a - \int \mu_1 < \varepsilon.\]

Note that if $\tau = \mu_a - \mu_1$ then we have

\[|E[\mu_a] - E[\mu_1]| = \left| \int U^{\mu_a-\mu_1}d\mu_a + \int U^{\mu_a-\mu_1}d\mu_1 \right| = \left| \int (U^{\mu_a} + U^{\mu_1})d\tau \right| = O(\varepsilon).\]

Let $\phi_n(y) = n^\frac{d}{2}e^{-n\pi|y|^2}$ be the sequence of normalised Gaussian kernels. It is well-known that $\phi_n$ is a mollification kernel for every $n \in \mathbb{N}$ and moreover $\hat{\phi}_n = e^{-\frac{1}{n}|\xi|^2}$. From the Parseval relation

\[(4.6)\quad \int (\phi_n * U^{\mu_1})d\mu_1 = \int \hat{\phi}_n \hat{K}|\mu_1|^2.\]

If we first send $n \to \infty$ and then $\varepsilon \to 0$ to conclude the identity

\[(4.7)\quad E[\mu_a] = \int \hat{K}|\mu_a|^2.\]

On the other hand $\hat{U}^{\mu_a} = \hat{K}\hat{\mu_a}$, which yields

\[(4.8)\quad E[\mu_a] = \int \hat{K}(\xi)\frac{|\hat{U}^{\mu_a}(\xi)|^2}{|\hat{K}(\xi)|^2}d\xi = \int \frac{4\pi^2|\xi|^2}{c_1(1 - J(2\pi r_0|\xi|))}|\hat{U}^{\mu_a}(\xi)|^2d\xi = \int_{|\xi| < \delta} + \int_{|\xi| \geq \delta}.\]

Using the expansion $J(t) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(s!)^2} \left( \frac{1}{2} \right) 2^s = 1 - \frac{t^2}{4} + \frac{t^4}{64} + \ldots$ we see that

\[\frac{4\pi^2|\xi|^2}{c_1(1 - J(2\pi r_0|\xi|))} = \frac{1}{r_0^2c_1} \left( 1 - \frac{2\pi r_0|\xi|}{t_0} + \frac{4}{16} \ldots \right)\]
hence the first integral is bounded below by \( C(\delta) \frac{1}{r_0} \int_{|\xi| < \delta} |\hat{U}^{\mu_a}(\xi)|^2 d\xi \) for sufficiently small \( \delta > 0 \). As for the second integral, we have

\[
(4.9) \quad \int_{|\xi| \geq \delta} \frac{4\pi^2|\xi|^2}{c_1(1 - J(2\pi r_0|\xi|))} |\hat{U}^{\mu_a}(\xi)|^2 d\xi \geq \frac{4\pi^2\delta^2}{c_1} \int_{|\xi| \geq \delta} |\hat{U}^{\mu_a}(\xi)|^2 d\xi.
\]

Combining we see that \( \hat{U}^{\mu_a} \in L^2(\mathbb{R}^2) \) which, after we apply Parseval’s relation again, yields \( U^{\mu_a} \in L^2(\mathbb{R}^2) \) and

\[
(4.10) \quad ||U^{\mu_a}||_{L^2(\mathbb{R}^2)} \leq C\mathcal{E}[\mu_a].
\]

To finish the proof we use that \( 4\pi^2|\xi|^2|\hat{U}^{\mu_a}|^2 = |\nabla \hat{U}^{\mu_a}|^2 \) which together with (4.8) implies that

\[
(4.11) \quad \mathcal{E}[\mu_a] = \int \frac{1}{c_1(1 - J(2\pi r_0|\xi|))} |\nabla \hat{U}^{\mu_a}(\xi)|^2 d\xi \geq \frac{1}{c_1} \int |\nabla \hat{U}^{\mu_a}(\xi)|^2 d\xi
\]

which finishes the proof.

\[ \square \]

**Corollary 4.3.** Let \( \mu_a \) be as in Theorem 2.1. Then there holds

\[
(4.12) \quad \mathcal{E}[\mu_a] = \int U^{\mu_a} d\mu_a > 0.
\]

5. **The thin obstacle problem**

From the \( H^1(\mathbb{R}^2) \) estimate for \( U^{\mu_a} \) it follows that \( U^{\mu_a} \) is a solution to some variational inequality, and hence \( U^{\mu_a} \) can be interpreted as a solution to an obstacle problem with a combination of both thin (on \( \Gamma \)) and ”thick” obstacles (on \( \mathbb{R}^2 \setminus \Gamma \)). It is convenient to define the obstacle as follows

\[
(5.1) \quad \psi(x) = \begin{cases} 
\frac{1}{2}(A_{\Gamma} - |x|^2) & \text{if } x \in \Gamma, \\
\frac{1}{2}(A_0 - |x|^2) & \text{if } x \in \mathbb{R}^2 \setminus \Gamma.
\end{cases}
\]

**Lemma 5.1.** Let \( U^{\mu_a} \) be the logarithmic potential of \( \mu_a \) and define

\[
\mathcal{K} = \{ v \in H^1_{loc}(\mathbb{R}^2) \text{ s.t. } v - U^{\mu_a} \text{ has bounded support in } \mathbb{R}^2, \, v \geq \psi \}.
\]

Then \( U^{\mu_a} \) solves the following obstacle problem:

\[
\int \nabla U^{\mu_a} \nabla (v - U^{\mu_a}) \geq 0, \quad \forall v \in \mathcal{K}.
\]

The proof is the same as in [2].

