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A NOTE ON EULER APPROXIMATIONS FOR SDES WITH HÖLDER CONTINUOUS DIFFUSION COEFFICIENTS

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ABSTRACT. We provide a rate for the strong convergence of Euler approximations for stochastic differential equations (SDEs) whose diffusion coefficient is not Lipschitz but only $(1/2 + \alpha)$ -Hölder continuous for some $\alpha \geq 0$.

Keywords. stochastic differential equation, Euler scheme, convergence speed, Hölder continuous

1. INTRODUCTION

In mathematical finance the SDE

$$dR(t) = (a - kR(t))dt + \sigma|R(t)|^{1/2}dW(t), \quad R(0) > 0 \quad (1.1)$$

with parameters $\sigma, a, k > 0$ is often used to describe the evolution of the interest rate (this is the so-called Cox–Ingersoll–Ross model, see e.g. section 4.6 of [4]). The diffusion coefficient here fails to be Lipschitz continuous near the origin, hence textbook results on the rate of strong convergence for the corresponding Euler scheme do not apply.

It was nevertheless claimed in [6] that the rate (of a slightly modified scheme) should be equal to the standard $n^{-1/2}$. This was proved in [2] when a is not too small. For small values of a numerical experiments showed very slow convergence, see [1].

In the present paper we prove a convergence speed estimate for Euler schemes corresponding to SDEs with $1/2$ -Hölder continuous diffusion coefficients (just like that of (1.1)) without any restrictions on the parameters. It is not surprising that only a slow rate $1/\ln n$ is established, in accordance with the results of [1]. We also regard the $(1/2 + \alpha)$ -Hölder continuous case with $\alpha > 0$ where the rate $n^{-\alpha}$ is obtained.

Let us fix $T > 0$ and consider the SDE

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \quad X(0) = \xi \quad (1.2)$$

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on the interval $[0, T]$, where $W(t), t \geq 0$ is a standard Brownian motion, ξ is independent of $W(t), t \geq 0$ and the coefficients satisfy the following condition.

Assumption 1.1. $\sigma, f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable; $g(t, \cdot)$ is monotone decreasing; $b = f + g$ and there exist $K > 0$, $\alpha \in [0, 1/2]$ and $\gamma \in (0, 1]$, such that for all $t \in [0, T]$ and $x, y \in \mathbb{R}$

$$\begin{aligned} |\sigma(t, x) - \sigma(t, y)| &\leq K|x - y|^{\frac{1}{2} + \alpha}, & |f(t, x) - f(t, y)| &\leq K|x - y| \\ |g(t, x) - g(t, y)| &\leq K|x - y|^\gamma, \end{aligned}$$

and

$$|b(t, 0)| + |\sigma(t, 0)| \leq K.$$

Remark 1.1. Assumption 1.1 implies that b, σ satisfy the linear growth condition (see (1.4) below). Under these conditions there exists a unique strong solution of (1.2), see, e.g. [10] and [12], hence it follows from [7] that the Euler scheme converges in probability, only the rate estimate of the present paper is a new contribution.

For integers $n \geq 1$, we define the functions $\kappa_n : [0, T] \rightarrow [0, T]$ by $\kappa_n(T) := \frac{n-1}{n}T$ and

$$\kappa_n(x) = \frac{iT}{n} \text{ for } \frac{iT}{n} \leq x < \frac{(i+1)T}{n}, \text{ for } i = 0, \dots, n-1.$$

We now define the Euler approximations of $X(t), t \in [0, T]$ as the solution of

$$\begin{aligned} dX_n(t) &= b(t, X_n(\kappa_n(t)))dt + \sigma(t, X_n(\kappa_n(t)))dW(t) \\ X_n(0) &= \xi, \end{aligned} \tag{1.3}$$

for each $n \geq 1$.

In this article we study the convergence speed of these Euler approximations. For more information about Euler schemes we refer to the books [3] and [13].

Before going on, let us recall some well-known facts.

