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Abstract

This paper addresses the role of manipulatives in learning by focusing on how their representational properties affect the strategies children employ in problem solving. Two studies examined the effect of physical materials (compared to no materials and pictorial materials) on children's (aged 4-7 years) problem solving strategies in a numerical (additive composition) task. The first study showed how children (n=32) not only identified more solutions using physical materials compared with no materials, but that using manipulatives fostered conceptually more developed strategies: relating consecutive solutions to each other systematically in exploring the space of permutations The second study demonstrated the unique benefits of physical manipulation by comparing children's (n=100) solutions and strategies using materials they could or could not spatially manipulate (physical v pictorial). As with the first study, children in the physical materials condition had more solutions and showed more conceptually developed strategies compared with the children in the pictorial condition. There was no advantage in providing children with a record of all their solutions. The paper discusses how this work focusing on the role of the representational properties of physical materials contributes to the wider debate about if and how manipulatives support learning.

1. Introduction

Despite considerable research on manipulatives and their continued use in education, it remains unclear if, and how, manipulatives support children's learning. In the present study, we sought to inform this debate by focusing on the way the representational properties of manipulatives influence young children's problem-solving strategies in an additive composition task. The two studies reported provide evidence that the representational properties of manipulatives can support children's strategies when exploring part-whole relationships.

1.1 The learning benefits of manipulatives

Manipulatives are physical materials such as blocks or tiles that are used pervasively in children's learning, particularly in mathematics. These materials have attracted considerable research interest over the last few decades, not least with the intention of informing teachers of how and when to use the materials to support children's learning.

In the last decade, there has been renewed research interest into manipulatives, largely attributable to two reasons. Firstly, empirical work is needed to contribute to theoretical developments surrounding children's manipulation and interpretation of symbolic representations (Martin, 2009; Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009), as well as claims that cognition may be inseparably linked to prior sensori-motoric experience (Lakoff & Núñez, 2000; Wilson, 2002). Secondly, understanding how physically manipulating representations supports learning is needed to address questions generated by recent interest in the use of digital learning materials for young children using touchscreens (Pitchford, 2015). Indeed, a common approach has been to compare the learning benefits of physical versus virtual manipulatives (Gire et al., 2010; Klahr, Triona, Strand-Cary, & Siler, 2008). Yet, such a dichotomy may be rendered invalid by the new and evolving forms of digital interaction: from devices such as Tablets (e.g. iPad) to digitally augmented physical materials ('Tangibles'), because of the close coupling between input (physical manipulation) and output (computer displays) – see BLIND FOR REVIEW]. These new digital opportunities emphasise a need for a more thorough grasp of how and when physically manipulating materials is important for learning.

In trying to understand the mechanisms by which manipulatives support learning, we can draw upon a range of relevant empirical studies. However, findings often seem contradictory, where manipulatives have been found to have a positive (Canobi, Reeve, & Pattison, 2003; Fuson & Briars, 1990), negative (e.g. Resnick & Omanson, 1987; Uttal, Scudder, & DeLoache, 1997), or insignificant effect on learning (compared to no materials or other materials, e.g. Baroody, 1989; Fennema, 1972). Although a recent meta-analysis suggests a slight positive effect (Carbonneau, Marley, & Selig, 2012), there remains a lack of consensus (McNeil & Jarvin, 2007). What these studies do highlight is that the effect of manipulatives on learning will depend upon the context in which they are used. And, in order to predict their effect in different contexts, we require a better understanding of *how* they influence children's thinking and learning.

Clearly, the potential effect of manipulatives will depend upon a range of interrelated factors, not least: the task at hand; the type of manipulative; children's initial understanding, and the teacher, who plays a pivotal role in structuring the environment and helping children connect concrete and abstract representations (Brown, McNeil, & Glenberg, 2009). The teacher's role will clearly be substantial, yet their approach will be informed by an improved understanding of how materials influence children's thinking. By furthering our understanding of *how* manipulatives support learning, we can offer teachers a clearer guide for *when* to use these materials.

1.2 Theories about how manipulatives support learning

Various authors have described ways in which manipulatives may support young children's learning. In a summary of the 'manipulatives debate', McNeil and Jarvin (2007) identify three reasons commonly put forward: manipulatives provide an additional channel for conveying information; activate real-world knowledge; and improve memory through physical actions. These proposed mechanisms are reflected in other work (Gravemeijer, 1991; Halford & Boulton-Lewis, 1992), however, because they are relatively high level, they lack sufficient explanatory power to predict *when* manipulatives will or will not support learning. More recently, the work of Martin and Schwartz (2005) has attempted to address this issue by proposing a theory focusing on the cognitive benefits of physical actions: the Theory of Physically Distributed Learning.

Although the Theory of Physically Distributed Learning (PDL) proposes to inform the wider debate on the relationship between action and cognition, the more specific empirical focus has been on the use of manipulatives in mathematics. To position their theory, the Martin and Schwartz put forward a framework for how individuals learn with physical objects using two dimensions: the stability or adaptability of 'the environment' (in this case physical objects), and the stability or adaptability of one's ideas (Figure 1). In this framing, the authors use the term 'offloading' to describe how the stable structure of materials may help thinking in a task, and distinguish this mechanism from actively manipulating materials to support thinking (Repurposing) or how the stable structure of the materials may foster changes in thinking (Induction). Physically Distributed Learning is the term given to describe when physically manipulating the environment can lead to changes in thinking (i.e. learning).

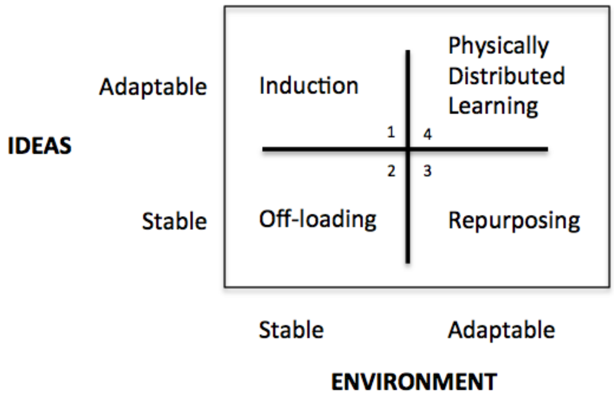

Fig 1: Four ways in which physical actions support learning by degree of stability of the environment and ideas (Martin & Schwartz, 2005, p. 588)

To test PDL, Martin and Schwartz compared children's (11-12 year old) learning of fraction concepts using two materials: one that could be physically manipulated (tiles and pie pieces), with one that had the same structure but could not be physically manipulated (squares on paper). Children solved fraction operator problems (such as one third of 12) using both materials in counterbalanced conditions. Each solution received an interpretation score that reflected the child's verbal answer and an adaptation score that reflected the child's physical arrangement of the pieces (*equal partitioning* of materials into spatial groups). It was found that physical materials conferred an advantage for both the number of adaptations (partitions) and interpretations (correct answers) thereby offering support for PDL. This theory has also received support in a study comparing younger children's (4-5 years old) use of physical and pictorial materials in addition and geometry tasks (Martin, Lukong, & Reaves, 2007).

PDL provides a theoretical framework for understanding how physical manipulation supports children's numerical development. Furthermore, the authors also suggest *when* PDL will occur: when children have '*incipient understanding'* in the task domain. Incipient understanding is framed in terms of children's familiarity and knowledge in the problem task. If children have too low knowledge, the physical environment may influence their actions but they may not be prepared to reinterpret and learn from the result of their actions. In contrast, if they have too high knowledge, they may simply use materials to enact their existing ideas. What is not clear, however, is whether PDL applies to all incipient concepts: whether there are times when physically manipulating materials may not affect, or even hinder, learning. Therefore, one aim of the studies reported in this paper is to apply the theory of PDL to a different task domain.

One limitation of PDL theory at present is that the cognitive processes by which interpreting different adaptations (configurations of manipulatives) will foster new ideas is fairly underspecified. If children's actions are not informed by domain knowledge, what is leading them to adaptations that are worth 'reinterpreting'? And what is the process by which children develop new ideas simply from perceiving particular adaptations (spatial configurations)? With respect to the fraction study, what is prompting children to create equally partitioned groups of objects? And how do these partitioned groups help children re-interpret their understanding of fractions? In order to begin to address these questions, we set out to study a simpler domain, related to the fractions domain and involving concepts underpinning understanding of fractions, which led us in turn to study a younger age group.

Thus, the contribution of this paper is to inform the debate surrounding the role of physical materials by extending the Theory of Physically Distributed Learning to a different concept and age group. The target concept is additive composition: how numbers can be composed in different ways. This concept is fundamental to younger children's numerical development and underpins later numerical concepts, including the fractions domain studied by Martin and Schwartz (2005).

