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Query Preserving Graph Compression

Wenfei Fan\textsuperscript{1,2} Jianzhong Li\textsuperscript{2} Xin Wang\textsuperscript{1} Yinghui Wu\textsuperscript{1,3}
\textsuperscript{1}University of Edinburgh \textsuperscript{2}Harbin Institute of Technology \textsuperscript{3}UC Santa Barbara
\{wenfei@inf, x.wang-36@sms, y.wu-18@sms\}.ed.ac.uk lizh@hit.edu.cn

ABSTRACT

It is common to find graphs with millions of nodes and billions of edges in, \textit{e.g.}, social networks. Queries on such graphs are often prohibitively expensive. These motivate us to propose \textit{query preserving graph compression}, to compress graphs relative to a class $Q$ of queries of users' choice. We compute a small $G_r$ from a graph $G$ such that (a) for \textit{any} query $Q \in Q$, $Q(G) = Q' (G_r)$, where $Q' \in Q$ can be efficiently computed from $Q$; and (b) any algorithm for computing $Q(G)$ can be \textit{directly} applied to evaluating $Q'$ on $G_r$ as is. That is, while we cannot lower the complexity of evaluating graph queries, we reduce data graphs while preserving the answers to all the queries in $Q$. To verify the effectiveness of this approach, (1) we develop compression strategies for two classes of queries: reachability and graph pattern queries via (bounded) simulation. We show that graphs can be efficiently compressed via a reachability equivalence relation and graph bisimulation, respectively, while preserving query answers. (2) We provide techniques for maintaining compressed graph $G_r$ in response to changes $\Delta G$ to the original graph $G$. We show that the incremental maintenance problems are \textit{unbounded} for the two classes of queries, \textit{i.e.}, their costs are not a function of the size of $\Delta G$ and changes in $G_r$. (3) Using real-life data, we experimentally verify that our compression techniques could reduce graphs in average by 95% for reachability and 57% for graph pattern matching, and that our incremental maintenance algorithms are efficient.

1. INTRODUCTION

It is increasingly common to find large graphs in, \textit{e.g.}, social networks [16], Web graphs [29] and recommendation networks [25]. For example, Facebook currently has more than 800 million users with 104 billion links\textsuperscript{1}. It is costly to query such large graphs. Indeed, graph pattern matching takes quadratic time (by simulation [12]) or cubic time (via bounded simulation [9]) to determine whether there exists a match in a data graph for a graph pattern. Worse still, it is \textit{NP}-complete when matching is defined in terms of subgraph isomorphism. Even for reachability queries that are $\textit{NP}$-complete when matching is defined in terms of subgraph isomorphism, it is costly to evaluate such queries in $G_r$ directly.


\textbf{Example 1:} Graph $G$ in Fig. 2 is a fraction of a multi-agent recommendation network. Each node denotes a customer (C), a book server agent (BSA), a music shop agent (MSA), or a facilitator agent (FA) assisting customers to find BSAs and MSAs. Each edge indicates a recommendation. To locate potential buyers, a bookstore owner issues a pattern query $Q_p$ depicted in Fig. 2. It is to find a set of BSAs such that they can reach a set of customers C who interact with a set of FAs, and moreover, the customers should be...

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image.png}
\caption{Compressing a real-life P2P network}
\end{figure}
within 2 hops from the BSAs. One may verify that the match of $Q_p$ in $G$ is a relation $S = \{ (X, X_i) \}$ for $X \in \{ \text{BSA, FA, C} \}$ and $i \in [1, 2]$. It is expensive to compute $S$ when $G$ is large. Among other things, one has to check the connectivity between all the $k$ customers and all the BSAs in $G$.

We can do better. Observe that BSAs and BSAs are of the same type of nodes (BSA), and both make recommendations to MSA and FA. Since they “simulate” the behavior of each other in the recommendation network $G$, they could be considered equivalent when evaluating $Q_p$. Similarly, the pairs $(\text{FA}1, \text{FA}2)$, $(C_1, C_2)$, and any pair $(C_i, C_j)$ of nodes for $i, j \in [3, k]$ can also be considered equivalent, among others.

This suggests that we build a compressed graph $G_r$ of $G$, also shown in Fig. 2. Graph $G_r$ consists of hypernodes $X_r$ for $X \in \{ \text{MSA, BSA, FA, FA’, C, C’} \}$, each denoting a class of equivalent nodes. Observe that (1) $G_r$ has less nodes and edges than $G$, (2) $Q_p$ can be directly evaluated on $G_r$; its result $S_r = \{ (X_r, X_i) \}$ can be converted to the original result $S$ by simply replacing $X_r$ with the set of nodes represented by $X_r$; and (3) the evaluation of $Q_p$ in $G_r$ is more efficient than in $G$, since, among other things, it only needs to check $C_r$ and $C'_r$ in $G_r$ to identify matches for the query node $C$.

One can verify that $G_r$ preserves the result for all pattern queries defined in terms of (bounded) simulation, not limited to $Q_p$. That is, for any such pattern query $Q$ on $G$, we can directly evaluate $Q$ on the much smaller $G_r$, instead. □

Contributions. Our main contributions are as follows.

1. We propose query preserving compression for querying large real-life graphs (Section 2). As opposed to previous graph compression strategies, it only preserves information needed for answering queries in a particular class $Q$ of users’ interest, and hence, achieves a better compression ratio. It is not yet another algorithm for evaluating graph queries; instead, any algorithms for evaluating queries of $Q$ on the original graphs can be directly applied to computing query answers in the compressed graphs, without decompression.

To verify the effectiveness of this approach, we develop query preserving compression strategies for two classes of queries commonly used in practice, namely, reachability queries and graph pattern queries via (bounded) simulation.

2. For reachability queries, we introduce reachability preserving compression (Section 3). We propose a notion of reachability equivalence relations, and based on this, we provide a compression function $R$ that, given a graph $G$, computes a small graph $G_r = R(G)$ in $O(|V||E|)$ time, where $|V|$ and $|E|$ are the number of the nodes and edges in $G$, respectively. We show that $G_r$ is reachability preserving: for any reachability query $Q$, one can find in constant time another reachability query $Q'$ such that $Q(G) = Q'(G_r)$.

3. For graph pattern queries defined in terms of (bounded) simulation $[9, 12]$, we define graph pattern preserving compression in terms of a bisimulation equivalence relation $[8]$ (Section 4). We show that graphs $G$ can be compressed into a smaller $G_r$ in $O(|E| \log |V|)$ time. We also show that the compression preserves pattern queries: for any graph pattern $Q$, $G$ matches $Q$ if and only if $G_r$ matches the same $Q$, and moreover, the match of $Q$ in $G$ can be computed in $G_r$.

4. Real-life graphs constantly change $[16]$. This highlights the need for studying incremental query preserving compression. Given a graph $G$, its compression $G_r$ via function $R$, and updates $\Delta G$ to $G$, it is to compute changes $\Delta G_r$ to $G_r$ such that $R(G_r \oplus \Delta G_r) = G_r \oplus \Delta G_r$, where $G_r \oplus \Delta G_r$ denotes $G_r$ updated by $\Delta G$. When $\Delta G$ is small as commonly found in practice, $\Delta G_r$ tends to be small as well and hence, is more efficient to find than recomputing $R(G_r \oplus \Delta G_r)$ starting from scratch. This allows us to compute compression $G_r$ once, and incrementally maintain it in response to changes to $G$.

We study this issue for reachability queries and graph pattern queries (Section 5). (a) We provide a complexity analysis for the problem, in terms of the size of the changes in the input ($\Delta G$) and output ($\Delta G_r$) characterized by affected area (AFF). We show that the problem is unbounded for both classes of queries, i.e., its cost is not a function of $|AFF|$. (b) Nevertheless, we develop incremental maintenance algorithms: (i) for reachability preserving compression, we show that compressed graphs can be maintained in $O(|AFF| \oplus |G_r|)$ time; and (ii) for graph pattern queries, we incrementally compress graphs in $O(|AFF|^2 + |G_r|)$ time. In both cases the algorithms are independent of the original graph $G$, and propagate changes without decompressing $G_r$.

5. We experimentally verify the effectiveness and efficiency of our (incremental) compression techniques using synthetic data and real-life data. We find that query preserving compression reduces the size of real-life graphs by 95% and 57% in average for reachability and pattern queries, respectively, and by 98% and 59%, respectively, for social networks. These lead to a reduction of 94% and 70% in query evaluation time, respectively. In addition, our incremental compression algorithms for reachability queries outperform their batch counterparts when changes are up to 20%.

We contend that query preserving compression yields a promising approach to querying real-life graphs. This work is among the first efforts to provide a complete package for query preserving compression, from complexity bounds to compression algorithms to incremental maintenance.

