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Citation for published version:

Nguyen, LTK & Smyth, NF 2021, 'Dispersive shock waves for the Boussinesq Benjamin-Ono equation', *Studies in Applied Mathematics*, vol. 147, no. 1, pp. 32-59. <https://doi.org/10.1111/sapm.12381>

Digital Object Identifier (DOI):

[10.1111/sapm.12381](https://doi.org/10.1111/sapm.12381)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Peer reviewed version

Published In:

Studies in Applied Mathematics

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DISPERSIVE SHOCK WAVES FOR THE BOUSSINESQ BENJAMIN-ONO EQUATION

A PREPRINT

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February 24, 2021

ABSTRACT

In this work the dispersive shock wave solution of a Boussinesq Benjamin-Ono (BBO) equation, the standard Boussinesq equation with dispersion replaced by nonlocal Benjamin-Ono dispersion, is derived. This dispersive shock wave solution is derived using two methods, dispersive shock wave fitting and from a simple wave solution of the Whitham modulation equations for the BBO equation. The first of these yields the two edges of the dispersive shock wave, while the second yields the complete dispersive shock wave solution. As the Whitham modulation equations could not be set in Riemann invariant form, the ordinary differential equations governing the simple wave are solved using a hybrid numerical method coupled to the dispersive shock fitting which provides a suitable boundary condition. The full dispersive shock wave solution is then determined, which is found to be in excellent agreement with numerical solutions of the BBO equation. This hybrid method is a suitable and relatively simple method to fully determine the dispersive shock wave solution of a nonlinear dispersive wave equation for which the (hyperbolic) Whitham modulation equations are known, but their Riemann invariant form is not.

Keywords dispersive shock wave; undular bore; modulation theory; dispersive shock fitting

1 Introduction

The two generic solutions of nonlinear dispersive wave equations are the solitary wave, soliton for integrable equations, and the dispersive shock wave (DSW), termed an undular bore in fluid mechanics. Of these, the solitary wave has been extensively studied as a solitary wave is a steady solution, particularly after the discovery that nonlinear dispersive wave equations with wide applicability, such as the Korteweg-de Vries (KdV) equation, the nonlinear Schrödinger (NLS) equation and the Sine-Gordon equation, have an exact solution via the method of inverse scattering [1, 2], which shows that a general (square integrable) initial condition will evolve into a finite number of solitons, plus scattered dispersive radiation.

The other generic nonlinear dispersive wave type, the DSW, is an unsteady wave form, a continuously expanding modulated periodic wave linking two distinct flow states. In its standard form it consists of solitary waves at one edge and linear dispersive waves at the other. The most well known physical examples of DSWs are a tidal bore and a tsunami. DSWs arise in a wide range of applications which can be modelled by nonlinear dispersive wave equations, including waves in fluids, both surface and internal [3, 4, 5], waves in the atmosphere [6, 7, 8], geophysics [9, 10, 11, 12], nonlinear optics [13, 14, 15, 16, 17], Bose-Einstein condensates [18] and Fermi fluids [19], see [20] for an appreciation of these various applications of DSWs in modelling natural phenomena. While DSWs are as widespread in nature as solitary waves, they initially received less attention. The main reason for this is that, as previously stated, DSWs are an unsteady wave form. The breakthrough for DSWs came through the development of Whitham modulation theory [1, 21]. Whitham modulation theory is, in essence, a sophisticated multiple scales method to determine equations for the properties, for instance amplitude, wavenumber, frequency and mean height, of a slowly varying periodic wavetrain. In a major feat, Whitham derived the modulation equations for the KdV equation as part of his development of modulation theory [1, 21]. These KdV modulation equations form a hyperbolic system, which implies that the underlying cnoidal wave solution of the KdV equation is modulationally stable. It was then realised that a simple wave solution of these hyperbolic modulation equations represents the DSW solution of the KdV equation, even though the initial condition for this solution is a step, which is not slowly varying [22].

The next major advance in the study of DSWs was the discovery that if the underlying nonlinear dispersive wave equation is integrable, then, using techniques from functional analysis, its Whitham modulation equations can be determined and set in Riemann invariant form [23], so that the simple wave DSW solution is then immediate. If the underlying nonlinear dispersive wave equation is not integrable, the calculation of its Whitham modulation equations, let alone its DSW solution if the underlying periodic wave solution is stable, is not straightforward. An additional problem for hyperbolic modulation equations is the determination of their Riemann invariant form, so that the DSW solution can be easily calculated, which is a manifestation of Pfaff's problem. This difficulty in calculating DSW solutions has been partly rectified [20, 24]. It was noted that hyperbolic Whitham modulation equations have degenerate forms in the linear (small amplitude) and solitary wave limits. As these two limits correspond to the two edges of a DSW, this degenerate form was used to derive general ordinary differential equations which determine the edges of a DSW, but not its interior. The method assumes that the DSW is of "KdV type," that is it consists of a modulated wavetrain with solitary waves at one edge and linear waves at the other and is given mathematically by a simple wave solution of the modulation equations.

The Benjamin-Ono equation is the nonlinear dispersive wave equation

$$\frac{\partial u}{\partial t} + 2u \frac{\partial u}{\partial x} + \mathcal{H} \left[\frac{\partial^2 u}{\partial x^2} \right] = 0. \quad (1)$$

Here,

$$\mathcal{H}[f] = \text{P.V.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy \quad (2)$$

is the Hilbert transform of $f = f(x)$ and P.V. denotes the principal value. It is the equivalent of the KdV equation for weakly nonlinear long waves in a fluid which overlies a deep fluid layer [25, 26] and is integrable using the method of inverse scattering [27], so that its DSW solution has been determined [28]. The dispersion relation for the Benjamin-Ono equation, $\omega = -k|k|$, $\omega = -k^2$ for right moving waves, is such that the dispersive shock fitting method, as developed for the KdV equation [20], does not work. At a basic level, the equation governing the periodic wave solution for a nonlinear dispersive wave equation which has a KdV-type DSW is of the form

$$u_\theta^2 = Q(u)r^2(u), \quad (3)$$

where $\theta = kx - \omega t$ is the phase, with k the wavenumber and ω the frequency, and Q is a polynomial in u with zeroes at $u = u_j$, $j = 1, \dots, N$. The function $r(u)$ is some smooth function which does not vanish at the $u = u_j$ [20, 29]. The periodic wave then oscillates between two appropriate zeroes of Q . The zeroes of $Q(u)$ play a key role in the Whitham modulation equations for this periodic wave, in fact giving the Riemann invariants for an integrable

equation [23]. The periodic wave solution of the Benjamin-Ono equation (1) and equations with Benjamin-Ono type dispersion do not have periodic wave solutions given by equations of the form (3), due to the Hilbert transform. This difference between equations with KdV and Benjamin-Ono dispersion is the basic reason that the dispersive shock fitting method, as originally developed, does not apply to equations with Benjamin-Ono dispersion. The dispersive shock fitting method has been extended and modified to apply to nonlinear dispersive wave equations with Benjamin-Ono dispersion [30]. This dispersive shock fitting method for Benjamin-Ono dispersion again determines the leading and trailing edges of the DSW, which were found to agree with the known DSW solution of the Benjamin-Ono equation [28], based on its known Whitham modulation equations in Riemann invariant form [27]. As well as arising from step initial conditions, DSWs also arise from the breaking of a smooth initial condition in the small dispersion limit of the governing nonlinear dispersive wave equation, which has been analysed for the Benjamin-Ono equation [31, 32], based on the similar problem for the Korteweg-de Vries equation [33] using the inverse scattering solution of the equation. The Benjamin-Ono equation has been discussed in terms of its applications in fluid mechanics. It has also been shown to arise in the weakly nonlinear interaction of incoherent random waves [34].

In the present work a Boussinesq Benjamin-Ono (BBO) equation and its Whitham modulation equations and related DSW solution are studied. The BBO equation is of similar form to the standard Boussinesq equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} \frac{\partial^2 u^2}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} = 0 \quad (4)$$

arising from weakly nonlinear long water wave theory [1, 35], but with the dispersion of this equation replaced by nonlocal Benjamin-Ono dispersion, as for the relation between the KdV and the Benjamin-Ono equations (1). This BBO equation is

$$\frac{\partial^2 u}{\partial t^2} - 10 \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} \frac{\partial^2 u^2}{\partial x^2} - \mathcal{H} \left[\frac{\partial^3 u}{\partial x^3} \right] = 0. \quad (5)$$

Note that in the BBO equation (5) the variables have been scaled so that the linear wave velocity (the square root of the coefficient of u_{xx}) has been normalised to $\sqrt{10}$. The reason for this choice is that it was found that a large linear wave velocity gave superior stability for numerical solutions. The BBO equation, a bidirectional form of the Benjamin-Ono equation (1), has applications in geophysics. It has been derived for Rossby waves generated by an external forcing in an inviscid, quasi-geostrophic stratified fluid [36] and for modelling squall lines in the atmosphere [37]. In this regard, it was found that the bidirectional nature of the BBO equation makes it better suited to modelling atmospheric phenomena than the uni-directional Benjamin-Ono equation [37]. The Whitham modulation equations for the BBO equation (5) will be derived from averaging the Lagrangian formulation of its potential form [1]. These Whitham modulation equations form a fourth order system which is hyperbolic for a wide range of wavetrain parameter values. Unfortunately, it was not found possible to set these equations in Riemann invariant form, a manifestation of Pfaff's problem, so that the BBO DSW solution, a simple wave solution of the Whitham modulation equations, could not be easily determined analytically. The equations governing the simple wave DSW solution were then solved numerically. To solve these ordinary differential equations, a boundary condition is needed, which was found from the trailing edge of the BBO DSW as determined using the dispersive shock fitting method [20, 24], extended to deal with Benjamin-Ono dispersion [30]. The determination of the DSW solution of the BBO equation of the present work is then a hybrid of the two standard methods to determine DSW solutions, Whitham modulation theory and dispersive shock fitting. If the Whitham modulation equations for a given nonlinear dispersive wave equation can be determined, it represents a viable method to determine the full DSW structure without the need for a Riemann invariant form of these modulation equations.