Our previous work [Blind for Review], discussed below, demonstrated how the manipulative properties of materials (ability to move multiple or single blocks at a time) influenced children's strategies in this partitioning problem. A limitation of this work was the absence of a control condition to examine how manipulatives influenced strategy compared to no materials, or a condition that offered a representation that could not be adapted spatially (represented by fixed squares on paper). The studies in this paper address this gap in our previously published studies and also address a further limitation of our work and that of Martin and Schwartz: evaluating the significance of a representational limitation of physical materials. Unlike some other external representations such as paper, manipulatives only provide a record of the last solution identified, not of previous solutions – in other words, manipulating physical materials leaves no record of previous solutions. This may be significant in a problem that requires children to identify and track multiple solutions.

1.3 The effect of representational properties on children's strategies

The Theory of Physical Distributed Learning can be related to earlier work of Scaife and Rogers (1996) who drew attention to how the environment can influence thinking. According to their theory of External Cognition, thinking can be described as a cognitive interplay between internal and external representations (Scaife & Rogers, 1996). Within a particular task, external representations change the cognitive demands of problem solving and hence affect how easily children are able to carry out particular strategies. For example, providing children with physical materials to solve '7 + 3' can help reduce the demands of counting (e.g. by helping keep track of items counted) using a 'count-all' strategy. Providing a particular quantity represented by the external representation may significantly reduce the demands of a problem: 12 tiles will likely support attempts to solve the problem of calculating 1/3 from 12 more than providing 11 or 13 tiles.

Therefore, the properties of materials, such as their spatial representation of groupings, can influence the demands of carrying out children's existing strategies: quadrants 2 and 3 in Figure 1 (Offloading and Repurposing). But what if children lack sufficient conceptual understanding to inform a (successful) strategy? We propose that in this case, the representational properties of materials are able to prompt and facilitate particular strategies. This is because materials have particular affordances, which can enable or facilitate certain physical actions or sensory perception (Gibson, 1977; see Hartson, 2003) In other words, physical materials can inform and influence children's strategies by prompting and facilitating certain actions, as well as children's ability to interpret these actions. The following section looks more closely at the manipulative and perceptual properties of physical representations to consider how they might influence children's actions and interpretations.

The affordances of physical materials can prompt particular actions. These might include: grasping multiple blocks, collating blocks into a single pile, moving blocks into different groups, stacking blocks, creating symmetrical groups, or simply touching (tagging) blocks. The context will influence children's actions: when given materials for a mathematical problem, children may be aware that moving blocks into different groups may be more appropriate than building a tower[1]. If children lack strategies with which to inform their actions with materials, the affordances of materials therefore offer a stimulus for action. In other words, the less developed are children's existing strategies, the more their actions with materials may be elicited by the affordances of materials.

Manipulatives may therefore prompt particular actions during problem solving. They also offer certain visual and tactile information that may affect what children interpret and how easily they can do so. Manipulating materials can create new spatial configurations, where spatial location offers information about the relatedness of information (Larkin & Simon, 1987), such as how objects are numerically grouped. For example, 12 objects can be partitioned into groups such as 6 and 6, or 5 and 7, or 4 groups of 3, depending on how they are spatially arranged. Spatial properties can even affect how easily groups are enumerated by providing a means to visually identify when to stop counting objects in a group. It is even possible to enumerate small groups perceptually (without counting) a process called 'subitizing' (Mandler & Shebo, 1982). Touching objects can offer a further mode to help keep track when counting (Alibali & DiRusso, 1999), and interestingly, subitizing can also be achieved tactilely (Riggs et al., 2006).

The perceptual affordances of manipulatives may therefore offload the demands of enumerating groups. This is important because it facilitates children's task of interpreting the result of their actions. Taking the aforementioned fraction problem as an example, having created four groups using 12 objects, it is relatively easy for children to enumerate that there are three objects in all four groups. Such interpretation may be significant if children have an incipient idea that fraction problems involve partitioning objects into equal groups.

In previously published work [blind for review], we investigated the representational properties of physical materials by comparing children's actions and strategies with physical blocks compared with virtual (on-screen) squares where manipulation was constrained through using a mouse. In order to predict the effect of constraining children's actions with materials, a small study was first carried out that video-recorded children's actions with physical materials when solving a partitioning problem requiring them to identify all the ways to partition a number into two groups (e.g. 9 and 1, 8 and 2, 7 and 3 etc.). The study helped to illustrate and explain the role of various representational properties in children's problem solving.

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[1] Although as, Uttal et al (1997) argue, children's prior experiences may lead them not to interpret physical materials as numerical representations, citing the example of using blocks to recreate notation in an addition problem.

In terms of perceptual properties, children often used tactile information: touching or covering objects to help keep track when counting (Figure 2a) or to remember what to move next. Children also used spatial information to support thinking, commonly moving blocks into different groups to identify new solutions, but also in other ways, such as moving blocks away from the body to keep track of what blocks to move next. As previously argued, creating small spatial groups of blocks may have helped children enumerate quantities perceptually (e.g., through subitizing). The perceptual properties of materials may therefore have helped children interpret the result of their actions with materials. The physical properties of the manipulatives properties, however, may have played a more significant role in children's strategies.

Video data analysed in our study illustrated how children were able to easily grasp and then manipulate single or multiple blocks at a time. Blocks could be slid, lifted or even dropped into different groups quickly and with seemingly little cognitive demands. Therefore, children were able to explore a range of spatial changes efficiently. Significantly, two strategies reflected two particular actions. For one strategy, children would use one hand to move one block from one group to another (Figure 2b). This strategy reflected consecutive solutions that differed by one (e.g. 7 and 2, then 8 and 1). In the other strategy, children would grab a group of blocks in each hand and swap over hands (Figure 2c). This strategy reflected consecutive solutions that were commutative (e.g. 7 and 2, then 2 and 7). This observation led to the prediction that constraining actions using the mouse (so children could only move one object at a time) would significantly increase the use of the strategy identifying consecutive solutions that differed by one (and reduce the use of the other strategy). This prediction was supported [REF]. This study therefore demonstrated the relationship between the representational properties of physical materials and the actions generated when solving certain numerical problems. The study did not, however, identify if and how children's strategies with manipulatives differ significantly compared to using no materials, or (as with the experimental set-up of the study by Martin and Schwatrz, 2005) a non-manipulable external representation such as squares on paper.

Figure 2: a) use of perceptual properties b) single block manipulation c) swapping groups of blocks

1.4 Summary

Physical materials, therefore, have particular manipulative and perceptual properties that may influence children's problem solving actions and how they interpret resulting representational states in relation to the problem. Whilst this influence on action and interpretation is relevant to the broader scope proposed by the theory of Physical Distributed Learning, the focus of this paper is on numerical problem solving where manipulatives reflect quantities. In this context, a key benefit of manipulatives may be the ability to create and recreate spatial configurations with ease. However, whilst manipulatives do provide information about the last representational state created, action with materials necessarily removes a record of this last state (because materials are physically moved from one place to another). Significantly, manipulatives do not provide a record of previous states, leading Kaput (1993) to refer to manipulatives being constrained to the 'eternal present'. In contrast, materials such as paper do provide a record of previous actions – through the trace of annotations. For this reason, children are able to 'show their working'. There may, therefore, be a representational 'trade-off' between spatial manipulation and record of previous actions. The roles of these representational properties are explored in the studies reported in this paper.

1.5 Aims

This paper examines if, and how, manipulatives can influence children's problem solving strategies, by reporting two studies examining the effect of physical materials on children's problem solving strategies within a numerical task. The studies aim to balance ecological validity (by employing materials and tasks familiar in a classroom) with experimental manipulation aiming to illuminate the role of particular representational properties.

The first study examines the effect of physical materials on children's problem solving in comparison to no materials. The research question addressed is:

- Do physical materials significantly affect the types of strategies children use in a numerical task compared to no materials?

By comparing physical materials to a no materials condition, it is not clear whether any differences found are attributable to physical manipulation or simply the presence of an external representation. The second study therefore focuses on the unique benefits or limitations of physically manipulating materials by comparing children's problem solving in the same task using physical materials with pictorial (diagrammatic) materials. Consequently, this study echoes the experimental design of PDL in a different domain but with a focus on representational properties and strategy. The second study further examines the effect of providing children with a record of all their previous solutions (representational states), thereby evaluating the trade-off between spatial manipulation and record of previous actions previously described. The research question addressed is:

- What are the benefits and limitations of spatial manipulation of materials on children's numerical strategies?