Lemma 1.1. *Let $b, \sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ have linear growth, i.e.*

$$|b(t, x)| + |\sigma(t, x)| \leq K(1 + |x|) \text{ for } (t, x) \in [0, T] \times \mathbb{R} \tag{1.4}$$

with some $K > 0$. Let $X(t), t \in [0, T]$ be a solution of

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \quad X(0) = \xi,$$

where ξ is independent of $W(t), t \geq 0$ and $E|\xi|^p < \infty$ for some $p > 0$. Then there is a constant $C > 0$ such that for $t \in [0, T]$

$$E \sup_{0 \leq s \leq t} |X(s) - X(0)|^p \leq C(1 + E|\xi|^p)t^{p/2}, \tag{1.5}$$

where C depends only on p , T and K .

Lemma 1.2. *Assume that b, σ satisfy the linear growth condition, see (1.4). Fix $p > 0$ and assume that $E|\xi|^p < \infty$. Then there is $C > 0$, independent of n , such that*

$$E \sup_{0 \leq s \leq T} |X_n(s)|^p \leq C(1 + E|\xi|^p),$$

for all n , where C is a constant depending on T , p and K .

The above lemmas can be easily found in textbooks and monographs when $p \geq 2$ in their formulation. (See, e.g., [14]). The case $0 < p < 2$ can be obtained from that of $p \geq 2$. For the convenience of the reader we prove them for all $p > 0$ in Remark 3.2 at the end of the paper.

Remark 1.2. Let the conditions of Lemma 1.1 hold. Then applying Lemma 1.1, by Lemma 1.2 we can easily get that for all $p > 0$,

$$E \sup_{s \leq T} |X_n(s) - X_n(\kappa_n(s))|^p \leq \frac{C}{n^{p/2}},$$

for some $C > 0$ depending only on p , T , $E|\xi|^p$ and K . Hence it is easy to see by Jensen's inequality that

$$E \left(\int_0^T |X_n(s) - X_n(\kappa_n(s))|^\delta ds \right)^p \leq \frac{C}{n^{\delta p/2}}$$

for all $\delta > 0$ with a constant C depending on δ and the same parameters as before.

2. ON ACCURACY IN L_1

The following theorem covers, in particular, equation (1.1) for which the same convergence rate has already been shown in [1].

Theorem 2.1. *Let Assumption 1.1 hold and let $E|\xi|^{1+2\alpha} < \infty$. Then there is a constant C depending only on K , T , γ and $E|\xi|^{1+2\alpha}$, such that*

$$E|X(\tau) - X_n(\tau)| \leq \begin{cases} \frac{C}{\ln n} & \text{if } \alpha = 0 \\ C \left(\frac{1}{n^\alpha} + \frac{1}{n^{\gamma/2}} \right) & \text{if } \alpha \in (0, 1/2] \end{cases} \quad (2.1)$$

for all $n \geq 2$ for every stopping time $\tau \leq T$.

Remark 2.1. In the so-called *constant elasticity of variance model* it is assumed that the price of a stock satisfies the SDE

$$dS(t) = S^\theta(t)dW(t), \quad S(0) > 0 \quad (2.2)$$

for some $\theta \in \mathbb{R}$. This model for stock prices first appeared in [5] and was extensively studied thereafter. In the cases $1/2 \leq \theta \leq 1$ Theorem 2.1 provides convergence rate for the Euler approximations corresponding to (2.2).

We obtain Theorem 2.1 from the following proposition for

$$Y_n(t) = X(t) - X_n(t) \quad \text{and} \quad U_n(t) := |X_n(t) - X_n(\kappa_n(t))|. \quad (2.3)$$

Proposition 2.2. *Let Assumption 1.1 hold. Then almost surely*

$$\begin{aligned} |Y_n|(t) \leq & K \int_0^t |Y_n|(s) ds + K \int_0^T (U_n(s) + U_n^\gamma(s)) ds \\ & + CR_n^\alpha + M_n(t) \quad \text{for all } t \in [0, T], \end{aligned} \quad (2.4)$$

where C is a constant depending only on K and T ,

$$R_n^\alpha = \begin{cases} \frac{1}{\ln n} + n^{1/3} \int_0^T U_n(s) ds & \text{if } \alpha = 0 \\ \frac{1}{n^\alpha} + \sqrt{n} \int_0^T U_n^{1+2\alpha}(s) ds & \text{if } \alpha \in (0, 1/2], \end{cases}$$

and M_n is a continuous local martingale starting from zero, such that

$$d\langle M_n \rangle(t) \leq 2K^2(|Y_n|^{1+2\alpha}(t) + U_n^{1+2\alpha}(t)) dt. \quad (2.5)$$

