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REMARKS ON THE THIN OBSTACLE PROBLEM AND CONSTRAINED GINIBRE ENSEMBLES

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Abstract. We consider the problem of constrained Ginibre ensemble with prescribed portion of eigenvalues on a given curve \( \Gamma \subset \mathbb{R}^2 \) and relate it to a thin obstacle problem. The key step in the proof is the \( H^1 \) estimate for the logarithmic potential of the equilibrium measure. The coincidence set has two components: one in \( \Gamma \) and another one in \( \mathbb{R}^2 \setminus \Gamma \) which are well separated. Our main result here asserts that this obstacle problem is well posed in \( H^1(\mathbb{R}^2) \) which improves previous results in \( H^1_{\text{loc}}(\mathbb{R}^2) \).

1. Introduction

Let \( \Gamma \) be a regular curve in \( \mathbb{R}^2 \) with locally finite length and \( \mathcal{M}_a \) the set of all probability measures such that

\[
\mu(\Gamma) \geq a, \quad a \in (0, 1).
\]

By an abuse of notation we let \( \Gamma : \mathbb{R} \rightarrow \mathbb{R}^2 \) be the arc-length parametrization of the curve such that

\[
|\dot{\Gamma}(t)| = 1, \quad t \in \mathbb{R}.
\]

In this paper we consider the minimizers of the energy

\[
I[\mu] = \int \int \log \frac{1}{|x-y|} d\mu(x)d\mu(y) + \int Qd\mu
\]

where \( Q(x) \) is a given function such that the weight function \( w = e^{-Q} \) on \( \mathbb{R}^2 \) is admissible (see Definition 1.1 p.26 [8]). This means that \( w \) satisfies the following three conditions:

(H1) \( w \) is upper semi-continuous;
(H2) \( \{w \in \mathbb{R}^2 \ s.t. \ w(z) > 0\} \) has positive capacity;
(H3) \( |z|w(z) \rightarrow 0 \) as \( |z| \rightarrow \infty. \)

In higher dimensions \( \mathbb{R}^d, d \geq 3 \) one can consider more general kernels

\[
K(x-y) = \begin{cases} \log \frac{1}{|x-y|}, & d = 2, \\ \frac{1}{|x-y|^{d-2}}, & d \geq 3, \end{cases}
\]

with \( \Gamma \) being a Lyapunov surface in \( \mathbb{R}^d \) and define the energy as follows

\[
I[\mu] = \int \int K(x-y)d\mu(x)d\mu(y) + \int Qd\mu.
\]

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In this note we mostly confine ourselves with quadratic potentials $Q(x) = |x|^2$ in $\mathbb{R}^2$, although all our results remain valid for more general $Q$ satisfying (H1) – (H3). Furthermore, our main result on global $L^2$ estimate of the gradient of the equilibrium potential with kernel $K(x - y) = |x - y|^{-d}$ remains valid in $\mathbb{R}^d, d \geq 3$, see Theorem 4.1.

The functional $I[\mu]$, with $Q = |x|^2, d = 2$, arises in the description of the convergence of the spectral measure of square $N \times N$ matrices with complex independent, standard Gaussian entries (i.e., the Ginibre ensemble) as $N \to \infty$. In case when there are no constraints imposed on the eigenvalues, it is well known that the eigenvalues spread evenly in the ball of radius $\sqrt{N}$, and after renormalization by a factor $\frac{1}{\sqrt{N}}$ the normalized spectral measure converges to the characteristic function of the unit disc. This is known as the circular law [4], [2]. In this context the functional $I$ is used to prove large deviation principles for the spectral measure.

If one demands that the eigenvalues are real (i.e. when $a = 1, \Gamma = \mathbb{R}$) we get the so called semicircle law. More generally, one can demand that a portion of eigenvalues is contained in a prescribed set $\Gamma$. This is considered in [2] when a portion of eigenvalues are contained in an open bounded subset of $\mathbb{R}^2$ and in [4] when $\Gamma$ is a line. These problems can be related to the thin obstacle and obstacle problems respectively. The key step in proving this is to establish $H^1_{loc}(\mathbb{R}^2)$ estimates for the logarithmic potential

$$U^{\mu_a} = K \ast \mu_a$$

of the corresponding equilibrium measure. The aim of this note is to show that the thin obstacle problem is well-posed in $H^1(\mathbb{R}^2)$ by showing that in fact $U^{\mu_a} \in H^1(\mathbb{R}^2)$, see Theorem 4.1. This improves the previous results in [2] and [4].