1.6 Task Domain – Additive Composition

With the aim of examining the effect of physical materials on children's numerical strategies, it is important to identify a task that a) reflects a significant concept in children's numerical development b) offers a range of strategies and c) can be approached using different materials including manipulatives. In this regard, the studies focused on the concept of additive composition: *an understanding that numbers can be composed and decomposed into small numbers.* This concept is significant in young children's numerical development (Baroody, 2004; Resnick, 1983), notably as a foundation to their understanding of the decade structure (Nunes & Bryant, 1996).

1.6.1 Additive Composition task

Various tasks have been identified for assessing additive composition (see Cowan, 2003), such as the use of decomposition strategies in addition. These assessment tasks are generally single answer problems where children's understanding is inferred from the strategies needed to solve them. These assessment tasks can be contrasted with activities where children focus on identifying how a number can be decomposed. Jones, Thornton, Putt, Hill, Mogill, & VanZoest (1996), for example, describe a partitioning task where children are asked to identify as many ways as possible to decompose a number. They present a problem using a story context and concrete materials as follows:

"*The man in the yellow hat shook 2 bags. 'I had 10 candies and put some in one bag and the rest in the other", he told George. How many could be in each bag?*"' (p. 316)

This task therefore seems ideal for our present purposes because a) it addresses an important numerical concept (additive composition) and b) offers a range of strategies for solving and c) can be studied using physical materials.

1.6.2 Task Strategies

In Jones et al.'s (1996) partitioning task, children are given the task of identifying all the different combinations of two parts (P1 and P2) for a given whole (W). For each valid solution, these parts combine to make the whole: P1 + P2 = W, for example: 2+6=8. As P1 or P2 can equal zero there are a total of W+1 solutions; for example, when partitioning the amount 3 into two parts, there are four solutions (3+0, 2+1, 1+2, 0+3).

The children's task is therefore to identify a valid solution for P1 and P2, to then to identify more solutions ensuring that the value of P1 and P2 are different each time (keeping track of what solutions have been given), and to continue so that all possible values of P1 and P2 have been identified (keeping track of solutions left to identify). There are at least five identifiable strategies for how a child might identify solutions mentally as illustrated in Table 1:

Table 1: Possible strategies in the partitioning task

Strategy	**Description**	**Example of expected verbal solution pattern**	**Strategy label in Studies**
1	Identify P1 such that P1 ≤ W. Then identify P2 through approximation	E.g. 2 & 5 following 4 & 3 (i.e. no clear relation)	*other*
2	Identify P1 such that P1 ≤ W. Then calculate P2 by counting down from W or up to W	E.g. 2 & 5 following 4 & 3 (i.e. no clear relation)	*other*
3	Recall P1 and P2 of previous solution and reverse such that P1=P2 and P2=P1	E.g. 2 & 5 following 5 & 2 (i.e. reverse parts)	*commutative*
4	Recall P1 and P2 of previous solution and change values by one (P1+/-1, P2 +/-1) maintaining P1+P2 = W	E.g. 2 & 5 following 1 & 6 (i.e. parts differ by one)	*compensation*
5	Recall solution from declarative memory	E.g. 2 & 5 following 4 & 3 (i.e. no clear relation)	*other (unlikely)*

This task therefore offers a range of possible strategies. Although this paper focuses on if (and how) physical materials influence children's strategies, it is also possible to consider whether they foster more or less conceptually developed strategies. Here is it possible to draw upon the work of Fuson (1992) who describes specific stages of children's numerical concepts. In her paper, Fuson describes a developmental step from the *Numerable Chain* level, where children understand that numbers form a sequence that can be broken (a whole into two parts), to the *Bi-directional chain* level, where the whole number sequence becomes a series of embedded cardinal amounts. Fuson also describes the relationship between these levels and the type of addition / subtraction strategies children might employ. The *Numerable chain* is reflected in children's ability to count up / down / on from one part to another. This procedural ability reflects strategy 2[2] (See Table 1). In contrast, the *Bi-directional chain* level is reflected in decomposition strategies: recomposing parts to facilitate addition (e.g. 6 + 5 into 5 + 5 + 1). The use of such a strategy has been proposed as evidence of understanding of additive composition. We argue that this process of recomposing parts is reflected in strategy 4, where children relate a solution to the previous one. Strategy 3 also reflects relating a solution to the previous one, but through re-ordering rather than recomposing (Commutativity). We therefore argue that by relating consecutive solutions, strategies 3 and 4 are conceptually more developed strategies than 1 and 2. Strategy 5 is not discussed, as children in our study were unfamiliar with the amounts to partition (therefore had not had the opportunity to commit to memory).

Consequently, the partitioning task offers the opportunity to examine and compare two measures: a) the number of solutions identified and b) the type of strategies used to identify solutions, where strategies 3 and 4 ('related') are considered more developed than 1 and 2 ('unrelated'). Whilst many studies focus on the first of these measures, the second is most relevant for this paper: can physical materials significantly affect children's numerical strategies?

2. Study 1: Do physical materials significantly affect the types of strategies children use in a numerical task compared to no materials?

2.1 Introduction

This study examined whether the use of physical materials significantly influence children's numerical strategies, and whether in this particular problem, they foster conceptually more developed or less developed strategies than no materials. One possibility is that the physical materials foster the use of less developed strategies: Strategies 1 and 2. This is because materials can offload the demands of calculating individual solutions in the partitioning task. Rather than counting each solution mentally, children need only to partition objects into two groups and count the amount in each (P1 and P2) to give a verbal solution. An alternative possibility is that physical materials foster the use of more developed strategies 2

[2] Strategy 2 is arguably more developed than Strategy 1 but is considered comparable, mainly for the practical difficulty of identifying when children are estimating as small quantities can be enumerated visually (subitized).

and 3. This is because providing a visual representation of each solution encourages children to identify a related solution: for example, having identified the solution 2 & 5, children need only swap over amounts to identify the solution 5 & 2 (*commutative*). To find such an effect would be significant, not just because physical materials are encouraging a more developed strategy, but such a strategy corresponds to one that children can employ in the later absence of materials. A final possibility is of course that using physical materials has no effect on strategy use.

2.2 Method

2.2.1 Design

In order to increase power and reduction in error variance for this study, a within subjects design was used with Condition (Physical/No Materials) as the within subjects independent variable. The dependent variable was the number of correct solutions. These solutions were then coded according to a scheme developed in this study in order to create a further dependent measure: the number of solutions identified using particular strategies.

2.2.2 Participants

Thirty-two children took part in this study (17 girls and 15 boys, age range 68 to 82 months; M=74.2; SD=3.86 months). Children were from two classes in the same year group in a local school in [blind for review] whose parents had signed and returned a consent form (56% response). The school is a larger than average primary school, with 345 pupils, and situated in a suburb that is recognised as having a high social, educational and economic level. This is reflected in the small proportion of children that receive free school meals (2% compared to national average of 16%). In this study, all but one child had English as their first language and one child was reported as having additional support needs. (This child was competent in the task and included in the analysis.)

2.2.3 Materials and Procedure

Sessions took place individually on a table in the corridor outside the class. They were held during lessons when noise levels in this area were acceptably low, and lasted between five and ten minutes. The sessions were presented as follows (always in this order):

1. Introduction to problem context
2. Condition 1 Example partitioning problem (with 3) with or without materials
3. Condition 1 Problem: partitioning 6
4. Condition 2 Example partitioning problem (with 3) without or with materials
5. Condition 2 Problem: partitioning 7

The order of condition (Physical/No Materials) for the problems was counterbalanced, changing for each child in turn. The order of children reflected an alphabetic class list, which made it easier for the class teacher to know who was next and was deemed sufficiently randomised for this within subjects design. The materials used in this study were small plastic *Unifix®* blocks (2cm^2) of the same colour (blue) (Figure. 3). These materials are common across early years classrooms in the UK.

Figure 3: Physical Materials in Study 1

2.2.3.1 Introduction to problem

The interviewer explained that the purpose of the research was to find out what children find easy and difficult about number questions. Children were then presented with the story context for the partitioning problems. They were introduced to a character: 'Jon' (Figure 4), and told how this character likes to buy bananas and put them in his two bowls. The bowls were different colours (red and green – no child was colour blind); this was done to emphasise the difference between *commutative* solutions (i.e. 3+5 is a different solution from 5+3). The interviewer explained that the aim was to try to help Jon by telling him all the different ways he could keep his bananas in the two bowls.