2 Dispersive shock waves governed by BBO equation

The easiest manner by which to derive Whitham modulation equations is by averaging a Lagrangian formulation of the governing equation [1, 21]. A Lagrangian for the BBO equation (5) can be found by using the potential ϕ , with

$u = \phi_x$. The BBO equation (5) can then be recast as

$$\phi_{tt} - 10\phi_{xx} - \phi_x\phi_{xx} + \sigma\mathcal{H}\phi_{xxx} = 0, \quad (6)$$

The dispersion coefficient σ has been introduced to generalise this equation, with $\sigma = -1$ for the BBO equation (5). The change in the sign of the dispersion, $\sigma = -1$ to $\sigma = 1$, is the key to determining the solitary wave edge of a DSW governed by a nonlinear dispersive wave equation with Benjamin-Ono-type dispersion using the dispersive shock fitting method [30]. Introducing the velocity field $v = \phi_t$, we can rewrite this bi-directional wave equation as a system of first-order equations as

$$\begin{aligned} v_t - (10 + u)u_x + \sigma\mathcal{H}u_{xx} &= 0, \\ u_t - v_x &= 0. \end{aligned} \quad (7)$$

The simplest initial condition which will generate a DSW is the jump initial condition

$$u(x, 0) = \begin{cases} u_- & x < 0, \\ u_+ & x > 0, \end{cases} \quad v(x, 0) = \begin{cases} v_- & x < 0, \\ v_+ & x > 0. \end{cases} \quad (8)$$

2.1 DSW fitting method

Before deriving the Whitham modulation equations for the BBO system (7) and their associated DSW solution, let us first apply the dispersive shock fitting method to determine the leading and trailing edges of the DSW [30]. This shock fitting solution will be found to be instructive for the determination of the full DSW solution via the Whitham modulation equations. In addition, this dispersive shock fitting solution for the linear wave edge will be used to determine the full DSW solution from the Whitham modulation equations as these modulation equations could not be set in Riemann invariant form.

The basic quantity needed for the DSW fitting method is the linear dispersion relation [20]. This can be determined by linearising the BBO system (7) about a background flow state $(\bar{u}, \bar{v})$ via

$$u = \bar{u} + \tilde{u}, \quad v = \bar{v} + \tilde{v}, \quad |\tilde{u}| \ll |\bar{u}|, \quad |\tilde{v}| \ll |\bar{v}|. \quad (9)$$

If we now seek a single phase periodic solution

$$\tilde{v} = V e^{i\theta}, \quad \tilde{u} = U e^{i\theta}, \quad \theta = kx - \omega t,$$

we obtain the linear dispersion relation

$$\omega^2(\bar{u}, k) = k^2(10 + \bar{u} - k). \quad (10)$$

In addition, we shall need the dispersionless system associated with the original BBO system (7), which is given in matrix form as

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ -(10 + u) & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = \mathbf{0}. \quad (11)$$

This hyperbolic system can be set in Riemann invariant form as

$$\begin{aligned} l = v + \frac{2}{3}(10 + u)^{3/2} = \text{constant} \quad \text{on} \quad C_l : \frac{dx}{dt} = V_l(u, v) = -\sqrt{10 + u}, \\ r = v - \frac{2}{3}(10 + u)^{3/2} = \text{constant} \quad \text{on} \quad C_r : \frac{dx}{dt} = V_r(u, v) = \sqrt{10 + u}. \end{aligned} \quad (12)$$

Note that the characteristic velocities $V_l(u, v)$ and $V_r(u, v)$ depend on u , but not v .

2.2 Intermediate level and expansion wave

As is standard for bidirectional nonlinear dispersive wave equations, the solution of the BBO system (7) for the step initial condition (8) consists of five distinct parts, which can be identified starting from left to right as [20]: (i) the

initial level (u_-, v_-) , followed by (ii) an expansion wave linking (u_-, v_-) to an intermediate level (u_i, v_i) , (iii) the intermediate level (u_i, v_i) , whose leading edge links to (iv) the DSW itself, which then links the intermediate level to (v) the initial level (u_+, v_+) ahead, see Figure 6. We then need to determine solutions for the slow expansion wave linking the levels $u = u_-$ and $v = v_-$ to the intermediate shelf $u = u_i$ and $v = v_i$ and for the fast DSW linking the intermediate shelf to the right levels $u = u_+$ and $v = v_+$. The intermediate level between the expansion wave and the DSW can be determined from the dispersionless system (11) and its Riemann invariant form (12). The slow expansion wave is a simple wave solution on the characteristic C_l with the Riemann invariant r constant through it. We then have

$$v_i - \frac{2}{3}(10 + u_i)^{3/2} = v_- - \frac{2}{3}(10 + u_-)^{3/2} = r_-. \quad (13)$$

In addition, the fast DSW zone is determined by the conservation of the dispersionless Riemann invariant l in (12) across the DSW [20], which implies

$$v_i + \frac{2}{3}(10 + u_i)^{3/2} = v_+ + \frac{2}{3}(10 + u_+)^{3/2} = l_+. \quad (14)$$

Solving, we obtain

$$\begin{aligned} u_i &= \left\{ \frac{3}{4}(v_+ - v_-) + \frac{1}{2}[(10 + u_-)^{3/2} + (10 + u_+)^{3/2}] \right\}^{2/3} - 10, \\ v_i &= \frac{v_- + v_+}{2} + \frac{1}{3}[(10 + u_+)^{3/2} - (10 + u_-)^{3/2}]. \end{aligned} \quad (15)$$

2.2.1 Trailing, linear DSW edge

As stated above, the trailing, linear edge of the DSW is linked to the intermediate shelf (u_i, v_i) , see Figure 6. Let us now determine this trailing edge of the DSW using the dispersive shock fitting method.

As we are considering a right propagating DSW, we use the positive dispersion relation from (10)

$$\omega(\bar{u}, k) = k\alpha(\bar{u}, k), \quad \alpha(\bar{u}, k) = \sqrt{10 + \bar{u} - k}. \quad (16)$$

Now the DSW fitting method [30] gives that the equation for the wavenumber $k = k(\bar{u})$ at the trailing edge of the DSW is

$$\frac{dk}{d\bar{u}} = \frac{\partial\omega/\partial\bar{u}}{V_r(\bar{u}) - \partial\omega/\partial k}, \quad (17)$$

which on using the linear dispersion relation (16) becomes

$$\frac{dk}{d\bar{u}} = \frac{k}{2\alpha\sqrt{10 + \bar{u}} - 2\alpha^2 + k}. \quad (18)$$

This differential equation can be solved using the change of variable

$$\tau = \frac{\alpha}{\sqrt{k}} = \sqrt{\frac{10 + \bar{u}}{k}} - 1, \quad (19)$$

which results in the transformed equation

$$\frac{2\tau}{1 + \tau^2} \frac{1}{1 - (1 + \tau^2)\Lambda(\tau)} \frac{d\tau}{d\bar{u}} = \frac{1}{10 + \bar{u}}, \quad (20)$$

where

$$\Lambda(\tau) = \frac{1}{2\tau\sqrt{\tau^2 + 1} - 2\tau^2 + 1}. \quad (21)$$

The easiest manner in which to evaluate the τ integral arising from this separable equation is to use the substitution $\tau = \cot \eta$, with $0 \leq \eta < \pi/2$. This results in the solution for the linear trailing edge of the DSW as

$$\cos^4\left(\frac{\eta}{2}\right)(3\cos\eta - 2)^3 = C_-(10 + \bar{u})^{-5/2}, \quad (22)$$

where C_- is an integration constant. The boundary condition $k(u_+) = 0$, or $\eta(u_+) = 0$ by the change of variable (19) and $\tau = \cot \eta$, which links the trailing edge of the DSW to its leading edge [20, 30], is now used to determine the

constant of integration as $C_- = (10 + u_+)^{5/2}$. This allows us to write the equation for determining η as a function of $\bar{u}$ as follows

$$g(\eta) = \cos^4\left(\frac{\eta}{2}\right)(3\cos\eta - 2)^3 = \left(\frac{10 + u_+}{10 + \bar{u}}\right)^{5/2}. \quad (23)$$

We now determine the wavenumber k_i at the trailing, linear wave edge of the DSW by setting $\bar{u} = u_i$ in (19) to obtain

$$k_i = \frac{10 + u_i}{\cot^2 \eta_i + 1} = (10 + u_i) \sin^2 \eta_i, \quad \eta_i = \eta(u_i), \quad (24)$$

where $\eta = \eta(\bar{u})$ is determined as the inverse function of (23). The velocity of the trailing, linear wave edge is given by the group velocity evaluated at $\bar{u} = u_i$ as

$$s_i = \frac{\partial \omega}{\partial k} \Big|_{\bar{u} \rightarrow u_i} = \alpha_i - \frac{k_i}{2\alpha_i} = \sqrt{10 + u_i} \cos \eta_i - \frac{1}{2} \sqrt{10 + u_i} \tan \eta_i \sin \eta_i, \quad (25)$$

in which we have used $\alpha_i = \sqrt{10 + u_i - k_i} = \sqrt{10 + u_i} \cos \eta_i$.