Related work. We categorize related work as follows.

General graph compression. Graph compression has been studied for e.g., Web graphs and social networks $[2, 5, 27]$. The idea is to encode a graph or its transitive closure into compact data structures via node ordering determined by, e.g., lexicographic URL and host $[27]$, linkage similarity $[3]$, and document similarity $[5]$. These general methods preserve the information of the entire graph, and highly depend on extrinsic information, coding mechanisms and application domains $[2]$. To overcome the limitations, $[2]$ proposes a compression-friendly node ordering but stops short of giving a compression strategy. Our work differs from these in the following: (a) our compression techniques rely only on intrinsic graph information that is relevant to a specific class of queries; (b) our compressed graphs can be directly queried without decompression; in contrast, even to answer simple queries, previous work requires the original graph to be re-
stored from compact structures [5], as observed in [2]; and (c) we provide efficient incremental maintenance algorithms.

Query-friendly compression. Closer to our work are compression methods developed for specific classes of queries.

(1) Neighborhood queries [18, 22, 27], to find nodes connected to a designated node in a graph. The idea of query-able compression (querying without decomposition) for such queries is advocated in [18], which adopts compressed data structures by exploiting Eulerian paths and multi-position linearization. A S-node representation is introduced in [27] for answering neighborhood queries on Web graphs. Graph summarization [22] aims to sketch graphs with small subgraphs and construct hypergraph abstraction. These methods construct compact data structures that have to be (partially) decompressed to answer the queries [2]. Moreover, the query evaluation algorithms on original graphs have to be modified to answer queries in their compact structures.

(2) Reachability queries [1, 10, 21, 32]. To answer such queries, [21] computes the minimum subgraphs with the same transitive closure as the original graphs, and [1] reduces graphs by substituting a simple cycle for each strongly connected component. These methods allow reachability queries to be evaluated on compressed graphs without decomposition. We show in Section 3 (and verify in Section 6) that our method achieves a better compression ratio, because (1) our compressed graphs do not have to be subgraphs of the original graphs, and (2) by merging nodes into hypernodes, we can further reduce edges. Bipartite compression [10] reduces graphs by introducing dummy nodes and compressing bicliques. However, (1) its compression is a bijection between graphs and their compressed graphs, such that they can be converted to each other. In contrast, we do not require that the original graphs can be restored; and (2) algorithms for reachability queries have to be modified before they can be applied to their compressed graphs [10]. [32] computes a compressed bit vector to encode the transitive closure of a graph. In contrast, we compute compressed graphs on which reachability algorithms and the compression scheme in [32] can be directly applied. The incremental maintenance of the bit vectors is not addressed in [32].

(3) Path queries [4]. There has also been work on compressing XML trees via bisimulation, to evaluate XPath queries. It is shown there that this may lead to exponential reduction, an observation that carries over to our setting. In contrast to [4], we consider compressing general graphs, to answer graph-structured queries rather than XPath. Moreover, we develop incremental techniques to maintain compressed graphs, which are not studied in [4].

We are not aware of any previous work on compressing graphs for answering graph pattern queries.

Graph indexing. There has been a host of work on building indexes on graphs to improve the query time [6, 11, 13–15, 19, 26, 34]. (1) 2-hop [6], PathTree [14], 3-hop [13], GRAIL [34] and HLSS [11] are developed for answering reachability queries. However, (a) these indexes come with high costs. For example, the construction time is biquadratic for 2-hop and 3-hop, cubic for HLSS, and quadratic for GRAIL and PathTree; the space costs of these indexes are all (near) quadratic [11, 32, 34, 34, 35]; and maintenance for 2-hop index easily degrades into recomputation [35]. (b) The algorithms for reachability queries on original graphs often do not run on these indexes. For example, it requires extra search or auxiliary data structures to answer the queries involving nodes that are not covered by PathTree [14, 32]. In contrast, all these algorithms can be directly applied to our compressed graphs. (2) 1-index [19], A(k)-index [15] and their generalization D(k)-index [26] yield index graphs as structure summarizations based on (parameterized) graph bisimulation. However, (a) only rooted graphs are considered for those indexes; and (b) these indexes are for regular path queries, instead of graph patterns and reachability queries. Indeed, none of these indexes preserves query results for reachability queries (shown in Section 3), and neither A(k)-index nor D(k)-index preserves query results for graph pattern queries (shown in Section 4); (c) these indexes are only accurate for those queries satisfying certain query load constraints (e.g., query templates [19], path lengths [15, 26]); in contrast, we compute compressed graphs that preserve results for all queries in a given query class; and (d) Incremental maintenance is not studied for 1-index and A(k)-index [15, 19]. The issue is addressed in [26], but the technique there depends on the query load constraints.

Incremental bisimulation. We use graph bisimulation to compress graphs for pattern queries. A bisimulation computation algorithm is given in [8]. Incremental computation of bisimulation for single edge insertions is studied in [7, 30]. Our work differs from those in (1) that we give complexity bounds (boundedness and unboundedness results) of incremental pattern preserving compression, of which incremental bisimulation is a subproblem, and (2) that we propose algorithms for batch updates instead of single updates.

2. PRELIMINARY

Below we first review graphs and graph queries. We then introduce the notion of query preserving graph compression.

2.1 Data Graphs and Graph Queries

Graphs. A labeled (directed) graph \( G = (V, E, L) \) consists of (1) a set \( V \) of nodes; (2) a set of edges \( E \subseteq V \times V \), where \((v, w) \in E\) denotes a directed edge from node \( v \) to \( w \); and (3) a function \( L \) defined on \( V \) such that for each node \( v \in V \), \( L(v) \) is a label from a set \( \Sigma \) of labels. Intuitively, the node labels may present e.g., keywords, social roles, ratings [16].

We use the following notations. A path \( \rho \) from node \( v \) to \( w \) in \( G \) is a sequence of nodes \((v = v_0, v_1, \ldots, v_n = w)\) such that for every \( i \in [1, n] \), \((v_{i-1}, v_i) \in E\). The length of path \( \rho \), denoted by \( \text{len}(\rho) \), is \( n \), i.e., the number of edges in \( \rho \). A path \( \rho \) is said to be nonempty if \( \text{len}(\rho) \geq 1 \). A node \( v \) can reach \( w \) (or \( w \) is reachable from \( v \)) if and only if (iff) there exists a path from \( v \) to \( w \) in \( G \). The distance between node \( v \) and \( w \) is the length of the shortest paths from \( v \) to \( w \).

Graph queries. In general, a graph query is a computable function from a graph to another object, e.g., a Boolean value, a graph, a relation, etc. It is independent of how the input data graphs are represented and therefore, ask for certain intrinsic properties of the graphs. In this paper, we consider two classes of queries commonly used in practice.

Reachability queries. A reachability query on a graph \( G \), denoted by \( Q_R(v, w) \), is a Boolean query that asks whether node \( v \) can reach node \( w \) in \( G \). For instance, \( Q_R(\text{BSA}_1, \text{FA}_2) \) is a reachability query on graph \( G \) of Fig. 2: the answer to the query is true, as there is a path from \( \text{BSA}_1 \) to \( \text{FA}_2 \) in \( G \).
We define graph pattern matching in terms of bounded simulation [9]. A graph pattern query is defined as \( Q_p = (V_p, E_p, \rho_p, f_p) \), where (1) \( (V_p, E_p, f_p) \) is a directed graph as defined above; and (2) \( f_p \) is a function defined on \( E_p \) such that for each edge \((u, u')\), \( f_p(u, u')\) is either a positive integer \( k \) or a symbol \( * \), called the bound of \((u, u')\).

A graph \( G = (V, E, L) \) matches \( Q_p \), denoted by \( G \prec Q_p \), if there exists a binary relation \( S \subseteq V \times V \) such that: (1) for each \( u \in V_p \), there exists \( v \in V \) such that \((u, v) \in S \); (2) for each \((u, v) \in S \), \( f_p(u, v) = L(v) \), and (b) for each edge \((u, u') \) in \( E_p \), there exists a nonempty path \( \rho \) from \( v \) to \( v' \) in \( G \) such that \((u', v') \in S \), and \( \text{len}(\rho) \leq k \) if \( f_p(u, u') = k \) is a constant \( k \). We refer to \( S \) as a match for \( P \) in \( G \).

