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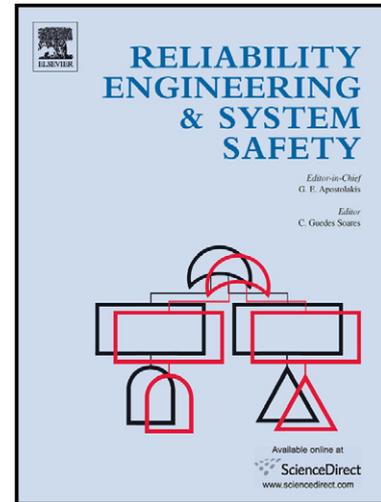
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Mission Reliability of Semi-Markov Systems under Generalized Operational Time Requirements

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Abstract

Mission reliability of a system depends on specific criteria for mission success. To evaluate the mission reliability of some mission systems that do not need to work normally for the whole mission time, two types of mission reliability for such systems are studied. The first type corresponds to the mission requirement that the system must remain operational continuously for a minimum time within the given mission time interval, while the second corresponds to the mission requirement that the total operational time of the system within the mission time window must be greater than a given value. Based on Markov renewal properties, matrix integral equations are derived for semi-Markov systems. Numerical algorithms and a simulation procedure are provided for both types of mission reliability. Two examples are used for illustration purposes. One is a one-unit repairable Markov system, and the other is a cold standby semi-Markov system consisting of two components. By the proposed approaches, the mission reliability of systems with time

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redundancy can be more precisely estimated to avoid possible unnecessary redundancy of system resources.

1. Introduction

In engineering applications, there exist many systems designed to support the accomplishment of critical missions. For example, during a space flight mission, it is necessary to use the spaceflight telemetry, tracking, and control (TT&C) system [1] to provide connection between the spacecraft and facilities on the ground, and to ensure that the spacecraft performs its mission correctly. Often, to avoid the risk of mission failure or waste of TT&C resources, space system engineers are interested in quantitatively assessing the mission reliability of the TT&C system which will support a planned spaceflight, so they can make reasonable decisions about the system design before practical execution of the mission.

Mission reliability of a system is the probability of successful completion of a stated mission by the system deployed in a given environment. Depending on different criteria of mission success, mission reliability may be defined more specifically. In some engineering applications, a mission must be successfully accomplished within a given time interval. Taking TT&C systems as an example, since the spacecraft orbits the earth, the time for which it is passing overhead a ground facility is limited to a specific interval (called the *time window*), so the facility can only provide TT&C services within this time window. However, for the mission to be successful, sometimes the system

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does not need to work normally for the whole time window. In this paper, we identify two specific cases. In the first case, to ensure mission success, the system needs only remain operational for a time period greater than a certain value within the mission time window. For example, to accomplish certain remote control instruction injection operations on a spacecraft, the ground facility only needs to function normally for a short period of time while the spacecraft passes over. We call this type of mission reliability, *mission reliability of type I*. In the second case, we require that the total sum of the system's operational periods within the given time window is greater than a given value. We call the mission reliability of this kind *mission reliability of type II*. For a TT&C system, if the mission is to transfer a certain amount of onboard data, as long as the total sum of transmission time is sufficient, the mission will be regarded as successfully completed.

Although there are papers on mission reliability for special systems of one mission phase [2][3], most existing literature on mission reliability focuses on phased-mission systems (PMS) that have multiple phases [4][5][6]. However, a commonly adopted assumption in existing research work is that for the mission to succeed in a phase, the system must remain operational throughout the whole time of the phase[3][7]. Recent theoretical research work on PMS has mainly focused on two fields. One is on the improvement of computational efficiency[8][9][7]; another is on modelling and analysis methods of various kinds of PMS with special features, such as demand-based PMS[10], PMS with common-cause failures [11][12], propagagted failures [13] and imperfect coverage [14]. For the two cases considered in our paper, if we apply the assumption that the system must remain operational throughout the

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whole time of the phase, we can only get a conservative estimate of the real mission reliability, because the assumption is stricter than really necessary. This is only acceptable if the mission time interval is short enough. For some TT&C services, the required TT&C task time may be just several minutes. For a low earth orbit (LEO) satellite, since the time of passing overhead a ground station (i.e. the time window) is of about the same magnitude, the errors due to such an assumption are insignificant. However, for some medium earth orbit (MEO) satellites or if inter-satellite links are used, the time window can be as long as several hours and the adoption of this assumption in mission reliability evaluation will unavoidably lead to serious underestimation of the real value, and may result in unnecessary redundancy in the deployment of expensive TT&C resources. Therefore, more precise modelling and solution methods are needed in such situations.

To the best of our knowledge, little previous research has been done on these two types of mission reliability. The first type was studied for the first time in [15], and a numerical method was given for its calculation based on the probabilities of mission success in mutually exclusive cases and order statistics. However, in that paper the system under study was restricted to a one-unit system with both exponential failure and repair times.

By a *Markov system* we mean a system whose behaviour can be described by its state evolution over a time horizon, and at any moment the future behaviour of the system, given its current state, is independent of its past history. A *semi-Markov system* is a generalization of a Markov system [16]. Compared with a Markov system, the main feature of a semi-Markov system is that the time required for each successive state transition can be a non-

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exponential random variable, which may depend on both its current state and the next state to be visited.