Proof. We use the method of Yamada and Watanabe [16] to approximate the function $\phi(x) = |x|$. Let $\delta > 1$ and $\varepsilon > 0$. Then

$$\int_{\varepsilon/\delta}^{\varepsilon} \frac{1}{x} dx = \ln \delta,$$

and therefore there is a continuous nonnegative function $\psi_{\delta\varepsilon}(x)$, $x \in [0, \infty)$, which is zero outside $[\varepsilon/\delta, \varepsilon]$, has integral 1 and satisfies

$$\psi_{\delta\varepsilon}(x) \leq \frac{2}{x \ln \delta}, \quad (2.6)$$

see, e.g., p. 168 of [10] or p. 291 of [12]. Define

$$\phi_{\delta\varepsilon}(x) := \int_0^{|x|} \int_0^y \psi_{\delta\varepsilon}(z) dz dy, \quad x \in \mathbb{R}.$$

Note that for all $x \in \mathbb{R}$, $\phi(x) \leq \phi_{\delta\varepsilon}(x) + \varepsilon$, and

$$0 \leq |\phi'_{\delta\varepsilon}(x)| \leq 1, \quad \phi''_{\delta\varepsilon}(x) = \psi_{\delta\varepsilon}(|x|) \leq \frac{2}{|x| \ln \delta} \mathbf{1}_{[\varepsilon/\delta, \varepsilon]}(|x|). \quad (2.7)$$

Itô's formula provides

$$\begin{aligned} |Y_n(t)| &\leq \varepsilon + \phi_{\delta\varepsilon}(Y_n(t)) \\ &= \varepsilon + \int_0^t I_{\delta\varepsilon n}(s) ds + \frac{1}{2} \int_0^t J_{\delta\varepsilon n}(s) ds + M_{\delta\varepsilon n}(t), \end{aligned}$$

where

$$\begin{aligned} I_{\delta\varepsilon n}(s) &:= \phi'_{\delta\varepsilon}(Y_n(s))(b(s, X(s)) - b(s, X_n(\kappa_n(s))), \\ J_{\delta\varepsilon n}(s) &:= \phi''_{\delta\varepsilon}(Y_n(s))(\sigma(s, X(s)) - \sigma(s, X_n(\kappa_n(s))))^2, \\ M_{\delta\varepsilon n}(t) &:= \int_0^t \phi'_{\delta\varepsilon}(Y_n(s))(\sigma(s, X(s)) - \sigma(s, X_n(\kappa_n(s)))) dW(s). \end{aligned}$$

By Assumption 1.1, using (2.7) we have

$$\begin{aligned} I_{\delta\varepsilon n}(s) &\leq K(|Y_n(s)| + U_n(s)) + \phi'_{\delta\varepsilon}(Y_n(s))(g(s, X(s)) - g(s, X_n(s))) \\ &\quad + KU_n^\gamma(s), \\ J_{\delta\varepsilon n}(s) &\leq \frac{4K^2\varepsilon^{2\alpha}}{\ln \delta} + \frac{4K^2\delta}{\varepsilon \ln \delta} U_n^{1+2\alpha}(s). \end{aligned}$$

Hence, noticing that

$$\phi'_{\delta\varepsilon}(X(s) - X_n(s))(g(s, X(s)) - g(s, X_n(s))) \leq 0,$$

we get

$$\begin{aligned} |Y_n(t)| &\leq K \int_0^t |Y_n(s)| ds + K \int_0^T (U_n(s) + U_n^\gamma(s)) ds \\ &\quad + R_{\delta\varepsilon n} + M_{\delta\varepsilon n}(t), \end{aligned} \tag{2.8}$$

with

$$R_{\delta\varepsilon n} := \varepsilon + \frac{2K^2\varepsilon^{2\alpha}}{\ln \delta} T + \frac{2K^2\delta}{\varepsilon \ln \delta} \int_0^T U_n^{1+2\alpha}(s) ds.$$