Intuitively, \((u, v) \in S\) if (1) node \( v \) in \( G \) satisfies the search condition specified by \( f_p(u) \) in \( Q_p \), and (2) each edge \((u, u')\) in \( E_p \) is mapped to a nonempty path \( \rho = (v, \ldots, v') \) in \( G \), such that \((u', v')\) is also in the match \( S \), and moreover, \( \text{len}(\rho) \) is bounded by \( k \) if \( f_p(u, u') = k \). If \( f_p(u, u') = * \), \( \text{len}(\rho) \) is not bounded. Observe that the child \( u' \) of \( u \) is mapped to a descendant \( v' \) of \( v \) via \( S \). For instance, relation \( S \) given in Example 1 is a match in graph \( G \) for pattern \( P \) of Fig. 2.

It has been shown [9] that there exists a unique maximum match \( S_M \) in \( G \) for \( Q_p \), if \( Q_p \preceq G \), i.e., for any match \( S \) in \( G \) for \( P \), \( S \subseteq S_M \). The answer to \( Q_p \) in \( G \) is defined as the maximum match \( S_M \) if \( Q_p \subseteq G \), and as \( \emptyset \) otherwise.

**Lemma 1** [9]: For any graph \( G \) and pattern \( Q_p \), if \( Q_p \preceq G \), then there is a unique maximum match in \( G \) for \( P \). □

There are two special cases of graph pattern queries. (1) A Boolean pattern query \( Q_p \) returns true if \( Q_p \preceq G \), and false otherwise. (2) A pattern query \( Q_p \) via graph simulation [12] is a query in which \( f_p(u, u') = 1 \) for each edge \((u, u') \in E_p \) of \( Q_p \), i.e., it maps edges in \( Q_p \) to edges in a data graph.

### 2.2 Query Preserving Graph Compression

For a class \( Q \) of queries, a query preserving graph compression is a triple \( <R, F, P> \), where \( R \) is a compression function, \( F : Q \rightarrow Q \) is a query rewriting function, and \( P \) is a post-processing function. For any graph \( G, G_r = R(G) \) is a graph computed from \( G \) by \( R \), referred to as the compressed graph of \( G \), such that \(|G_r| \leq |G| \), and for any query \( Q \in Q \),

- \( Q(G) = P(Q(G_r)) \), and
- any algorithm for evaluating \( Q \) queries can be directly used to compute \( Q(G_r) \), without decompressing \( G_r \). Here \( Q' = F(Q) \), \( Q(G) \) is the answer to \( Q \) in \( G \), \( Q'(G_r) \) is the answer to \( Q' \) in \( G_r \), and \( P(Q'(G_r)) \) is the result of post-processing the answer to \( Q' \) in the compressed \( G_r \).

As shown in Fig. 3(a), (1) for any query \( Q \in Q \), the answer \( Q(G) \) to \( Q \) in \( G \) can be computed by evaluating \( Q' \) in the (smaller) compressed graph \( G_r \) of \( G \); (2) the compression is generic: any data structures and indexing techniques for the original graph can be directly applied to \( G_r \) (e.g., the 2-hop technique of [6], see Section 6); (3) the post-processing function finds the answer in the original \( G \) by only accessing the query answer \( Q'(G_r) \) and an index on the inverse of node mappings of \( R \); (4) in contrast to generic lossless compression schemes (e.g., [10]), we do not need to restore the original graph \( G \) from \( G_r \), and moreover, the compressed graph \( G_r \) is not necessarily a subgraph of \( G \).

For instance, a query preserving compression for graph pattern queries is described in Example 1, where the compression function \( R \) groups nodes into hypernodes based on graph bisimulation; the query rewriting function \( F \) is the identity mapping: for any pattern query \( Q \), \( F(Q) = Q \); and the post-processing function \( P \) simply replaces each hypernode with the set of equivalent nodes it represents.

In Sections 3 and 4, we show that there exist query preserving compressions with efficient \( R, F \) and \( P \) functions.

(1) For reachability queries, \( R \) reduces graph \( G \) by 95% in average, in \( O(|V||E|) \) time; and \( F \) is in \( O(1) \) time. Moreover, as shown in Fig. 3(b), post-processing \( P \) is not needed at all.

(2) For pattern queries, \( R \) reduces the size of \( G \) by 57% in average, in \( O(|E| \log |V|) \) time; \( F \) is the identity mapping, and \( P \) is in linear time in the size of the query answer, a cost necessary for any evaluation algorithm (see Fig. 3(c)). Better still, for Boolean pattern queries, \( P \) is no longer needed.

We remark that for each graph \( G \), its compression \( G_r = R(G) \) is computed once for all queries \( Q \) in \( Q \), and is incrementally maintained in response to updates to \( G \) (Section 5).

### 3. COMPRESSION FOR REACHABILITY

In this section we study query preserving compression for reachability queries, referred to as reachability preserving compression. The main result of the section is as follows.

**Theorem 2**: There exists a reachability preserving compression \( <R, F> \), where \( R \) is in quadratic time, and \( F \) is in constant time, while no post-processing \( P \) is required. □

As a proof of the theorem, we first define the compression \( <R, F> \) in Section 3.1. We then provide an algorithm for implementing the compression function \( R \) in Section 3.2.

#### 3.1 Reachability Equivalence Relations

Our compression is based on the following notion.

**Reachability equivalence relations**. We first define a reachability relation on a graph \( G = (V, E, L) \) to be a binary relation \( R_e \subseteq V \times V \) such that for each \((u, v) \in R_e \) and any node \( x \in V \), \( x \) can reach \( u \) iff \( x \) can reach \( v \); and (2) \( u \) can reach \( x \) iff \( v \) can reach \( x \). Intuitively, \((u, v) \in R_e \) if and only if they have the same set of ancestors and the same set of descendants. One can readily verify the following.

**Lemma 3**: For any graph \( G \), (1) there is a unique maximum reachability relation \( R_e \) on \( G \), and (2) \( R_e \) is an equivalence relation, i.e., it is reflexive, symmetric and transitive. □

The reachability equivalence relation \( G \) is the maximum reachability relation of \( G \), denoted by \( R_e(G) \) or simply \( R_e \). We denote by \([v]_{R_e} \) the equivalence class containing \( v \).
Reachability preserving compression. Based on reachability equivalence relations we define $< R, F >$ as follows.

1) Compression function $R$. Given $G = (V, E, L)$, we define $R(G) = G_r = (V_r, E_r, L_r)$, where (a) $V_r = \{ [v]^r | v \in V \}$; (b) $E_r$ consists of all edges $([v]^r, [w]^r)$ if there exist nodes $v' \in [v]^r$ and $w' \in [w]^r$ such that $(v', w') \in E$; and (c) for each $u \in V_r$, $L_r(u) = \sigma$, where $\sigma$ is a fixed label in $\Sigma$. Here $R_r(G)$ is the reachability equivalence relation of $G$.

Intuitively, (a) for each node $v \in V_r$ there exists a node $[v]^r \in V_r$; (b) $R_r$ in $R_r(G)$ is a subrelation of $R(G)$; and (c) $R_r(G)$ is fixed to be a symbol $\sigma$ in $\Sigma$ since node labels are irrelevant to reachability queries.

2) Query rewriting function $F$. We define $F$ such that for any reachability query $Q_R(v, w) \in G$, $F(Q_R(v, w)) = Q_r$, where $Q_r = Q_r(R(v), R(w))$ is a reachability query on $G_r$. It simply asks whether there is a path from $[v]^r$ to $[w]^r$ in $G_r$. Using index structures for the equivalence classes of $R_r$, $Q_r$ can be computed from $Q_R$ in constant time.

Correctness. One can easily verify that $< R, F >$ is a reachability preserving compression. Indeed, $|G_r| \leq |G|$ since $|V_r| \leq |V|$ and $|E_r| \leq |E|$. Moreover, for any reachability query $Q_R(v, w)$ posed on $G$, one can show by contradiction that there exists a path from $[v]^r \in V_r$ to $[w]^r \in V_r$ in $G$ if and only if $R(v)$ can reach $R(w)$ in $G_r$. Hence, given $Q_R(v, w)$ on $G$, one can find its answer in $G$ by evaluating $Q_R(R(v), R(w))$ in the smaller compressed graph $G_r$, as shown in Fig. 3(b).

Example 3: Recall graph $G$ of Fig. 2. Using the reachability preserving compression $< R, F >$, one can get $G_r = R(G)$ shown in Fig. 4, in which e.g., $R(C_1) = R(C_2) = R(F(A) = CFA_r)$. Given a reachability query $Q_R(\{S\}, C_1) \in G$, $F(Q_R(v, w)) = Q_r(MB_1, CFA_r)$ on the smaller compressed graph $G_r$. As another example, $G_{r1}$ and $G_{r2}$ in Fig. 4 are the compressed graphs generated by $R$ for $C_1$ and $C_2$ of $G$ in Fig. 4, respectively.

