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REMARKS ON THE THIN OBSTACLE PROBLEM AND CONSTRAINED GINIBRE ENSEMBLES

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Abstract. We consider the problem of constrained Ginibre ensemble with prescribed portion of eigenvalues on a given curve $\Gamma \subset \mathbb{R}^2$ and relate it to a thin obstacle problem. The key step in the proof is the $H^1$ estimate for the logarithmic potential of the equilibrium measure. The coincidence set has two components: one in $\Gamma$ and another one in $\mathbb{R}^2 \setminus \Gamma$ which are well separated. Our main result here asserts that this obstacle problem is well posed in $H^1(\mathbb{R}^2)$ which improves previous results in $H^1_{locc}(\mathbb{R}^2)$.

1. Introduction

Let $\Gamma$ be a regular curve in $\mathbb{R}^2$ with locally finite length and $\mathcal{M}_a$ the set of all probability measures such that

$$\mu(\Gamma) \geq a, \quad a \in (0, 1).$$

By an abuse of notation we let $\Gamma : \mathbb{R} \to \mathbb{R}^2$ be the arc-length parametrization of the curve such that $|\dot{\Gamma}(t)| = 1, \quad t \in \mathbb{R}$.

In this paper we consider the minimizers of the energy

$$I[\mu] = \int \int \log \frac{1}{|x-y|} d\mu(x)d\mu(y) + \int Qd\mu$$

where $Q(x)$ is a given function such that the weight function $w = e^{-Q}$ on $\mathbb{R}^2$ is admissible (see Definition 1.1 p.26 [8]). This means that $w$ satisfies the following three conditions:

- (H1) $w$ is upper semi-continuous;
- (H2) $\{w \in \mathbb{R}^2 \text{ s.t. } w(z) > 0\}$ has positive capacity;
- (H3) $|z|w(z) \to 0$ as $|z| \to \infty$.

In higher dimensions $\mathbb{R}^d, d \geq 3$ one can consider more general kernels

$$K(x-y) = \begin{cases} \log \frac{1}{|x-y|}, & d = 2, \\ \frac{1}{|x-y|^{d-2}}, & d \geq 3, \end{cases}$$

with $\Gamma$ being a Lyapunov surface in $\mathbb{R}^d$ and define the energy as follows

$$I[\mu] = \int \int K(x-y)d\mu(x)d\mu(y) + \int Qd\mu.$$
In this note we mostly confine ourselves with quadratic potentials $Q(x) = |x|^2$ in $\mathbb{R}^2$, although all our results remain valid for more general $Q$ satisfying $\text{(H1)} - \text{(H3)}$. Furthermore, our main result on global $L^2$ estimate of the gradient of the equilibrium potential with kernel $K(x - y) = |x - y|^{-d}$ remains valid in $\mathbb{R}^d, d \geq 3$, see Theorem 4.1.

The functional $I[\mu]$, with $Q = |x|^2, d = 2$, arises in the description of the convergence of the spectral measure of square $N \times N$ matrices with complex independent, standard Gaussian entries (i.e., the Ginibre ensemble) as $N \to \infty$. In case when there are no constraints imposed on the eigenvalues, it is well known that the eigenvalues spread evenly in the ball of radius $\sqrt{N}$, and after renormalization by a factor $\frac{1}{\sqrt{N}}$ the normalized spectral measure converges to the characteristic function of the unit disc. This is known as the circular law [4], [2]. In this context the functional $I$ is used to prove large deviation principles for the spectral measure.

If one demands that the eigenvalues are real (i.e. when $a = 1, \Gamma = \mathbb{R}$) we get the so called semicircle law. More generally, one can demand that a portion of eigenvalues is contained in a prescribed set $\Gamma$. This is considered in [2] when a portion of eigenvalues are contained in an open bounded subset of $\mathbb{R}^2$ and in [4] when $\Gamma$ is a line. These problems can be related to the thin obstacle and obstacle problems respectively. The key step in proving this is to establish $H^1_{loc}(\mathbb{R}^2)$ estimates for the logarithmic potential

$$U^{\mu_a} = K * \mu_a$$

of the corresponding equilibrium measure. The aim of this note is to show that the thin obstacle problem is well-posed in $H^1(\mathbb{R}^2)$ by showing that in fact $U^{\mu_a} \in H^1(\mathbb{R}^2)$, see Theorem 4.1. This improves the previous results in [2] and [4].

The paper is organized as follows: In the next section we prove the existence and uniqueness of the equilibrium measure $\mu_a$ minimizing the energy $I[\mu]$. In section 3 we discuss some basic properties of $\mu_a$. In particular we show that there are two positive constants $A_\Gamma$ and $A_0$ such that $2U^{\mu_a} + Q = A_\Gamma$ on supp $\mu_a \cap \Gamma$ and $2U^{\mu_a} + Q = A_0$ on supp $\mu_a \setminus \Gamma$. Furthermore, $A_\Gamma > A_0$. This fact will be used later to show that supp $\mu_a \setminus \Gamma$ and supp $\mu_a \cap \Gamma$ are disjoint.