The paper is organized as follows: In the next section we prove the existence and uniqueness of the equilibrium measure $\mu_a$ minimizing the energy $I[\mu]$. In section 3 we discuss some basic properties of $\mu_a$. In particular we show that there are two positive constants $A_\Gamma$ and $A_0$ such that $2U^{\mu_a} + Q = A_\Gamma$ on supp $\mu_a \cap \Gamma$ and $2U^{\mu_a} + Q = A_0$ on supp $\mu_a \setminus \Gamma$. Furthermore, $A_\Gamma > A_0$. This fact will be used later to show that supp $\mu_a \setminus \Gamma$ and supp $\mu_a \cap \Gamma$ are disjoint.

Our main result Theorem 4.1 is contained in section 4. To prove it we study the Fourier transformations of $U^{\mu_a}$ and $\mu_a$. It leads to some integral identity involving Bessel functions. This approach is based on a method of L. Carleson [3]. Finally, combining the results obtained, in section 5 we show that $U^{\mu_a}$ solves the obstacle problem where the obstacle is given by

$$\psi(x) = \begin{cases} \frac{1}{2}(A_\Gamma - |x|^2) & \text{if } x \in \Gamma, \\ \frac{1}{2}(A_0 - |x|^2) & \text{if } x \in \mathbb{R}^2 \setminus \Gamma. \end{cases}$$

(1.5)

2. Existence of minimizers

In this section we show the existence of a unique equilibrium measure.

**Theorem 2.1.** Suppose $d = 2, \Gamma \subset \mathbb{R}^2$ is a regular $C^{1,\alpha}$ smooth planar curve without self-intersections. There is a unique minimizer $\mu_a \in \mathcal{M}_a$ of $I[\mu]$ such that

$$I[\mu_a] = \inf_{\mu \in \mathcal{M}_a} I[\mu].$$

**Proof.** Observe that the uniqueness follows from the convexity of $\mathcal{M}_a$ and can be proved as in [4]. Moreover, $I[\mu]$ is also semicontinuous. Thus, we have to show that $I[\mu]$ is bounded by below for all $\mu \in \mathcal{M}_a$.
and there is at least one $\mu_0$ such that $I[\mu]$ is finite. The lower bound follows as in the proof of Theorem 1.3 (a) p. 27 [8].

It remains to check that the first integral is finite. Let us fix $s \in [0, L]$ Then we have that

$$\int_0^L \log \frac{1}{|x-y|} d\mu(x) = \int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} dt + \int_0^L \log \frac{1}{|\Gamma(t) - y|} dt + \int_B \log \frac{1}{|x-y|} d\mu(x).$$

Assuming that $\Gamma$ is given by arc-length parametrization we have for the logarithmic energy

\begin{equation}
\mathcal{L}[\mu] = \frac{a^2}{L^2} \int_0^L \int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} dt ds + \frac{2a(1-a)}{L|B|} \int_0^L \int_B \log \frac{1}{|\Gamma(t) - y|} dt dy + \frac{(1-a)^2}{|B|^2} \int_B \int_B \log \frac{1}{|x-y|} dx dy.
\end{equation}

Since $\text{dist}(\Gamma, B) > 0$ then the second integral is bounded. As for the last integral then after change of variables $x - y = \xi$ we have

$$\int_{B_{\rho(z)}} \log \frac{1}{|x-y|} dx = \int_{B_{\rho(z-y)}} \log \frac{1}{|\xi|} d\xi \leq \int_{B_{2\rho(0)}} \log \frac{1}{|\xi|} dx < \infty$$

where we used $|z-y| \leq \rho$ and the fact that $\rho$ is small by construction.

It remains to check that the first integral is finite. Let us fix $s \in [0, L]$ Then we have that

$$\int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} dt = \int_{s}^{L-s} \log \frac{1}{|\Gamma(t + s) - \Gamma(s)|} d\tau = \tau \log \frac{1}{|\Gamma(t + s) - \Gamma(s)|} \bigg|_{-s}^{L-s} - \int_{-s}^{L-s} \frac{\dot{\Gamma}(t + s) \cdot (\Gamma(t + s) - \Gamma(s))}{|\Gamma(t + s) - \Gamma(s)|^2} d\tau = (L-s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} - I_0$$

where $I_0$ is the last integral. Using the crude estimate

\begin{equation}
|I_0| \leq \int_{-s}^{L-s} \frac{|\tau|}{|\Gamma(t + s) - \Gamma(s)|} d\tau = \int_{-s}^{L-s} \frac{|\tau|}{|\Gamma(t + s) - \Gamma(s)|} d\tau = \int_{-s,L-s} \frac{|\tau|}{|\Gamma(t + s) - \Gamma(s)|} d\tau + \int_{-\delta}^{\delta} \frac{|\tau|}{|\Gamma(t + s) - \Gamma(s)|} d\tau \leq \frac{4L^2}{C_\delta} + \int_{-\delta}^{\delta} \frac{|\tau|}{|\Gamma(t + s) - \Gamma(s)|} d\tau
\end{equation}