As remarked earlier, there has been work on index graphs based on bisimulation $[15, 19, 26]$. However, such indexes do not preserve reachability. To see this, consider the index graph $G_{r1}$ of $G$ shown in Fig. 4, where $\{C_1, C_2\}$ and $\{E_1, E_2\}$ are bisimilar and thus merged $[19]$. However, $G_{r2}$ cannot be directly queried to answer e.g., $Q_R(\{C_1, E_2\})$ posed on $G_2$, i.e., one cannot find its equivalent reachability query on $G_{r2}$. Indeed, $C_2$ can reach $E_2$ in $G_2$ but $C_1$ does not, while in $G_{r2}$, $C_1$ and $C_2$ are merged into a single node.

3.2 Compression Method for Reachability Queries

We next present an algorithm that, given a graph $G = (V, E, L)$, computes its compressed graph $G_r = R(G)$ based on the compression function $R$ given earlier. The algorithm, denoted as $compress_r$, is shown in Fig. 5.

Given a graph $G$, the algorithm first computes its reachability equivalence relation $R_r$ and the induced partition $Par$ by $R_r$ over the node set $V$ (lines 2-3). Here $R_r$ is found as follows (details omitted): for each node in $V$, it computes its ancestors and descendants, via forward (resp. backward) BFS traversals, respectively; it identifies those nodes with the same ancestors and descendants. After this, for each equivalence class $S \in Par$, it creates a node $r_S$ representing $S$, assigns a fixed label $\sigma$ to $r_S$, and adds $r_S$ to $V_r$ (lines 4-5). It constructs the edge set $E_r$ by connecting nodes $(v, v') \in E$ if $(v, v') \in E$ of $G$, where $v$ and $w$ are in the equivalence classes represented by $S$ and $S'$, respectively, and (2) $v_S$ does not reach $v_{S'}$ via $E_r$ (lines 6-8). Condition (2) assures that $compress_r$ inserts no redundant edges, e.g., if $(v_S, v_{S'})$ and $(v_{S'}, v_{S''})$ are already in $E_r$, then $(v_S, v_{S''})$ is not added to $E_r$. While it is a departure from the reachability equivalence relation $R_r$, it is an optimization without losing reachability information, as noted for transitivity equivalent graphs $[1]$ (lines 6-8). The compressed graph $G_r$ is then constructed and returned (line 9).

Correctness & Complexity. One can verify that the algorithm correctly computes $G_r$ by the definition of $R$ given above. In addition, $compress_r$ is in $O(|V|^2 + |V||E|)$ time. Indeed, $R_r$ and $Par$ can be computed in $O(|V||E|)$ time (lines 2-3). The construction of $G_r$ is in $O(|V|(|V| + |E|))$ time (lines 4-8). This completes the proof of Theorem 2.

Optimizations. Instead of compressing $G$ directly, we first compute its SCC graph $G_{SCC}$, which collapses each strongly connected component into a single node without losing reachability information. We then apply $compress_r$ to $G_{SCC}$, which is often much smaller than $G$ (see Section 6).

Note that $|G_r|$ is much smaller than reachability matrices $[35]$, which take $O(|V|^2)$ space. Further, $G_r$ takes substantially less construction time (quadratic) and space (linear) as opposed to 2-hop indexing $[6]$, which is biquadratic.

4. COMPRESSION FOR GRAPH PATTERN

We next present a query preserving compression for graph pattern queries, referred to as graph pattern preserving compression. The main result of the section is as follows.

Theorem 4: There exists a graph pattern preserving compression $< R, F, P >$ in which for any graph $G = (V, E, L)$, $R$ is in $O(|E| \log |V|)$ time, $F$ is the identity mapping, and $P$ is in linear time in the size of the query answer.
Query rewriting function

4.1 Compressing Graphs via Bisimilarity

We construct a graph pattern preserving compression in terms of bisimulation relations, which are defined as follows.

Bisimulation relations [8]. A bisimulation relation on a graph $G = (V, E, L)$ is a binary relation $B \subseteq V \times V$, such that for each $(u, v) \in B$, (1) $L(u) = L(v)$; (2) for each edge $(u, u') \in E$, there exists an edge $(v, v') \in E$, such that $(u', v') \in B$; and (3) for each edge $(v, v') \in E$, there exists an edge $(u, u') \in E$ such that $(u', v') \in B$.

Intuitively, $(u, v) \in B$ if and only if for each child $u'$ of $u$ there exists an child $v'$ of $v$ such that $(u', v') \in B$, and vice versa. Similar to Lemma 3, one can verify the following.

Lemma 5: For any graph $G$, (1) there is a unique maximum bisimulation relation $R_b$ on $G$, and (2) $R_b$ is an equivalence relation, i.e., it is reflexive, symmetric and transitive.

We define the bisimulation equivalence relation $R_t$ to be the maximum bisimulation relation of $G$, denoted by $R_t(G)$ or simply $R_t$. We denote by $[v]_{R_t}$ the equivalence class containing node $v$. We say that nodes $u$ and $v$ are bisimilar if $(u, v) \in R_t$. Since for any nodes $v$ and $v'$ in $[v]_{R_t}$, $L(v') = L(v)$, we simply call $L(v)$ the label of $[v]_{R_t}$.

Example 4: Recall the graph $G$ given in Fig. 2. One can verify that $FA_3$ and $FA_4$ are bisimilar. In contrast, $FA_2$ and $FA_3$ are not bisimilar; indeed, $FA_2$ has a child $C_2$, which is not bisimilar to any $C$ child of $FA_3$.

Consider graphs given in Fig. 6. Note that $A_1$ and $A_2$ in $G_1$ are not bisimilar as there is no child of $A_1$ bisimilar to child $B_2$ or $B_3$ of $A_2$. Similarly, $A_1$ and $A_3$ in $G_1$ are not bisimilar. In contrast, $A_2$ and $A_3$ in $G_2$ are bisimilar.

Note that $A_2$ and $A_3$ in $G_2$ are not bisimilar, but they are in the same reachability equivalence class; while $A_2$ and $A_3$ are bisimilar, they are not reachability equivalent. This illustrates the difference between the reachability equivalence relation and the bisimulation equivalence relation.

Graph pattern preserving compression. Based on bisimulation equivalence relations, we define $<R, F, P>$. P

(1) Compression function $R$. Given a graph $G = (V, E, L)$, we define $R(G) = G_b = (V_b, E_b, L_b)$, where (a) $V_b = \{[v]_{R_t} | v \in V\}$; (b) an edge $([v]_{R_t}, [w]_{R_t})$ in $E_b$ as long as there exist nodes $v', w' \in [v]_{R_t}$ such that $(v', w') \in E$, and (c) for each $[v]_{R_t} \in V_b$, $L_b([v]_{R_t})$ is its label $L(v)$. Intuitively, (a) for each node $v \in V$, there exists a node $[v]_{R_t} \in V_b$; (b) for each edge $(v, w) \in E$, $([v]_{R_t}, [w]_{R_t})$ is an edge in $E_b$; and (c) each $[v]_{R_t}$ has the same label as $L(v)$.

(2) Query rewriting function $F$ is simply the identity mapping, i.e., $F(Q_p) = Q_p$.

(3) Post processing function $P$. Recall that $Q_p(G)$ is the maximum match in $G$ for pattern $Q_p$. We define $P$ such that $P(Q_p(G)) = Q_p(G)$ as follows. For each $(v_p, [v]_{R_t}) \in Q_p(G)$ and each $v' \in [v]_{R_t}$, $(v_p, v') \in Q_p(G)$. Intuitively,

Input: A graph $G = (V, E, L)$.
Output: A compressed graph $G_c = R(G) = (V_c, E_c, L_c)$.

1. $V_c := \emptyset$; $E_c := \emptyset$;
2. compute the maximum bisimulation relation $R_b$ of $G$;
3. compute the partition $Par := V/R_b$;
4. for each $S \in Par$ do
   5. create a node $v_S$ and set $L(v_S) := L(v)$ where $v \in S$;
6. $V_c := V_c \cup \{v_S\}$;
7. for each $v_S, v_{S'} \in V_c$ do
   8. if there exist $u \in S$ and $v \in S'$ such that $(u, v) \in E$ then $E_c := E_c \cup \{(v_S, v_{S'})\}$;
9. return $G_c = (V_c, E_c, L_c)$.

Figure 6: Examples of bisimulation relations

Figure 7: Algorithm compress for pattern queries

if $[v]_{R_t}$ simulates $v_p$ in $G$, then do each $v' \in [v]_{R_t}$ in $G$. Hence, $P$ expands $Q_p(G)$ by replacing $[v]_{R_t}$ with all the nodes $v'$ in the class $[v]_{R_t}$, in $O(|Q_p(G)|)$ time via an index structure for the inverse node mapping of $R$. When $Q_p$ is a Boolean pattern query, $P$ is not needed.