Our main result Theorem 4.1 is contained in section 4. To prove it we study the Fourier transformations of $U^{\mu_a}$ and $\mu_a$. It leads to some integral identity involving Bessel functions. This approach is based on a method of L. Carleson [3]. Finally, combining the results obtained, in section 5 we show that $U^{\mu_a}$ solves the obstacle problem where the obstacle is given by

\begin{equation}
\psi(x) = \begin{cases} \frac{1}{2}(A_\Gamma - |x|^2) & \text{if } x \in \Gamma, \\
\frac{1}{2}(A_0 - |x|^2) & \text{if } x \in \mathbb{R}^2 \setminus \Gamma. \end{cases}
\end{equation}

2. Existence of minimizers

In this section we show the existence of a unique equilibrium measure.

**Theorem 2.1.** Suppose $d = 2, \Gamma \subset \mathbb{R}^2$ is a regular $C^{1,\alpha}$ smooth planar curve without self-intersections. There is a unique minimizer $\mu_a \in \mathcal{M}_a$ of $I[\mu]$ such that

$$I[\mu_a] = \inf_{\mu \in \mathcal{M}_a} I[\mu].$$

**Proof.** Observe that the uniqueness follows from the convexity of $\mathcal{M}_a$ and can be proved as in [4]. Moreover, $I[\mu]$ is also semicontinuous. Thus, we have to show that $I[\mu]$ is bounded below for all $\mu \in \mathcal{M}_a$.
and there is at least one \( \mu_0 \) such that \( I[\mu] \) is finite. The lower bound follows as in the proof of Theorem 1.3 (a) p. 27 [8].

It remains to check that the first integral is finite. Let us fix \( \hat{\mu} \) and \( 0 \leq 0 \leq L \) such that

\[
\mu = a \frac{1}{L} \mathcal{H}^1(\Gamma \cap \Omega) + (1 - a) \frac{1}{|B|} \chi_B
\]

where \( B = B_\rho(z) = \{ x \in \mathbb{R}^2 : |x - z| < \rho \} \) with small \( \rho \) such that \( B \subset \Omega, \Omega \subset \mathbb{R}^2 \) is a compact, \( L = \mathcal{H}^1(\Gamma \cap \Omega) > 0 \), and \( \text{dist}(\Gamma, B) > 0 \). Observe that for this choice of \( \mu \) we have

\[
\int_0^L \log \frac{1}{|x-y|} d\mu(x) = \frac{1}{L} \int_0^L \log \frac{1}{|\Gamma(t) - y|} dt + \frac{1}{|B|} \int_B \log \frac{1}{|x-y|} d\mu(x).
\]

Assuming that \( \Gamma \) is given by arc-length parametrization we have for the logarithmic energy

\[
\mathcal{L}[\mu] = \frac{a^2}{L^2} \int_0^L \int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} dtds + \frac{2a(1-a)}{L|B|} \int_0^L \int_B \log \frac{1}{|\Gamma(t) - y|} dtdy + \frac{(1-a)^2}{|B|^2} \int_B \int_B \log \frac{1}{|x-y|} dx dy.
\]

Since \( \text{dist}(\Gamma, B) > 0 \) then the second integral is bounded. As for the last integral then after change of variables \( x - y = \xi \) we have

\[
\int_{B^\rho(z)} \log \frac{1}{|x-y|} dx = \int_{B^\rho(z-y)} \log \frac{1}{|\xi|} d\xi \leq \int_{B^\rho(0)} \log \frac{1}{|\xi|} dx < \infty
\]

where we used \( |z-y| \leq \rho \) and the fact that \( \rho \) is small by construction.

It remains to check that the first integral is finite. Let us fix \( s \in [0, L] \) Then we have that

\[
\int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} dt = \int_{-s}^{L-s} \log \frac{1}{|\Gamma(\tau + s) - \Gamma(s)|} d\tau = \\
= \int_{-s}^{L-s} \log \frac{1}{|\Gamma(\tau + s) - \Gamma(s)|} \cdot \int_{-s}^{L-s} \frac{\dot{\Gamma}(\tau + s) \cdot (\Gamma(\tau + s) - \Gamma(s))}{|\Gamma(\tau + s) - \Gamma(s)|^2} d\tau = \\
= (L - s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} - I_0
\]