2.2.2 Leading, solitary wave DSW edge

For KdV type DSWs, the leading, solitary wave edge of the DSW is determined in a similar manner to the trailing, linear edge with this edge determined from the linear dispersion relation (16) on changing to a “conjugate” wavenumber $\tilde{k}$ and a “conjugate” frequency $\tilde{\omega}$ such that $k \rightarrow i\tilde{k}$, $\omega \rightarrow i\tilde{\omega}$ [20]. However, this identification does not hold for BO type DSWs. For BO type DSWs, the correct transformation of the linear dispersion relation to determine the solitary wave edge is to reverse the sign of the dispersion in the BBO system (7) [30], that is $\sigma = 1$ in (7). This results in the conjugate dispersion relation

$$\tilde{\omega}(\bar{u}, \tilde{k}) = \tilde{k} \tilde{\alpha}(\bar{u}, \tilde{k}), \quad \tilde{\alpha}(\bar{u}, \tilde{k}) = \sqrt{10 + \bar{u} + \tilde{k}}. \quad (26)$$

As the velocity of the leading, solitary wave edge of the DSW governed by the original BBO equation can be identified with that of the leading, linear wave edge of the conjugate DSW, we only need to solve the differential equation (18) with k and ω replaced by $\tilde{k}$ and $\tilde{\omega}$. Computing the derivatives $\partial \tilde{\omega} / \partial \bar{u}$ and $\partial \tilde{\omega} / \partial \tilde{k}$, we obtain

$$\frac{d\tilde{k}}{d\bar{u}} = \frac{\tilde{k}}{2\tilde{\alpha}\sqrt{10 + \bar{u}} - 2\tilde{\alpha}^2 - \tilde{k}}. \quad (27)$$

This differential equation can be solved using a similar change of variable as for the trailing edge,

$$\coth \tilde{\eta} = \frac{\tilde{\alpha}}{\sqrt{\tilde{k}}} = \sqrt{\frac{10 + \bar{u}}{\tilde{k}}} + 1, \quad \tilde{\eta} \geq 0, \quad (28)$$

which gives the solution

$$\cosh^4\left(\frac{\tilde{\eta}}{2}\right)(3\cosh \tilde{\eta} - 2)^3 = C_+(10 + \bar{u})^{-5/2}. \quad (29)$$

The integration constant C_+ must be determined by linking with the trailing edge. This condition is $\tilde{\eta}(u_i) = \tilde{\eta}_i$, so that $\tilde{k} = 0$, or $\tilde{\eta} = 0$ by (28), giving $C_+ = (10 + u_i)^{5/2}$. Thus, we obtain the solution of the differential equation for determining the linear wave edge of the conjugate DSW, the leading edge of the real DSW, as

$$\tilde{g}(\tilde{\eta}) = \cosh^4\left(\frac{\tilde{\eta}}{2}\right)(3\cosh \tilde{\eta} - 2)^3 = \left(\frac{10 + u_i}{10 + \bar{u}}\right)^{5/2}. \quad (30)$$

Finally setting $\bar{u} = u_+$, the conjugate wavenumber at the solitary wave edge $\tilde{k}_+ = \tilde{k}(u_+)$ is given by

$$\tilde{k}_+ = \frac{10 + u_+}{\coth^2 \tilde{\eta}_+ - 1} = (1 + u_+) \sinh^2 \tilde{\eta}_+, \quad (31)$$

where $\tilde{\eta}_+$ is the solution of

$$\cosh^4\left(\frac{\tilde{\eta}_+}{2}\right)(3\cosh \tilde{\eta}_+ - 2)^3 = \left(\frac{10 + u_i}{10 + u_+}\right)^{5/2}. \quad (32)$$

The velocity of the linear wave edge of the conjugate DSW is then [30]

$$\tilde{s}_+ = \frac{\partial \tilde{\omega}}{\partial k} \Big|_{\tilde{u} \rightarrow u_+} = \sqrt{10 + u_+} \cosh \tilde{\eta}_+ + \frac{1}{2} \sqrt{10 + u_+} \tanh \tilde{\eta}_+ \sinh \tilde{\eta}_+. \quad (33)$$

This velocity is also the velocity of the leading, solitary wave edge of the DSW governed by the original BBO equation, that is $s_+ = \tilde{s}_+$.

2.3 Modulation solution for resolution of the DSW

The dispersive shock fitting solution for the BBO DSW of the previous section gives a rapid and simple derivation of its leading and trailing edges. However, it does not give the details of the interior of the DSW, for which the full Whitham modulation equations for the BBO system need to be found. This will now be done. Accordingly, we first find the exact single phase periodic wave solution of the BBO system (7), from which the Whitham modulation equations for single phase periodic modulated waves governed by the BBO equation can be found. Unfortunately, the simple wave solution of these modulation equations for the BBO DSW cannot be found analytically. This solution would have been immediate if the modulation equations could have been set in Riemann invariant form. This lack of a Riemann invariant form is linked, to a certain extent, to the BBO equation not being integrable [23]. Instead, we propose a hybrid numerical approach to solve the ordinary differential equations for the DSW similarity solution of the modulation equations. This approach employs the DSW fitting method applied to the linear wave edge to determine a set of boundary conditions for this similarity solution, which is why this dispersive shock fitting solution was found.

2.3.1 Exact periodic solution of the BBO system

Let us seek a travelling wave solution of the BBO system (7) of the form

$$u = u(\theta), \quad v = v(\theta), \quad \theta = kx - \omega t, \quad (34)$$

which are 2π periodic in θ , where k and ω are the wavenumber and frequency. Substituting these forms into the BBO system and integrating once gives

$$c^2 u - \left(10u + \frac{1}{2}u^2\right) + k\mathcal{H}u_\theta = B, \quad (35)$$

where B is an integration constant and $c = \omega/k$ is the phase speed. Based on the form of the periodic wave solution of the BO equation, we seek a solution of this ordinary differential equation of the form

$$u(\theta) = \beta' + \zeta(\theta), \quad \zeta(\theta) = \frac{P}{Q + R \cos \theta}. \quad (36)$$

By choosing $R = \sqrt{A^2 + k^2} > 0$ and $B = (c^2 - 10 - \beta'/2)\beta'$ and noting that $k > 0$, the periodic single phase solution can then be obtained as

$$u(x, t) = \beta' + \frac{2k^2}{c^2 - 10 - \beta' + R \cos(kx - \omega t)}, \quad R = \sqrt{(c^2 - 10 - \beta')^2 - k^2}. \quad (37)$$

For the derivation of the Whitham modulation equations in the next section it is convenient to set u in the form $u = \beta + \psi$, where β is the mean of u over a period and so ψ has zero mean over a period. Since the average of u over a period is $2k$,

$$u = \beta + \psi, \quad \psi = \frac{2k^2}{c^2 - 10 - \beta + 2k + R \cos(kx - \omega t)} - 2k \quad (38)$$

and the phase velocity is given by

$$c^2 = \beta + \sqrt{R^2 + k^2} + 10 - 2k. \quad (39)$$

This periodic wave solution has three free parameters, the mean height β , wavenumber k and amplitude $A = 4R$, as for the KdV cnoidal wave solution.