Example 5: Recall the graph $G$ of Fig. 2. Using the graph pattern preserving compression $<R, F, P>$, one can get the compressed graph $G_c$ of $G$, shown in Fig. 2, in which e.g., $R(FA_1) = R(FA_2) = FA_4$, where $FA_4$ is the equivalence class containing $FA_1$ and $FA_2$. For the graph $G_2$ of Fig. 6, its compressed graph $R(G_2)$ is $G_{2c}$, as shown in Fig. 6.

Correctness. We show that $<R, F, P>$ given above is indeed a graph pattern preserving compression. (1) $|V_c| \leq |G|$, as $|V_c| \leq |V|$ and $|E_c| \leq |E|$. (2) For any pattern query $Q_p$, $Q_p(G) = P(Q_p(G))$. To see this, it suffices to verify that $(u, v) \in Q_p(G)$ if and only if $(u, [v]_{R_t}) \in Q_p(G)$. If $(u, [v]_{R_t}) \in Q_p(G)$, then for any child $u'$ of $u$, there is a node $[v']_{R_t}$ such that $(u', [v']_{R_t}) \in Q_p(G)$, and there is a bounded path $p$ from $[v]_{R_t}$ to $[v']_{R_t}$. By the definition of $R$, we can show that for each node $w \in [v]_{R_t}$, there is a node $w' \in [v']_{R_t}$ to which there is a path $p'$ from $w$ such that $|p'| = |p|$. Moreover, for each query edge $(u, u')$, $[v]_{R_t}$ has a bounded path to a node $[v']_{R_t}$ in $G$, with a path $p'$ from $w$ such that $|p'| = |p|$. Hence $(u, [v]_{R_t}) \in Q_p(G)$. From these it also follows that $P(Q_p(G))$ is indeed the unique maximum match in $G$ for $Q_p$. In light of this, as shown in Fig. 3(c), we can find the match of $Q_p$ in $G$ by computing $P(Q_p(G))$ via any algorithm for answering $Q_p$. As remarked earlier, $A(k)$-index and $D(k)$-index [15, 26] may not preserve the answers to graph pattern queries. To see this, consider graph $G_1$ of Fig. 6 and its index graph $G^*_2$, of $A(k)$-index when $k = 1$, also shown in Fig. 6. Although $A_1, A_2$ and $A_3$ are not bisimilar, they all have and only have $B$ children; as such, they are 1-bisimilar [26], and are merged into a single node in $G^*_2$. However, $G^*_2$ cannot be directly queried by e.g., a $Q_p$ consisting of two query edges $\{(B, C),(B, D)\}$, both with bound 1. Indeed, for $Q_p$, $G^*_2$ returns all the $B$ nodes in $G$ as matches for query node $B$ in $Q_p$, while only $B_1$ and $B_3$ are the true matches in $G$.

4.2 Compression Algorithm for Graph Patterns

We next present an algorithm that computes the compressed graph $G_c = R(G)$ for a given graph $G = (V, E, L)$, where $R$ is the compression function given earlier.

The algorithm, denoted as compress, is shown in Fig. 7. Given a graph $G = (V, E, L)$, compress first computes the maximum bisimulation relation $R_b$ of $G$, and finds the in-
duced partition $\mathcal{P}$ by $R_0$ over the node set $V$ (lines 2-3). To do this, it follows [8]: it first partitions $V$ into $\{S_1, \ldots, S_t\}$, where each set $S_i$ consists of nodes with the same label; the algorithm then iteratively refines $\mathcal{P}$ by splitting $S_i$ if it does not represent an equivalence class of $R_0$, until a fixpoint is reached (details omitted). For each class $S \in \mathcal{P}$, $\text{compress}_R$ then creates a node $v_S$, assigns the label of a node $u \in S$ to $v_S$, and adds $v_S$ to $V_r$ (lines 4-6). For each edge $(u, v) \in E_r$, it adds an edge $(v_S, v_{S'})$, where $u$ and $v$ are in the equivalence classes represented by $v_S$ and $v_{S'}$, respectively (lines 7-9). Finally $G_r = (V_r, E_r, L_r)$ is returned (lines 10).

Correctness & Complexity. Algorithm $\text{compress}_R$ indeed computes the compressed graph $G_r$ by the definition of $R$ (Section 4.1). In addition, $\text{compress}_R$ is in $O(|E| \log |V|)$ time: $R_0$ and $\mathcal{P}$ can be computed in $O(|E| \log |V|)$ time [8] (lines 2-3), and $G_r$ can be constructed in $O(|V_r| + |E_r|)$ time (lines 4-9). This completes the proof of Theorem 4.

5. INCREMENTAL COMPRESSION

To cope with the dynamic nature of social networks and Web graphs, incremental techniques have to be developed to maintain compressed graphs. Given a query preserving compression $<R, F, P>$ for a class $Q$ of queries, a graph $G$, a compressed graph $G_C = R(G)$ of $G$, and batch updates $\Delta G$ (a list of edge deletions and insertions) to $G$, the incremental query preserving compression problem is to compute changes $\Delta G_r$ to $G_r$ such that $G_r \oplus \Delta G_r = R(G \oplus \Delta G)$, i.e., the updated compressed graph $G_r \oplus \Delta G_r$ is the compressed graph of the updated graph $G \oplus \Delta G$. It is known that while real-life graphs are constantly updated, the changes are typically minor [23]. As remarked earlier, when $\Delta G$ is small, $\Delta G_r$ is often small as well. It is thus often more efficient to compute $\Delta G_r$ than compressing $G \oplus \Delta G$ starting from scratch, by minimizing unnecessary recomputation.

As observed in [28], it is no longer adequate to measure the complexity of incremental algorithms by using the traditional complexity analysis for batch algorithms. Following [28], we characterize the complexity of an incremental compression algorithm in terms of the size of the affected area (AFF), which indicates the changes in the input $\Delta G$ and the output $\Delta G_r$, i.e., $|\text{AFF}| = |\Delta G| + |\Delta G_r|$. An incremental algorithm is said to be $\text{bounded}$ if its time complexity can be expressed as a function $f(|\text{AFF}|)$, i.e., it depends only on $|\Delta G| + |\Delta G_r|$ rather than the entire input $G$. An incremental problem is $\text{bounded}$ if there exists a bounded incremental algorithm for it, and is $\text{unbounded}$ otherwise.

5.1 Incremental Maintenance for Reachability

We first study the incremental graph compression problem for reachability queries, referred to as $\text{incremental reachability compression}$ and denoted as RCM. One may want to develop a bounded algorithm for incremental reachability compression. The problem is, however, nontrivial.

Theorem 6: RCM is $\text{unbounded}$ even for unit update, i.e., a single edge insertion or deletion.

Proof sketch: We verify this by reduction from the single source reachability problem (SSR). Given a graph $G_s$, a fixed source node $s$ and updates $\Delta G_s$, SSR is to decide whether for all $u \in G_s$, $s$ reaches $u$ in $G_s \oplus \Delta G_s$. It is known that SSR is $\text{unbounded}$ [28]. We show that SSR is bounded if RCM with unit update is bounded.

Incremental algorithm. Despite the unbounded result, we present an incremental algorithm for RCM that is in $O(|\text{AFF}| + |G_r|)$ time, i.e., it only depends on $|\text{AFF}|$ and $|G_r|$ instead of $|G|$, and solves RCM without decompressing $G_r$.

To present the algorithm, we need the following notations.

(1) A strongly connected component (SCC) graph $G_{\text{SCC}} = (V_{\text{SCC}}, E_{\text{SCC}})$ merges each strongly connected component into a single node without self cycle. We use $\text{V}_{\text{SCC}}$ to denote an SCC node containing $v$, and $E_{\text{SCC}}$ the edges between SCC nodes.

(2) The topological rank $r(s)$ of a node $s$ in $G$ is defined as follows: (a) $r(s) = 0$ if $s$ has no child in $G$, i.e., $s_{\text{SCC}}$ has no child in $G_{\text{SCC}}$; (b) $r(s) = r(s') + 1$ when $s'$ ranges over the children of $s$. We also define $r(\varepsilon) = r(s)$ for an edge update $e = (s, v)$.

Lemma 7: In any graph $G$, $r(u) = r(v)$ if $(u, v) \notin R_r$.

Leveraging Lemma 7, we present the algorithm, denoted as incRCM and shown in Fig. 8. It has three steps.