where \( I_0 \) is the last integral. Using the crude estimate

\[
|I_0| \leq \int_{-s}^{L-s} \left| \frac{\dot{\Gamma}(\tau + s)}{|\Gamma(\tau + s) - \Gamma(s)|} \right| d\tau = \int_{-s}^{L-s} \left| \frac{\dot{\Gamma}(\tau + s) \cdot (\Gamma(\tau + s) - \Gamma(s))}{|\Gamma(\tau + s) - \Gamma(s)|^2} \right| d\tau = \\
\leq \int_{-s}^{L-s} \frac{|\tau|}{|\Gamma(\tau + s) - \Gamma(s)|} d\tau + \int_{-s}^{L-s} \frac{|\tau|}{|\Gamma(\tau + s) - \Gamma(s)|} d\tau = \\
\leq \frac{4L^2}{C_\delta} + \int_{-\delta}^{\delta} \frac{|\tau|}{|\Gamma(\tau + s) - \Gamma(s)|} d\tau
\]

because \( |\Gamma(\tau + s) - \Gamma(s)| \geq C_\delta \) if \( |\tau| \geq \delta \). Finally, from \( C^{1,\alpha} \) regularity of \( \Gamma \) we get

\[
|\Gamma(\tau + s) - \Gamma(s)| \geq |\tau| \left( \int_0^1 \dot{\Gamma}(\sigma \tau + s) d\sigma \right) \geq \\
\geq |\tau| \left( \int_0^1 \dot{\Gamma}(\sigma \tau + s) - \dot{\Gamma}(\tau + s) d\sigma \right) \geq |\tau| (1 - \delta^\alpha).
\]
Combining (2.3) with (2.2) we get
\[ |I_0| \leq \frac{4L^2}{C_\delta} + 2\delta (1 - \delta^\alpha) < \infty. \]

Returning to the first integral in (2.1) we infer
\[
\int_0^L \int_0^L \log \frac{1}{|\Gamma(t) - \Gamma(s)|} \, dt \, ds \leq \int_0^L \left\{ (L - s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} + \frac{4L^2}{C_\delta} + 2\delta (1 - \delta^\alpha) \right\} ds
\]
\[
\leq L \left[ \frac{4L^2}{C_\delta} + 2\delta (1 - \delta^\alpha) \right] + L \log \frac{1}{C_\delta} + \int_\delta^{L-\delta} \left\{ (L - s) \log \frac{1}{|\Gamma(L) - \Gamma(s)|} + s \log \frac{1}{|\Gamma(0) - \Gamma(s)|} \right\} ds
\]
\[
\leq C(\delta, L)
\]
if we choose \( \delta > 0 \) suitably small. This finishes the proof for \( d = 2 \). \qed

**Remark 2.2.** If \( d \geq 3 \), \( Q(x) = |x|^2 \) then clearly \( I[\mu] \geq 0 \). The upper estimate for \( I[\mu] \) follows from a similar argument if we assume that \( \Gamma \) is a Lyapunov surface and take \( \mu = a \frac{1}{L} \mathcal{H}^{d-1}(\Gamma \cap \Omega) + (1 - a) \frac{1}{|B|} \chi_B \) with \( L = \mathcal{H}^{d-1}(\Gamma \cap \Omega) \) and \( \text{dist}(B, \Gamma) > 0 \). Therefore Theorem 2.1 remains valid for \( d \geq 3 \).

3. Basic properties of minimizers

In this section we prove some basic properties of the equilibrium measure. The arguments are along the line of those in [2]. Therefore, we mostly focus on those aspects of the proofs which are new or differ essentially. The results to follow are valid in \( \mathbb{R}^d, d \geq 2 \) unless otherwise stated.

**Lemma 3.1.** Let \( \mu_a \) be as in Theorem 2.1. Then \( \mu_a(\Gamma) = a \).

**Proof.** If the claim fails then \( \mu_a(\Gamma) > a \). Fix \( \delta \in (0, a) \) and let \( \mu_{a-\delta} \) be the minimizer of \( I[\cdot] \) over \( \mathcal{M}_{a-\delta} \supset \mathcal{M}_a \). Form \( \mu = (1 - \epsilon)\mu_a + \epsilon \mu_{a-\delta}, \epsilon \in [0, 1] \). Clearly, \( \mu \in \mathcal{M}_a \) if we choose \( \epsilon \delta \) sufficiently small because
\[ \mu(\Gamma) > a + [\mu_a(\Gamma) - a] - \epsilon \delta. \]

Consequently, we have from the strict convexity of \( I \)
\[ I[(1 - \epsilon)\mu_a + \epsilon \mu_{a-\delta}] < (1 - \epsilon)I[\mu_a] + \epsilon I[\mu_{a-\delta}] = I[\mu_a] + \epsilon (I[\mu_{a-\delta}] - I[\mu_a]) \]
\[ \leq I[\mu_a] \]
which is in contradiction with the fact that \( \mu_a \) is a minimizer. \qed