because $|\Gamma(t + s) - \Gamma(s)| \geq C_\delta$ if $|\tau| \geq \delta$. Finally, from $C^{1,\alpha}$ regularity of $\Gamma$ we get

\begin{equation}
|\Gamma(t + s) - \Gamma(s)| = |\tau| \left| \int_0^1 \dot{\Gamma}(\sigma t + s) d\sigma \right| \geq |\tau| \left| \int_0^1 (\dot{\Gamma}(s) - \dot{\Gamma}(s) + \dot{\Gamma}(s)) d\sigma \right| \geq |\tau| (1 - \delta^\alpha).
\end{equation}
Combining (2.3) with (2.2) we get
\[ |I_0| \leq \frac{4L^2}{C_\delta} + 2\delta (1 - \delta^\alpha) < \infty. \]

Returning to the first integral in (2.1) we infer
\[
\begin{align*}
\int_0^L \int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} \, dt \, ds &\leq \int_0^L \left\{ (L-s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} + \frac{4L^2}{C_\delta} + 2\delta (1 - \delta^\alpha) \right\} \, ds \\
&\leq L \left[ \frac{4L^2}{C_\delta} + 2\delta (1 - \delta^\alpha) \right] + L \log \frac{1}{C_\delta} + \\
&\quad + \int_\delta^{L-\delta} \left\{ (L-s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} \right\} \, ds \\
&\leq C(\delta,L)
\end{align*}
\]
if we choose \( \delta > 0 \) suitably small. This finishes the proof for \( d = 2 \).

\[ \square \]

**Remark 2.2.** If \( d \geq 3 \), \( Q(x) = |x|^2 \) then clearly \( I[\mu] \geq 0 \). The upper estimate for \( I[\mu] \) follows from a similar argument if we assume that \( \Gamma \) is a Lyapunov surface and take \( \mu = a \frac{1}{L} \mathcal{H}^{d-1}(\Gamma \cap \Omega) + (1-a) \chi_B \) with \( L = \mathcal{H}^{d-1}(\Gamma \cap \Omega) \) and \( \text{dist}(B, \Gamma) > 0 \). Therefore, Theorem 2.1 remains valid for \( d \geq 3 \).

### 3. Basic properties of minimizers

In this section we prove some basic properties of the equilibrium measure. The arguments are along the line of those in [2]. Therefore, we mostly focus on those aspects of the proofs which are new or differ essentially. The results to follow are valid in \( \mathbb{R}^d, d \geq 2 \) unless otherwise stated.

**Lemma 3.1.** Let \( \mu_a \) be as in Theorem 2.1. Then \( \mu_a(\Gamma) = a \).

**Proof.** If the claim fails then \( \mu_a(\Gamma) > a \). Fix \( \delta \in (0,a) \) and let \( \mu_{a-\delta} \) be the minimizer of \( I[\cdot] \) over \( \mathcal{M}_{a-\delta} \supset \mathcal{M}_a \). Form \( \mu = (1-\varepsilon)\mu_a + \varepsilon \mu_{a-\delta}, \varepsilon \in [0,1] \). Clearly, \( \mu \in \mathcal{M}_a \) if we choose \( \varepsilon \delta \) sufficiently small because
\[
\mu(\Gamma) > a + [\mu_a(\Gamma) - a] - \varepsilon \delta.
\]
Consequently, we have from the strict convexity of \( I \)
\[
\begin{align*}
I[(1-\varepsilon)\mu_a + \varepsilon \mu_{a-\delta}] &< (1-\varepsilon)I[\mu_a] + \varepsilon I[\mu_{a-\delta}] = I[\mu_a] + \varepsilon(I[\mu_{a-\delta}] - I[\mu_a]) \\
&\leq I[\mu_a]
\end{align*}
\]
which is in contradiction with the fact that \( \mu_a \) is a minimizer. \[ \square \]

Observe that the Fréchet derivative of \( I[\mu] \) is \( 2U^{\mu_a} + Q \) where
\[
U^{\mu_a}(y) = \int K(x-y) d\mu_a(x).
\]
It is convenient to consider variations of the equilibrium measure in terms of affine combinations. More precisely, let \( \mu = (1 - \varepsilon)\mu_a + \varepsilon \nu, \nu \in \mathcal{M}_a, \varepsilon \in [0, 1] \), then by direct computation we have that

\[
(3.1) \quad I[\mu] = (1 - \varepsilon)^2 \int \int K(x - y)d\mu_a(x)d\mu_a(y) + 2\varepsilon(1 - \varepsilon) \int \int K(x - y)d\mu_a(x)d\nu(y) + \varepsilon^2 \int \int K(x - y)d\nu(x)d\nu(y) + (1 - \varepsilon) \int Qd\mu_a + \varepsilon \int Qd\nu \\
= I[\mu_a] + \varepsilon \left( 2 \int \int K(x - y)d\mu_a(x)d(\nu(y) - \mu_a) + \int Qd(\nu - \mu_a) \right) + O(\varepsilon^2) = \nonumber \\
= I[\mu_a] + \varepsilon \int (2U^{\mu_a} + Q)d(\nu - \mu_a) + O(\varepsilon^2). 
\]