(1) Preprocessing. The algorithm first preprocesses updates $\Delta G$ and compressed graph $G_r$ (lines 1–2). (a) It first removes redundant changes in $\Delta G_r$ that have no impact on reachability (line 1). More specifically, it removes (i) edge insertions $(u, u')$ where $[u]_{R_r} \neq [u']_{R_r}$, and $[u]_{R_r}$ can reach $[u']_{R_r}$ in $G_r$; and (ii) edge deletions $(u, u')$ if either $[u]_{R_r}$ reaches $[u']_{R_r}$ via a path of length no less than 2 in $G_r$, or if $[u]_{R_r} = [u']_{R_r}$, and there is a child $u''$ of $u$ such that $(u, u'') \notin \Delta G_r$ and $[u]_{R_r} = [u'']_{R_r}$ (line 2). It then identifies a set of nodes $u$ with $r(u)$ changed in $G_r$, for each edge update $(u, u') \in \Delta G_r$; it updates the rank of $u$ in $G_r$ accordingly.

(2) Updating. The algorithm then updates $G_r$ based on $r$ (line 3). It first splits those nodes $[u]_{R_r}$ of $G_r$ in which there exist nodes with different ranks. By Lemma 7, these nodes are not in the same equivalence class, thus $[u]_{R_r}$ must be split. Then it finds all the newly formed SCCs in $G_r$ and introduce a new node for each of them in $G_r$. These two steps identify an initial area affected by updates $\Delta G_r$.

(3) Propagation. The algorithm then locates $\Delta G_r$ by propagating changes from the initial affected area identified in step (2). It processes updates $e = (u, u')$ in the ascending topological rank (line 4). It first finds $[u]_{R_r}$ and $[u']_{R_r}$, the (revised) equivalence classes of $u$ and $u'$ in the current compressed graph $G_r$. It then invokes procedure incRCM$^+$ (resp. incRCM$^-\text{m}$) to update $G_r$ when $e$ is to be inserted (resp. deleted) (lines 5–8). Updating $G_r$ may make some updates in $\Delta G_r$ redundant, which are removed from $\Delta G_r$ (line 9). After all updates in $\Delta G_r$ are processed, the updated compressed graph $G_r$ is returned (line 10).

Given an edge $e = (u, u')$ to be inserted into $G_r$ and their corresponding nodes $[u]_{R_r}$ and $[u']_{R_r}$ in $G_r$, procedure incRCM$^+$ updates $G_r$ as follows. First, note that since $(u, u')$ is not redundant (by lines 1 and 9 of incRCM$^-$), $u$ cannot reach $u'$ in $G_r$, but after the insertion of $e$, $u'$ becomes a child of $u$. Moreover, no nodes in $[u]_{R_r} \setminus \{u\}$ can reach $u'$ in $G_r$. Hence $u$ and nodes in $[u]_{R_r} \setminus \{u\}$ can no longer be in the same equivalence class after the insertion of $e$. Thus incRCM$^+$ splits $[u]_{R_r}$ into two nodes representing $\{u\}$ and $[u]_{R_r} \setminus \{u\}$, respectively; similarly for $[u']_{R_r}$ (line 1). This is done by invoking procedure Split (omitted).

In addition, nodes may also have to be merged (lines 2–8).
We denote the set of children (resp. parents) of a node u as C(u) (resp. P(u)), and use B(u) to denote the set of nodes having the same parents as u. By Lemma 7, consider r(u) and r(u') in the updated G. Observe that r(u) ≥ r(u') since u' is a child of u after the insertion of e. (1) If r(u) > r(u'), i.e., u and u' are not in the same SCC, then there may only be merged with those nodes v' ∈ B[[u]rG] such that C({u}) = C({v'}); similarly for u' (lines 2-4). Hence we invoke procedure Merge (omitted) that works on G', given nodes w and w', it checks whether P(w) = P(w') and C(w) = C(w'); if so, it merges w and w' into one that shares the same parents and children as w and w' (2). When r(u) = r(u'), as e is non-redundant, u and u' may not be in the same SCC. Thus {u} (resp. {u'}) may only be merged with a parent of [u]rG (resp. a child of [u]rG; lines 5-7).

Similarly, procedure incRMC updates G', by using Split and Merge in response to the deletion of an edge (omitted). Here when a node is split, its parents may need to be split as well, i.e., the changes are propagated upward.

Example 6: Recall graph G of Fig 2. A subgraph G_1 (excluding e_1 and e_2) of G and its compressed graph G_r are shown in Fig 9. (1) Suppose that edges e_1 and e_2 are inserted into G_r. Algorithm incRMC first identifies e_1 as a redundant insertion, since FA_1 can reach v in G_r (line 1). It then updates the rank r of FA_1 to be 0 due to the insertion of e_2 (line 2), by traversing G_r to identify a newly formed SCC. It next invokes procedure incRMC (line 6), which merges FA_1 to the node v in G_r, and constructs G_r' as the compressed graph, shown in Fig 9. The affected area AFF includes nodes v, v, and edge (v, v). (2) Now suppose that edges e_3 and e_4 are removed. The algorithm first identifies e_3 as a redundant update, since FA_1 has a child C_2 in the nodes V_r. It then processes update e_4 by updating the rank of FA_2, and splits the node v, in G_r' into FA_2 and v via incRMC (line 8). This yields G_r' update by updating G_r (see Fig 9). The AFF includes nodes v, v', C_1, and their edges.

Correctness & Complexity. Algorithm incRMC correctly maintains the compressed graph G_r. Indeed, one can verify that the loop (lines 3-7) guarantees that for any nodes u and u' of G_r, u can reach u' if and only if [u]rG reaches [u']rG in G_r when G_r is updated in response to ∆G. In particular, procedure Merge is justified by the following: nodes can be merged if they share same parents and children after non-redundant updates. This can be verified by contradiction.

For the complexity, one can show that the first two steps of the algorithm (lines 1-3) are in O(|AFF|G_r|) time. Indeed, (1) it takes O(|AFF|G_r|) time to identify redundant updates by testing the reachability of the nodes in G_r, which accesses R but does not search G; and (2) it takes O(|AFF|G_r|) time to identify the nodes and their changed rank for each update in ∆G, and updates G_r accordingly. Procedures incRMC and incRMC' are in O(|AFF|G_r|) time. Thus incRMC is in O(|AFF|G_r|) time. As will be verified by our experimental study, |G_r| and |AFF| are typically small in practice.

5.2 Incremental Maintenance for Graph Patterns

We next study the incremental graph compression problem for graph pattern queries, referred to as incremental graph pattern preserving compression and denoted as PCM. Like RCM, PCM is also unbounded and hard.

Theorem 8: PCM is unbounded even for unit update.

Proof sketch: We show that SSR is bounded iff PCM with unit update is bounded, also by reduction from SSR.

Incremental algorithm. Despite this, we develop an incremental algorithm for PCM that is in O(|AFF|^2 + |G_r|) time. Like incRMC, the complexity of the algorithm is independent of |G|. It solves PCM without degressing G.

We first define some notions. (1) A strongly connected component graph G_scc is as defined in Section 5.1. (2) Following [8], we define the well founded set WF to be the set of nodes that cannot reach any cycle in G, and the non-well-founded set NWF to be \( \nabla \) \( \\WNWF \). (3) Based on (1) and (2), we define the rank \( r_{\WNWF} (v) \) of nodes v in G: (a) \( r_{\WNWF} (v) = 0 \) if v has no child; (b) \( r_{\WNWF} (v) = -\infty \) if \( v_{\WNWF} \) has no child in \( G_{scc} \) but v has children in G; and (c) \( r_{\WNWF} (v) = \max (\{ r_{\WNWF} (v') + 1 \} \cup \{ r_{\WNWF} (v') \} ) \) when \( v_{\WNWF} <_{\WNWF} v' \) and \( v_{\WNWF} <_{\WNWF} v' \) are in \( E_{scc} \) for all v' \( \in \WNWF \).

We also define \( r_{\WNWF} (v) = r_{\WNWF} (u) \) for a node \( u_{\WNWF} \in G_r \), and \( r_{\WNWF} (u) = r_{\WNWF} (v) \) for an update e = (u, v).

Analogous to Lemma 7, we show the lemma below.

Lemma 9: For any graph G and its compressed graph G_r, \( r_{\WNWF} (u) = r_{\WNWF} (v) \) if (u, v) \( \in \WNWF \), and (2) each node u in G_r can only be affected by updates e with \( r_{\WNWF} (e) < r_{\WNWF} (u) \).

For PCM, the affected area AFF includes (1) the nodes in G with their ranks changed after G is modified, as well as the edges attached to them, and (2) the changes to G_r, including the updated nodes and the edges attached to them.

Our incremental algorithm is based on Lemma 9, denoted as incPCM and shown in Fig. 10. It has two steps.