Observe that the Fréchet derivative of \( I[\mu] \) is \( 2U^{\mu_a} + Q \) where
\[ U^{\mu_a}(y) = \int K(x - y) d\mu_a(x). \]
It is convenient to consider variations of the equilibrium measure in terms of affine combinations. More precisely, let \( \mu_\varepsilon = (1 - \varepsilon)\mu_a + \varepsilon \nu, \nu \in \mathcal{M}_a, \varepsilon \in [0, 1] \), then by direct computation we have that

\[
I[\mu_\varepsilon] = (1 - \varepsilon)^2 \int \int K(x - y) d\mu_a(x) d\mu_a(y) \\
+ 2\varepsilon(1 - \varepsilon) \int \int K(x - y) d\mu_a(x) d\nu(y) + \varepsilon^2 \int \int K(x - y) d\nu(x) d\nu(y) \\
+ (1 - \varepsilon) \int Q d\mu_a + \varepsilon \int Q d\nu
\]

\[
= I[\mu_a] + \varepsilon \left( 2 \int \int K(x - y) d\mu_a(x) d(\nu(y) - \mu_a) + \int Q d(\nu - \mu_a) \right) + O(\varepsilon^2) = \\
= I[\mu_a] + \varepsilon \int (2U^{\mu_a} + Q) d(\nu - \mu_a) + O(\varepsilon^2).
\]

Since \( \mu_a \) is the minimizer then \( I[\mu_a] \leq I[\mu] \), and after sending \( \varepsilon \to 0 \) it follows that

\[
\int (2U^{\mu_a} + Q) d(\nu - \mu_a) \geq 0, \quad \forall \nu \in \mathcal{M}_a.
\]

**Lemma 3.2.** Let \( A_\Gamma = \frac{1}{a} \int_\Gamma (2U^{\mu_a} + Q) d\mu_a \) then quasi everywhere

\[
(3.3)
2U^{\mu_a} + Q = A_\Gamma \quad \text{on} \quad \Gamma \cap \text{supp} \mu_a, \\
\geq A_\Gamma \quad \text{on} \quad \Gamma.
\]

Similarly, let us denote \( A_0 = \frac{1}{1-a} \int_{\mathbb{R}^2 \setminus \Gamma} (2U^{\mu_a} + Q) d\mu_a \) then

\[
(3.4)
2U^{\mu_a} + Q = A_0 \quad \text{on} \quad \text{supp} \mu_a \setminus \Gamma, \\
\geq A_0 \quad \text{on} \quad \mathbb{R}^2 \setminus (\text{supp} \mu_a \setminus \Gamma).
\]

Furthermore,

\[
(3.5)
A_\Gamma > A_0.
\]

**Proof.** We first prove (3.3). Suppose that there is a set capacitable \( E \) of positive capacity such that \( \Gamma \cap E \) has zero capacity and

\[
2U^{\mu_a} + Q < A_\Gamma - \delta \quad \text{q.e. on} \quad E
\]

for some positive \( \delta \). Let \( \mu_E \) be the equilibrium measure of \( E \) and form \( \nu = \mu_a\lfloor (\mathbb{R}^2 \setminus \Gamma) + a\mu_E \). Clearly \( \nu \in \mathcal{M}_a \). Therefore, in view of (3.1) for the measure \( \mu_\varepsilon = \varepsilon \mu_a + (1 - \varepsilon)\nu \in \mathcal{M}_a \) we get

\[
I[\mu_\varepsilon] = I[\mu_a] + \varepsilon \left( 2 \int \int K(x - y) d\mu_a(x) d(\nu(y) - \mu_a) + \int Q d(\nu - \mu_a) \right) + O(\varepsilon^2) = \\
= I[\mu_a] + \varepsilon \int_\Gamma (2U^{\mu_a} + Q) d(a\mu_E - \mu_a) + O(\varepsilon^2) = \\
= I[\mu_a] + \varepsilon \left( a \int_\Gamma (2U^{\mu_a} + Q) d\mu_E - aA_\Gamma \right) + O(\varepsilon^2) < \\
I[\mu_a] - a\varepsilon \delta + O(\varepsilon^2) < \\
I[\mu_a]
\]
if $\varepsilon$ and $\delta$ are sufficiently small. This will be in contradiction with the fact that $\mu_a$ is the minimizer. Thus we have proved that $2U^{\mu_a} + Q \geq A_\Gamma$ q.e. on $\Gamma$.

Next we show that on $\text{supp} \mu_a \cap \Gamma$ we have $2U^{\mu_a} + Q = A_\Gamma$ q.e. Indeed, from the definition of $A_\Gamma$ it follows

$$aA_\Gamma = \int_\Gamma (2U^{\mu_a} + Q) d\mu_a \geq aA_\Gamma$$

where the last inequality follows from the first inequality in (3.3). The proof of (3.4) is similar. In order to prove the last claim $A_\Gamma > A_0$ we first observe that there exists a measure $\nu \in \mathcal{M}_a$ such that

- $a > \nu(\Gamma)$,
- $I[\nu] \leq I[\mu_a]$.