Since \( \mu_a \) is the minimizer then \( I[\mu_a] \leq I[\mu] \), and after sending \( \varepsilon \to 0 \) it follows that

\[
(3.2) \quad \int (2U^{\mu_a} + Q)d(\nu - \mu_a) \geq 0, \quad \forall \nu \in \mathcal{M}_a.
\]

**Lemma 3.2.** Let \( A_\Gamma = \frac{1}{a} \int_\Gamma (2U^{\mu_a} + Q)d\mu_a \) then quasi everywhere

\[
(3.3) \quad 2U^{\mu_a} + Q = A_\Gamma \quad \text{on} \quad \Gamma \cap \text{supp} \mu_a, \nonumber \\
\geq A_\Gamma \quad \text{on} \quad \Gamma.
\]

Similarly, let us denote \( A_0 = \frac{1}{1 - a} \int_{\Gamma}(2U^{\mu_a} + Q)d\mu_a \) then

\[
(3.4) \quad 2U^{\mu_a} + Q = A_0 \quad \text{on} \quad \text{supp} \mu_a \setminus \Gamma, \nonumber \\
\geq A_0 \quad \text{on} \quad \mathbb{R}^2 \setminus (\text{supp} \mu_a \setminus \Gamma).
\]

Furthermore,

\[
(3.5) \quad A_\Gamma > A_0.
\]

**Proof.** We first prove (3.3). Suppose that there is a set capacitable \( E \) of positive capacity such that \( \Gamma \cap E \) has zero capacity and

\[
2U^{\mu_a} + Q < A_\Gamma - \delta \quad \text{q.e. on} \quad E
\]

for some positive \( \delta \). Let \( \mu_E \) be the equilibrium measure of \( E \) and form \( \nu = \mu_a\lfloor (\mathbb{R}^2 \setminus \Gamma) + a\mu_E \). Clearly \( \nu \in \mathcal{M}_a \). Therefore, in view of (3.1) for the measure \( \mu_\varepsilon = \varepsilon \mu_a + (1 - \varepsilon)\nu \in \mathcal{M}_a \) we get

\[
(3.6) \quad I[\mu_\varepsilon] = I[\mu_a] + \varepsilon \left( 2 \int \int K(x - y)d\mu_a(x)d(\nu(y) - \mu_a) + \int Qd(\nu - \mu_a) \right) + O(\varepsilon^2) \\
= I[\mu_a] + \varepsilon \int_\Gamma (2U^{\mu_a} + Q)d(\mu_a - \mu_a) + O(\varepsilon^2) = \nonumber \\
= I[\mu_a] + \varepsilon \left( a \int_\Gamma (2U^{\mu_a} + Q)d\mu_E - aA_\Gamma \right) + O(\varepsilon^2) < \nonumber \\
= I[\mu_a] - a\varepsilon \delta + O(\varepsilon^2) < I[\mu_a]
\]
if $\varepsilon$ and $\delta$ are sufficiently small. This will be in contradiction with the fact that $\mu_a$ is the minimizer. Thus we have proved that $2U^{\mu_a} + Q \geq A_\Gamma$ q.e. on $\Gamma$.

Next we show that on $\text{supp} \, \mu_a \cap \Gamma$ we have $2U^{\mu_a} + Q = A_\Gamma$ q.e. Indeed, from the definition of $A_\Gamma$ it follows

$$a A_\Gamma = \int_\Gamma (2U^{\mu_a} + Q) d\mu_a \geq a A_\Gamma$$

where the last inequality follows from the first inequality in (3.3). The proof of (3.4) is similar. In order to prove the last claim $A_\Gamma > A_0$ we first observe that there exists a measure $\nu \in M_a$ such that

- $a > \nu(\Gamma)$,
- $I[\nu] \leq I[\mu_a]$.

First notice that $M_a \subset M_{a-\delta}$ for $\delta \in (0, a)$. Fix such $\delta > 0$ and let $\mu_{a-\delta}$ be the minimizer of $I[\cdot]$ over $M_{a-\delta}$. Then by Lemma 3.1 $\mu_{a-\delta}(\Gamma) = a - \delta < a$ and $I[\mu_{a-\delta}] = \inf M_{a-\delta} I[\mu] \leq I[\mu_a] = \inf M_a I[\mu]$. Therefore one can take $\nu = \mu_{a-\delta}$.