(1) Preprocessing. The algorithm first finds an initial af-
Input: a graph $G$, a compressed graph $G_r$, batch updates $\Delta G$; Output: an updated $G_r$.
1. AFF := $\emptyset$;
2. incr($G, G_r, \Delta G$); /* update rank and $G_r$ */
3. for each $i \in \{-\infty\} \cup [0, \max(r_G(v)))$ do
   AFF := AFF.add (AFF, $i$), where AFF is the set of new nodes $v$ with $r_G(v) < i$
4. for each AFF, of ascending rank order do
   PT(AFF); /* update compressed graph at rank $i$ */
5. minDelta(AFF, $G_r, \Delta G$); update AFF;
6. for each $[u'R]_i \in$ AFF, and $e = (u, u') \in \Delta G$ do
   SplitMerge($[u'R]_i, G_r, e, \text{AFF}$);
9. return $G_r$;

Procedure SplitMerge
Input: Compressed graph $G_r = (V_r, E_r, L_r)$, an update $(u, v)$, node $[u']_R$, AFF;
Output: An updated $G_r$.
1. Boolean flag := true; AFF$_p$ := $\emptyset$;
2. AFF$_p$ := AFF$_p \cup \{[u']_R\} \cup P([u']_R);
3. for each node $[v']_R \notin$ AFF$_p$ with $r([v']_R) > r([u']_R)$ do
   /* split $[v']_R$, w.r.t. $v$ into $[v_1']_R$ and $[v_2']_R$ */
4. flag := bSplit ($[v_1']_R$, $[v_2']_R$);
5. if flag then
6. AFF$_{r[x]} ([v_1']_R) :=$ AFF$_{r[x]} ([v_1']_R) \cup \{(v_1', [v_2']_R)\};$
7. for each $v''$ with $r(v'') = r([v_2']_R)$ do
8. if mergeCon ($v''$, $[v_1']_R$) then bMerge ($v''$, $[v_1']_R$);
9. for each $v''$ with $r(v'') = r([v_2']_R)$ do
10. if mergeCon ($v''$, $[v_2']_R$) then bMerge ($v''$, $[v_2']_R$);
11. update AFF; return $G_r$;

Figure 10: Algorithm incPCM

Example 7: Recall $G$ and its compressed graph $G_r$ from Fig 2. Consider removing $e_1$ and $e_3$ from $G$, followed by the insertion of $e_2$, as indicated in Fig 11. When $e_1$ is removed, the algorithm incPCM first updates the rank of $C_1$ (line 2), and adds $C_1$ to AFF (line 4). Since $C_1$ has a different rank from $C_2$, it is split from ($C_1, C_2$) at the same time (line 4). The algorithm then invokes PT to merge $C_1$ and ($C_3, \ldots, C_6$) (line 6), and uses SplitMerge to (a) remove $FA_1$ from ($FA_3, FA_2$), and (b) merges $FA_3$ with ($FA_3, FA_2$) (line 9). Observe that the deletion of $e_3$ becomes redundant, as identified by minDelta (line 7). The updated compressed graph $G_r$ is shown in Fig 11, in which AFF is marked.

Correctness & Complexity. One can verify that incPCM correctly maintains compressed graphs, by induction on the rank of nodes in $G_r$ processed by the algorithm. For its complexity, note that procedure incr is in $O(|AFF| \log |AFF|)$ time. Moreover, procedures minDelta, PT and SplitMerge take $O(|AFF|)$ time, $O(|AFF| \log |AFF| + |G_r|)$, and $O(|AFF|^2)$ time in total. Hence incPCM is in $O(|AFF|^2 + |G_r|)$ time. The algorithm accesses $R$ and $G_r$, without searching $G$.

6. EXPERIMENTAL EVALUATION

We next present an experimental study using both real-life and synthetic data. For reachability and graph pattern queries, we conducted four sets of experiments to evaluate:

(1) the effectiveness of the query preserving compressions proposed, measured by compression ratio, i.e., the ratio of the compressed graph size to the original graph size,
(2) query evaluation time over original and compressed graphs,
(3) the efficiency of the incremental compression algorithms, and
(4) the effectiveness of incremental compression.

Experimental setting. We used the following datasets.

(a) Real-life data. For graph pattern queries, we used the following graphs with attributes and labels on the nodes:

\begin{itemize}
  \item [[A] Youtube\textsuperscript{2}] where nodes are videos labeled with their category;
  \item California\textsuperscript{3}, a Web graph in which each node is a host labeled with its domain;
\end{itemize}

(2) California\textsuperscript{3}, a Web graph in which each node is a host labeled with its domain; (c) Citation\textsuperscript{31}, a citation

\textsuperscript{2}http://netsg.cs.sunysf.edu/youtube/data/
\textsuperscript{3}http://www.cs.cornell.edu/courses/cs685/2002fa/
network in which nodes represent papers, labeled with their publishing information; and (d) Internet where a node represents an autonomous system labeled with its location.

For reachability queries, we used a [a] six social networks: a Wikipedia voting network wikiVote, a Wikipedia communication network wikiTalk, an online social network a product co-purchasing network amazon, socEpinions, a fragment of facebook, and Youtube: (b) three Web graphs: a peer-to-peer network P2P, a Web graph NotreDame, and Internet; and (c) a citation network citHepTh.

The sizes of these graphs (the number $|V|$ of nodes and the number $|E|$ of edges) are shown in Tables 1 and 2.

(2) Synthetic data. We designed a graph generator to produce synthetic graphs. Graph generation was controlled by three parameters: the number of nodes $|V|$, the number of edges $|E|$, and the size $|L|$ of the label set $L$.

(3) Pattern generator. We implemented a generator for graph pattern queries controlled by four parameters: the number of query nodes $V_p$, the number of edges $E_p$, label set $L_p$ along the same lines as their counterpart $L$ for data graphs, and an upper bound $k$ for edge constraints.

(4) Implementation. We implemented the following algorithms, in Java. (1) our compression algorithms compressRC (Section 3) and compressPC (Section 4); (2) AHO [1] which, as a comparison to compressRC, computes transitive reduced graphs; (3) our incremental compression algorithms incRCM and incPCM for batch updates (Section 5); we also implemented IncBsim, an algorithm that invokes the algorithm of [30] (for a single update) multiple times when processing batch updates; (4) query evaluation algorithms: for reachability queries, the breadth-first (resp. bidirectional) search algorithm BFS (resp. BIBFS); for pattern queries, algorithm Match and its incremental version IncBMatch [9]; and (5) algorithms for building 2-hop indexes [6].

All experiments were run on a machine powered by an Intel Core(TM)2 Duo 3.00GHz CPU with 4GB of memory, using scientific linux. Each experiment was run 5 times and the average is reported here.

**Experimental results**. We next present our findings.

**Exp-1: Effectiveness: Compression ratio.** We first evaluate the compression ratios of our methods using real-life data. We define the compression ratio of compressRC to be $RC = |G_r|/|G|$, where $G$ is the original graph and $G_r$ is its compressed graph by compressRC. Similarly, we define $PC$, of compressPC, and $R_{ahd}$ of AHO [1] in which $G_r$ denotes the transitive reduced graph. We also consider SCC graphs $G_{sc}$ (Section 3), and define $R_{sc}$ as $|G_r|/|G_{sc}|$ to evaluate the effectiveness of compressRC on SCC graphs.

Observe the following. (1) The smaller the compression ratio is, the more effective the compressing scheme used is. (2) We treat the compression ratio as a measurement for representation compression, which differs from the ratio measuring the memory cost reduction (to be discussed shortly).

The compression ratios of reachability preserving compression compressRC are reported in Table 1. We find the following. (1) Real-life graphs can be highly compressed for reachability queries. Indeed, $RC$ is in average $3\%$ over these datasets. In other words, it reduces real-life graphs by $95\%$. (2) Algorithm compressRC performs significantly better than AHO. It also reduces SCC graphs by $81\%$ in average. (3) The compression algorithms perform best on social networks e.g., wikiVote, socEpinions, facebook and Youtube. The average $RC$ is $2\%$, $8\%$ and $14.7\%$ for (six) social networks, (three) Web graphs and the citation network, respectively. This is because social networks have higher connectivity.

The effectiveness of compressPC is reported in Table 2. We find that (1) graphs can also be effectively compressed by pattern preserving compression, with $PC$ of $43\%$ in average, i.e., it reduces graphs by $57\%$; (2) Internet can be better compressed for graph pattern queries than social networks (Youtube) and citation networks (Citation), since the latter two have more diverse topological structures than the former, as observed in [22]; and (3) compressPC performs better than compressRC over all the datasets. This is because it is more difficult to merge nodes due to the requirements on topological structures and label equivalence imposed by pattern queries, compared to reachability queries.