First notice that $\mathcal{M}_a \subset \mathcal{M}_a - \delta$ for $\delta \in (0, a)$. Fix such $\delta > 0$ and let $\mu_a - \delta$ be the minimizer of $I[\cdot]$ over $\mathcal{M}_a - \delta$. Then by Lemma 3.1 $\mu_a - \delta(\Gamma) = a - \delta < a$ and $I[\mu_a - \delta] = \inf_{\mathcal{M}_a - \delta} I[\mu] \leq I[\mu_a] = \inf_{\mathcal{M}_a} I[\mu]$. Therefore one can take $\nu = \mu_a - \delta$.

From the strict convexity of $I$ it follows that

$$I[\nu] > I[\mu_a] + \langle DI[\mu_a], \nu - \mu_a \rangle$$

where $DI[\mu] = 2U^{\mu} + Q$ is the Fréchet derivative of $I[\mu]$. Therefore, from the properties of $\nu$ we infer (3.7)

$$0 \geq I[\nu] - I[\mu_a] > \langle DI[\mu_a], \nu - \mu_a \rangle$$

or equivalently

$$\langle 2U^{\mu_a} + Q, \nu - \mu_a \rangle < 0.$$

On the other hand

(3.8) $$\int (2U^{\mu_a} + Q) d\mu_a = aA_\Gamma + (1 - a)A_0$$

while

$$\int (2U^{\mu_a} + Q) d\nu = \int_\Gamma (2U^{\mu_a} + Q) d\nu + \int_{\mathbb{R}^2 \setminus \Gamma} (2U^{\mu_a} + Q) d\nu \geq \nu(\Gamma)A_\Gamma + \nu(\mathbb{R}^2 \setminus \Gamma)A_0.$$

This together with (3.8), (3.7) yields

$$aA_\Gamma + (1 - a)A_0 > \nu(\Gamma)A_\Gamma + (1 - \nu(\Gamma))A_0 \Rightarrow A_0(\nu(\Gamma) - a) > A_\Gamma(\nu(\Gamma) - a).$$

Finally, the property $\nu(\Gamma) < a$ implies that $A_\Gamma > A_0$. \qed

**Corollary 3.3.** $\text{supp} \mu_a$ is compact.

**Proof.** If $d \geq 3$ then $K(x - y) \geq 0$, hence by Lemma 3.2 for $x \in \text{supp} \mu_a$ we have

(3.9) $$\max(A_\Gamma, A_0) \geq 2U^{\mu_a}(x) + Q(x) \geq Q(x) \to \infty \quad \text{if} \quad |x| \to \infty$$

which is a contradiction. If $d = 2$ then from the triangle inequality we get that

(3.10) $$K(x - y) \geq -\log |x| - \log \left(1 + \frac{|y|}{|x|}\right).$$

Consequently, for $x \in \text{supp} \mu_a$

$$\max(A_\Gamma, A_0) \geq 2U^{\mu_a}(x) + Q(x) \geq Q(x) - 2\log |x| - \int \log \left(1 + \frac{|y|}{|x|}\right) d\mu_a$$

$$= Q(x) - 2\log |x| + O(1) \to \infty \quad \text{if} \quad |x| \to \infty$$
for sufficiently large $|x|$, where the last inequality follows from (4.12) and $\int Q d\mu_a < I[\mu_a] < \infty$. Since $Q = |x|^2$ (of for the general case from the hypotheses on $Q$ (H1) – (H3)) it again follows that supp $\mu_a$ is bounded.

4. Global $L^2$ estimates for $U^{\mu_a}$ and $\nabla U^{\mu_a}$

Our main result is contained in the following

**Theorem 4.1.** Let $U^{\mu_a}(y) = \int K(x-y) d\mu_a$, if $d \geq 3$ then $\nabla U^{\mu_a} \in L^2(\mathbb{R}^d)$. If $d = 2$ then $U^{\mu_a} \in H^1(\mathbb{R}^2)$.

Furthermore, there holds

$$\|U^{\mu_a}\|_{H^1(\mathbb{R}^2)} \leq C\mathcal{E}[\mu_a].$$

Here $\mathcal{E}[\mu]$ is the energy of $\mu$ defined as $\int \int K(x-y) d\mu(x) d\mu(y)$.

**Remark 4.2.** It is shown in [3] that $\mathcal{E}[\mu] > 0$ for any probability measure $\mu$ and $d \geq 2$. In fact, this can be seen from the proof to follow (see also Corollary 4.3).