Figure 4: Image used to provide story context to problem

2.2.3.2 Example problem

Before each partitioning problem in both conditions, the interviewer presented an example to help children understand the task demands and what constituted a valid solution. The interviewer explained: "*One day, Jon bought 3 bananas* [interviewer shows image of 3 bananas]. *Watch how I use* [my head/these blocks] *to help me find all the ways the three bananas could be in the two bowls.*" In the Physical condition the interviewer placed three blocks on the table. In the No Materials condition, the interviewer pointed to their own head (the teacher of the class had explained how this prompt was used when children were being asked to solve problems mentally).

The interviewer then identified the four ways to partition three in the following order: 3+0, 1+2, 2+1, and 0+3. This order was chosen to reduce the likelihood of prompting a particular strategy. These solutions also intended to highlight that commutative solutions were considered unique and that zero was a valid solution (we expected children to understand that zero was valid from the study context and our prior work, as well as the literature (Clarke, Cheeseman, & Clarke, 2006)).

In the Physical condition, the interviewer partitioned the blocks before identifying the solution. Partitioning blocks involved moving the blocks into left and right groups in front of the interviewer. The blocks were not re-collected after each solution but moved directly from one solution to the next[3]. In the No Materials condition, the interviewer simply pointed to the corresponding bowls when stating the verbal solutions. In the demonstration, the interviewer explained that there could be "*three in the red bowl and none in the green*", "*one in the red bowls and two in the green*", "*two in the red bowl and ...*" On this third solution, the interviewer purposefully paused and looked at the child to prompt the child to say the solution (two in the green). If the child did not answer, the interviewer used the image of the bananas and repeated "*two in the red bowl and …*" All children were able to complete this, as well as the final solution which again the interviewer prompted "*and none in the red bowl and ...*" (three in the green). The prompts for children to complete the solution were to ensure understanding and for children to practise giving numerical answers for each part.

2.2.3.3 Partitioning problems

After the demonstration problem, the interviewer removed the picture of the three bananas but kept the picture of the stick figure and the two bowls. The children were then told that on another day Jon went shopping and bought 6 (then 7) bananas. The order of total amount to partition was the same for all children: 6 followed by 7[4]. Similarly to the above example, in the Physical condition, children were presented with the correct total number of blocks to partition, which were placed in a line in front of the child.[5]

Children were then asked to use the blocks (or "use their heads" in the No Materials condition) to tell the interviewer all the ways in which Jon could put the 6 or 7 bananas in the two bowls. The interviewer did not touch the blocks during children's problem solving. The children were reminded that, for each solution, they were to say how many there were altogether in each bowl so that the interviewer could write down their answers. After solving the first partitioning problem, the interviewer presented the example and partitioning problem in the other condition. Condition order was counterbalanced between children.

2.2.3.4 Prompts given during problem solving

For all problems, if children did not respond after 10 seconds they were prompted by the interviewer: "*can you think of any ways that Jon can put the [6/7] bananas in the two bags?*" If there were significant pauses after children had identified the first solution, the interviewer prompted by saying "*is that all the ways or can you think of any more ways?*" The session ended after two prompts had been given or if the child indicated that

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[3] The blocks therefore provided a record of the last solution created. It was decided not to recollect blocks as a) this was considered more ecologically valid and b) this was considered an important representational affordance of manipulatives not to design out of the study.

[4] This was not counter-balanced due to sample size. We recognize this limitation, although counterbalanced conditions should have mitigated possible effects.

[5] Providing this amount likely offloaded the demands of using physical materials; however, the focus of the study was on if and how materials influenced children's strategies (where offloading demands might arguably encourage a less efficient strategy). It is interesting to reflect upon how various manipulatives offer this representational benefit (e.g. bead string or abacus when partitioning ten).

he/she had finished. If a child used non specific words such as 'some' or 'the rest' when identifying solutions, the interviewer prompted by asking "so how many is 'some'/'the rest'?

The interviewer wrote down all solutions given by the children so that they could see that their answers were being recorded (and that they were therefore important to the task) although they could not see what was actually being written down. Children generally said or pointed to the bowl to which they were referring (e.g., 'three in that one') but if it was not clear the interviewer prompted "*three in which bowl?*" The interviewer recorded the left bag as referring to the first part and right as the second.

2.3 Results

2.3.1 Correct solutions

Solutions were initially coded as correct or incorrect. Correct solutions were then further coded as being unique or repeated (see Figure 5). A repeated solution was any solution that had been given previously (in the same addend order). Each child received a score for the number of unique correct solutions identified in each condition[6]. Henceforth, unique correct solutions will simply be referred to as correct solutions and repeated correct solutions will be referred to as repeated solutions. If a score was incorrect, it did not matter whether it was repeated or not. The distribution of group data was tested (Kolmogorov-Smirnov) and revealed significant departures from normality for scores on the first problem, partitioning 6 ($D(32)=0.17$, $DF=32$, $p<0.05$), although not the second ($D(32)=0.13$, $p=ns$). A Wilcoxon test was therefore carried out and showed there were no significant differences for correct solutions between the first (Mdn=5) and second problems (Mdn=5) ($Z=-0.70$, $p=ns$).

The distribution of group data was tested (Kolmogorov-Smirnov) and revealed no significant departures from normality for scores in the Physical condition (Kolmogorov-Smirnov: $D(32)=0.161$, $p<0.05$). A Wilcoxon test revealed that children identified significantly more correct solutions in the Physical condition (Mdn=6) than the No Materials condition (Mdn=4) ($Z=-4.50$, $p<0.0005$). In addition, the effect size was found to be fairly large ($d=1.09$, $r=0.48$) using Cohen's *d* for paired samples (Cohen, 1988). Children typically only gave one incorrect solution if any, but were more likely to do so in the No Materials (18 children) than Physical condition (4 children).

2.3.2 Strategy

In order to examine differences in the possible strategies used between conditions, a coding scheme was first developed for correct solutions.

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[6] In all studies carried out on this problem, scores were categorized to compare performance on problems of different total amount accordingly: no solutions, single solution, more than one but less than half total solutions, and more than half solutions. It was found that analyses using these coded scores revealed differences in the same direction and magnitude. Therefore, the analyses reported henceforth just examined the absolute number of correct solutions.

2.3.2.1 Coding Scheme

Two key strategies for partitioning were previously identified: *commutative* and *compensation*. A *commutative* strategy was defined as reversing the order of parts of the previous solution. A *compensation* strategy was defined when adding one to one part and taking one from the other. It is thereby possible to examine each solution children gave (after the first solution) in terms of its relationship to the previous solution and use this to infer strategy. For example, the solution '1+6' after '6+1' might arguably reflect a *commutative* strategy. Similarly, the solution '2+5' after '1+6' might reflect a *compensation* strategy.[7]

Clearly, this form of coding allows both type 1 and 2 errors: a solution identified using a strategy might not be coded because children did not actually verbalise the initial solution. Equally a solution might be coded although it only followed the previous by chance. However, as these errors should be equally as likely to occur in each condition, it should be possible to compare conditions to examine any significant differences.

It is important to note that a solution that is coded as neither *compensation* nor *commutative* does not mean that children were not relating successive solutions. Indeed a couple of children seemed to apply a combination of *commutative* and *compensation* at the same time (e.g., swapping over and moving one object: e.g., '1+6' following '7+0'). However, these were less clear and not frequent, and any solution after the first that was not coded as *compensation* or *commutative* was coded as *other*. The coding flow diagram is presented in Figure 5.

[7] Identifying 3+4 following 4+3 (or visa versa) falls under *compensation* and *commutative*; yet it is difficult to know which strategy children are employing. Solutions of this pattern were coded as *compensation* for several reasons: a) this was suggested by the patterns of solutions (e.g. preceded by and/or followed by a compensation solution) b) observations in the physical condition (moving one block not swapping groups of blocks as associated with other commutative solutions c) we had no reason to believe that any coding errors would not be as likely to occur in either condition.