2.3.2 Solitary wave solution of the BBO system

To derive the solitary wave solution of the BBO system (7), we add a phase shift π to the periodic wave solution (37) so that a maximum falls at $x = ct$, which will then be the location of the solitary wave maximum. This results in

$$u = \beta + \frac{2k^2}{c^2 - 10 - \beta + 2k + R \cos[k(x - ct) + \pi]} - 2k. \quad (40)$$

To find the solitary wave solution, we take the limit $k \rightarrow 0$ and expand in Taylor series. In this limit

$$R = c^2 - 10 - \beta + 2k - \frac{k^2}{2(c^2 - 10 - \beta)} + \mathcal{O}(k^3), \quad \cos[k(x - ct) + \pi] = -1 + \frac{k^2(x - ct)^2}{2} + \mathcal{O}(k^3). \quad (41)$$

In this manner we find the solitary wave solution

$$u = \beta + \frac{4(c^2 - 10 - \beta)}{1 + (c^2 - 10 - \beta)^2(x - ct)^2},$$

of the BBO equation, where $c = v_s$ is now the solitary wave velocity and β is the background level. This solitary wave solution can be rewritten as

$$u_{\text{soliton}} = \beta + \frac{A_s}{1 + A_s^2/16(x - v_s t)^2}, \quad A_s = 4(v_s^2 - 10 - \beta), \quad (42)$$

where A_s and v_s are the amplitude and solitary wave velocity, respectively. Combining the amplitude-velocity relation for the solitary wave solution and the dispersive shock fitting solution (33) for the velocity of the lead wave of the BBO DSW, the amplitude of the leading solitary wave of the BBO DSW is

$$A_+ = 4(s_+^2 - 10 - u_+). \quad (43)$$

2.3.3 Whitham modulation equations

The simplest method to obtain the Whitham modulation equations for the BBO equation (6) is to average the Lagrangian formulation of this equation in order to average out the fast phase and leave equations for the slowly varying amplitude, wavenumber and mean [1]. This BBO equation is the critical point of the variational problem

$$\delta \int \int L dx dt = 0, \quad \text{where } L = \frac{1}{2}\phi_t^2 - 5\phi_x^2 - \frac{1}{6}\phi_x^3 + \frac{1}{2}\phi_x \mathcal{H}\phi_{xx}. \quad (44)$$

The key point with modulations of nonlinear dispersive waves is that modulations of the mean of the wavetrain couple with the modulations of its amplitude and wavenumber. As for the modulation equations of the KdV equation, to capture this effect we set $\phi = \chi + \varphi(\theta)$, where χ is a pseudo-potential and φ is 2π periodic in θ . We define $\beta = \chi_x$, with the mean β arising from the periodic wave solution (38), and $\gamma = -\chi_t$ [1, 21]. If we set $\varphi_\theta = \psi/k$, we have

$$v = \phi_t = -\gamma - c\psi, \quad u = \phi_x = \beta + \psi, \quad \phi_{xx} = k\psi_\theta. \quad (45)$$

To calculate the averaged Lagrangian, we note that equation (35) for the periodic wave solution gives

$$(c^2 - 10 - \beta)\psi - \frac{1}{2}\psi^2 + k\mathcal{H}\psi_\theta = -2k(c^2 + k - 10 - \beta), \quad (46)$$

which shall be used to relate $\mathcal{H}\psi_\theta$ to ψ and the wave parameters. The averaged Lagrangian for the single phase periodic wave (37) now is computed as

$$\begin{aligned} \mathcal{L} = \langle L \rangle &= \frac{1}{2\pi} \int_0^{2\pi} L d\theta \\ &= \frac{1}{12} \langle \psi^3 \rangle + \frac{\beta}{4} \langle \psi^2 \rangle + \left[\gamma c - 5\beta - \frac{1}{2}c^2\beta - k(c^2 - 10 - \beta + k) \right] \langle \psi \rangle + \frac{1}{2} \left(\gamma^2 - 10\beta^2 - \frac{1}{3}\beta^3 \right) \\ &\quad - k\beta(c^2 - 10 - \beta + k). \end{aligned}$$

The BBO periodic wave solution (38) can now be used to calculate the required averages

$$\langle \psi \rangle = 0, \quad \langle \psi^2 \rangle = 4k(c^2 + k - 10 - \beta), \quad \langle \psi^3 \rangle = 12k(c^2 + k - 10 - \beta)^2. \quad (47)$$

Hence, the averaged Lagrangian is finally

$$\mathcal{L} = -\frac{\beta^3}{6} - 5\beta^2 + \frac{\gamma^2}{2} + \frac{\omega^4}{k^3} + 2\omega^2 + k^3 + \beta^2 k - 2\beta k^2 + 20\beta k - 20k^2 - \frac{2\beta\omega^2}{k} + 100k - \frac{20\omega^2}{k}. \quad (48)$$

Taking variations of this averaged Lagrangian with respect to θ and χ yields

$$\begin{aligned} \frac{\partial}{\partial t} [4c(c^2 + k - 10 - \beta)] + \frac{\partial}{\partial x} [(c^2 + k - 10 - \beta)(3c^2 - 3k + 10 + \beta)] &= 0, \\ \frac{\partial \gamma}{\partial t} + \frac{\partial}{\partial x} \left[2c^2 k + \frac{1}{2}(\beta - 2k)(\beta - 2k + 20) \right] &= 0, \end{aligned} \quad (49)$$

on noting that $\chi_t = -\gamma$, $\chi_x = \beta$ and $\theta_t = -\omega$, $\theta_x = k$. The modulation equations are completed with the consistency relations

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0, \quad \frac{\partial \beta}{\partial t} + \frac{\partial \gamma}{\partial x} = 0. \quad (50)$$

The modulation equations (49)-(50) can be given in the matrix form $\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = \mathbf{0}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 10 - 2k + \beta & 0 & 2(c^2 + 2k - 10 - \beta) & 4ck \\ 0 & 0 & c & k \\ \frac{c^2 - 2k + \beta + 10}{2(10 + \beta - 3c^2 - k)} & \frac{c}{3c^2 + k - 10 - \beta} & -1 + \frac{4c^2 - k}{2(3c^2 + k - 10 - \beta)} & c\left(1 - \frac{2k}{3c^2 + k - p - \beta}\right) \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \beta \\ \gamma \\ k \\ c \end{bmatrix} \quad (51)$$

In theory, we can compute the characteristics and Riemann variables for the modulation equations from the eigenvalues of $\mathbf{A}$. However, such an exact determination is algebraically involved and of little use. Hence, the genuine nonlinearity of the modulation equations cannot be rigorously shown [20]. Nevertheless, extensive numerical experiments did not show any degeneracy over a wide parameter range. In addition, these numerical experiments showed that $\mathbf{A}$ can have complex characteristics, so that the modulation equations are not always hyperbolic, but become of mixed type (elliptic/hyperbolic) so that the periodic wave solution (37) can be modulationally unstable, see Section 15.3 of [1]. It should be noted that nonlinear dispersive wave equations possessing modulationally unstable periodic wave solutions have DSW-like solutions which link two states by a modulated wavetrain [38]. The difference with the generic DSW is that due to the instability, the waves inside the DSW are not rank ordered and have a random appearance without a monotonic envelope. Further discussion of the characteristic velocities is given in the next subsection.

2.3.4 Numerical solution of the Whitham modulation equations

Unfortunately, the modulation equations (49) and (50), while they form a hyperbolic system, could not be set in Riemann invariant form. In general, it is only guaranteed that modulation equations can be set in Riemann invariant form if the underlying equation is integrable [20, 23]. A DSW solution is derived from Whitham modulation equations as a simple wave solution [20], a solution in the similarity variable $\xi = x/t$ as the step initial condition (8) has no length scale, which is easiest to determine if these equations are in Riemann invariant form. A different, hybrid approach will be taken here. A similarity solution in x/t of the modulation equations (49) and (50) will be sought, with the resulting ordinary differential equations solved numerically. The consistency relations (50) then become

$$\frac{d\gamma}{d\xi} = \xi \frac{d\beta}{d\xi}, \quad (c - \xi) \frac{dk}{d\xi} + k \frac{dc}{d\xi} = 0. \quad (52)$$

If we then eliminate γ from the modulation equations (49), they reduce to the ordinary differential equations

$$\begin{aligned} (c^2 + \beta - 2k + 10 - 2\xi c) \frac{d\beta}{d\xi} + (2\xi c + 3k - 20 - 2\beta) \frac{dk}{d\xi} + [6\xi c^2 - 6c^3 + 2c(10 + \beta) - 2\xi(10 - k + \beta)] \frac{dc}{d\xi} &= 0, \\ (\beta - 2k + 10) \frac{d\beta}{d\xi} + 2(c^2 - \beta + 2k - 10) \frac{dk}{d\xi} + 4kc \frac{dc}{d\xi} &= 0. \end{aligned} \quad (53)$$

Subtracting the first equation from the second equation, we obtain

$$(2\xi - c)c\frac{d\beta}{d\xi} + (2c^2 - 2\xi c + k)\frac{dk}{d\xi} + [2\xi(\beta - k + 10) - 2c(\beta - 2k + 10 + 3\xi c - 3c^2)]\frac{dc}{d\xi} = 0. \quad (54)$$