Due to $|\phi'_{\delta\varepsilon}| \leq 1$ and Assumption 1.1, we have

$$d\langle M_{\delta\varepsilon n}(t) \rangle \leq 2K^2(|Y_n(t)|^{1+2\alpha} + U_n^{1+2\alpha}(t)) dt. \tag{2.9}$$

If $\alpha = 0$ then choosing $\varepsilon = 1/\ln n$ and $\delta = n^{1/3}$ we get

$$R_{\delta\varepsilon n} \leq \frac{C}{\ln n} + Cn^{1/3} \int_0^t U_n(s) ds = CR_n^{(0)}, \tag{2.10}$$

and if $\alpha \in (0, 1/2]$ then taking $\varepsilon = 1/\sqrt{n}$ and $\delta = 2$ we get

$$R_{\delta\varepsilon n} \leq \frac{C}{n^\alpha} + C\sqrt{n} \int_0^T U_n^{1+2\alpha}(s) ds = CR_n^{(\alpha)}(t) \tag{2.11}$$

for $t \in [0, T]$ for all $n \geq 2$, where C is a constant depending only on K and T . Let $M_n(t)$ denote $M_{\delta\varepsilon n}(t)$ with $\delta = n^{1/3}$ and $\varepsilon = 1/\ln n$ when $\alpha = 0$, and with $\delta = 2$ and $\varepsilon = 1/\sqrt{n}$ when $\alpha \in (0, 1/2]$. Then

M_n is a local martingale, starting from 0, and by virtue of (2.9) it satisfies (2.5). Thus from (2.8), taking into account (2.10)-(2.11), we get (2.4). \square

Proof of Theorem 2.1. Let τ be a stopping time, bounded by T . Then by Proposition 2.2 for

$$Z_n(t) = |Y_n(t \wedge \tau)| = |X - X_n|(t \wedge \tau)$$

we have that almost surely

$$\begin{aligned} Z_n(t) \leq & K \int_0^t Z_n(s) ds + K \int_0^T (U_n(s) + U_n^\gamma(s)) ds \\ & + CR_n^{(\alpha)} + M_n(t \wedge \tau) \end{aligned} \quad (2.12)$$

for all $t \leq T$. By virtue of Lemmas 1.1, 1.2,

$$E \sup_{t \leq T} |X(t) - X_n(t)| < \infty,$$

and by Remark 1.2 for every $\gamma \in (0, 1]$ there is a constant depending only on K, T, γ and $E|\xi|$, such that

$$\sup_{s \leq T} EU_n^\gamma(s) \leq \frac{C}{n^{\gamma/2}}. \quad (2.13)$$

Thus, from (2.12) we see that the local martingale $(M_n(t \wedge \tau))_{t \geq 0}$ has an integrable lower bound, which by Fatou's lemma implies

$$EM_n(t \wedge \tau) \leq 0.$$

Moreover, we have

$$ER_n^{(\alpha)} = \begin{cases} \frac{C}{\ln n} & \text{if } \alpha = 0 \\ \frac{C}{n^\alpha} & \text{if } \alpha \in (0, 1/2], \end{cases}$$

where C is a constant depending only on K, T and $E|\xi|^{1+2\alpha}$. Thus taking expectation on both sides of (2.12) we obtain

$$EZ_n(t) \leq K \int_0^t EZ_n(s) ds + \frac{C}{n^{\gamma/2}} + CER_n^\alpha, \quad t \in [0, T],$$

where C is a constant depending on K, T, γ and $E|\xi|^{1+2\alpha}$. Hence by Gronwall's lemma

$$E|X - X_n|(t \wedge \tau) \leq Ce^{KT} \left(\frac{1}{n^{\gamma/2}} + ER_n^{(\alpha)} \right), \quad \text{for } t \in [0, T],$$

and we can finish the proof by letting $t \rightarrow \infty$ and using Fatou's lemma. \square