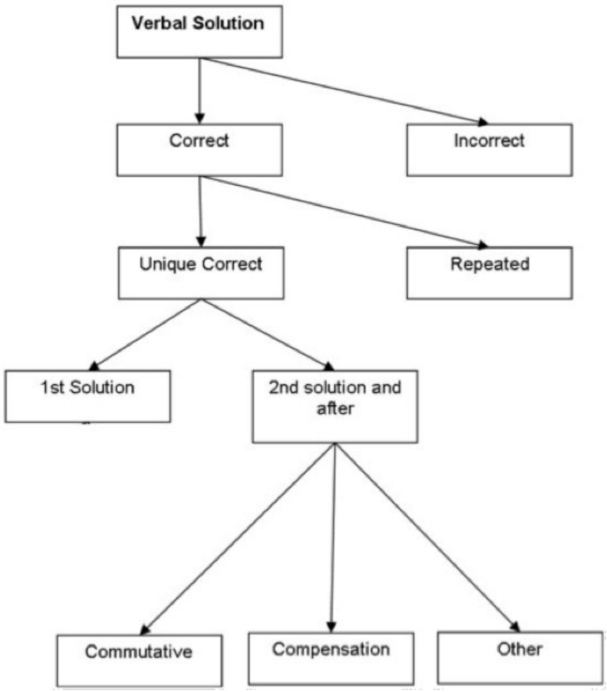

Figure 5 Coding of Strategies

2.3.2.2 Differences in strategy use between conditions

Using the coding scheme, it was possible to give each child a score in each condition for the number of *compensation*, *commutative* and *other* solutions given. The maximum number of *commutative* solutions possible for partitioning 6 and 7 was three. The maximum number of *compensation* and *other* solutions for partitioning 6 was six, and for partitioning 7 was seven. The median and interquartile scores are shown in Table 4. Whilst 19 children identified at least one *commutative* solution in the Physical condition, less than half (10) did so when solving the partitioning problems without materials. Similarly, whilst most children (28) identified at least one *compensation* solution in the Physical condition, only 14 did so in the No Materials condition. Wilcoxon tests[8] showed that children identified significantly more *commutative* solutions ($Z=-2.25$, $p<0.05$) and significantly more *compensation* ($Z=-3.69$, $p<0.01$) solutions in the Physical condition than the No Materials condition. There were no significant differences between conditions for the number of *other* solutions ($Z=-0.39$, p=ns).

Table 2: Medians (IQR) for strategy solutions in the Physical and No Materials conditions

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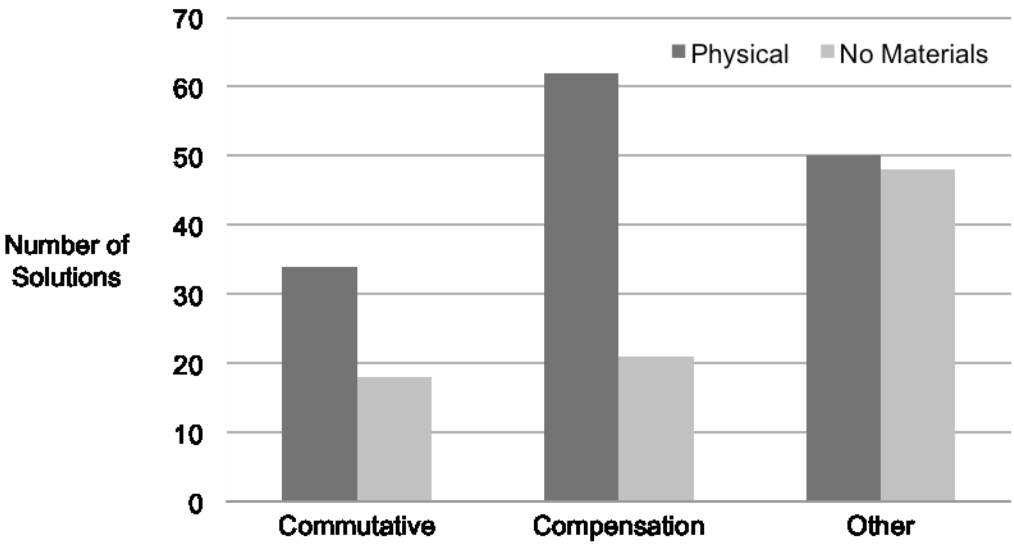
[8] Considering the median scores of zero in the No Materials condition, another way to approach analysis would have been to categorise scores according to whether children identified at least one solution or not, and then carry out paired sampled tests on the binomial distributions. However, Wilcoxon tests will be reported in this paper as a) significance levels for differences between conditions were unchanged, and b) this acknowledges the interval data for the majority of children in one of the within subjects conditions.

	Commutative	*Compensation*	*Other*
Physical	1* (0, 2)	1.5* (1, 2.75)	2 (1, 2)
No Materials	0 (0, 1)	0 (0, 1)	1 (0.25, 2)

*Significant differences between conditions ($p<0.05$)

Although these tests revealed a significantly greater number of *commutative* and *compensation* solutions in the Physical condition, it might be argued that this can be explained by the fact that children in the Physical condition simply identified more correct solutions overall (although the difference in *other* solutions was not significant). Indeed, Spearman Rank order correlations revealed significant positive relationships between *compensation* solutions and overall solutions in the No Materials ($r=0.465$, $p<0.01$) and Physical conditions ($r=0.506$, $p<0.005$), and similarly, significant positive relationships between *commutative* solutions and overall solutions in the No Materials ($r=0.606$, $p<0.001$) and Physical ($r=0.471$, $p<0.01$) conditions. However, whilst the correlation between *other* solutions and overall solutions was large in the No Materials condition ($r=0.718$, $p<0.001$), this was not significant in the Physical condition ($r=0.231$, p=ns). In other words, when children used materials, a greater number of correct solutions reflected a greater number of related (*compensation* and *commutative*) but not unrelated (*other*) solutions.

Therefore, analysis of the relationship between the number of strategy solutions and overall solutions supports the prior analysis of differences between conditions for the strategies used. Children identified more solutions overall in the Physical condition and this is reflected in a greater number of *compensation* and *commutative* solutions but not in *other* solutions. The differences between conditions are more clearly illustrated in Figure 6 according to the total number of strategy solutions identified in each condition.



	Commutative	Compensation	Other
Physical	34	62	50
No Materials	18	21	48

Figure 6: Total number of coded strategies identified by children in each condition[9]

2.3.2.3 Initial Solution – Equal Partitioning

The strategies analyzed above were for solutions given after the first. However, it was interesting to notice differences in the pattern of first solutions given. For many children, the first solutions given for partitioning 6 were 3 & 3: and *equal partitioning.* For partitioning 7, many children identified an initial solution that was as close to equal as possible: 3 & 4 or 4 & 3. By coding such first solutions as '*equal partitioning*', it was possible to examine differences between the two conditions. A signed ranked test was carried out to test differences between binomial data for each condition and found significantly more *equal partitioning* solutions in the Physical condition (+ve=18, -ve=4, ties=10, $p<0.005$). This lends further support that the representational properties of physical material significantly affected children's strategies.

2.4. Discussion

This study examined the effect of physical representations on children's partitioning strategies. As expected, children identified more partitioning solutions with Physical materials than without. It is not uncommon for studies to interpret this advantage as support for the use of materials. In this paper, however, it is the effect on strategies that is considered significant. This study demonstrated that a) physical materials did significantly change what strategies children used with materials than without and b) physical materials fostered the use of conceptually more developed strategies. In other words, rather than using physical materials to offload the demands of identifying solutions independently of each other, the materials prompted children to identify solutions that were related to each other. This is significant. An important stage of children's numerical development is developing an understanding of how number can be decomposed and recomposed in different ways and the relation between these ways. This understanding can be directly related to a common curriculum objective of identifying number bonds to ten (e.g. 1 & 9, 2 & 8, etc.). Baroody (2006) has argued that mastery of number bonds requires understanding of how numbers are related.

A further finding from this study was the influence of physical materials on children's initial solution. With materials, children were much more likely to identify an '*equal partitioning*' solution. This is particularly significant because this is arguably *not* the most efficient way to start this particular problem as there is then no commutative solution when parts are equal (e.g. 4 and 4), and a compensation strategy would only identify about half of all solutions (i.e. requires children to subsequently find a way to identify remaining solutions). This lends further support that, in the absence of a planned strategy, children's actions are prompted by the affordances of materials. This finding may also help explain the benefit of physical materials in Martin and Schwartz' (2005) fraction study: by prompting children to partition materials into equal parts, manipulatives may have encouraged effective strategies for solving fraction problems (where partitions are equal).

[9] As described, the total possible number of *compensation* and *other* solutions is greater than the total number of *commutative* solutions

Whilst this study shows the benefits of materials in a particular problem, it is unclear what representational properties of physical materials supported problem solving. It is possible that simply providing an external representation of the amount to partition was sufficient. The following study examines the unique benefits of spatial manipulation by comparing physical and pictorial materials in the same task. As well as examining the benefits of spatial manipulation, the study looks at the limitations of spatial manipulation of physical materials: that actions necessarily removed any record of previous actions/solutions.

3. Study 2: What are the benefits and limitations of spatial manipulation of materials on children's numerical strategies?