The three ordinary differential equations (52)₂, (53)₂ and (54) for $k = k(\xi)$, $c = c(\xi)$, $\beta = \beta(\xi)$ can be set in matrix form as

$$\mathbf{M}(\xi)\frac{d\mathbf{v}}{d\xi} = 0, \quad (55)$$

where

$$\mathbf{M}(\xi) = \begin{bmatrix} -(c - \xi)^2 & 2c(c - \xi) + k & 2\xi(\kappa + k) - 2c(\kappa + 3\xi c - 3c^2) \\ \kappa - \xi^2 & 2(c^2 - \kappa) & 4kc \\ 0 & c - \xi & k \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \beta \\ k \\ c \end{bmatrix},$$

and $\kappa(\xi) = \beta(\xi) - 2k(\xi) + 10$. In order for this system to have a non-trivial solution, we require

$$\det \mathbf{M}(\xi) = 2(3c^2 - \kappa)(c - \xi)^2(\kappa - \xi^2) + 2k(c - \xi)^4 + k^2(\xi^2 - \kappa) = 0. \quad (56)$$

We can use this constraint to eliminate one of the three equations in the system (55) and solve the remaining equations together with the constraint $\det \mathbf{M}(\xi) = 0$. For this purpose, we choose the last two equations in (55). In summary, we shall solve the system of differential-algebraic equations

$$\begin{aligned} (\kappa - \xi^2)\frac{d\beta}{d\xi} + 2(c^2 - \kappa)\frac{dk}{d\xi} + 4ck\frac{dc}{d\xi} &= 0, \\ (c - \xi)\frac{dk}{d\xi} + k\frac{dc}{d\xi} &= 0, \\ 2(3c^2 - \kappa)(c - \xi)^2(\kappa - \xi^2) + 2k(c - \xi)^4 + k^2(\xi^2 - \kappa) &= 0 \end{aligned} \quad (57)$$

to determine the full DSW solution of the BBO equation (5).

At this point, we note that $\det \mathbf{M}(\xi) = 0$ is equivalent to $\det(\mathbf{A} - \xi\mathbf{I}) = 0$, with $\mathbf{A}$ given by (51). Assuming that the roots of this quartic equation are ordered in ascending order, the third root was used in our numerical simulations as it corresponds to the 3-rarefaction wave. Again, we note that a closed form expression for this root is too complicated to be useful. As mentioned at the end of Section 2.3.3 the characteristics of the Whitham modulation equations (51) can be complex on suitable choices of c , k and β . However, numerical evaluation of the eigenvalues of $\mathbf{M}$ shows that for the particular, simple wave solution of the modulation equations for which c , k and β are not independent, these characteristic velocities are real, so that the DSW is stable. This conclusion is supported by extensive numerical solutions of the BBO equation for which no unstable DSWs were found.

Boundary conditions To solve the system (57) for the DSW solution of the BBO equation, we need boundary conditions at one edge of the DSW. The easiest manner in which to obtain these is to use the dispersive shock fitting solution of Section 2.2.1 for the linear, trailing edge of the DSW. This solution gives values for the wavenumber k_i and phase velocity c_i at the trailing edge, with the mean height $\beta_i = u_i$ given by (15). We then have the boundary conditions

$$k_i = k(\xi_i), \quad c_i = c(\xi_i), \quad \beta_i = u_i = \beta(\xi_i), \quad (58)$$

where $\xi_i = s_i$ is the (group) velocity of the trailing, linear wave edge. Hence, the linear edge solution (25) gives

$$k_i = (10 + u_i) \sin^2 \eta_i, \quad \xi_i = \sqrt{10 + u_i} \cos \eta_i - \frac{1}{2} \sqrt{10 + u_i} \tan \eta_i \sin \eta_i, \quad (59)$$

where η_i is defined by

$$\eta_i = g^{-1} \left[\left(\frac{10 + u_i}{10 + u_i} \right)^{5/2} \right], \quad \text{with } g(\eta) = \cos^4 \left(\frac{\eta}{2} \right) (3 \cos \eta - 2)^3.$$

The boundary condition for the phase velocity c is then

$$c_i = \frac{\omega}{k} \Big|_{\xi \rightarrow s_i} = [\sqrt{R^2 + k^2} + 10 + \beta - 2k]^{1/2} \Big|_{R \rightarrow 0, k \rightarrow k_i, \beta \rightarrow u_i} = \sqrt{10 + u_i - k_i} = \sqrt{10 + u_i} \cos \eta_i. \quad (60)$$

As we are dealing with a system of differential-algebraic equations, it is important that the boundary conditions are consistent with the algebraic constraint (56), or (57). Indeed, if we substitute the values from (59)–(60) into the matrix $\mathbf{M}$, we obtain

$$\mathbf{M}(u_i, \eta_i) = \begin{bmatrix} -\frac{1}{4}(10 + u_i) \sin^2 \eta_i \tan^2 \eta_i & 2(10 + u_i) \sin^2 \eta_i & 4(10 + u)^{3/2} \cos \eta_i \sin^2 \eta_i \\ -\frac{1}{4}(10 + u_i) \sin^2 \eta_i \tan^2 \eta_i & 2(10 + u_i) \sin^2 \eta_i & 4(10 + u)^{3/2} \cos \eta_i \sin^2 \eta_i \\ 0 & \frac{1}{2} \sqrt{10 + u_i} \sin \eta_i \tan \eta_i & (10 + u_i) \sin^2 \eta_i \end{bmatrix}, \quad (61)$$

which clearly has zero determinant, so that the constraint is satisfied by the boundary conditions. This must be the case, as otherwise the dispersive shock fitting method would not be consistent. This result also implies that we may vary u_+ and u_- , so that u_i and η_i can be arbitrary. Thus, the necessary condition $\det \mathbf{M} = 0$ is fulfilled *a priori* for the set of boundary conditions defined above. Moreover, as mentioned above, the third root of the equation $\det \mathbf{M}(\xi) = 0$ is used because it satisfies the boundary conditions (58). The modulation theory for the BBO equation results in a fourth order system, rather than the third order system for the KdV equation [1], as γ is retained as a variable. The simple wave solution is then shifted to the third root for the BBO modulation equations, rather than the second root for the KdV modulation equations.

Figure 1: Numerical solutions of the system of differential-algebraic equations (57) for the BBO DSW for the wavenumber k , phase velocity c and mean height β . The initial conditions $\{u_-, u_+, v_-, v_+\} = \{3, 0, 0, 0\}$ have been used.

To illustrate this method, let us use the values $\mathcal{I} = \{u_- = 3, u_+ = 0, v_- = 0, v_+ = 0\}$, so that the mean height at the trailing edge (the level of the intermediate shelf) (15) is

$$u_i = -10 + \left(5\sqrt{10} + \frac{13\sqrt{13}}{2} \right)^{2/3}. \quad (62)$$

The algebraic-differential system (57) with the boundary conditions (59)–(60) was solved numerically using the Mathematica package. The resulting solution is shown in Fig. 1. In this figure, we show the value of ξ , $\xi_{(k=0)}$, at which the

Figure 2: DSW of the BBO equation. Numerical solution: blue line; modulation theory envelope: black (dashed) line. Left figure: $t = 480$, right figure: $t = 600$. The initial conditions $\{u_-, u_+, v_+, v_-\} = \{3, 0, 0, 0\}$ have been used. The envelopes are $\beta \pm 4R$ for the periodic wave solution (40). The red (dashed) line is the intermediate shelf u_i , which terminates at the linear wave edge. The magenta (dashed) line corresponds to the full modulation solution up to the point $\xi_{(k=0)} \times t$.

wavenumber k vanishes. The two values $\xi_i = s_i$ and $\xi_{(k=0)}$ of the similarity variable then determine the trailing and leading edges of the DSW, respectively. At time t , the DSW then lies in the region $\Omega_{\text{DSW}} = [\xi_i t, \xi_{(k=0)} t]$.

After obtaining numerical solutions for the modulation variables in terms of the characteristic velocity ξ , we calculated $\partial_u \xi \cdot \mathbf{p}^{(3)}$ (see Fig. 2), where $\mathbf{p}^{(3)}$ denotes the third eigenvector of $\mathbf{A}$ corresponding to the characteristic velocity used to construct the DSW solution. This dot product was found to be positive, verifying genuine nonlinearity of the modulation equations on this characteristic [20].

A DSW is the dispersive resolution of a discontinuity, such as the initial condition (8). The other generic manner in which a discontinuity is resolved is through viscosity, as in a compressible flow shock wave. It is instructive to consider the differences between the current dispersive resolution and a standard shock wave. Let us now consider the dispersionless form of the BBO system (7), that is the Hilbert transform dispersion is now neglected. The initial condition (8) generates a right propagating shock wave with velocity U , ahead of which $u = u_+$ and $v = v_+$. Behind the shock, there is an intermediate level with $u = u_{is}$ and $v = v_{is}$, noting that this intermediate level will not be the same as that for the DSW solution. There is then a left propagating expansion fan linking the intermediate level to the level behind $u = u_-$ and $v = v_-$. This shock solution is the equivalent of the compressible flow shock tube problem [1].