From the strict convexity of $I$ it follows that

$$I[\nu] > I[\mu_a] + \langle DI[\mu_a], \nu - \mu_a \rangle$$

where $DI[\mu] = 2U^\mu + Q$ is the Fréchet derivative of $I[\mu]$. Therefore, from the properties of $\nu$ we infer (3.7)

$$0 \geq I[\nu] - I[\mu_a] > \langle DI[\mu_a], \nu - \mu_a \rangle$$

or equivalently

$$\langle 2U^{\mu_a} + Q, \nu - \mu_a \rangle < 0.$$ 

On the other hand

(3.8)

$$\int_\Gamma (2U^{\mu_a} + Q) d\mu_a = a A_\Gamma + (1 - a) A_0$$

while

$$\int_\Gamma (2U^{\mu_a} + Q) d\nu = \int_\Gamma (2U^{\mu_a} + Q) d\nu + \int_{\mathbb{R}^2 \setminus \Gamma} (2U^{\mu_a} + Q) d\nu \geq \nu(\Gamma) A_\Gamma + \nu(\mathbb{R}^2 \setminus \Gamma) A_0.$$ 

This together with (3.8), (3.7) yields

$$a A_\Gamma + (1 - a) A_0 > \nu(\Gamma) A_\Gamma + (1 - \nu(\Gamma)) A_0 \Rightarrow A_0(\nu(\Gamma) - a) > A_\Gamma(\nu(\Gamma) - a).$$

Finally, the property $\nu(\Gamma) < a$ implies that $A_\Gamma > A_0$.

\[\square\]

**Corollary 3.3.** $\text{supp} \, \mu_a$ is compact.

**Proof.** If $d \geq 3$ then $K(x - y) \geq 0$, hence by Lemma 3.2 for $x \in \text{supp} \, \mu_a$ we have

(3.9) $\max(A_\Gamma, A_0) \geq 2U^{\mu_a}(x) + Q(x) \geq Q(x) \to \infty$ if $|x| \to \infty$

which is a contradiction. If $d = 2$ then from the triangle inequality we get that

(3.10) $K(x - y) \geq -\log |x| - \log \left(1 + \frac{|y|}{|x|}\right)$.

Consequently, for $x \in \text{supp} \, \mu_a$

$$\max(A_\Gamma, A_0) \quad \geq \quad 2U^{\mu_a}(x) + Q(x) \geq Q(x) - 2 \log |x| - \int \log \left(1 + \frac{|y|}{|x|}\right) d\mu_a$$

$$= \quad Q(x) - 2 \log |x| + O(1) \to \infty \quad \text{if} \quad |x| \to \infty$$
for sufficiently large $|x|$, where the last inequality follows from (4.12) and $\int Q d\mu_a < I[\mu_a] < \infty$. Since $Q = |x|^2$ (or for the general case from the hypotheses on $Q$ $(H1) - (H3)$) it again follows that supp $\mu_a$ is bounded.

4. Global $L^2$ estimates for $U^{\mu_a}$ and $\nabla U^{\mu_a}$

Our main result is contained in the following

**Theorem 4.1.** Let $U^{\mu_a}(y) = \int K(x - y) d\mu_a$, if $d \geq 3$ then $\nabla U^{\mu_a} \in L^2(\mathbb{R}^d)$. If $d = 2$ then $U^{\mu_a} \in H^1(\mathbb{R}^2)$.

Furthermore, there holds

$$\|U^{\mu_a}\|_{H^1(\mathbb{R}^2)} \leq C\mathcal{E}[\mu_a].$$

Here $\mathcal{E}[\mu]$ is the energy of $\mu$ defined as $\int \int K(x - y) d\mu(x) d\mu(y)$.

**Remark 4.2.** It is shown in [3] that $\mathcal{E}[\mu] > 0$ for any probability measure $\mu$ and $d \geq 2$. In fact, this can be seen from the proof to follow (see also Corollary 4.3).