Whilst we believe our type I and type II mission reliability measures to be novel, a similar measure has previously appeared in the literature as *interval reliability* [17][18], *remaining probability* [19] or *interval availability* [20]. Interval reliability is defined as the probability that a system will work normally for a specific time interval of given length without failure given that it begins to work at a fixed start moment. Barlow gave a general formula for computing interval reliability by use of a renewal property [18] and obtained its limit solution as time tends to infinity. It has been shown that for repairable semi-Markov systems, either a double Laplace transform or an integral equation approach can be used to obtain interval reliability [2]. For semi-Markov systems with general state space (not limited to finite or countable state space), in both continuous time and discrete time cases, Markov renewal equations (MRE) can be built to give the formulae of interval reliability and its limiting expression [20][21][22].

However, interval reliability is defined only for a fixed interval time horizon, whereas our mission reliability of type I is defined for a mission that can be executed in an interval that is not prescribed prior to the mission, within a given mission time window. Moreover, by definition, the meaning of interval reliability is totally different from that of mission reliability of type II.

In this paper, we define two types of mission reliability. Furthermore, for the general case of semi-Markov systems, by the renewal property of semi-Markov processes, we derive matrix integral equations and provide numerical algorithms for calculating these two types of mission reliability.

The remainder of this paper is organized as follows. Section 2 introduces the equations for sojourn time distributions of semi-Markov systems, and gives numerical methods for their solution. Section 3 establishes the integral matrix equations for mission reliability of type I . Section 4 is devoted to mission reliability of type II: a group of matrix integral equations is derived, and algorithms are provided for their solution. Section 5 presents a simulation procedure for estimating both types of mission reliability for semi-Markov systems. Two numerical examples are developed to verify our proposed analytical solution methods, and the results are compared with simulation. In the last section, some concluding remarks are given.

2. Sojourn Time Distributions

2.1. Semi-Markov Systems

Suppose we have a system whose state changes only at discrete time moments and takes values in a finite space S. Let S_n denote the time of nth state transition, $n \geq 0$, and the corresponding state of the system is Z_n. Assume the sequence $\{(Z_n, S_n), n \geq 0\}$ is a *Markov renewal sequence* [23], define $N(t) = \sup\{n \geq 0 \mid S_n \leq t\}$, then the state of the system at time t will be $Y(t) = Z_{N(t)}$, which is a *continuous time semi-Markov process* (SMP) [23][24]. In this case, the system is defined as a *semi-Markov system*.

For convenience, we will use Y_t, Y to denote $Y(t)$, $\{Y_t, t \geq 0\}$ respectively in the sequel.

For any $i, j \in S$, let

$$K_{i,j}^n(t) = P\{Z_{n+1} = j, S_{n+1} - S_n \leq t \mid Z_n = i\}.$$

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In this paper, we only consider time-homogeneous semi-Markov systems. So, we can define the semi-Markov kernel of Y as $Q(t) = [q_{i,j}(t)]$, where

$$q_{i,j}(t) = \begin{cases} K_{i,j}^0(t) & i \neq j \\ 0 & i = j \end{cases} \qquad \forall i, j \in S \tag{1}$$

Notice that we assume that the system already has a minimal representation [23], so all Markov renewal moments represent real state transitions, which is why in equation (1) we define $q_{i,i}(t) \equiv 0, \forall i \in S$.

Suppose S is partitioned into U and D, where U is the set of *up states*, in which the system is operational, and D is the set of *down states*, in which the system has failed and is under repair. Hence from a reliability point of view, the state of the system alternates between U and D during system evolution.

For convenience, we will also use U, D to denote the tuples of the corresponding sets of state;

$$U = (u_1, u_2, \ldots, u_{|U|})$$
$$D = (d_1, d_2, \ldots, d_{|D|})$$

where, $|U|, |D|$ stand for the number of elements in the corresponding sets .

2.2. Equations for Sojourn Time Distributions

For later use, here we introduce the main results by Csenki [2] [25] about the sojourn time of a semi-Markov system before it makes a state transition.

Let $k_{u,d}(t)$ denote the distribution function of the sojourn time of the system holding in state $u \in U$ before first entering or re-entering into down state $d \in D$, and $k_{d,u}(t)$ denote the distribution function of the sojourn time

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of the system holding in state $d \in D$ before first entering or re-entering up state $u \in U$.

Thus, letting $K_{UD}(t) = [k_{u,d}(t)]_{u \in U, d \in D}$, $K_{DU}(t) = [k_{d,u}(t)]_{d \in D, u \in U}$, the following system of integral equations can be established [2][25].

$$K_{UD}(t) = \int_{[0,t]} Q_{UU}(\mathrm{d}w) K_{UD}(t-w) + Q_{UD}(t) \tag{2}$$

$$K_{DU}(t) = \int_{[0,t]} Q_{DD}(\mathrm{d}w) K_{DU}(t-w) + Q_{DU}(t) \tag{3}$$

The above equations are derived by Csenki [2] based on renewal arguments. For instance, $k_{u,d}(t)$ can be obtained as the sum of the following two parts:

- the probability that the system first enters or re-enters state $s \in U, s \neq u$ from state $u \in U$ at time $w < t$, and then from state s first enters or re-enters state $d \in D$ within time length $t-w$.