As the expansion fan is on the C_l characteristics of the Riemann invariant form (12), the C_r Riemann invariant gives

$$v_{is} = v_- + \frac{2}{3} (10 + u_{is})^{3/2} - \frac{2}{3} (10 + u_-)^{3/2} \quad (63)$$

as it is constant through the expansion fan. The jump conditions across the shock for the dispersionless BBO system (7) are

$$U [v_{is} - v_+] + \left[10u_{is} - 10u_+ + \frac{1}{2}u_{is}^2 - \frac{1}{2}u_+^2 \right] = 0, \quad (64)$$

$$U [u_{is} - u_+] + [v_{is} - v_+] = 0. \quad (65)$$

The intermediate level v_{is} can be eliminated from these jump conditions using (63), resulting in

$$U^2 = 10 + \frac{1}{2} (u_{is} + u_+) \quad (66)$$

for the shock velocity, with the intermediate level u_{is} the solution of

$$v_- + \frac{2}{3}(10 + u_{is})^{3/2} - \frac{2}{3}(10 + u_-)^{3/2} = v_+ - \frac{u_{is} - u_+}{\sqrt{10 + \frac{1}{2}(u_{is} + u_+)}}. \quad (67)$$

2.4 Comparison with direct numerical solutions

In this section, the DSW solution of the BBO equation (5) derived using the dispersive shock fitting method [20, 24] and directly from the Whitham modulation equations for this equation will be compared with full numerical solutions of the BBO equation (5). These full numerical solutions will be obtained using the standard pseudo-spectral method, first introduced by Fornberg and Whitham [39, 40], as this is the most suitable method due to the Hilbert Transform in the BBO equation. To this end, we adopt the implementation of this method in MATLAB [41]. Applying the Fourier transform to the BBO equation (7), we obtain

$$\frac{d}{dt}\widehat{v} = 10ik\widehat{u} + \frac{1}{2}ik\widehat{u^2} + \sigma k|k|\widehat{u}, \quad \frac{d}{dt}\widehat{u} = ik\widehat{v}, \quad (68)$$

where the hat superscript denotes the Fourier transform, which can be implemented using the fast Fourier transform (FFT). This discrete system of ODEs can be then propagated by any numerical time integration scheme in Fourier space. The solution in real space is retrieved via the inverse FFT. In the present work, the fourth order Runge-Kutta method was used for time propagation. The discontinuity of the step initial condition was smoothed using a hyperbolic tangent linking the two levels at the jump location for both u and v using

$$\begin{aligned} u_0(x) &= \frac{1}{2}(u_- - u_+) [\tanh(\delta_1(x - j_1)) - \tanh(\delta_2(x - j_2))] + u_+, \\ v_0(x) &= \frac{1}{2}(v_- - v_+) [\tanh(\delta_1(x - j_1)) - \tanh(\delta_2(x - j_2))] + v_+, \end{aligned} \quad (69)$$

where $j_1 = -L$ and $j_2 = 0$ are the two jump locations in the initial conditions and $\delta_{1,2}$ are related to the width of regularization. In all of our numerical simulations we used $\delta_{1,2} = 1.5$ as these values were found to give sharp jumps without resulting in instability. Additionally, as the pseudo-spectral method implicitly gives periodic boundary conditions, a large domain in x was used and the numerical comparisons were based on the central portion of the solution which does not interact with waves from the boundaries. All comparisons of numerical solutions with the theoretical DSW solutions were done when the lead waves of the numerical DSW settled to steady states, with the final time $T = 1500$. The number of modes used was $N_{\text{modes}} = 2^{16} = 65536$ and the computational domain was $[-L, L]$, with $L = 3500\pi$ to 3800π . The numerical scheme was tested by propagating the exact solitary wave solution (42).

Using this DSW solution of the modulation equations in the one phase periodic wave solution (38), which gives the envelope of the DSW, we have that the full DSW solution for the modulated periodic wavetrain is

$$u(x, t) = \beta(\xi) + \frac{2k^2(\xi)}{c^2(\xi) - 10 - \beta(\xi) + 2k(\xi) + R(\xi) \cos[k(\xi)x - c(\xi)k(\xi)t]} - 2k(\xi), \quad \xi = x/t, \quad (70)$$

where $R(\xi) = \sqrt{[c^2(\xi) - 10 - \beta(\xi) + 2k(\xi)]^2 - k^2(\xi)}$.

The modulation DSW solution for the data set $\mathcal{I}$, shown in Fig. 1, is compared with the full numerical solution of the BBO equation (5) in Fig. 2. The envelope of the DSW as given by modulation theory is shown in the figure. This envelope consists of two sections. The full modulation theory envelope is shown by both the black and magenta lines. However, the modulation solution terminated at the last peak of the numerical solution is shown by the black envelope alone. This terminated envelope is in excellent agreement with the numerical solution. The full modulation theory envelope extends beyond the numerical solution. If the underlying modulated periodic wave (40) is added to the modulation envelope to show the full theoretical DSW, it is found that the modulation solution has a half oscillation extending beyond the numerical solution. This disagreement between modulation theory and numerical solutions for the DSW solution of BO-type equations due to a leading “missing half wave” can be observed for the DSW solution of the BO equation itself [42]. Indeed, if one substitutes the DSW modulation solution of this work into the underlying

periodic wave of the BO equation the modulation DSW solution extends a half wave beyond the numerical DSW. It can be seen from Figure 2 that the truncated modulation theory amplitude is in excellent agreement with the numerical solutions. In addition, the numerical solution shows a modulated wavetrain extending upstream of the linear edge of the modulation solution DSW. Again, this is a known feature of DSW solutions based on Whitham modulation theory [20] and has been noted since the first studies of modulation theory DSW solutions [39]. Modulation theory does not capture all of the linear wave edge of a DSW.

Figure 3: Velocity of the leading solitary wave s_+ . Left: $v_- = v_+ = u_+ = 0$ and u_- varies from 1 to 3 with the step size $\Delta u_- = 0.25$ and then from 3 to 6 with the step size $\Delta u_- = 1$. Middle: Intermediate shelf according to (15) and classical shock theory. Right: $v_- = v_+ = 0, u_- = 2$ and u_+ varies from -1 to 1 with the step size $\Delta u_+ = 0.25$. Dispersive shock fitting amplitude (43): black line with triangles; modulation theory amplitude (70): blue lines with circle; numerical solution: red line with squares.

More detailed comparisons between the DSW solutions and full numerical solutions can be obtained by comparing the velocity s_+ of the leading solitary wave of the DSW, as in Figure 3. The theoretical DSW velocity is shown for both the shock fitting formulation (33) and from full Whitham modulation theory (57), with both giving the same velocity, apart from the numerical errors inherited in the numerical integration of (57), as expected. To determine the leading solitary wave velocity from the modulation solution, we computed $\xi_{(k=0)}$ at which k vanishes and then evaluated $c(\xi_{(k=0)})$. In addition, the classical shock velocity (66) is also compared in the left hand figure of Figure 3. For small jump heights $u_- - u_+$ the modulation and numerical leading edge velocities and the shock velocity are essentially the same. Even though a DSW is not a classical shock, the agreement between the modulation velocities and the shock velocity is expected for low jump heights as the shock is then weak, so that the Riemann invariant on the characteristic C_l (12) is conserved through the shock to leading order [1], which is the same as for the DSW solution. As the jump height increases, the numerical lead wave velocity converges to the modulation velocity and diverges from the shock velocity, as expected. The middle figure of Figure 3 shows comparisons for the intermediate level u_i (15) given by modulation theory, the intermediate level u_{is} (67) from the shock solution and numerical solutions. It can be seen that the shock solution intermediate level is far from the modulation and numerical levels, with the deviation growing with the jump height. The disagreement for the intermediate level is much greater than for the lead wave velocity. The modulation and numerical intermediate levels are in exact agreement.

Comparisons for the amplitude of the leading solitary wave of the BBO DSW obtained by the dispersive shock fitting method (43), the modulation theory solution (70) and direct numerical simulations are given in Fig. 4. It should be noted that due to the “missing half wave” issue noted above, the amplitude termed “modulation solution” is that at the leading wave of the numerical solution. If the amplitude at the leading edge of the full modulation theory simple wave is chosen, then the “modulation solution” amplitude is the same as the “fitting method” solution. The truncation of the modulation theory solution results in excellent agreement with the numerical solution for the amplitude, as deduced for the special case shown in Figure 2.

It is remarked that the large difference between the amplitudes given by the dispersive shock fitting method (black) and the modulation solution (blue) of Figure 4 is not contradictory. It is due to the phase of the modulated wavetrain,

Figure 4: Amplitude of the lead solitary wave. Left: $v_- = v_+ = u_+ = 0$ and u_- varies from 1 to 3 with the step size $\Delta u_- = 0.25$. Right: $v_- = v_+ = 0, u_- = 2$ and u_+ varies from -1 to 1 with a step size $\Delta u_+ = 0.25$. See the caption of Fig. 3 for the meaning of each curve.

which is not determined by Whitham modulation theory at leading order. It is for this reason that the π phase shift was added to the solitary wave solution (40) so that the modulation solution solitary wave front agreed with the numerical solution. The DSW fitting method does not have this phase shift added, resulting in the discrepancy of Fig. 4.