**Proof.** The case $d \geq 3$ follows from Lemma 1.6 p. 92 [7] (see also Lemma 17 p. 95), which assert that

$$\frac{\partial U^{\mu_a}(x)}{\partial x_i} = \int \frac{\partial K(x - y)}{\partial x_i} d\mu_a$$

almost everywhere and moreover

$$\frac{1}{4\pi^2} \int_{\mathbb{R}^d} |\nabla U^{\mu_a}|^2 \leq \int \int K(x - y) d\mu_a(x) d\mu_a(y) = \mathcal{E}[\mu_a].$$

The case of the logarithmic potential follows from a modification of the argument by L. Carleson [3] Lemma 3 page 22. We begin with computing the Fourier transformation of $K$. Note that since supp $\mu_a$ is compact we can assume that $K(r) = 0$ for $r \geq r_0$ for some fixed $r_0 > 0$. We have

$$\hat{K}(\xi) = \int K(x) e^{-2\pi i(x, \xi)} dx = \int K(x) e^{-2\pi i(x|\xi|, \frac{x}{|\xi|})} dx$$

$$= \frac{1}{4\pi^2|\xi|^2} \int K \left( \frac{y}{2\pi|\xi|} \right) e^{i<y, \frac{x}{|\xi|}} dx.$$

Let us denote $K_0(y) = K \left( \frac{y}{2\pi|\xi|} \right)$ and define

$$F(\eta) = \int K_0(y) e^{i\eta<y, \xi>}, \quad \eta = \frac{\xi}{|\xi|}.$$

From Lemma 2 p. 21 [3] it follows that there is a universal constant $c_1$ such that

$$F(\eta) = c_1 \int_0^\infty K_0(r) J(r) r dr, \quad |\eta| = 1$$

where $J$ is the Bessel function

$$J(r) = -J''(r) - \frac{J'(r)}{r}, \quad J(0) = 1, J'(0) = 0, \quad J(r) < 1, r \neq 0.$$
Therefore $F(\eta)$ can be further simplified as follows

$$
(4.3) \quad F(\eta) = -c_1 \int_0^\infty K_0(r)(rJ(r))' dr = c_1 \int_0^{2\pi|\xi|\eta_0} rJ'(r)K'_0(r) dr
$$

because from the definition of $K_0$ we have $\text{supp } K_0 \subset [0, 2\pi|\xi|\eta_0]$. Moreover, $K'_0(r) = -\frac{1}{r}$ hence

$$
(4.4) \quad F(\eta) = c_1(1 - J(2\pi|\xi|\eta_0)).
$$

Consequently,

$$
(4.5) \quad \hat{K}(\xi) = \frac{c_1}{4\pi^2|\xi|^2}(1 - J(2\pi|\xi|\eta_0)).
$$

Next we restrict $\mu_1 = \mu_a \sqcap C$ where $C \subset \text{supp } \mu_a$ is a compact such that $U^{\mu_1}$ is continuous. Observe that $\int U^{\mu_a} d\mu_a$ is finite hence $U^{\mu_a}$ is finite $\mu_a$ almost everywhere. By Theorem 1.8 p. 70 [7] for every $\varepsilon > 0$ small there is a restriction of $\mu_a$ such that

$$
0 < \int \mu_a - \int \mu_1 < \varepsilon.
$$

Note that if $\tau = \mu_a - \mu_1$ then we have

$$
|\mathcal{E}[\mu_a] - \mathcal{E}[\mu_1]| = \left| \int U^{\mu_a - \mu_1} d\mu_a + \int U^{\mu_a - \mu_1} d\mu_1 \right| = \left| \int (U^{\mu_a} + U^{\mu_1}) d\tau \right| = O(\varepsilon).
$$

Let $\phi_n(y) = n^{\frac{d}{2}} e^{-n|y|^2}$ be the sequence of normalised Gaussian kernels. It is well-known that $\phi_n$ is a mollification kernel for every $n \in \mathbb{N}$ and moreover $\widehat{\phi_n} = e^{-\frac{|\xi|^2}{n}}$. From the Parseval relation

$$
(4.6) \quad \int (\phi_n * U^{\mu_1}) d\mu_1 = \int \widehat{\phi_n} \widehat{K} |\widehat{\mu_1}|^2.
$$

If we first send $n \to \infty$ and then $\varepsilon \to 0$ to conclude the identity

$$
(4.7) \quad \mathcal{E}[\mu_a] = \int \widehat{K} |\widehat{\mu_a}|^2.
$$

On the other hand $\widehat{U^{\mu_a}} = \widehat{K} \widehat{\mu_a}$, which yields

$$
(4.8) \quad \mathcal{E}[\mu_a] = \int \widehat{K}(\xi) \frac{|\widehat{U^{\mu_a}}(\xi)|^2}{|\widehat{K}(\xi)|^2} d\xi
$$

$$
= \int \frac{4\pi^2|\xi|^2}{c_1(1 - J(2\pi r_0|\xi|))} |\widehat{U^{\mu_a}}(\xi)|^2 d\xi
$$

$$
= \int_{|\xi|<\delta} + \int_{|\xi|\geq\delta}.
$$

Using the expansion $J(t) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s)!} \left(\frac{t}{2}\right)^{2s} = 1 - \frac{t^2}{4} + \frac{t^4}{64} + \ldots$ we see that