Corollary 2.3. *Under the conditions of the previous theorem there is a constant C depending on K, T, γ and $E|\xi|^{1+2\alpha}$ such that for all $n \geq 2$ we have*

$$E \sup_{0 \leq t \leq T} |X(t) - X_n(t)| \leq \begin{cases} \frac{C}{\ln^{1/2} n} & \text{if } \alpha = 0 \\ C \left(n^{-2\alpha^2} + n^{-\alpha\gamma} \right) & \text{if } \alpha \in (0, 1/2]. \end{cases} \quad (2.14)$$

Moreover, for each $0 < \delta < 1$ we have

$$E \sup_{0 \leq t \leq T} |X_n(t) - X(t)|^\delta \leq \begin{cases} C_\delta \frac{1}{\ln^\delta n} & \text{if } \alpha = 0 \\ C_\delta \left(n^{-\delta\alpha} + n^{-\delta\gamma/2} \right) & \text{if } \alpha \in (0, 1/2], \end{cases} \quad (2.15)$$

where C_δ is a constant depending on δ, K, T, γ and $E|\xi|^{1+2\alpha}$.

Proof. We use the notation given in (2.3). By Proposition 2.2 for

$$Z_n^*(t) := \sup_{0 \leq s \leq t} |X(s) - X_n(s)|$$

we get that almost surely

$$\begin{aligned} Z_n^*(t) &\leq K \int_0^t Z_n^*(s) ds + KT \int_0^T (U_n(s) + U_n^\gamma(s)) ds \\ &\quad + CR_n^{(\alpha)} + \sup_{s \leq t} |M_n(s)|, \end{aligned} \quad (2.16)$$

for all $t \in [0, T]$. Using (2.5), by Davis's inequality we have

$$E \sup_{s \leq t} |M_n(s)| \leq 3E\langle M_n \rangle^{1/2}(t) \leq 5K(EA_n(t) + EB_n(T)), \quad (2.17)$$

where

$$\begin{aligned} A_n(t) &:= \left(\int_0^t |Y_n|^{1+2\alpha}(s) ds \right)^{1/2}, \\ B_n(t) &:= \left(\int_0^t |X_n - X_n(\kappa_n)|^{1+2\alpha}(s) ds \right)^{1/2}. \end{aligned}$$

By Jensen's inequality and by Remark 1.2 we have

$$EB_n(T) \leq C \left(\int_0^T E|X_n - X_n(\kappa_n)|^{1+2\alpha}(s) ds \right)^{1/2} \leq Cn^{-\frac{\alpha}{2} - \frac{1}{4}}. \quad (2.18)$$

If $\alpha = 0$ then by Jensen's inequality and the previous theorem we get

$$EA_n(t) \leq C \left(\int_0^T E|Y_n(t)| dt \right)^{1/2} \leq \frac{C}{\ln^{1/2} n},$$

which by virtue of (2.16)-(2.18) proves (2.14) for $\alpha = 0$. If $\alpha \in (0, 1/2]$ then by Young's and Jensen's inequalities and by the previous theorem we have

$$\begin{aligned} EA_n(t) &\leq E \left((Z_n^*(t))^{1/2} \int_0^t |Y_n|^{2\alpha}(s) ds \right)^{1/2} \\ &\leq \frac{1}{10K} EZ_n^*(t) + C \int_0^T E|Y_n(s)|^{2\alpha} ds \\ &\leq \frac{1}{10K} EZ_n^*(t) + C(n^{-2\alpha^2} + n^{-\alpha\gamma}). \end{aligned} \quad (2.19)$$

Note that for $0 \leq \alpha \leq 1/2$ one has $2\alpha^2 \leq \min(\alpha/2 + 1/4, \alpha)$. Thus from (2.16) by (2.17), (2.19) and (2.18) we get

$$Z_n^*(t) \leq 2K \int_0^t EZ_n^*(s) ds + C(n^{-2\alpha^2} + n^{-\alpha\gamma}), \quad \text{for } t \in [0, T],$$

with a constant C , which proves (2.14) by Gronwall's lemma. For the second statement, notice that, owing to Theorem 2.1, we may apply Lemma 3.2 of [8] (see also Theorem 8 on p. 108 of [14]), which yields (2.15). \square

Remark 2.2. We thus obtained a weaker convergence rate in the uniform norm which we could not improve. Notice that if $g = 0$ (i.e. Lipschitz-continuous drift) and $\alpha = 1/2$ (i.e. Lipschitz-continuous diffusion coefficient) then the ‘‘canonical’’ rate $n^{-1/2} = n^{-2\alpha^2}$ is established in Corollary 2.3.