**Corollary 5.2.** \( \text{dist}(\Gamma, \text{supp}(\mu_a \setminus \Gamma)) > 0. \)

**Proof.** This follows from the estimate \( A_{\Gamma} > A_0. \) Indeed, let us assume that \( x_0 \in \Gamma \cap \text{supp} \mu_a \) and there is a sequence \( \{x_k\}_{k=1}^\infty, x_k \in \text{supp} \mu_a \setminus \Gamma \) such that \( \lim_{k \to \infty} x_k = x_0. \) Using the lower semicontinuity of \( U^{\mu_a} \) (see Lemma 1 p.15 [3]) we see that

\[
(5.2) \quad \frac{1}{2}(A_0 - |x_0|^2) = \liminf_{x_k \to x_0} U^{\mu_a}(x_k) \geq U^{\mu_a}(x_0).
\]

Let \( \rho > 0 \) be such that \( \{x_k\} \subset B_{\rho}(x_0). \) If \( \rho \) is small then \( \Gamma \) divides \( B_{\rho}(x_0) \) into two parts \( D^+ \) and \( D^- \). To fix the ideas let us suppose that \( D^+ \) contains a subsequence \( \{x_k\}. \) Let \( h \) be the harmonic function in \( D^+ \)
such that \( h = \psi \) on \( \partial D^+ \). Observe that \( h \) is continuous at \( x_0 \) because \( \Gamma \in C^{1,\alpha} \). Since \( U^{\mu_a} \) is superharmonic and on \( \partial D^+ \) we have \( U^{\mu_a} \geq \psi = h \) then the comparison principle implies that

\[
U^{\mu_a}(x_0) \geq h(x_0) = \frac{1}{2}(A_\Gamma - |x_0|^2).
\]

Combining (5.2) and (5.3) we see that \( A_0 \geq A_\Gamma \) which is a contradiction in view of (3.5). \( \square \)

From Corollary 5.2 it follows that near \( \Gamma \) the potential \( U^{\mu_a} \) is a solution to a thin obstacle problem in the following sense, see [5] p. 108:

\[
U^{\mu_a} \geq \frac{1}{2}(A_\Gamma - Q)
\]

\[
\frac{\partial U^{\mu_a}}{\partial n^+} + \frac{\partial U^{\mu_a}}{\partial n^-} \geq 0
\]

\[
(u - \frac{1}{2}(A_\Gamma - Q)) \left( \frac{\partial U^{\mu_a}}{\partial n^+} + \frac{\partial U^{\mu_a}}{\partial n^-} \right) = 0
\]

on \( \Gamma \)

where \( n^\pm \) are the outward normals on the \( \Gamma \) corresponding to the domains that \( \Gamma \) separates. In particular, if \( \Gamma \) is \( C^3 \) regular then \( U^{\mu_a} \) is \( C^{1,\alpha} \) up to \( \Gamma \) from each of its side, see Theorem 11.4 p.111 [5].

A particular case is \( \Gamma = \mathbb{R} \) [4]. Using a simple symmetrization argument (see e.g. [6] p. 119 Theorem 4.6) we can show that the potential \( U^{\mu_a} \) is symmetric w.r.t. the real line and hence we get the Signorini problem near \( \mathbb{R} \) [5] p. 111.

One can make the connections with the obstacle problem more explicit by using the \( H^1(\mathbb{R}^2) \) estimate in Theorem 4.1 and transforming the energy \( I[\mu_a] \). Let \( R > 0 \) be fixed then using the divergence theorem

\[
\int_{B_R} U^{\mu_a} d\mu_a = -\frac{1}{2\pi} \int_{\partial B_R} U^{\mu_a} \Delta U^{\mu_a} = -\frac{1}{2\pi} \int_{\partial B_R} U^{\mu_a} \frac{\partial}{\partial n} U^{\mu_a}.
\]

For a.e. \( R > 0 \) the last integral can be estimated as follows

\[
\left| \int_{\partial B_R} U^{\mu_a} \frac{\partial}{\partial n} U^{\mu_a} \right| \leq \int_{\partial B_R} |U^{\mu_a}| |\nabla U^{\mu_a}| \leq \int_{\partial B_R} |U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2.
\]

From Theorem 4.1 and Fubini’s theorem it follows that

\[
\int_{\mathbb{R}^2} (|U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2) = \int_0^\infty \int_{\partial B_R} (|U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2) dR.
\]

Consequently,

\[
\int_{\partial B_R} |U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2 \to 0 \quad R \to \infty
\]

and we infer from (5.5) that

\[
\int_{\mathbb{R}^2} U^{\mu_a} d\mu_a = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla U^{\mu_a}|^2.
\]

Recalling that by Corollary 3.3 \( \text{supp} \mu_a \subset B_{r_0} \) for some \( r_0 > 0 \) and using the divergence theorem again we conclude

\[
\int_{B_{r_0}} |x|^2 d\mu_a = -\frac{1}{2\pi} \int_{B_{r_0}} |x|^2 \Delta U^{\mu_a} = -\frac{1}{2\pi} \int_{B_{r_0}} U^{\mu_a} \Delta |x|^2 + \frac{1}{2\pi} \int_{\partial B_{r_0}} (2r_0 U^{\mu_a} - r_0^2 \frac{\partial}{\partial n} U^{\mu_a})
\]

\[
\int_{\partial B_{r_0}} U^{\mu_a} + \frac{r_0}{\pi} \int_{\partial B_{r_0}} U^{\mu_a} + r_0^2.
\]
Combining these we have that the energy can be rewritten in terms of $U^\mu_a$ in the following form

$$I[\mu_a] = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla U^\mu_a|^2 - \frac{2}{\pi} \int_{B_{r_0}} U^\mu_a + \frac{r_0}{\pi} \int_{\partial B_{r_0}} U^\mu_a + r_0^2.$$  

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