- the probability that the system first enters or re-enters state $d \in D$ before time t.

Therefore, we have

$$k_{u,d} = \sum_{\substack{s \in U \\ s \neq u}} \int_{[0,t]} k_{s,d}(t-w) q_{u,s}(dw) + q_{u,d}(t)$$

The matrix form of this will give equation (2). Equation (3) can be derived similarly.

Furthermore, letting $H(t) = K_{UD}(t)$, $L(t) = Q_{UD}(t)$, and

$$J(t) = Q_{UU}(t) \tag{4}$$

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equation (2) can be rewritten in matrix form as

$$H(t) = \int_{[0,t]} J(\mathrm{d}w)H(t-w) + L(t) \tag{5}$$

Analogously, letting $H(t) = K_{DU}(t)$, $L(t) = Q_{DU}(t)$, and

$$J(t) = Q_{DD}(t) \tag{6}$$

equation (3) can also be rewritten as (5).

2.3. Algorithm for Sojourn Time Distributions

To solve (5), we can use the two-point trapezoidal rule for computing Stieltjes integrals [2][26] as follows

$$\int_a^b f(x)\mathrm{d}g(x) \approx \frac{1}{2} \sum_{k=0}^{n-1} [f(x_k) + f(x_{k+1})][g(x_{k+1}) - g(x_k)] \tag{7}$$

where $[a,b]$ is divided into n segments of equal length.

Assume $[0,t]$ is equally divided into n intervals by $w_i = i\delta$, $i \in \{0, 1, \ldots, n\}$, $\delta = \frac{1}{n}t$, namely

$$0 = w_0 < w_1 < \cdots < w_{n-1} < w_n = t \tag{8}$$

then, by equation (7), approximately, (5) has the form

$$H(n\delta) = \frac{1}{2} \sum_{i=0}^{n-1} [J((i+1)\delta) - J(i\delta)] \\ \times [H((n-i)\delta) + H((n-i-1)\delta)] + L(n\delta) \tag{9}$$

Based on the definition of Q in (1), we can verify that $J(0) = \mathbf{0}$ in (4),(6). Therefore, from (5), we have initial conditions: $H(0) = L(0) = J(0) = \mathbf{0}$.

In what follows, it will be convenient to denote $H(i\delta)$, $J(i\delta)$, $L(i\delta)$ as H_i, J_i, L_i, $i = 0, 1, \ldots$, respectively.

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Consequently, (9) can be written as

$$H_n = \frac{1}{2}\sum_{i=0}^{n-1}(J_{i+1} - J_i)(H_{n-i} + H_{n-i-1}) + L_n. \qquad (10)$$

By which, H_n can be calculated recursively [2].

3. Mission reliability of Type I

3.1. Equations for Mission Reliability

Suppose a mission requires the minimum operational time T_d within time window $[0, T]$. Assume that x is a Markov renewal time, then we define

$R_u^I(x,t)$: the probability that there is an operational time span greater than or equal to T_d within $[x, t]$, and the system enters into state $u \in U$ immediately after x.

$R_d^I(x,t)$: the probability that there is an operational time span greater than or equal to T_d within $[x, t]$, and the system enters into state $d \in D$ immediately after x.

Suppose the length of the system's first operational time span is w, by renewal theory, we have

$$
\begin{aligned}
R_u^I(0,t) &= \sum_{d \in D}\int_0^\infty \left[I_{w \ge T_d} + I_{w < T_d}R_d^I(w,t) \right] k_{u,d}(\mathrm{d}w) \\
&= \sum_{d \in D}\left[\int_{T_d}^\infty k_{u,d}(\mathrm{d}w) + \int_0^{T_d-0} R_d^I(w,t)k_{u,d}(\mathrm{d}w) \right] \qquad (11) \\
&= \sum_{d \in D}\left[\int_{T_d}^\infty k_{u,d}(\mathrm{d}w) + \int_0^{T_d-0} R_d^I(0,t-w)k_{u,d}(\mathrm{d}w) \right]
\end{aligned}
$$

with the condition $R_u^I(0,t) = 0 \quad \forall t < T_d$.

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In the same way, letting the length of Y's first down time span be w, we obtain

$$
\begin{aligned}
R_d^I(0,t) &= \sum_{u \in U} \int_0^\infty R_u^I(w,t) k_{d,u}(\mathrm{d}w) \\
&= \sum_{u \in U} \int_0^{t-T_d} R_u^I(0, t-w) k_{d,u}(\mathrm{d}w)
\end{aligned}
\tag{12}
$$

where the upper bound of the integral becomes $t - T_d$ because $R_u^I(w,t) = 0$ when $t - w < T_d$, by definition. The initial condition is $R_d^I(0,t) = 0 \quad \forall t \leq T_d$.