3.1 Introduction

As articulated in PDL theory, adapting physical materials, in contrast to pictorial materials, can create new spatial configurations. Whilst the materials therefore provide visual information on the last representational state created, further action necessarily remove all evidence of this last configuration: manipulatives do not provide a record of prior actions. As Kaput (1993) states: manipulatives are constrained by the 'eternal present'. In contrast, with materials such as paper, actions are recorded through annotation. This 'cognitive trace' offers the possibility to review prior actions to inform plans of subsequent actions. Yet, the extent to which this supports children's problem solving is not clear. Not only will there be demands in revisiting and interpreting prior actions, but in order to recognise the value of this record, children arguably already possess a good conceptual overview of the task at hand.

Examining the benefits of a 'representational record' is important, not simply to evaluate the relative limitations of physical materials over other representations, but because it is possible to address such limitations through digital design. As argued by Kaput and others (e.g. Sarama & Clements, 2009) have argued, digital materials are able to overcome the drawbacks of their physical counterparts by providing a means to record and revisit prior actions on materials. Consequently, this study can help evaluate the relative merits of this design possibility in this specific domain.

3.2 Method

3.2.1 Design

A 2x2 between subjects design was used with Material (Physical/Pictorial) and Record[10] (No Record/Record) as the two independent variables, resulting in four independent groups: Physical Record, Physical No Record, Pictorial Record and Pictorial No Record. The primary dependent measure was the verbal solutions provided by children for three partitioning problems, which were then coded according to strategy using the previous coding scheme.

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[10] We use the term 'record' to describe the representational feature enabling children to refer back to previous solutions during action; however, we recognise that manipulatives do provide a record of the last solution before subsequent action, and later discuss this study design limitation.

3.2.2 Participants

One hundred children took part in this study (54 girls and 46 boys; age range 53 months to 87 months; M=70.79 months; SD=9.98 months). Children were from three consecutive year group classes at a local primary school in the [Blind for Review] area. The percentage of children receiving free school meals is slightly above the national average (a measure of Social Economic Status). There were 2 children with English as a second language and 1 with additional support needs. These children did not have significant difficulties with the problems so they were included in the analysis. Children were randomly assigned (using a random number generator) to one of the four conditions.

3.2.3 Materials and Procedure

In this study, children solved three partitioning problems: partitioning 7[11], then 8, then 9 in all conditions; however, as they used the same materials, they were only given one example partitioning problem (with 3) before problems. The procedure for this study was identical to Study 1 apart from two key differences: the materials provided and a different story context.

3.2.3.1 Story context

It was decided to present the children in this study with a different story context from that given to the children in Study 1. The problem structure was isomorphic but used cows in fields rather than fruit in bags for two main reasons. Firstly, because some children were younger, it was felt that a clear visual image of the two partitioning areas would support children's understanding. Secondly, it was expected that this problem was less hypothetical: cows can change fields over time, whereas a person is not likely to change objects in two bags (or reflect on the change). Importantly, it is also less logical for cows to be equally partitioned between two fields than fruit in bags.

The interviewer then explained the problem: the farmer kept cows in the fields but, because the gate was open, the cows kept wandering from one field to the other. The interviewer then explained what was required: to help the farmer by telling him *"all the different ways the cows can be in the two fields",* and then told the children to watch an example showing them what this meant. The materials used in the demonstration and problem are described below as they differed according to which condition the child was in.

3.2.3.2 Materials in each Condition

Physical No Record

Similarly to Study 1, children in this group were presented with a line of counted out red blocks ($2cm^3$ wooden blocks) in front of them for each problem. The interviewer did not recollect these blocks during problem solving.

Physical Record

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[11] Unlike Study 1, this meant children began with an odd number to partition; however, Study 1 found children partitioned 'equally' for odd and even numbers. Moreover, children in Study 2 received the same problem order in all four conditions.

This used the same set-up to the Physical No Record condition, however, whenever children verbally identified a solution, the interviewer quickly recreated the configuration of the blocks children had made on the right hand side of their workspace using black wooden blocks (as illustrated in Figure 7). It was decided that the interviewer, not child, would create this record, and not use the blocks children had just manipulated, in order not to interrupt children's use of the physical representation. The interviewer would start at the top of this space and create successive configurations under each other so that a maximum total of 13 configurations would fit in this space. As the maximum number of correct solutions was 10 it was decided to stop children after 13 solutions (where children would have given at least four incorrect or repeated solutions).

Figure 7: Example of Record solutions created in the Physical Record condition

Pictorial No Record

Children in this group were provided with a sheet of paper with rows of squares (equal to the partitioning amount). The squares were $2cm^2$ white with a black border separated by a 1.5cm gap (see Figure 8). Each sheet of paper was 6cm by 30cm. In the example, the interviewer demonstrated annotating around the squares for partitioning (similar to Martin and Schwartz, 2005). After each verbal solution, the interviewer removed (and concealed) this piece, and replaced with an identical set of squares for their next solution.

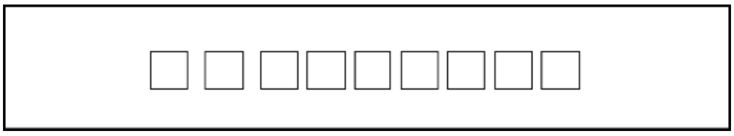

Figure 8: Pictorial materials used in conditions

Pictorial Record

Children in this group were provided with an A3 (Portrait) sheet of paper with 13 aligned rows of the number of squares to partition (Figure 9). The squares were identical to the Pictorial No Record condition, and were aligned in order to facilitate comparison between solutions. In all conditions, it was decided to set a maximum number of solutions for the children.

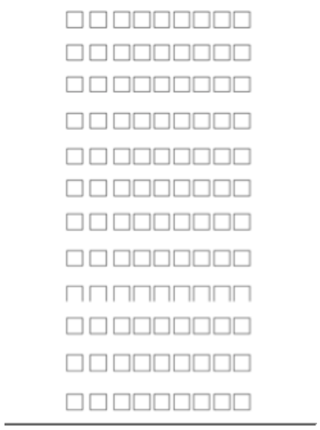

Figure 9: Pictorial materials used in Pictorial Record condition (13 rows)

3.3 Results

3.3.1 Correct solutions

All children therefore received a score between 0 and 27 for the number of correct solutions identified (maximum score of 8, 9,and 10 for partitioning 7,8,9 respectively). The distribution of group data was tested (Kolmogorov-Smirnov) and revealed no significant departures from normality for scores on any of the conditions: Physical No Record (D(25)=0.12, p=ns); Physical Record (D(25)=0.12, p=ns); Pictorial No Record (D(25)=0.14, p=ns); and Pictorial Record (D(25)=0.17, p=ns). Analysis of Variance was therefore carried out with Material (Physical/Pictorial) and Record (Record /No Record) as between-subjects variables.

Analysis revealed a significant main effect for Materials (F(3,96)=4.29, p<0.01) with Cohen's (1988) effect size value (d = .70) suggested a moderate to high practical significance, but failed to reveal a main effect for Record (F(1,96)=0.64, p=ns). There were also no significant interaction effects (F(1,96)=0.05, p=ns). The means for each condition and factor are shown in Figure 10. A Freidman test showed that there were no significant differences in the total number of solutions identified between the three partitioning problems (χ2=0.88, DF=2 p=ns).

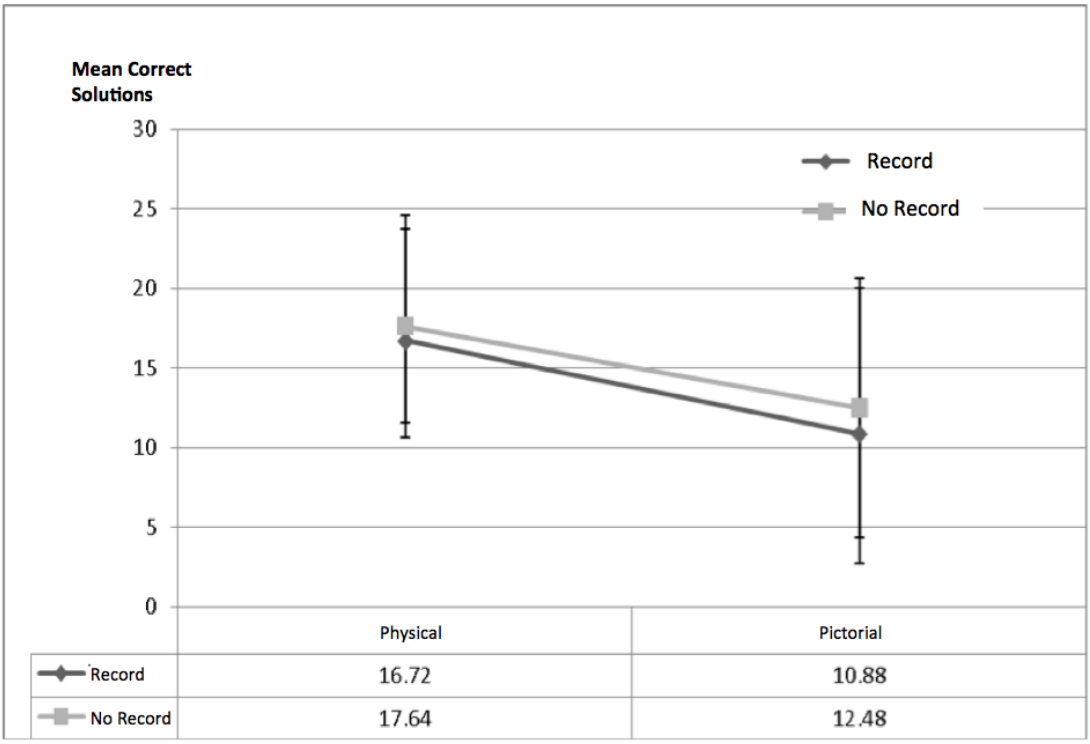

Figure 10: Mean Correct Solutions in the four conditions (Physical/Pictorial – Record/No Record)