3. ESTIMATES OF MOMENTS

Theorem 3.1. *Let Assumption 1.1 hold. Let $p \geq 2$ and assume that $E|\xi|^p < \infty$. Then the following estimates hold for all integers $n \geq 2$.*

(i) *If $\alpha = 0$ in Assumption 1.1 then*

$$E \sup_{s \leq T} |X(s) - X_n(s)|^p \leq C \frac{1}{\ln n}, \quad (3.1)$$

where C is a constant depending only on K, T, p and $E|\xi|^p$.

(ii) *If $\alpha \in (0, 1/2)$ then*

$$E \sup_{s \leq T} |X(s) - X_n(s)|^p \leq C \left(\frac{1}{n^\alpha} + \frac{1}{n^{\gamma/2}} \right),$$

where C depends only on K, T, p, α, γ and $E|\xi|^p$.

(iii) *If $\alpha = 1/2$ then*

$$E \sup_{s \leq T} |X(s) - X_n(s)|^p \leq C \left(\frac{1}{n^{p/2}} + \frac{1}{n^{\gamma p/2}} \right),$$

where C depends on K, T, p, γ and $E|\xi|^p$.

To prove this theorem we need the following lemma.

Lemma 3.2. *Let $(Z(t))_{t \geq 0}$ be a nonnegative stochastic process and set $V(t) = \sup_{s \leq t} Z(s)$. Assume that for some $p > 0, q \geq 1, \rho \in [1, q]$ and constants K and $\delta \geq 0$*

$$EV^p(t) \leq KE \left(\int_0^t V(s) ds \right)^p + KE \left(\int_0^t Z^\rho(s) ds \right)^{p/q} + \delta < \infty \quad (3.2)$$

for all $t \geq 0$. Then for each $T \geq 0$ the following statements hold.

(i) If $\rho = q$ then there is a constant C_T such that

$$EV^p(T) \leq C_T \delta. \quad (3.3)$$

The constant C_T depends only on K, p, q and T . It increases in T .

(ii) If $p \geq q$ or both $\rho < q$ and $p > q + 1 - \rho$ hold, then there exist constants C_1 and C_2 , depending on K, T, ρ, q and p , such that

$$EV^p(T) \leq C_1 \delta + C_2 \int_0^T EZ(s) ds. \quad (3.4)$$

Proof. To prove (i) notice that for $t \in [0, T]$

$$\left(\int_0^t Z(s) ds \right)^p \leq \frac{1}{4K} V^p(t) + C \int_0^t Z^p(s) ds \quad (3.5)$$

where C depends only on p, K and T . Indeed, if $p \geq 1$ then this follows immediately by Jensen's inequality, and if $p \in (0, 1)$ then

$$\left(\int_0^t Z(s) ds \right)^p \leq V^{p(1-p)}(t) \left(\int_0^t Z^p(s) ds \right)^p,$$

and (3.5) follows by Young's inequality. Using (3.5) with Z^q and p/q in place of Z and p , respectively, we get

$$\left(\int_0^t Z^q(s) ds \right)^{p/q} \leq \frac{1}{4K} V^p(t) + C \int_0^t Z^p(s) ds \quad (3.6)$$

with a constant depending on p, T, q and K . Using (3.5), with V in place of Z , and (3.6), from (3.2) we obtain

$$EV^p(t) \leq C \int_0^t EV^p(s) ds + 2\delta, \quad \text{for } t \leq T,$$

with a constant C depending on T, K, q and p , which implies (i) by Gronwall's lemma.