In matrix form, (11) can be rewritten as

$$
\begin{aligned}
R_U^I(0,t) &= \int_{T_d}^\infty K_{UD}(\mathrm{d}w)\mathbf{1}_D + \int_0^{T_d-0} K_{UD}(\mathrm{d}w) R_D^I(0, t-w) \\
&= \int_0^{T_d-0} K_{UD}(\mathrm{d}w) R_D^I(0, t-w) + J_U
\end{aligned}
\tag{13}
$$

where $\mathbf{1}_D$ is the column vector of ones corresponding to the dimension of K_{UD}, and

$$
J_U = \int_{T_d}^\infty K_{UD}(\mathrm{d}w)\mathbf{1}_D = K_{UD}(\infty)\mathbf{1}_D - K_{UD}(T_d)\mathbf{1}_D
$$

Similarly, the matrix form of (12) is

$$
R_D^I(0,t) = \int_0^{t-T_d} K_{DU}(\mathrm{d}w) R_U^I(0, t-w)
\tag{14}
$$

Letting $H(t) = R_U^I(0,t)$, $\bar{H}(t) = R_D^I(0,t)$, and

$$
K(t) = K_{UD}(t)
\tag{15}
$$

$$
\bar{K}(t) = K_{DU}(t)
\tag{16}
$$

we can write (13) and (14) as

$$
H(t) = \int_0^{T_d-0} K(dw)\bar{H}(t-w) + J_U
\tag{17}
$$

$$
\bar{H}(t) = \int_0^{t-T_d} \bar{K}(dw)H(t-w)
\tag{18}
$$

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respectively, with boundary conditions

$$H(t) = \begin{cases} J_U & t = T_d \\ 0 & t < T_d \end{cases} \quad \text{and} \quad \bar{H}(t) = 0 \quad t \le T_d\,.$$

3.2. Calculation of Mission Reliability

Suppose the initial state distribution of the system is $\pi = (\pi_U, \pi_D)$. Let δ be the step for the discretization of the time horizon.

Assume $d = \arg\min_i |i\delta - T_d|$, $n = \arg\min_i |i\delta - t|$, and approximating (17),(18) by (7), we get :

$$H_n = \frac{1}{2} \sum_{i=0}^{d-1} G_i \left(\bar{H}_{n-i} + \bar{H}_{n-i-1} \right) + J_U \tag{19}$$

$$\bar{H}_n = \frac{1}{2} \sum_{i=0}^{n-d-1} \bar{G}_i \left(H_{n-i} + H_{n-i-1} \right) \tag{20}$$

where

$$G_i = K_{i+1} - K_i \tag{21}$$

$$\bar{G}_i = \bar{K}_{i+1} - \bar{K}_i \tag{22}$$

The above equations can be solved by iterations.

In summary, the solution procedure for mission reliability of type I may consist of the following main steps:

Step 1. Solve equation (5) for $K_{UD}(t), K_{DU}(t)$ by solving (10).

Step 2. Using the obtained results, solve (17),(18) for $R_U^I(0, t)$, $R_D^I(0, t)$ by solving (19) and (20).

Step 3. Compute the mission reliability of type I as

$$(\pi_U, \pi_D) \begin{bmatrix} R_U^I(0, T) \\ R_D^I(0, T) \end{bmatrix}$$

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4. Mission Reliability of Type II

4.1. Equations for Mission Reliability

Suppose a mission requires that the sum of the operational time spans be greater than a given value T_σ within mission time window $[0, T]$.

Define $R_u^{II}(x, t, \sigma)$ to be the probability that the system starts from state $u \in U$ at time x, and the total length of the operational time spans is greater than or equal to σ within time window $[x, t]$.

By the above definition, we have

$$
\begin{aligned}
R_u^{II}&(0, t, \sigma) \\
&= \sum_{d \in D} \int_0^\infty \left[I_{w \geq \sigma} + I_{w < \sigma} R_d^{II}(w, t, \sigma - w) \right] k_{u,d}(\mathrm{d}w) \\
&= \sum_{d \in D} \left[\int_\sigma^\infty k_{u,d}(\mathrm{d}w) + \int_0^{\sigma - 0} R_d^{II}(w, t, \sigma - w) k_{u,d}(\mathrm{d}w) \right]
\end{aligned}
$$

By the renewal property of semi-Markov processes, it follows that

$$
\begin{aligned}
R_u^{II}(0, t, \sigma) = \sum_{d \in D} \bigg[& \int_\sigma^\infty k_{u,d}(\mathrm{d}w) \\
& + \int_0^{\sigma - 0} R_d^{II}(0, t - w, \sigma - w) k_{u,d}(\mathrm{d}w) \bigg]
\end{aligned}
\tag{23}
$$

Similarly, define $R_d^{II}(x, t, \sigma)$ as the probability that the system starts from state $d \in D$ at time x, and the total length of the operational time spans is greater than σ within time window $[x, t]$. We have

$$
\begin{aligned}
R_d^{II}(0, t, \sigma) &= \sum_{d \in D} \int_0^\infty R_u^{II}(w, t, \sigma) k_{d,u}(\mathrm{d}w) \\
&= \sum_{d \in D} \int_0^{t - \sigma} R_u^{II}(0, t - w, \sigma) k_{d,u}(\mathrm{d}w)
\end{aligned}
\tag{24}
$$

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where the upper bound of the integral is changed to $t-\sigma$ because if $w > t-\sigma$, $R_u^{II}(0, t-w, \sigma)$ will become zero by its definition.