**Proof.** The case $d \geq 3$ follows from Lemma 1.6 p. 92 [7] (see also Lemma 17 p. 95), which assert that

$$\frac{\partial U^{\mu_a}(x)}{\partial x_i} = \int \frac{\partial K(x-y)}{\partial x_i} d\mu_a$$

almost everywhere and moreover

$$\frac{1}{4\pi^2} \int_{\mathbb{R}^d} |\nabla U^{\mu_a}|^2 \leq \int \int K(x-y) d\mu_a(x) d\mu_a(y) = \mathcal{E}[\mu_a].$$

The case of the logarithmic potential follows from a modification of the argument by L. Carleson [3] Lemma 3 page 22. We begin with computing the Fourier transformation of $K$. Note that since supp $\mu_a$ is compact we can assume that $K(r) = 0$ for $r \geq r_0$ for some fixed $r_0 > 0$. We have

$$\hat{K}(\xi) = \int K(x) e^{-2\pi i \langle x, \xi \rangle} dx = \int K(x) e^{-2\pi i \langle x|\xi|, \xi \rangle} dx$$

$$= \frac{1}{4\pi^2|\xi|^2} \int K \left( \frac{y}{2\pi|\xi|} \right) e^{i\langle y, \frac{\xi}{|\xi|} \rangle} dy.$$

Let us denote $K_0(y) = K \left( \frac{y}{2\pi|\xi|} \right)$ and define

$$F(\eta) = \int K_0(y) e^{i\pi \langle y, \eta \rangle}, \quad \eta = \frac{\xi}{|\xi|}.$$

From Lemma 2 p. 21 [3] it follows that there is a universal constant $c_1$ such that

$$F(\eta) = c_1 \int_0^\infty K_0(r) J(r) rdr, \quad |\eta| = 1$$

where $J$ is the Bessel function

$$J(r) = -J''(r) - \frac{J'(r)}{r}, \quad J(0) = 1, J'(0) = 0, \quad J(r) < 1, r \neq 0.$$
Therefore $F(\eta)$ can be further simplified as follows

$$
F(\eta) = -c_1 \int_0^\infty K_0(r)(rJ(r))'dr = c_1 \int_0^{2\pi|\xi|\eta_0} rJ'(r)K_0'(r)dr
$$

because from the definition of $K_0$ we have supp $K_0 \subset [0, 2\pi|\xi|\eta_0]$. Moreover, $K_0'(r) = -\frac{1}{r}$ hence

$$
F(\eta) = c_1(1 - J(2\pi|\xi|\eta_0))
$$

Consequently,

$$
\hat{K}(\xi) = \frac{c_1}{4\pi^2|\xi|^2}(1 - J(2\pi|\xi|\eta_0)).
$$

Next we restrict $\mu_1 = \mu_a - \mu_1$ where $C \subset$ supp $\mu_a$ is a compact such that $U^{\mu_1}$ is continuous. Observe that $\int U^{\mu_1}d\mu_a$ is finite hence $U^{\mu_1}$ is finite $\mu_a$ almost everywhere. By Theorem 1.8 p. 70 [7] for every $\varepsilon > 0$ small there is a restriction of $\mu_a$ such that

$$
0 \leq \int \mu_a - \int \mu_1 < \varepsilon.
$$

Note that if $\tau = \mu_a - \mu_1$ then we have

$$
|\mathcal{E}[\mu_a] - \mathcal{E}[\mu_1]| = \left|\int U^{\mu_a}d\mu_a + \int U^{\mu_1}d\mu_1\right| = \left|\int (U^{\mu_a} + U^{\mu_1})d\tau\right| = O(\varepsilon).
$$

Let $\phi_n(y) = n^d e^{-ng^2|y|^2}$ be the sequence of normalised Gaussian kernels. It is well-known that $\phi_n$ is a mollification kernel for every $n \in \mathbb{N}$ and moreover $\hat{\phi_n} = e^{-\frac{\xi^2}{n}}$. From the Parseval relation

$$
\int (\phi_n * U^{\mu_1})d\mu_1 = \int \hat{\phi_n} \hat{K}d\mu_1.
$$

If we first send $n \to \infty$ and then $\varepsilon \to 0$ to conclude the identity

$$
\mathcal{E}[\mu_a] = \int \hat{K}d\mu_a.
$$

On the other hand $\hat{U^{\mu_1}} = \hat{K}\hat{\mu_a}$, which yields

$$
\mathcal{E}[\mu_a] = \int \hat{K}(\xi)\frac{|\hat{U^{\mu_a}}(\xi)|^2}{|\hat{K}(\xi)|^2}d\xi = \int \frac{4\pi^2|\xi|^2}{c_1(1 - J(2\pi|\xi|\eta_0))}|\hat{U^{\mu_a}}(\xi)|^2d\xi = \int_{|\xi|<\delta} + \int_{|\xi|\geq\delta}.
$$