To prove (ii) we show that

$$\left(\int_0^t Z^\rho(s) ds \right)^{p/q} \leq \frac{1}{4K} V^p(t) + C \int_0^t Z(s) ds + C \int_0^t V^p(s) ds \quad (3.7)$$

with a constant C depending only on K, T, ρ, q and p . If $q + 1 - \rho < p < q$ then by Young's inequality

$$\begin{aligned} \left(\int_0^t Z^\rho(s) ds \right)^{p/q} &\leq V^{(q-p)p/q}(t) \left(\int_0^t Z^{\rho-(q-p)}(s) ds \right)^{p/q} \\ &\leq \frac{1}{4K} V^p(t) + C \int_0^t Z^{\rho-(q-p)}(s) ds, \end{aligned}$$

and

$$\begin{aligned} \int_0^t Z^{\rho-(q-p)}(s) ds &\leq \int_0^t Z^{\frac{q-p}{p-1}} V^{p(1-\frac{q-p}{p-1})}(s) ds \\ &\leq C_1 \int_0^t Z(s) ds + C_2 \int_0^t V^p(s) ds, \end{aligned}$$

where C is a constant depending on K, q and p , and C_1, C_2 are constants, depending on p, q and ρ . Hence (3.7) clearly follows when $q + 1 - \rho < p < q$.

If $p \geq q$ then by Jensen's and Young's inequalities for $t \in [0, T]$ we have

$$\begin{aligned} \left(\int_0^t Z^\rho(s) ds \right)^{p/q} &\leq T^{\frac{p-q}{q}} \int_0^t Z^{\rho p/q}(s) ds \\ &\leq T^{\frac{p-q}{q}} \int_0^t Z^{\frac{p}{p-1}(1-\frac{\rho}{q})} V^{\frac{p}{p-1}(\frac{p\rho}{q}-1)}(s) ds \\ &\leq C \int_0^t Z(s) ds + C \int_0^t V^p(s) ds \quad (3.8) \end{aligned}$$

with a constant $C = C(T, \rho, p, q)$, which finishes the proof of (3.7). By (3.5) (with V in place of Z) and (3.7), from (3.2) we have

$$EV^p(t) \leq CE \int_0^t V^p(s) ds + 2\delta + CE \int_0^T Z(s) ds < \infty,$$

which gives (3.4) by Gronwall's lemma. By a quick inspection we see that we can take the constants above increasing in T . \square

Proof of Theorem 3.1. Set $U_n(t) = |X_n(t) - X_n(\kappa_n(t))|$ and

$$Z_n(t) = |X(t) - X_n(t)|, \quad V_n(t) = \sup_{s \leq t} Z_n(t).$$

To prove (i) note that by Proposition 2.2 for any $p \geq 1$,

$$\begin{aligned} EV_n^p(t) &\leq C' E \left(\int_0^t V_n(s) ds \right)^p + \frac{C}{\ln^p n} \\ &\quad + 3^{p-1} E \sup_{s \leq t} |M_n(s)|^p, \end{aligned} \quad (3.9)$$

where C' depends only on K and p ; C depends on K, T, γ, p and $E|\xi|$, and M_n is a continuous local martingale, such that $M(0) = 0$ and (2.5) holds. By the Davis-Burkholder-Gundy inequality, by (2.5) and Remark 1.2

$$\begin{aligned} E \sup_{s \leq t} |M_n(s)|^p &\leq C(p) E \langle M_n \rangle^{p/2}(t) \\ &\leq KC(p) E \left(\int_0^t Z_n(s) ds \right)^{p/2} + Cn^{-p/4}. \end{aligned} \quad (3.10)$$

Thus from inequality (3.9) we have a constant C such that for all $n \geq 2$

$$EV_n^p(t) \leq CE \left(\int_0^t V_n(s) ds \right)^p + CE \left(\int_0^t Z_n(s) ds \right)^{p/2} + \frac{C}{\ln^p n} < \infty$$

for $t \in [0, T]$. Hence, using Theorem 2.1 we can easily obtain estimate (3.1) for $p = 2$ and can get it also for $p > 2$ by using part (ii) of Lemma 3.2 (with $q = 2$).