3.3.2 Strategy

Using the Coding scheme developed in Study 1, children's solutions were coded according to *commutative* and *compensation* (related) and *other* (not commutative/compensation) solutions. Mann-Whitney tests revealed no differences in the number of strategy solutions identified between the Record and No Record conditions for Compensation: ($U=1208.5$, $Z=-0.28$, p=ns), but a significant difference for Commutative ($U=940.5$, $Z=2.13$, $p<0.05$), with children identify more commutative solutions in the No Record condition. On further inspection this seems to be explained by greater number of *commutative* solutions identified in the Physical No Record condition (total of 65 *commutative* solutions) compared to the Physical Record condition (total of 39 *commutative* solutions), rather than between Pictorial conditions (Totals of 13 and 14 accordingly). In contrast, there were significantly more *compensation* ($U=937.5$, $Z=-2.18$, $p<0.05$) solutions identified in the Physical conditions than Pictorial. Similarly, there were significantly more *commutative* solutions in the Physical condition ($U=722.00$, $Z=-3.98$, $p<0.01$). Whilst 32 out of 50 children identified at least 1 *commutative* solution in the Physical condition, only 14 out of 50 did in the Pictorial conditions and half of these only identified 1 *commutative* solution.

As well as related solutions, it was found that children in the Physical condition also identified significantly more *other* solutions than children in the Pictorial conditions ($U=941.5$, $Z=-2.14$, $p<0.05$); there were no differences in *other* solutions between the Record and No Record conditions ($U=1013$, $Z=1.63$, p=ns). Median scores for strategies in the Physical and Pictorial conditions are shown in Table 3.

Table 3: Median (IQR) scores for coded strategies in the Physical and Pictorial conditions

	Commutative	*Compensation*	*Other*

Physical	1 (0, 3)	6 (1.75, 9)	7 (2, 9)
Pictorial	0 (0, 1)	3 (0, 6.25)	4 (0, 8)

3.3.3. Initial Solution: Equal partitioning

Similarly to Study 1, children's first solution using blocks was commonly equal partitioning; however, there were no significant differences in the number of equally partitioned first solutions between the groups or main conditions.

3.4. Discussion

This study supports the predictions of PDL by finding that children interpreted significantly more partitioning solutions using physical materials than pictorial materials. It was further found that providing children with a record of previous representational states they had created did not support problem solving in this study, despite children being explicitly shown how this record showed what solutions they have previously identified. Indeed, the only difference found was that children identified significantly more commutative solutions when using blocks without a record of previous solutions than with such a record. This finding is difficult to interpret, but might be that children were slightly distracted by the record, thereby mitigating the beneficial effects of manipulatives on this particular type of strategy.

It seems therefore, in this problem, children did not perceive any benefit from having a record of *all* their previous actions to plan their subsequent actions. However, a possible limitation was that children in the Physical No Record condition did have a record of their last solution (until they acted upon the materials). It may therefore have been more balanced to have provided children in the Pictorial No Record condition with visual access to their last annotated solution until they started creating their next solution. Alternatively, the interviewer could have recollected children's blocks after each solution in the Physical No Record condition. It was decided not to do this, as this would have eliminated a key affordance of the materials (although an additional physical condition could have been created). It is possible therefore that the study design unfairly favoured physical materials over pictorial materials. However, if the ability to see the last solution created before starting the next solution was significant, we would have expected performance in the Pictorial Record condition to have been better. Instead, it seems that it is the ability to *manipulate* the previous representational state that is significant, thereby demonstrating the iterative relationship between action and interpretation.

It is interesting that children did not seem to benefit from a record of their previous solutions, given that this record could at least inform them of what solutions they had and had not identified. However, young children can find planning difficult (Ellis & Siegler, 1997), and may have lacked sufficient problem understanding to know how to use this record. Moreover, adopting a successful strategy (e.g. *compensation*) would render this record less necessary. Nevertheless, it is important to note that the record in this study referred to an iconic representation; it is possible that asking children to create a symbolic representation to record their solutions (e.g. using written numbers) would have been more beneficial, not least because it is much quicker to refer to and compare previous solutions.

Children in this study also identified significantly more solutions that were related to the previous solutions: *compensation* and *commutative,* when using physical materials. This suggests that in using physical materials, children had more opportunity to recognise these important numerical relations. In contrast to Study 1, however, they also identified significantly more *other* solutions. Consequently, it is possible that the benefits of physical materials simply helped children to identify more solutions, perhaps for motivational reasons.

It is not clear how PDL accounts for possible motivational effects. Using physical, as opposed to other representations, might encourage children to adapt the materials more, thereby leading them to develop more ideas. Nevertheless, there was reason to believe that the advantage of the materials in this study was not purely motivational. Firstly, there were no clear signs of loss of motivation in either condition (e.g., loss of visual concentration). Secondly, sessions were relatively short (around 12 minutes on average), especially for the older children where the advantage of physical materials was still clear. Finally, if children were losing motivation, a fall in performance over the three problems might have been expected, yet there were no such differences in either condition. Therefore, although it is not possible to rule out motivation as a key factor in differences between conditions, it is unlikely to be the only factor.

Interestingly, there were no differences in the number of equal (or the closest to equal possible, e.g. 3 and 4) partitioning solutions between physical and pictorial conditions. Therefore, this suggests that the prompt to partition equally was not attributable to the manipulative properties of the external representation. This raises the question of why children did not partition pictorial materials equally in Martin and Schwartz' fraction study. Two possibilities are that, firstly, children were required to partition into varying types of equal groups in Martin and Schwartz's study (e.g. 3 partitions for 12), in contrast, in this study they only partitioned amounts into two groups. Perhaps more importantly was the decision in this study to present pictorial materials in a line (affording symmetry); they were presented randomly in the fraction study. The decision to present materials linearly in this study aimed to not disadvantage the pictorial condition unnecessarily. Furthermore, it is arguably more typical, and hence ecologically valid to present pictorial materials in such a linearly arranged configuration.

4. Conclusions

We currently lack an explanatory model that allows us to predict if and when physical materials, or 'manipulatives', will support children's learning. This paper has focused on the representational properties of manipulatives, helping to explain similar work that has found that children provide more solutions using manipulatives than using a pictorial representation or with no external representation. However, the more significant focus of this paper concerns how physical representations can encourage the use of more efficient strategies - strategies that can be employed in the later absence of materials. By further examining the manipulatives properties of materials, the studies in this paper both support and help to elucidate an existing theoretical model - PDL.

4.1 Implications for the Theory of Physically Distributed Learning

According to PDL, physically manipulating the environment can lead to changes in thinking (i.e. learning) when children have incipient ideas. However, it is not clear if and how the representational properties of manipulatives can foster particular actions that can subsequently be interpreted. The studies in this paper suggest that manipulatives can encourage particular actions such as moving objects into equal groups, swapping over groups of objects, or moving objects on by one between groups. Such actions represent numerically significant part-whole relationships. Consequently, the representational properties of manipulatives may lead to changes in children's thinking through the iterative relationship between external and internal representations (Scaife & Rogers, 1996)

In support of PDL, the second study showed how manipulatives supported children more than paper-based representations – allowing them to identify significantly more consecutive solutions that were related. However, it was not quite clear if this was a generalised effect as children identified more solutions using all strategies including '*other*'. It is important to note a clear methodological difference in the set up of this study compared to Martin and Schwartz: children were provided with the initial amount, and the materials in physical and pictorial conditions were presented linearly. In contrast, in Martin and Schwartz's study, children were not given the correct initial amount and representations were presented in random configurations. This may have favoured the physical condition, where manipulation allows children to create more ordered configurations (indeed, many of our children lined up blocks), and also to count out and remove unrequired materials.