In matrix notation, (23) and (24) become

$$R_U^{II}(0, t, \sigma) = \int_0^{\sigma-0} K_{UD}(\mathrm{d}w) R_D^{II}(0, t-w, \sigma-w) + J(\sigma) \qquad (25)$$

$$R_D^{II}(0, t, \sigma) = \int_0^{t-\sigma} K_{DU}(\mathrm{d}w) R_U^{II}(0, t-w, \sigma) \qquad (26)$$

respectively, where

$$J(\sigma) = \int_\sigma^\infty K_{UD}(\mathrm{d}w) \mathbf{1}_D = K_{UD}(\infty) \mathbf{1}_D - K_{UD}(\sigma) \mathbf{1}_D$$

and $\mathbf{1}_U, \mathbf{1}_D$ are the column vectors of ones with lengths $|U|, |D|$ respectively.

Let $H(t, \sigma) = R_U^{II}(0, t, \sigma)$, $\bar{H}(t, \sigma) = R_D^{II}(0, t, \sigma)$, then (25), (26) can be rewritten as

$$H(t, \sigma) = \int_0^{\sigma-0} K(\mathrm{d}w) \bar{H}(t-w, \sigma-w) + J(\sigma) \qquad (27)$$

$$\bar{H}(t, \sigma) = \int_0^{t-\sigma} \bar{K}(\mathrm{d}w) H(t-w, \sigma) \qquad (28)$$

respectively, where $K, \bar{K}$ are defined by (15), (16). The initial conditions are

$$H(t, 0) = \mathbf{1} \qquad \forall t \geq 0 \qquad\qquad H(t, \sigma) = 0 \quad \forall t < \sigma$$
$$H(t, \sigma) = J(\sigma) \quad \forall t = \sigma \qquad\qquad \bar{H}(t, \sigma) = 0 \quad \forall t \leq \sigma$$

4.2. Algorithm for Solving Equations

Suppose that $[0, t]$ is divided by points $w_i, i = 0, 1, \ldots, n$ into n segments of equal length, $0 = w_0 < w_1 < \cdots < w_n = t$, and $m = \arg\min_i |i\delta - \sigma|$.

Using similar notation as for mission reliability of Type I, by approximation using (7), equations (27), (28) can be rewritten as

$$H_{n,m} = \frac{1}{2} \sum_{i=0}^{m-1} \left[G_i(\bar{H}_{n-i,m-i} + \bar{H}_{n-i-1,m-i-1}) \right] + J_m \qquad (29)$$

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$$\bar{H}_{n,m} = \frac{1}{2} \sum_{i=0}^{n-m-1} \left[\bar{G}_i(H_{n-i,m} + H_{n-i-1,m}) \right] \qquad n, m = 1, 2, \ldots \qquad (30)$$

where $J_m = J(m\delta) = J(w_m)$, and $G_i, \bar{G}_i$ are defined by (21), (22).

Equations (29), (30) can be solved by iteration.

4.3. Calculation of Mission Reliability

Assume that the mission time window is $[0, T]$, and the minimum total operational time is T_σ, the initial state distribution is $\pi = (\pi_U, \pi_D)$. Set a large enough n for the discretization of the time horizon, as we do for (8).

Based on the previous results, we can now give the main steps for obtaining mission reliability of type II:

Step 1. Solve equation (5) for $K_{UD}(t), K_{DU}(t)$ by solving equation (10).

Step 2. Using the obtained results, solve (27), (28) for $R_U^{II}(0, t, \sigma)$, $R_D^{II}(0, t, \sigma)$ in the form of equations (29), (30).

Step 3. Compute the mission reliability of type II as

$$(\pi_U, \pi_D) \begin{bmatrix} R_U^{II}(0, T, T_\sigma) \\ R_D^{II}(0, T, T_\sigma) \end{bmatrix}$$

5. Simulation and Numerical Study

To illustrate and verify our approach, we study two example systems and give a simulation procedure. All the algorithms and the simulation procedure have been coded in Python 3, and run on a Macbook Air laptop with an Intel 1.3GHz processor and 4GB of memory.

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5.1. Simulation Procedure

Let $Q = [q_{i,j}]$ be the semi-Markov kernel of the system under study; then the transition probability matrix of the embedded Markov chain, $P = [p_{ij}]$, is defined as

$$p_{i,j} = \lim_{t \to \infty} q_{i,j}(t) \qquad \forall i, j \in S \tag{31}$$

Let T_{s_1,s_2} be the time the system has spent in state s_1 since it last entered s_1 prior to jumping to state s_2. Then the sojourn time distribution matrix $W(t) = [w_{s_1,s_2}(t)]$ is defined as [24]:

$$
\begin{aligned}
w_{s_1,s_2}(t) &= P\{T_{s_1,s_2} \le t\} \\
&= P\{S_{n+1} - S_n \le t \mid X_n = s_1, X_{n+1} = s_2\} \\
&= \begin{cases} \frac{q_{s_1,s_2}(t)}{p_{s_1,s_2}} & p_{s_1,s_2} > 0 \\ 1 & p_{s_1,s_2} = 0 \end{cases}
\end{aligned}
\tag{32}
$$

The simulation procedure to estimate both types of mission reliability for a semi-Markov system is given below (see [16]).

ALGORITHM 1 MissionReliabilitySim

Step 1. (*Initialization*) Assume T_m is the mission time, and T_d, T_σ are the minimum operational times corresponding the mission requirements for mission reliability of type I and type II respectively.

Let the total number of simulation runs be N_{sim}; let the two lists $simRs_1, simRs_2$ for storing the results of simulation runs both be set initially as empty lists.