To prove (ii) and (iii) we can use Proposition 2.2 to get, by similar arguments as before, that for $p > 0$

$$EV_n^p(t) \leq CE \left(\int_0^t V_n(s) ds \right)^p + CE \left(\int_0^t Z_n^{1+2\alpha}(s) ds \right)^{p/2} + C\delta_n < \infty$$

for all $t \in [0, T]$, with

$$\delta_n = (n^{-p/2} + n^{-\alpha p} + n^{-\gamma p/2} + n^{-p(1+2\alpha)/4}).$$

If $\alpha \in (0, 1/2)$ then hence we get (ii) by using part (ii) of Lemma 3.2, and when $\alpha = 1/2$ then we obtain (iii) by using part (i) of Lemma 3.2 (with $q = 2$). \square

Remark 3.1. If a solution of an SDE does not leave a domain D , where the coefficients are locally Lipschitz, then Euler's approximations almost surely converge to this solution with any order of accuracy $\delta < 1/2$ (see [9]). Moreover, if the coefficients are smooth in D then higher order schemes can also be used, and higher order almost sure convergence can be obtained (see [11]). In particular, it follows from Theorem 2.4 in [9] that if $2a \geq \sigma^2$ then the solution $(X_t)_{t \geq 0}$ of equation (1.1) is

positive for all t , and for every $\delta < 1/2$ there exists a finite random variable η such that almost surely

$$\sup_{t \leq T} |X(t) - X_n(t)| \leq \eta n^{-\delta} \quad \text{for all } n \geq 1,$$

where X_n are the Euler approximations.

Remark 3.2. One can easily get Lemmas 1.1 and 1.2 by applying part (i) of Lemma 3.2. To prove Lemma 1.1, note that for $\bar{X}(t) := X(t) - X(0)$ we have

$$|\bar{X}(t)| \leq \int_0^t K(1 + |\xi| + |\bar{X}(s)|) ds + \left| \int_0^t \sigma(s, \xi + \bar{X}(s)) dW(s) \right|$$

Set

$$\tau_k := \inf\{t \in [0, T] : |\bar{X}(t)| \geq k\} \quad (3.11)$$

for $k > 0$. Then using the linear growth condition and the Burkholder-Gundy-Davis inequality, for $Z_k(t) = |\bar{X}(t \wedge \tau_k)|$, $t \in [0, T]$, we have

$$\begin{aligned} E \sup_{s \leq t} Z_k^p(s) &\leq C(1 + E|\xi|^p)t^p + CE \left(\int_0^t Z_k(s) ds \right)^p \\ &\quad + CE \left(\int_0^t \mathbf{1}_{\tau_n > 0} (1 + |\xi|^2 + Z_k^2(s)) ds \right)^{p/2} \\ &\leq C'(1 + E|\xi|^p)(t^p + t^{p/2}) + C'E \left(\int_0^t Z_k(s) ds \right)^p \\ &\quad + C'E \left(\int_0^t Z_k^2(s) ds \right)^{p/2} < \infty \end{aligned}$$

Hence using part (i) of Lemma 3.2 with $q = 2$ and with each $t \leq T$ in place of T , we have for all $t \in [0, T]$

$$\begin{aligned} E \sup_{s \leq t} |X(s \wedge \tau_k) - X(0)|^p &\leq C_t(1 + E|\xi|^p)(t^p + t^{p/2}) \\ &\leq C_T(1 + E|\xi|^p)(t^p + t^{p/2}) < \infty, \end{aligned}$$

where the constant C_t depends only on M , t and p , and it is an increasing function of t . Hence $\tau_k \rightarrow \infty$ as $k \rightarrow \infty$, and letting $k \rightarrow \infty$ we obtain (1.5).

We can prove Lemma 1.2 similarly. Set $Z_{nk}(t) = |X_n(t \wedge \tau_k)|$, where X_n denotes the Euler approximation defined by (1.3), and τ_k is defined in (3.11) with $X_n(t) - X_n(0)$ in place of $\bar{X}(t)$. Then proceeding in the

same way as before we get

$$\begin{aligned} E \sup_{s \leq t} Z_{nk}^p(s) &\leq C(E|\xi|^p + 1) + CE \left(\int_0^t \sup_{r \leq s} Z_{nk}(r) ds \right)^p \\ &\quad + CE \left(\int_0^t \sup_{r \leq s} Z_{nk}^2(r) ds \right)^{p/2} < \infty. \end{aligned}$$

Hence we can finish the proof of Lemma 1.2 by applying part (i) of Lemma 3.2 with $q = 2$, and then letting $k \rightarrow \infty$ as before.

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