**Exp-2: Effectiveness: query processing.** In this set of experiments, we evaluated the performance of the algorithms for reachability and pattern queries on original and compressed graphs, respectively. We used exactly the same algorithms in both settings, without decompressing graphs.

For a pair of randomly selected nodes, we queried their reachability and evaluated the running time of BFS and BIBFS on the original graph $G$ and its compressed graph $G_r$. As shown in Fig. 12(a), the evaluation time on the compressed $G_r$ is much less than that on $G$, when either BFS or BIBFS is used. Indeed, for socEpinions the running time of BFS on $G_r$ is only $2\%$ of the cost on $G$ in average.

For graph pattern queries, Figure 12(b) shows the running time of Match on Youtube and Citation, and on their compressed counterparts $L_p$, is the same as $L$; see Table 2. In addition, we conducted the same experiments on synthetic graphs with $|V| = 50K$, $|E| = 435K$ while $|L| = 10$ or $|L| = 20$, and on compressed graphs. Fixing $L_p = 10$, we varied $(V_p, E_p, k)$ of these queries from (3, 3, 3) to (8, 8, 3), as reported in Fig. 12(c). These results tell us the following: (a) the running time of Match on compressed graphs is only $30\%$ of that on their original graphs; and (b) when $|L|$ is changed from 10 to 20 on synthetic data, Match runs faster as the compressed graphs contain more node labels.

As remarked earlier, the compression ratio of Table 1 only measures graph representation. In Fig. 12(d) we compare

| dataset | $(|V|, |E|, |L|)$ | $PC_{\text{c}}$ |
|---------|----------------|---------------|
| California | $(10K, 16K, 95)$ | 45.9\% |
| Internet | $(52K, 105K, 247)$ | 29.8\% |
| Youtube | $(155K, 796K, 16)$ | 41.3\% |
| Citation | $(500K, 660K, 67)$ | 48.2\% |

| dataset | $(|V|, |E|)$ | $RC_{\text{c}}$ | $R_{\text{sc}}$ | $RC_{\text{ahd}}$ |
|---------|-----------|--------|--------|---------|
| facebook | 1.6M (64K, 1.5M) | 13.19\% | 5.89\% | 0.028\% |
| amazon | 1.5M (262K, 1.2M) | 30.09\% | 18.94\% | 0.18\% |
| Youtube | 931K (155K, 796K) | 41.60\% | 17.02\% | 1.77\% |
| wikiVote | 111K (7K, 101K) | 63.56\% | 8.33\% | 1.91\% |
| wikiTalk | 1.4M (72M, 61M) | 43.21\% | 18.82\% | 5.77\% |
| socEpinions | 558K (76K, 509K) | 29.53\% | 19.59\% | 2.85\% |
| NotreDame | 1.8M (362K, 1.5M) | 43.22\% | 10.75\% | 2.01\% |
| P2P | 27K (6K, 21K) | 73.24\% | 17.02\% | 5.97\% |
| Internet | 155K (52K, 105K) | 58.32\% | 30.89\% | 16.08\% |
| citHepTh | 381K (29K, 353K) | 71.32\% | 37.15\% | 14.70\% |

**Table 1: Reachability preserving: compression ratio**

| dataset | $(|V|, |E|, |L|)$ | $PC_{\text{c}}$ |
|---------|----------------|---------------|
| California | $(10K, 16K, 95)$ | 45.9\% |
| Internet | $(52K, 105K, 247)$ | 29.8\% |
| Youtube | $(155K, 796K, 16)$ | 41.3\% |
| Citation | $(500K, 660K, 67)$ | 48.2\% |

**Table 2: Pattern preserving: compression ratio**
the memory cost of the original graph $G$, the compressed graph $G_c$, by reachability preserving compression, and their 2-hop indexes [6], for real-life datasets. The result tells us the following: (a) at least 92% of the memory cost of $G$ is reduced by $G_c$; (b) the 2-hop indexes have higher space cost than $G$ and $G_c$; e.g., 2-hop on wikiVote took 234MB memory, while its original graph took 8.9MB and the compressed graph took 0.2MB; and (c) 2-hop indexes can be built over small compressed graphs, but may not be feasible over large original graphs, e.g., facebook, due to its high cost.

The results of the same experiments on other real-life graphs are consistent and hence, are not reported here.

Exp-3: Efficiency of incremental compression. We next evaluate the efficiency of incRCM and incPCM. Fixing the number of nodes in the social network socEpinions, we varied the number of edges from 509K to 617K (resp. from 509K to 374K) by inserting (resp. deleting) edges in 12K increments (resp. 15K decrements). The results in Figures 12(e) and 12(f) tell us that incRCM outperforms compress$_\text{RC}$ when insertions are up to 20% and deletions are up to 22% of the original graph.

Figure 12(g) shows the performance of incPCM on Youtube compared with compress$_\text{PC}$ and IncBsim in response to mixed updates, where we fixed the node size, and varied the size of the updates $|\Delta E|$ in 0.8K increments. The result shows that incPCM is more efficient than compress$_\text{PC}$ when the total updates are up to 5K, and consistently outperforms IncBsim, due to the removal of redundant updates by incPCM.

Figure 12(h) compares the performance of the following two approaches, both for incrementally evaluating pattern queries over Citation: (1) we used IncBMatch to incrementally update the query result, and alternatively, (2) we first used incPCM to update the compressed graph, and then ran Match over the updated compressed graph to get the result. The total running times, reported in Figure 12(h), tell us that once the updates are more than 8K, it is more efficient to update and query the compressed graphs than to incrementally update the query results.

We also conducted the same experiments on other real-life datasets. The results are consistent and hence not reported.

Exp-4: Effectiveness of incremental compression. We evaluated the effectiveness of incRCM and incPCM, in terms of compression ratios $\text{RC}$ and $\text{PC}$, respectively. (1) Fixing $|L|=10$ and starting with $|V_0|=1M$, we varied the size of synthetic graphs $G$ by simulating the densification law [17]: for a synthetic graph $G_s$ with $|V_i|$ nodes and $|E_i|=|V_i|^\alpha$ edges at iteration $i$, we increased its nodes to $|V_{i+1}|=\beta|V_i|$, and edges to $|E_{i+1}|=|V_{i+1}|^\alpha$ in the next iteration. (2) We varied the size of real-life graphs following power-law [20], where the edge growth rate was fixed to be 5%, and an edge was attached to the high degree nodes with 80% probability.

Figure 12(i) shows that for reachability queries, RC, varies from 2.2% to 0.2% with $\alpha =1.05$, and decreases from 1.4% to 0.05% with $\alpha = 1.1$, when $\beta$ is fixed to be 1.2. This shows that the more edges are inserted into dense graph, the better the graph can be compressed for reachability queries. Indeed, when edges are increased, more nodes may become reachability equivalent, as expected (Section 3). The results over real-life graphs in Fig. 12(j) also verify this observation.

The results in Fig. 12(k) tell us that for graph pattern queries, PC, is not sensitive to the changes of the size of graphs. On the other hand, Figure 12(l) shows the following. (1) When more edges are inserted into the real-life graphs, PC, increases; this is because when new edges are added, the bisimilar nodes may have diverse topological structures and hence are no longer bisimilar; and (2) PC, is more sensitive to the changes of the size of Web graphs (e.g., California, Internet) than social networks (e.g., Youtube), because the
high connectivity of social networks makes most of the insertions redundant, i.e., having less impact on $P_G$.

Summary. From the experimental results we find the following. (1) Real-life graphs can be effectively and efficiently compressed by reachability and graph pattern preserving compressions. (2) Evaluating queries on compressed graphs is far more efficient than on the original graphs, and is less sensitive to the query sizes. Moreover, existing index techniques can be directly applied to compressed graphs, e.g., 2-hop index. (3) Compressed graphs by query preserving compressions can be efficiently maintained in response to batch updates. Better still, it is more efficient to evaluate queries on incrementally updated compressed graphs than incrementally evaluate queries on updated original graphs.

7. CONCLUSION

We have proposed query preserving graph compression for querying large real-life graphs. For queries of users’ choice, the compressed graphs can be directly queried without decompression, using any available evaluation algorithms for the queries. As examples, we have developed efficient compression schemes for reachability queries and graph pattern queries. We have also provided incremental techniques for maintaining the compressed graphs, from boundedness results to algorithms. Our experimental results have verified that our methods are able to achieve high compression ratios, and reduce both storage space and query processing time; moreover, our compressed graphs can be efficiently maintained in response to updates to the original graphs.

We are currently experimenting with real-life graphs in various domains. We are also studying compression methods for other queries, e.g., pattern queries with embedded regular expressions. We are also to extend our compression and maintenance techniques to query distributed graphs.

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8. REFERENCES