3 Classification of Riemann problems

The detailed form of the solution of the BBO equation (5) with the step initial condition (8) depends critically on the relationship between the values (u_-, u_+) and (v_-, v_+) ahead and behind this initial jump [30, 45]. The solution found in Section 2.3, consisting of an expansion wave linking the level u_- behind to an intermediate level u_i , which is preceded by a DSW linking u_i to the level ahead u_+ , is just one case. In this section, all possible solutions will be detailed, together with their existence regions in terms of (u_-, u_+) and (v_-, v_+) . In particular, it will be shown that the solutions consist of four possible combinations of expansion waves and DSWs, which have existence domains partitioned by the two curves $u_i = u_+$ and $u_i = u_-$. To illustrate this, a classification diagramme is given in Fig. 5, based on the conditions ahead $u_+ = v_+ = 0$. Note that it is sufficient to take the level ahead to vanish to obtain all possible solution types. If u is set to $\beta + U$ in the BBO equation (5), where β is a constant, the same BBO equation in U is obtained with the squared linear wave speed 10 replaced by $10 + \beta$. The variables in the equation can then be rescaled so that the (squared) wavespeed is again 10 .

The boundaries of Figure 5 are determined by the relationship between u_+ and u_- and the intermediate level u_i given by (15), as jumps down from u_i to u_+ generate a DSW and jumps up generate an expansion wave, the same general result also holding for the relationship between u_- and u_i . Hence, as u_i is determined by one of the Riemann invariants r and l constant across the expansion wave and one constant across the DSW, these boundaries are determined by the non-dispersive equations (12).

The solution of the non-dispersive Riemann problem is the classical shock tube problem of compressible gas flow [1] and bears a close connection with the dispersive Riemann problem. With this correspondence, the dispersive shock wave is replaced by a gas dynamic shock wave, a jump in level, and the expansion wave remains as it is non-dispersive.

Figure 5: Riemann problem classification for the BBO system (7). The initial levels u_+ and v_+ are set to 0. The curves $u_i = u_+$ (blue) and $u_i = u_-$ (red) partition the (u_-, v_-) -plane into four regions corresponding to four possible cases.

3.1 Case (i): fast DSW and slow expansion wave

The solution found in Section 2.3, which will be termed case (i), comprises a fast DSW propagating to the right and an expansion wave propagating to the left, as shown in Fig. 6. As already noted, the DSW solution of the BBO equation was derived in Section 2. In this subsection, we shall derive the expansion wave solution, a simple wave solution of the dispersionless system (12), corresponding to the expansion fan linking the level u_- to the intermediate shelf $u = u_i$. This solution is a similarity solution with $r = \text{const}$ on the C_l characteristics of the dispersionless system. On using the intermediate shelf solution (15), this expansion fan solution is

$$u = \begin{cases} u_-, & \xi < -\sqrt{10 + u_-}, \\ \xi^2 - p, & -\sqrt{10 + u_-} < \xi < -\sqrt{10 + u_i}, \\ u_i, & -\sqrt{10 + u_i} < \xi, \end{cases} \quad v = \begin{cases} v_-, & \xi < -\sqrt{10 + u_-}, \\ -\frac{2}{3}\xi^3 + r_-, & -\sqrt{10 + u_-} < \xi < -\sqrt{10 + u_i}, \\ v_i, & -\sqrt{10 + u_i} < \xi, \end{cases} \quad (71)$$

where $\xi = x/t$ is the self-similar variable and $r_- = r(u_-, v_-)$. We see that this expansion fan solution is valid for $V_l(u_-, v_-) < V_l(u_i, v_i)$, or simply $u_- > u_i$. This slow expansion wave solution is compared with the numerical solution of the BBO equation (5) in Fig. 6. It can be seen that there is near exact agreement, except for the dispersive wavetrain generated at the junction between the shelf and the expansion wave. The details of this wavetrain are partly due to Gibbs' phenomenon for the pseudo-spectral scheme at the junction [46]. In addition, a study of the similar wavetrain generated by the DSW solution of the Serre equations [47] shows that its details are highly dependent on the numerical scheme used, whether pseudo-spectral or finite difference, but there is an underlying wavetrain which smooths the corner where the expansion wave meets the shelf.

As for the existence of the fast DSW solution, we require $s_i < s_+$, where s_i and s_+ are given by (25) and (33), respectively. The trailing edge velocity s_i of the DSW with $u = u_i$ is determined by equation (23). This equation has a real root if

$$0 < \left(\frac{10 + u_+}{10 + u_i} \right)^{5/2} < 1,$$

see Figure 7. Thus, the admissibility condition for the fast DSW is

$$\sqrt{10 + u_+} < \sqrt{10 + u_i}, \quad \text{or} \quad u_+ < u_i. \quad (72)$$

Figure 6: Case (i). Slow expansion wave and fast DSW. Black line: numerical DSW solution; blue line: initial condition (8); red line: intermediate level u_i (15) and expansion wave. The initial conditions $\{u_-, u_+, v_-, v_+\} = \{3, 0, 0, 0\}$ have been used.

Figure 7: Function $g(\eta) = \cos^4(\eta/2)(3 \cos \eta - 2)^3$ and $\tilde{g}(\eta) = \cosh^4(\eta/2)(3 \cosh \eta - 2)^3$

In summary, case (i), consisting of a right propagating DSW and a left propagating expansion wave, occurs for $u_+ < u_i < u_-$. Using expression (15) for the intermediate level u_i , we have that case (i) exists for

$$\frac{2}{3}[(10 + u_-)^{3/2} - (10 + u_+)^{3/2}] < v_+ - v_- < \frac{2}{3}[(10 + u_+)^{3/2} - (10 + u_-)^{3/2}]. \quad (73)$$

3.2 Case (ii): two non-interacting DSWs

When $V_l(u_-, v_-)$ tends to $V_l(u_i, v_i)$ from below, namely u_- approaches u_i from above, the expansion wave of case (i) gradually disappears and completely vanishes at $u_- = u_i$. This limiting scenario is illustrated in Fig. 8, in which it can be seen that the intermediate shelf satisfies $u_i = u_-$, as well as $v_i = v_-$. In this limiting case, the BBO equation DSW is essentially the BO equation DSW [28].

In the case $V_l(u_-, v_-) > V_l(u_i, v_i)$, namely $u_i > u_-$, the intermediate shelf u_i will rise above both initial levels u_- and u_+ . Thus, this case (ii) consists of two DSWs propagating in opposite directions on either side of the intermediate self, as illustrated in Fig. 9. The solution for the fast right propagating DSW is the same as for case (i),

Figure 8: Right propagating DSW only for $u_i = u_-$. Black line: numerical DSW solution; blue line: initial condition (8); red line: intermediate level u_i (15). The initial conditions are $\{u_-, u_+, v_-, v_+\} = \{2.9538, 0, 0, 10\}$.

Figure 9: Case (ii). Two DSWs propagating in opposite directions. Black line: numerical DSW solution; blue line: initial condition (8); red line: intermediate level u_i (15). The initial conditions are $\{u_-, u_+, v_-, v_+\} = \{1, 0, 0, 10\}$.

while the slow left propagating DSW can be determined in a similar manner, essentially just reversing the propagation direction.

Let us denote by the superscripts *slow* and *fast* the values of a quantity associated with the slow (left propagating) and the fast (right propagating) DSW, respectively. The velocities of the harmonic and solitary wave edges of the fast DSW are given by equations (25) and (33), so that

$$s_i^{\text{fast}} = \sqrt{10 + u_i} \cos \eta_i^+ - \frac{1}{2} \sqrt{10 + u_i} \tan \eta_i^+ \sin \eta_i^+, \quad (74)$$

and

$$s_+^{\text{fast}} = \sqrt{10 + u_+} \cosh \eta_+ + \frac{1}{2} \sqrt{10 + u_+} \tanh \eta_+ \sinh \eta_+, \quad (75)$$

where η_i^+ and η_+ are the solutions of

$$\cos^4 \left(\frac{\eta_i^+}{2} \right) (3 \cos \eta_i^+ - 2)^3 = \left(\frac{10 + u_+}{10 + u_i} \right)^{5/2}, \quad \cosh^4 \left(\frac{\eta_+}{2} \right) (3 \cosh \eta_+ - 2)^3 = \left(\frac{10 + u_i}{10 + u_+} \right)^{5/2}. \quad (76)$$

For the left propagating slow DSW we have

$$s_i^{\text{slow}} = -\sqrt{10 + u_i} \cos \eta_i^- + \frac{1}{2} \sqrt{10 + u_i} \tan \eta_i^- \sin \eta_i^-, \quad (77)$$

and

$$s_-^{\text{slow}} = -\sqrt{10 + u_-} \cosh \eta_- - \frac{1}{2} \sqrt{10 + u_-} \tanh \eta_- \sinh \eta_-, \quad (78)$$

where η_i^- and η_- are the solutions of

$$\cos^4 \left(\frac{\eta_i^-}{2} \right) (3 \cos \eta_i^- - 2)^3 = \left(\frac{10 + u_-}{10 + u_i} \right)^{5/2}, \quad \cosh^4 \left(\frac{\eta_-}{2} \right) (3 \cosh \eta_- - 2)^3 = \left(\frac{10 + u_i}{10 + u_-} \right)^{5/2}. \quad (79)$$

This solution for case (ii) holds for $u_- < u_i$ and $u_+ < u_i$.