Step 2. (*Loops to collect simulation results*) For $k = 1, 2, \ldots, N_{sim}$, do

1) Get the result of one simulation run, rs_1, rs_2, by calling function

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OneRunSim (see ALGORITHM 2 given below).

rs_1 = 'Success'/rs_1 ='fail' means that the type I mission is successful/failed respectively. Similarly, rs_2 = 'Success'/rs_2 = 'fail' corresponds to the success/failure of the type II mission in the current run.

2) Append rs_1 to $simRs_1$, and rs_2 to $simRs_2$.

Step 3. (*Estimating mission reliabilities*)

1) Count the number of items with value *'Success'* in $simRs_1$, denote it as n_{s1}.

2) Count the number of of items with value *'Success'* in $simRs_2$, denote it as n_{s2}.

3) Type I mission reliability is estimated as $R = n_{s1}/N_{sim}$.

4) Type II mission reliability is estimated as $R = n_{s2}/N_{sim}$.

ALGORITHM 2 OneRunSim

Step 1. (*Initialization*) Set s_0 as the initial state, current simulation time $t = 0$, let $s_1 := s_0$, and set the list of operational spans, L_u, to be the empty list.

Step 2. (*Sampling the next state*) Select the next state to visit from s_1 by the sampling result according to transition probability matrix P defined by (31).

Step 3. (*Sampling the sojourn time*) If state s_2 is selected, then a sojourn time T_{s_1,s_2} is sampled from the distribution $w_{s_1,s_2}(t)$ defined by (32).

Step 4. (*Recording operational time interval*) Let $t_w = t + T_{s_1,s_2}$, (note that for a semi-Markov system, the system will remain in s_1 until the end of its holding time T_{s_1,s_2}) and do

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1) If $t_w < T_m$

    if $s_1 \in U$ append T_{s_1,s_2} to L_u, doing appropriate merging if they are consecutively connected operational spans.

    set $t := t_w$, $s_1 := s_2$

    return to Step 2.

2) If $t_w \geq T_m$

    if $s_1 \in U$ append $t_w - T_m$ to L_u, doing appropriate merging with previous span if they are consecutively connected operational spans.

Step 5. (*Judgement of mission success*)

1) Search L_u to check each of its items; if there exists an item in L_u that is greater than T_d, then set $rs_1 :=$ '*Success*', otherwise, $rs_1 :=$ '*fail*'.

2) Let L_σ be the total sum of the length of all the items in L_u; if $L_\sigma \geq T_\sigma$ then $rs_2 :=$ '*Success*', otherwise, $rs_2 :=$ '*fail*'.

3) return rs_1, rs_2.

5.2. One Unit System

Now, we take as example the one-unit system studied in [15]. For this system, the life time and repair time are both assumed to follow exponential distributions, with failure rate $\lambda = 1/60$, repair rate $\mu = 1/10$. The mission time window is $[0, 100]$, within which, for mission reliability of type I, the minimum required operational time is 60 time units. In addition, we assume that for mission reliability of type II, the minimum required total operational time is also 60 time units.

With the above assumptions, the mission reliability of type I was obtained

as 0.53347 using the analytical method presented in [15] (we have corrected errors in [15], so the result here differs).

For this system, let $U = \{1\}$, $D = \{0\}$, where 1 denotes the working state, and 0 denotes the down state (i.e. under repair).

This system is obviously a Markov system (therefore a special case of a semi-Markov system); its semi-Markov kernel can be easily built as

$$Q = \begin{bmatrix} Q_{UU} & Q_{UD} \\ Q_{DU} & Q_{DD} \end{bmatrix} = \begin{bmatrix} 0 & 1 - e^{-\lambda_A t} \\ 1 - e^{-\mu_A t} & 0 \end{bmatrix}$$

The initial state distribution is $\pi = (1, 0)$.

By (31), (32), we have

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$W(t) = \begin{bmatrix} 1 & 1 - e^{-\lambda_A t} \\ 1 - e^{-\mu_A t} & 1 \end{bmatrix}$$

Using our simulation procedure, with the given parameters, we obtained estimates of both types of mission reliability for this system. Table 1 shows selected estimated values of mission reliability with different numbers of simulation runs, where for convenience, we use the following notation:

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N_R: number of simulation runs.

R_{U1}:   mission reliability of type I;

system starts in operational state.

R_{U2}:   mission reliability of type II;

system starts in operational state.

R_{D1}:   mission reliability of type I;

system starts in down state.

R_{D2}:   mission reliability of type II;

system starts in down state.

P_T: simulation processing time in seconds.

Table 1: Estimated mission reliabilities by simulation

N_R	R_{U1}	R_{D1}	R_{U2}	R_{D2}	P_T
10000	0.537	0.475	0.951	0.878	1.49
100000	0.532	0.478	0.953	0.877	13.08
200000	0.532	0.480	0.953	0.875	26.05
250000	0.532	0.479	0.953	0.875	32.98
300000	0.533	0.478	0.953	0.875	39.09
390000	0.533	0.477	0.952	0.876	51.78

We also used the analytical models and algorithms presented in this paper, setting $\delta = 100/120$. By solving the systems of linear equations for mission reliability of both types, we obtain $R_{U1} = 0.533, R_{D1} = 0.478$, $R_{U2} = 0.952, R_{D2} = 0.876$. The total computer processing time is 30.2944 seconds with 15 numerical iterations.