4.2 Generalizability of findings

It was predicted in this paper that the representational properties of physical materials would influence children's strategies. Rather than simply creating more unrelated solutions, manipulatives encouraged children to create configurations that related to the previous configurations: moving a single block from one group to another or swapping over groups. Manipulatives may therefore support problem solving when they encourage actions that correspond to more particular procedures or concepts; for example, exploring the way quantities can be partitioned equally in different ways for fraction problems, exploring odd and even numbers, or even exploring multiplication as repeated addition of equal amounts. The ways in which children were able to partition and then recollect groups of blocks may be important in children's early strategies for combining amounts in addition, or separating amounts for subtraction. Indeed, it has been proposed that children's early experiences with objects provides the foundation for such thinking later (Resnick, 1983).

It is important to consider when the actions encouraged by manipulatives may not be the most beneficial for learning. For example, in our studies, partitioning equally to begin with was not the most efficient strategy. Similarly, in our previous work, we found that children were more likely to employ a less developed count-all strategy for addition problems when using manipulatives compared to paper or no materials [blind for review]. There are also some numerical strategies that are not easily represented through actions with physical materials, such as doubling the amount of objects or combining a collection of ten objects when exploring tens and units. This is where virtual manipulatives may offer particular benefits by enabling designers to create certain perceptual and manipulative properties that are not easily produced physically (e.g. Figure 11). Indeed, Sarama & Clements (2009) describe such benefits of virtual materials alongside other benefits such as linking concrete and symbolic representations with feedback. Recording and replaying students' actions may be another benefit of virtual materials.

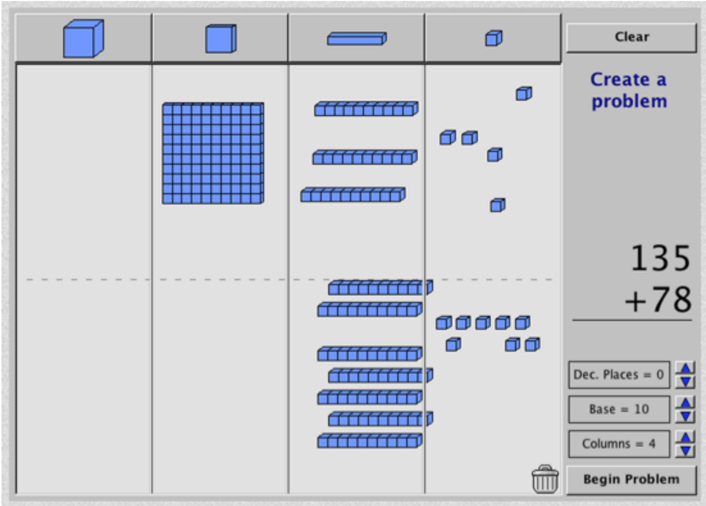

Figure 11: Virtual Manipulatives[12]

4.3 From Problem solving to Learning

The studies in this paper focus on problem solving in a short-term context. Therefore it is difficult to evaluate claims about longer term effects on learning and conceptual development. Nevertheless, there are several observations that are significant. Firstly, physical materials fostered the use of strategies that children can employ without materials (and older children do in the form of 'decomposition' strategies in addition, see Martins-Mourao & Cowan, 1998). Secondly, a couple of children changed to these more developed strategies using materials during the session and continued to do so. This suggests a certain amount of self-evaluation. Thirdly, several children were observed to enumerate blocks without seeming to count, even when looking away, whilst identifying related solutions – suggesting they may have employed the strategy mentally. Finally, the pattern of solutions of one child was of particular interest. When partitioning 8 using physical objects, the child changed to a *compensation* strategy, moving objects one at a time. In the first couple of solutions (7 & 1, 6 & 2), they clearly counted out the larger amount (touching blocks to support counting). After this, they continued moving one block at a time but without any clear counting. Significantly they identified the following solutions: 5 & 3, 4 & 2, 3 & 1. This tentatively suggests that the child was attempting to identify solutions mentally (making the error of subtracting from both parts), whilst continuing with the same physical actions. Further work could examine how these observed behaviours play out over time: whether or not children move toward more developed strategies over repeated sessions and whether they display more signs of independence from materials.

The iterative relationship between procedural and conceptual knowledge (Rittle-Johnson, Siegler, & Alibali, 1999) suggests that the result of encouraging more developed strategies may be significant for learning in this domain. However, as Rittle-Johnson et al. (1999) argue, whilst there is evidence demonstrating how conceptual knowledge can develop from children's improved procedural knowledge, the relationship is not clearly defined and there are examples where one does not lead to the other. The authors therefore suggest that: "*procedural knowledge may only lead to greater conceptual knowledge under certain circumstances, such as after extensive*

[12] National Library of Virtual Manipulatives: http://nlvm.usu.edu

experience using the procedure, or when the relation between the procedure and the underlying concepts is relatively transparent." (Rittle-Johnson et al., 1999, p. 177).

The degree of transparency between physical materials and the concepts they are meant to represent is one of significant debate (Stacey, Helme, Archer, & Condon, 2001) but does raise an important question concerning the generalizability of findings from this study: to what extent do manipulatives represent different types of numerical concepts? Indeed, the problems presented in this study referred to physical things (bananas/cows) and may have unfairly benefited the use of a physical representation. It is possible that the benefits of manipulatives are significantly mitigated in numerical problems that are not as easily represented through collections of objects (e.g. time or distance problems). An interesting question for further work is the change of value of physical representations in this task when the problems are presented only symbolically.

4.4 Limitations

A challenge of research into educational materials such as manipulatives is that of balancing ecological validity with an experimental approach aiming to elucidate a complex relationship of variables. The first study intended to maintain ecological validity by employing a common manipulative, in a curriculum relevant problem with an interviewer's role of simulating a teacher who has set up a learning environment but is trying to encourage children's independent problem solving. Yet, many aspects of the study are less familiar, from the interviewer's capacity to record children solutions to the demonstration and prompts that were balanced across children. A myriad of decisions, such as providing children with the initial amount of materials to the use of a concrete problem context, is likely to have influenced children's performance (and possibly exaggerated the benefits of manipulatives). For this reason, the effect of materials on children's strategies is perhaps more illuminating than the finding that children provided more solutions.

The challenge of maintaining ecological validity was more pronounced in the second study, where the nature of each condition may not have been familiar to children, and findings may have been quite different if children had been provided with an opportunity to familiarise themselves with the materials. Significantly, the study examined the role of providing a record of solutions to children, yet the decision to let children manipulate the same set of blocks meant that they did have a record of the last solution created. In the Pictorial No Record condition, children did not have such a record, although the lack of differences between the Pictorial No Record and Pictorial Record conditions suggests this may not have been too influential. This methodological issue does raise pertinent questions about the role played by the representational properties of manipulatives, and the ease in which they may be 'designed out' in comparison studies.

4.5 Implications for Education

This paper has intended to contribute toward the goal of informing teachers about how and when to use manipulatives in the classroom. A clear limitation is that the study has examined the use of materials in a particular context, using a particular problem, and a particular type of material; it is likely that changing these variables would have impacted children's performance. Nevertheless, we believe the findings are informative for two main reasons.

Firstly, the study was designed to be experimental but not unfamiliar to a typical classroom. The materials (Unifix Blocks) are widespread with limited extraneous features intended to reduce the risk of distracting children (Uttal et al., 1997) and support transfer (Kaminski, Sloutsky, & Heckler, 2009). The problem task used is highly relevant to a common numerical goal of learning number bonds to ten. Therefore, on a more specific level, the studies contribute to other work (e.g. Baroody, 2006) aiming to support this more specific educational goal.

A more significant contribution, however, is the broader message of aiming to develop children's thinking with materials to support their later ability without materials. The findings from this study demonstrate how physical materials are able to make important numerical relationships explicit, such as how amounts can be added in any order, and how the materials can be presented in a way to help children independently explore these relationships.

This paper aims to contribute to the on-going debate about if and how manipulatives support learning by drawing attention to how the representational properties of the materials can prompt particular problem solving actions. It has been emphasised throughout the paper that the influence of materials will very much be mediated by a host of factors, most significantly the teacher's role. Through greater understanding in this area, it may be possible to offer clearer guidance to teachers on how and when to use these.

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