$$
\frac{4\pi^2|\xi|^2}{c_1(1 - J(2\pi r_0|\xi|))} = \frac{1}{r_0^2c_1} \left(1 - \frac{(2\pi r_0|\xi|)^2}{16} + \ldots\right)
$$
hence the first integral is bounded below by $C(\delta) \frac{1}{\tau_0 c_1} \int_{|\xi| < \delta} |\hat{U}^{\mu_a}(\xi)|^2 d\xi$ for sufficiently small $\delta > 0$. As for the second integral, we have
\begin{equation}
(4.9) \int_{|\xi| \geq \delta} \frac{4\pi^2 |\xi|^2}{c_1(1 - J(2\pi r_0|\xi|))} |\hat{U}^{\mu_a}(\xi)|^2 d\xi \geq \frac{4\pi^2 \delta^2}{c_1} \int_{|\xi| \geq \delta} |\hat{U}^{\mu_a}(\xi)|^2 d\xi.
\end{equation}
Combining we see that $\hat{U}^{\mu_a} \in L^2(\mathbb{R}^2)$ which, after we apply Parseval’s relation again, yields $U^{\mu_a} \in L^2(\mathbb{R}^2)$ and
\begin{equation}
(4.10) \|U^{\mu_a}\|_{L^2(\mathbb{R}^2)} \leq C\mathcal{E}[\mu_a].
\end{equation}
To finish the proof we use that $4\pi^2|\xi|^2|\hat{U}^{\mu_a}|^2 = |\nabla U^{\mu_a}|^2$ which together with (4.8) implies that
\begin{equation}
(4.11) \mathcal{E}[\mu_a] = \int \frac{1}{c_1(1 - J(2\pi r_0|\xi|))} |\nabla U^{\mu_a}(\xi)|^2 d\xi \geq \frac{1}{c_1} \int |\nabla U^{\mu_a}(\xi)|^2 d\xi
\end{equation}
which finishes the proof. ∎

**Corollary 4.3.** Let $\mu_a$ be as in Theorem 2.1. Then there holds
\begin{equation}
(4.12) \mathcal{E}[\mu_a] = \int U^{\mu_a} d\mu_a > 0.
\end{equation}

5. **The Thin Obstacle Problem**

From the $H^1(\mathbb{R}^2)$ estimate for $U^{\mu_a}$ it follows that $U^{\mu_a}$ is a solution to some variational inequality, and hence $U^{\mu_a}$ can be interpreted as a solution to an obstacle problem with a combination of both thin (on $\Gamma$) and ”thick” obstacles (on $\mathbb{R}^2 \setminus \Gamma$). It is convenient to define the obstacle as follows
\begin{equation}
(5.1) \psi(x) = \begin{cases} \frac{1}{2}(A_\Gamma - |x|^2) & \text{if } x \in \Gamma, \\
\frac{1}{2}(A_0 - |x|^2) & \text{if } x \in \mathbb{R}^2 \setminus \Gamma.
\end{cases}
\end{equation}

**Lemma 5.1.** Let $U^{\mu_a}$ be the logarithmic potential of $\mu_a$ and define
\[ \mathcal{K} = \{ v \in H^1_{loc}(\mathbb{R}^2) \text{ s.t. } v - U^{\mu_a} \text{ has bounded support in } \mathbb{R}^2, \: v \geq \psi \}. \]
Then $U^{\mu_a}$ solves the following obstacle problem:
\[ \int \nabla U^{\mu_a} \nabla (v - U^{\mu_a}) \geq 0, \quad \forall v \in \mathcal{K}. \]

The proof is the same as in [2].

**Corollary 5.2.** $\text{dist}(\Gamma, \text{supp}(\mu_a \setminus \Gamma)) > 0$.

**Proof.** This follows from the estimate $A_\Gamma > A_0$. Indeed, let us assume that $x_0 \in \Gamma \cap \text{supp } \mu_a$ and there is a sequence $\{x_k\}_{k=1}^\infty, x_k \in \text{supp } \mu_a \setminus \Gamma$ such that $\lim_{k \to \infty} x_k = x_0$. Using the lower semicontinuity of $U^{\mu_a}$ (see Lemma 1 p.15 [3]) we see that
\begin{equation}
(5.2) \frac{1}{2}(A_0 - |x_0|^2) = \liminf_{x_k \to x_0} U^{\mu_a}(x_k) \geq U^{\mu_a}(x_0).
\end{equation}
Let $\rho > 0$ be such that $\{x_k\} \subset B_\rho(x_0)$. If $\rho$ is small then $\Gamma$ divides $B_\rho(x_0)$ into two parts $D^+$ and $D^-$. To fix the ideas let us suppose that $D^+$ contains a subsequence $\{x_k\}$. Let $h$ be the harmonic function in $D^+$
such that \( h = \psi \) on \( \partial D^+ \). Observe that \( h \) is continuous at \( x_0 \) because \( \Gamma \in C^{1,\alpha} \). Since \( U^{\mu_a} \) is superharmonic and on \( \partial D^+ \) we have \( U^{\mu_a} \geq \psi = h \) then the comparison principle implies that

\[
U^{\mu_a}(x_0) \geq h(x_0) = \frac{1}{2}(A_\Gamma - |x_0|^2).
\]

(5.3)

Combining (5.2) and (5.3) we see that \( A_0 \geq A_\Gamma \) which is a contradiction in view of (3.5).