We can see that the results of mission reliability of type I match well

with the results obtained using the analytical method in [15]. Comparing the derived results with the results obtained via simulation we see very good agreement. Due to the numerical imprecision introduced by discretization in solving the integral equations and in running simulation, we consider that the difference between them is sufficiently small to be considered insignificant.

In addition, we can understand that with $T_d = T_\sigma$, mission reliability of type II is higher than mission reliability of type I. The mission reliability of type II is defined as the probability that the total length of operational time of the system within the mission time window is no less than the given value, while the mission reliability of type I is defined as the probability that there is a operational time span within the mission time window whose length is no less than a given value. Therefore, the condition of mission reliability of type I is more stringent than that of mission reliability of type II. So, mission reliability of type II is higher than that of type I provided the required time length is the same. Moreover, for both types of mission reliability, the reliability corresponding to when the system starts from the operational state is higher than that corresponding to when the system starts from the down state.

By setting different required operational lengths T_d, T_σ, Table 2 gives the obtained mission reliabilities and processing times (in seconds).

We can see from Table 2 that for the method in Ref.[15], the computational time P_T decreases as the required time length $T_d (T_\sigma)$ increases. This is because the probability of mission success in each mutually exclusive case will decrease fast enough to make the numerical integration involved converge quickly. For the simulation method of this paper, the processing times P_T are

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Table 2: Reliabilities and processing times of different $T_d\,(T_\sigma)$

T_d/T_σ	R_{U1}	30	60	90
Method	R_{U1}	0.9487	0.5335	0.2362
of Ref.[15]	P_T	130.9950	94.5621	56.1107
Simlation (390,000 runs)	R_{U1}	0.9485	0.5333	0.2356
	R_{D1}	0.9219	0.4773	0.1439
	R_{U2}	0.9986	0.9523	0.5120
	R_{D2}	0.9937	0.8762	0.2526
	P_T	54.2361	53.5522	54.1446
Analytical Method	R_{U1}	0.9486	0.5334	0.2361
	R_{D1}	0.9219	0.4779	0.1448
	R_{U2}	0.9985	0.9515	0.5119
	R_{D2}	0.9935	0.8758	0.2520
	P_T	16.5478	30.2944	35.1067

approximately the same. This is because no matter how large the required time length is, the same number of samples are needed for the simulation. For the analytical method of this paper, we can observe that the processing time P_T becomes longer if the required time length increases, mainly due to an increase in the number of terms m of equations (29) and (30) in computing the mission reliability of type II. Finally, it can be seen that there are no significant differences in the accuracy of the mission reliabilities with different required time lengths for all these methods, as the reliability results are sufficiently close to each other.

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5.3. Cold Standby System

Suppose we have a cold standby system with perfect switching [27]. We assume that the system consists of two components, A and B, and the switching between components always works without failure. A is on-site repairable. However, B is not repairable on-site. Assume that the times to failure of A and B are independent. For each component, the failure and repair times are also independent. Furthermore, the failure time of B follows an exponential distribution with rate $\lambda_B = 1/30$. The repair time of A follows an exponential distribution with rate $\mu_A = 1/10$, and its failure time follows a two-parameter Weibull distribution [16] with distribution function $G_A(t)$ given as

$$G_A(t) = P(T \le t) = 1 - \exp\left(-\left(\frac{t}{\eta}\right)^\beta\right) \tag{33}$$

where $\eta = 60$, $\beta = 2.0$.

We assume that once A fails, it will immediately be under repair. Moreover, the switchover time is negligible. For instance, once component A fails, if B is in the standby state, then B will be put into service, and the repair of A starts immediately. After A has been repaired, it will enter into the standby state if B is in the working state. Otherwise, A will enter the working state immediately.

The state transitions of the system are shown in Fig. 1, where f_A, f_B denote the failure of A and B respectively, and r_A denotes the repair of A, and the states are indexed using integers, explained as follows:

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state 1: A is working(w), and B is in standby(b).

state 2: A is down(d) and under repair, B is working(w).

state 3: A is down and under repair(d), B is down(d).

state 4: A is in standby(b), B is working(w).

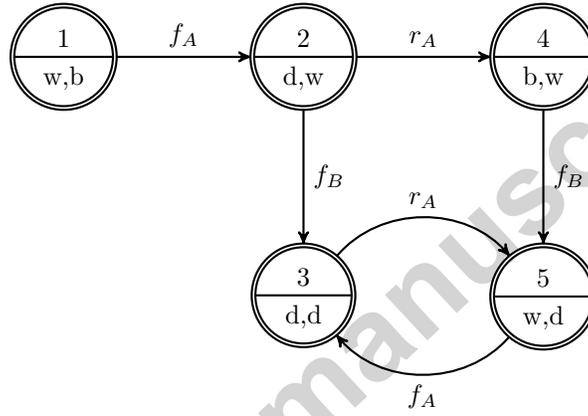state 5: A is working(w), and B is down(d).

Figure 1: State transition of the example cold standby system

It can be easily concluded that all state transitions correspond to Markov renewal moments. Therefore, the system is identified as semi-Markov with the above 5 states. For this system, $U = \{1, 2, 4, 5\}$, and $D = \{3\}$.