As expected, in the limiting case $u_i = u_+$ the fast DSW disappears, leaving only the slow DSW propagating to the left. This is actually a mirror of the limiting case $u_i = u_-$ and thus corresponds to a downstream DSW only for the BO equation, see Figs. 5 and (8).

For the equivalent case for the NLS equation [18, 45] and the Calogero-Sutherland (CS) system [30] the two DSWs can interact, in that their trailing edges can overlap and so a multiphase wavetrain forms. However, such a multiphase wavetrain of two interacting DSWs cannot occur for the BBO equation. This can be seen from the following argument. To obtain two interacting DSWs we need $s_i^{\text{fast}} < s_i^{\text{slow}}$, which is equivalent to

$$\cos \eta_i^+ + \cos \eta_i^- < \frac{1}{2} (\tan \eta_i^+ \sin \eta_i^+ + \tan \eta_i^- \sin \eta_i^-). \quad (80)$$

We recall that η_i^+ and η_i^- are the roots of the equations (76)₁ and (79)₁, respectively. These values η_i^+ and η_i^- are defined in the interval $(0, \arccos(2/3))$, so that the right hand side values in (76)₁ and (79)₁ are in $(0, 1)$ (cf. Fig. 7). However, as

$$\cos a - \frac{1}{2} \tan a \sin a > 0 \quad \forall a \in (0, \arccos(2/3)), \quad (81)$$

the condition (80) for two interacting DSWs cannot be fulfilled for any η_i^+ and η_i^- in their domains of validity. The two DSWs then propagate in opposite directions, with the slow to the left and the fast to the right, without their tails interacting.

3.3 Case (iii): downstream propagating expansion wave and DSW

This case occurs as a mirror image of case (i) for which the DSW propagates upstream (to the left) and the expansion wave propagates downstream (to the right), which occurs for $u_- < u_+$. That is, case (iii) occurs for $u_- < u_i < u_+$. The expansion fan is now on the C_r characteristic with the Riemann invariant $l = \text{const}$. Furthermore, the Riemann invariant on the C_r characteristic is constant through the left propagating DSW. Combining these two conditions, the intermediate level is still given by equation (15), the same as for case (i). With these results, the expansion wave solution is then

$$u = \begin{cases} u_i, & \xi < \sqrt{10 + u_i}, \\ \xi^2 - p, & \sqrt{10 + u_i} < \xi < \sqrt{10 + u_+}, \\ u_+, & \sqrt{10 + u_+} < \xi, \end{cases} \quad v = \begin{cases} v_i, & \xi < \sqrt{10 + u_i}, \\ -\frac{2}{3} \xi^2 + l_+, & \sqrt{10 + u_i} < \xi < \sqrt{10 + u_+}, \\ v_+, & \sqrt{10 + u_+} < \xi, \end{cases} \quad (82)$$

where $l_+ = l(u_+, v_+)$. The trailing and leading edges of the slow DSW are again given by equations (77) and (78), so that case (iii) is valid for $u_- < u_i < u_+$. A typical solution for case (iii) is shown in Fig. 10. As for case (i), there is near exact agreement for the expansion wave solution with the numerical solution, except for the dispersive wavetrain in the region where the expansion wave joins the intermediate level.

3.4 Case (iv): two expansion waves connected by a constant plateau

Case (iv) consists of two expansion waves connected by a plateau. The DSW solution for case (i) is valid if $s_i < s_+$, so that the DSW is properly ordered, where s_i and s_+ are given by equations (74) and (75), respectively. In the opposite case $s_i > s_+$, the DSW disappears and is replaced by an expansion wave. Therefore, in this case (iv) there are two

Figure 10: Case (iii). Upstream propagating DSW and downstream propagating expansion wave. Black line: numerical DSW solution; blue line: initial condition (8); red line: intermediate level u_i (15) and expansion wave. The initial conditions $\{u_-, u_+, v_-, v_+\} = \{0, 3, 0, 0\}$ have been used.

expansion waves, one linking u_- to u_i on the C_l characteristic and the other linking u_i to u_+ on the C_r characteristic. The solution for the left propagating expansion fan is still given by (71). The right propagating expansion fan on the characteristic C_r is determined by taking the Riemann invariant l on the characteristic C_l constant through the expansion fan, resulting in

$$u = \begin{cases} u_i, & \xi < \sqrt{10 + u_i}, \\ \xi^2 - p, & \sqrt{10 + u_i} < \xi < \sqrt{10 + u_+}, \\ u_+, & \xi > \sqrt{10 + u_+}, \end{cases} \quad v = \begin{cases} v_i, & \xi < \sqrt{10 + u_i}, \\ \frac{2}{3}\xi^3 + l_i, & \sqrt{10 + u_i} < \xi < \sqrt{10 + u_+}, \\ v_+, & \xi > \sqrt{10 + u_+} \end{cases} \quad (83)$$

for this right propagating expansion fan, where $l_i = l(u_i, v_i)$. This new expansion fan solution is valid for $u_i < u_+$, which is the complement of condition (72) for case (i). Combining the expansion fan solutions (71) and (83) gives the full solution

$$u = \begin{cases} u_-, & \xi < -\sqrt{10 + u_-}, \\ \xi^2 - 10, & -\sqrt{10 + u_-} < \xi < -\sqrt{10 + u_i}, \\ u_i, & -\sqrt{10 + u_i} < \xi < \sqrt{10 + u_i}, \\ \xi^2 - 10, & \sqrt{10 + u_i} < \xi < \sqrt{10 + u_+}, \\ u_+, & \xi > \sqrt{10 + u_+}, \end{cases} \quad (84)$$

and

$$v = \begin{cases} v_-, & \xi < -\sqrt{10 + u_-}, \\ -\frac{2}{3}\xi^3 + v_- - \frac{2}{3}(10 + u_-)^{3/2}, & -\sqrt{10 + u_-} < \xi < -\sqrt{10 + u_i}, \\ v_i, & -\sqrt{10 + u_i} < \xi < \sqrt{10 + u_i}, \\ \frac{2}{3}\xi^3 + v_+ + \frac{2}{3}(10 + u_+)^{3/2}, & \sqrt{10 + u_i} < \xi < \sqrt{10 + u_+}, \\ v_+, & \xi > \sqrt{10 + u_+} \end{cases} \quad (85)$$

for case (iv). To obtain these solutions we have used the relation

$$l_i = l_+ \quad \Leftrightarrow \quad v_i + \frac{2}{3}(10 + u_i)^{3/2} = v_+ + \frac{2}{3}(10 + u_+)^{3/2}.$$

These solutions are valid for $u_i < u_-$ and $u_i < u_+$. A typical solution for case (iv) is illustrated in Fig. 11, where, again, excellent agreement with the numerical solution is seen, again except for the dispersive wavetrains generated at the joins between the expansion waves and the intermediate level.

Figure 11: Case (iv). Two expansion waves connected by a plateau. Black line: numerical DSW solution; red line: initial condition (8); blue line: initial condition (8); red line: intermediate level u_i (15) and expansion wave. The initial conditions $\{u_-, u_+, v_-, v_+\} = \{1, 0, 0, -10\}$ have been used.

4 Conclusions

While full DSW solutions of nonlinear dispersive wave equations can, in principle, be obtained from their (hyperbolic) Whitham modulation equations, in practice, even if the Whitham modulation equations can be derived, the DSW solutions correspond to centred simple waves and the ODEs for these cannot be solved analytically in general. Such DSW solutions are immediate if the modulation equations can be set in Riemann invariant form. The calculation of the Riemann invariant form is only guaranteed for a second order system, for which there is always an integrating factor. To overcome this limitation, the dispersive shock fitting method [20, 24] was developed as it does not require a knowledge of the full Whitham modulation equations. Its drawback is that it only determines the leading and trailing edges of the DSW, with no determination of its interior. While there have been recent extensions to the dispersive shock fitting method to extend its range of validity at the linear wave edge [48], in principle it is still not valid through the DSW up to the solitary wave edge. In the present work the Whitham modulation equations for the BBO equation were derived and its DSW solution found using a combination of the dispersive shock fitting method and modulation theory. The ordinary differential equations for the simple wave DSW solution were solved numerically with a boundary condition derived from the dispersive shock fitting method applied at the linear wave edge. While this hybrid method was applied to an equation with Benjamin-Ono dispersion, it is applicable to a nonlinear dispersive wave equation with other types of dispersion, including KdV-type dispersion, if its Whitham modulation equations can be derived and are hyperbolic.

Acknowledgments

This work was done during a research visit by L.T.K.N to the University of Edinburgh, the visit funded by Germany's Excellence Initiative [DFG GSC 98/3].

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