□

From Corollary 5.2 it follows that near \( \Gamma \) the potential \( U^{\mu_a} \) is a solution to a thin obstacle problem in the following sense, see [5] p. 108:

\[
\begin{align*}
\frac{\partial U^{\mu_a}}{\partial \nu^+} + \frac{\partial U^{\mu_a}}{\partial \nu^-} \geq 0 \\
\left(u - \frac{1}{2}(A_\Gamma - Q)\right) \left(\frac{\partial U^{\mu_a}}{\partial \nu^+} + \frac{\partial U^{\mu_a}}{\partial \nu^-}\right) = 0
\end{align*}
\]

(5.4)
on \( \Gamma \)

where \( n^\pm \) are the outward normals on the \( \Gamma \) corresponding to the domains that \( \Gamma \) separates. In particular, if \( \Gamma \) is \( C^3 \) regular then \( U^{\mu_a} \) is \( C^{1,\alpha} \) up to \( \Gamma \) from each of its side, see Theorem 11.4 p.111 [5].

A particular case is \( \Gamma = \mathbb{R} \) [4]. Using a simple symmetrization argument (see e.g. [6] p. 119 Theorem 4.6) we can show that the potential \( U^{\mu_a} \) is symmetric w.r.t. the real line and hence we get the Signorini problem near \( \mathbb{R} \) [5] p. 111.

One can make the connections with the obstacle problem more explicit by using the \( H^1(\mathbb{R}^2) \) estimate in Theorem 4.1 and transforming the energy \( I[\mu_a] \). Let \( R > 0 \) be fixed then using the divergence theorem

\[
\int_{B_R} U^{\mu_a} d\mu_a = -\frac{1}{2\pi} \int_{B_R} U^{\mu_a} \Delta U^{\mu_a} =
\]

\[
\frac{1}{2\pi} \int_{B_R} |\nabla U^{\mu_a}|^2 - \frac{1}{2\pi} \int_{\partial B_R} U^{\mu_a} \partial_n U^{\mu_a}.
\]

(5.5)

For a.e. \( R > 0 \) the last integral can be estimated as follows

\[
\left| \int_{\partial B_R} U^{\mu_a} \partial_n U^{\mu_a} \right| \leq \int_{\partial B_R} |U^{\mu_a}||\nabla U^{\mu_a}| \leq \int_{\partial B_R} |U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2.
\]

From Theorem 4.1 and Fubini’s theorem it follows that

\[
\int_{\mathbb{R}^2} (|U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2) = \int_0^\infty \int_{\partial B_R} (|U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2) dR.
\]

Consequently,

\[
\int_{\partial B_R} |U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2 \to 0 \quad R \to \infty
\]

and we infer from (5.5) that

\[
\int_{\mathbb{R}^2} U^{\mu_a} d\mu_a = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla U^{\mu_a}|^2.
\]

Recalling that by Corollary 3.3 \( \text{supp} \mu_a \subset B_{r_0} \) for some \( r_0 > 0 \) and using the divergence theorem again we conclude

\[
\int_{B_{r_0}} |x|^2 d\mu_a = -\frac{1}{2\pi} \int_{B_{r_0}} |x|^2 \Delta U^{\mu_a} = -\frac{1}{2\pi} \int_{B_{r_0}} U^{\mu_a} \Delta |x|^2 + \frac{1}{2\pi} \int_{\partial B_{r_0}} (2r_0 U^{\mu_a} - r_0^2 \partial_n U^{\mu_a})
\]

\[
= -\frac{2}{\pi} \int_{B_{r_0}} U^{\mu_a} + \frac{r_0}{\pi} \int_{\partial B_{r_0}} U^{\mu_a} + r_0^2.
\]

(5.6)
Combining these we have that the energy can be rewritten in terms of $U^\mu_a$ in the following form

$$I[\mu_a] = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla U^\mu_a|^2 - \frac{2}{\pi} \int_{B_{r_0}} U^\mu_a + \frac{r_0}{\pi} \int_{\partial B_{r_0}} U^\mu_a + r_0^2.$$ 

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