From above, we can get the elements of the semi-Markov kernel as

$$q_{1,2}(t) = P\{A \text{ fails up to } t\} = G_A(t)$$

$q_{2,4}(t) = P\{A\text{'s repair is finished by } t$ and

B does not fail by that time$\}$

$$= \frac{\mu_A}{\mu_A + \lambda_B} \left(1 - e^{-(\mu_A + \lambda_B)t}\right)$$

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$q_{2,3}(t) = P\{B$ fails before t and

A's repair does not finish by that time$\}$

$$= \frac{\lambda_B}{\mu_A + \lambda_B}\left(1 - e^{-(\mu_A + \lambda_B)t}\right)$$

$q_{4,5}(t) = P\{B$ fails by time $t\} = 1 - e^{-\lambda_B t}$

$q_{3,5}(t) = P\{A$"s repair is finished by $t\} = 1 - e^{-\mu_A t}$

$q_{5,3}(t) = P\{A$ fails by $t\} = G_A(t)$

So, the semi-Markov kernel is

$Q_{UU} =$

$$\begin{bmatrix} 1 - G_A(t) & G_A(t) & 0 & 0 \\ 0 & e^{-(\mu_A + \lambda_B)t} & \frac{\mu_A}{\mu_A + \lambda_B}\left(1 - e^{-(\mu_A + \lambda_B)t}\right) & 0 \\ 0 & 0 & e^{\lambda_B t} & 1 - e^{-\lambda_B t} \\ 0 & 0 & 0 & 1 - G_A(t) \end{bmatrix}$$

$$Q_{UD} = \begin{bmatrix} 0 \\ \frac{\lambda_B}{\mu_A + \lambda_B}\left(1 - e^{-(\mu_A + \lambda_B)t}\right) \\ 0 \\ G_A(t) \end{bmatrix}$$

$$Q_{DU} = \begin{bmatrix} 0 & 0 & 0 & 1 - e^{-\mu_A t} \end{bmatrix} \qquad Q_{DD} = \begin{bmatrix} 0 \end{bmatrix}$$

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By (31), (32), we have

$$P = \begin{bmatrix} P_{UU} & P_{UD} \\ P_{DU} & P_{DD} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mu_A}{\mu_A+\lambda_B} & 0 & \frac{\lambda_B}{\mu_A+\lambda_B} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$W(t) = \begin{bmatrix} W_{UU}(t) & W_{UD}(t) \\ W_{DU}(t) & W_{DD}(t) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & G_A & 1 & 1 & 1 \\ 1 & 1 & 1-e^{-(\mu_A+\lambda_B)t} & 1 & 1-e^{-(\mu_A+\lambda_B)t} \\ 1 & 1 & 1 & 1-e^{-\lambda_B t} & 1 \\ 1 & 1 & 1 & 1 & G_A(t) \\ 1 & 1 & 1 & 1-e^{-\mu_A t} & 1 \end{bmatrix}$$

The results obtained by simulation and by analytical algorithms are given in Table 3. The number of simulation runs is 400,000, and the computer processing time is 136.023 seconds. For numerical solution, the discretization interval length is $\delta = 100/120 = 0.8333$, number of iterations is 15, and the computer processing time is 107.436 seconds. In Table 3, the following notation is used for convenience.

St: the starting state of the system.

$R1$: mission reliability of type I.

$R2$: mission reliability of type II.

Sim: results by simulation method.

Ana: results by solving equations.

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Table 3: Comparison of mission reliabilities

St	$R1(Sim)$	$R1(Ana)$	$R2(Sim)$	$R2(Ana)$
1	0.864	0.865	0.995	0.996
2	0.675	0.675	0.978	0.978
4	0.713	0.712	0.991	0.991
5	0.457	0.457	0.979	0.979
3	0.414	0.414	0.916	0.917

From Table 3, we can see that there is good agreement between the results obtained by the two approaches, considering the numerical precision of simulation and solution of the integral equations.

In this example, we have used a Weibull distribution for the failure time of component A. Note that if we set $\beta = 1$ in equation (33), the Weibull distribution becomes an exponential distribution with rate $\lambda = 1/\eta$; that is, the same distribution as the one-unit system in the first example of this section. Moreover, from Fig. 1, we see that if the system starts in either state 3 or state 5, the behavior of this system will be exactly the same as the one-unit system in the first example with corresponding starting states. Consequently, the results of mission reliability should be the same. We have confirmed this observation with numerical results obtained setting $\beta = 1$.

6. Conclusions

Based on mission requirements in engineering applications, we have presented two new reliability measures for mission systems under generalized operational time requirements within a given mission time window. One

measure is for the case when the mission requires a minimum operational time interval. The other measure is for the case when the mission requires the sum of operational time to be greater than a given value. For semi-Markov systems, matrix integral equations for calculating both types of mission reliability have been derived and numerical solution algorithms were presented. By the results of this paper, we can more precisely evaluate the mission reliability to reduce unnecessary redundancy of system resources for such mission systems.

Since a Markov system is a special case of a semi-Markov system, the results can be easily applied to mission systems consisting of multiple components if all of them have exponential failure and repair time distributions. However, careful checking of the semi-Markov properties are often necessary to ensure a system under study is really semi-Markov. Therefore, it will be useful in future research work to give a systematic approach to identifying an embedded semi-Markov process for systems with typical structures.

7. Acknowlegement

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