Using the expansion $J(t) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2^s \pi)^s} \left(\frac{t}{2}\right)^{2s} = 1 - \frac{t^2}{4} + \frac{t^4}{64} + \ldots$ we see that

$$
\frac{4\pi^2|\xi|^2}{c_1(1 - J(2\pi|\xi|\eta_0))} = \frac{1}{r_0^2c_1(1 - \frac{(2\pi|\xi|\eta_0)^2}{16}) + \ldots}
$$
hence the first integral is bounded below by $C(\delta) \frac{1}{\delta c_1} \int_{|\xi| < \delta} |\hat{U}^{\mu_a}(\xi)|^2 d\xi$ for sufficiently small $\delta > 0$. As for the second integral, we have

\begin{equation}
(4.9) \int_{|\xi| \geq \delta} \frac{4\pi^2 |\xi|^2}{c_1(1 - J(2\pi r_0|\xi|))} |\hat{U}^{\mu_a}(\xi)|^2 d\xi \geq \frac{4\pi^2 \delta^2}{c_1} \int_{|\xi| \geq \delta} |\hat{U}^{\mu_a}(\xi)|^2 d\xi.
\end{equation}

Combining we see that $\hat{U}^{\mu_a} \in L^2(\mathbb{R}^2)$ which, after we apply Parseval’s relation again, yields $U^{\mu_a} \in L^2(\mathbb{R}^2)$ and

\begin{equation}
(4.10) \|U^{\mu_a}\|_{L^2(\mathbb{R}^2)} \leq C\mathcal{E}[\mu_a].
\end{equation}

To finish the proof we use that $4\pi^2 |\xi|^2 |\hat{U}^{\mu_a}|^2 = |\nabla U^{\mu_a}|^2$ which together with (4.8) implies that

\begin{equation}
(4.11) \mathcal{E}[\mu_a] = \int \frac{1}{c_1(1 - J(2\pi r_0|\xi|))} |\nabla U^{\mu_a}(\xi)|^2 d\xi \geq \frac{1}{c_1} \int |\nabla U^{\mu_a}(\xi)|^2 d\xi
\end{equation}

which finishes the proof. \hfill \Box

**Corollary 4.3.** Let $\mu_a$ be as in Theorem 2.1. Then there holds

\begin{equation}
(4.12) \mathcal{E}[\mu_a] = \int U^{\mu_a} d\mu_a > 0.
\end{equation}

5. THE THIN OBSTACLE PROBLEM

From the $H^1(\mathbb{R}^2)$ estimate for $U^{\mu_a}$ it follows that $U^{\mu_a}$ is a solution to some variational inequality, and hence $U^{\mu_a}$ can be interpreted as a solution to an obstacle problem with a combination of both thin (on $\Gamma$) and ”thick” obstacles (on $\mathbb{R}^2 \setminus \Gamma$). It is convenient to define the obstacle as follows

\begin{equation}
(5.1) \psi(x) = \begin{cases} \frac{1}{2} (A \Gamma - |x|^2) & \text{if } x \in \Gamma, \\ \frac{1}{2} (A_0 - |x|^2) & \text{if } x \in \mathbb{R}^2 \setminus \Gamma. \end{cases}
\end{equation}

**Lemma 5.1.** Let $U^{\mu_a}$ be the logarithmic potential of $\mu_a$ and define

$$\mathcal{K} = \{ v \in H^1_{loc}(\mathbb{R}^2) \text{ s.t. } v - U^{\mu_a} \text{ has bounded support in } \mathbb{R}^2, \ v \geq \psi \}. $$

Then $U^{\mu_a}$ solves the following obstacle problem:

$$\int \nabla U^{\mu_a} \nabla (v - U^{\mu_a}) \geq 0, \ \forall v \in \mathcal{K}. $$

The proof is the same as in [2].

**Corollary 5.2.** $\text{dist}(\Gamma, \text{supp}(\mu_a \setminus \Gamma)) > 0$.

**Proof.** This follows from the estimate $A \Gamma > A_0$. Indeed, let us assume that $x_0 \in \Gamma \cap \text{supp} \mu_a$ and there is a sequence $\{x_k\}_{k=1}^\infty, x_k \in \text{supp} \mu_a \setminus \Gamma$ such that $\lim_{k \to \infty} x_k \to x_0$. Using the lower semicontinuity of $U^{\mu_a}$ (see Lemma 1 p.15 [3]) we see that

\begin{equation}
(5.2) \frac{1}{2} (A_0 - |x_0|^2) = \liminf_{x_k \to x_0} U^{\mu_a}(x_k) \geq U^{\mu_a}(x_0).
\end{equation}

Let $\rho > 0$ be such that $\{x_k\} \subset B_\rho(x_0)$. If $\rho$ is small then $\Gamma$ divides $B_\rho(x_0)$ into two parts $D^+$ and $D^-$. To fix the ideas let us suppose that $D^+$ contains a subsequence $\{x_k\}$. Let $h$ be the harmonic function in $D^+$
such that $h = \psi$ on $\partial D^+$. Observe that $h$ is continuous at $x_0$ because $\Gamma \in C^{1,\alpha}$. Since $U^{\mu_a}$ is superharmonic and on $\partial D^+$ we have $U^{\mu_a} \geq \psi = h$ then the comparison principle implies that

$$U^{\mu_a}(x_0) \geq h(x_0) = \frac{1}{2}(A\Gamma - |x|^2).$$

Combining (5.2) and (5.3) we see that $A_0 \geq A\Gamma$ which is a contradiction in view of (3.5). \hfill \Box

From Corollary 5.2 it follows that near $\Gamma$ the potential $U^{\mu_a}$ is a solution to a thin obstacle problem in the following sense, see [5] p. 108:

$$U^{\mu_a} \geq \frac{1}{2}(A\Gamma - Q),$$

$$\frac{\partial U^{\mu_a}}{\partial n^+} + \frac{\partial U^{\mu_a}}{\partial n^-} \geq 0$$

$$\left( u - \frac{1}{2}(A\Gamma - Q) \right) \left( \frac{\partial U^{\mu_a}}{\partial n^+} + \frac{\partial U^{\mu_a}}{\partial n^-} \right) = 0$$

on $\Gamma$

where $n^\pm$ are the outward normals on the $\Gamma$ corresponding to the domains that $\Gamma$ separates. In particular, if $\Gamma$ is $C^3$ regular then $U^{\mu_a}$ is $C^{1,\alpha}$ up to $\Gamma$ from each of its side, see Theorem 11.4 p.111 [5].

A particular case is $\Gamma = \mathbb{R}$ [4]. Using a simple symmetrization argument (see e.g. [6] p. 119 Theorem 4.6) we can show that the potential $U^{\mu_a}$ is symmetric w.r.t. the real line and hence we get the Signorini problem near $\mathbb{R}$ [5] p. 111.

One can make the connections with the obstacle problem more explicit by using the $H^1(\mathbb{R}^2)$ estimate in Theorem 4.1 and transforming the energy $I[\mu_a]$. Let $R > 0$ be fixed then using the divergence theorem

$$\int_{B_R} U^{\mu_a} \mu_a = -\frac{1}{2\pi} \int_{B_R} U^{\mu_a} \Delta U^{\mu_a} =$$

$$= \frac{1}{2\pi} \int_{B_R} |\nabla U^{\mu_a}|^2 - \frac{1}{2\pi} \int_{\partial B_R} U^{\mu_a} \partial_n U^{\mu_a}.$$  

For a.e. $R > 0$ the last integral can be estimated as follows

$$\left| \int_{\partial B_R} U^{\mu_a} \partial_n U^{\mu_a} \right| \leq \int_{\partial B_R} |U^{\mu_a}||\nabla U^{\mu_a}| \leq \int_{\partial B_R} |U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2.$$

From Theorem 4.1 and Fubini’s theorem it follows that

$$\int_{\mathbb{R}^2} (|U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2) = \int_0^\infty \int_{\partial B_R} (|U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2) dR.$$

Consequently,

$$\int_{\partial B_R} |U^{\mu_a}|^2 + |\nabla U^{\mu_a}|^2 \to 0 \quad R \to \infty$$

and we infer from (5.5) that

$$\int_{\mathbb{R}^2} U^{\mu_a} \mu_a = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla U^{\mu_a}|^2.$$

Recalling that by Corollary 3.3 $\text{supp} \mu_a \subset B_{r_0}$ for some $r_0 > 0$ and using the divergence theorem again we conclude

$$\int_{B_{r_0}} |x|^2 \mu_a = -\frac{1}{2\pi} \int_{B_{r_0}} |x|^2 \Delta U^{\mu_a} = -\frac{1}{2\pi} \int_{B_{r_0}} U^{\mu_a} \Delta |x|^2 + \frac{1}{2\pi} \int_{\partial B_{r_0}} (2r_0 U^{\mu_a} - r_0^2 \partial_n U^{\mu_a})$$

$$= -\frac{2}{\pi} \int_{B_{r_0}} U^{\mu_a} + \frac{r_0}{\pi} \int_{\partial B_{r_0}} U^{\mu_a} + r_0^2.$$
Combining these we have that the energy can be rewritten in terms of $U^\mu_a$ in the following form

$$I[\mu_a] = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla U^\mu_a|^2 - \frac{2}{\pi} \int_{B_{r_0}} U^\mu_a + \frac{r_0}{\pi} \int_{\partial B_{r_0}} U^\mu_a + r_0^2.$$  

REFERENCES


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