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DEL PEZZO ZOO

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ABSTRACT. We study del Pezzo surfaces that are quasismooth and well-formed weighted hypersurfaces. In particular, we find all such surfaces whose α-invariant of Tian is greater than $2/3$.

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All varieties are assumed to be complex, projective and normal.

1. Introduction

Let X be a hypersurface in $\mathbb{P}(a_0, \ldots, a_n)$ of degree d, where $a_0 \leqslant \ldots \leqslant a_n$. Then X is given by

$$\phi\big(x_0, \ldots, x_n\big) = 0 \subset \mathbb{P}\big(a_0, \ldots, a_n\big) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x_0, \ldots, x_n\big]\Big),$$

where $\mathrm{wt}(x_i) = a_i$, and ϕ is a quasihomogeneous polynomial of degree d. The equation

$$\phi\big(x_0, \ldots, x_n\big) = 0 \subset \mathbb{C}^{n+1} \cong \mathrm{Spec}\Big(\mathbb{C}\big[x_0, \ldots, x_n\big]\Big),$$

defines a quasihomogeneous singularity (V, O), where O is the origin of $\mathbb{C}^{n+1}$.

Definition 1.1. The hypersurface X is quasismooth if the singularity (V, O) is isolated.

Suppose that X is quasismooth.

Remark 1.2. It follows from [15, Theorem 7.9], [15, Proposition 8.13] and [15, Remark 8.14.1] that $\sum_{i=0}^{n} a_i > d$ if and only if the singularity (V, O) is canonical. Moreover, since (V, O) is Gorenstein, it is canonical if and only if it is rational (see [15, Theorem 11.1]).

Definition 1.3. The hypersurface $X \subset \mathbb{P}(a_0, \ldots, a_n)$ is well-formed if

$$\gcd\big(a_0, \ldots, \widehat{a}_i, \ldots, \widehat{a}_j, \ldots, a_n\big) \mid d$$

and $\gcd(a_0, \ldots, \widehat{a}_i, \ldots, a_n) = 1$ for every $i \neq j$.

Suppose that X is well-formed. Then $\sum_{i=0}^{n} a_i > d$ if and only if X is a Fano variety. Put

$$I = \sum_{i=0}^{n} a_i - d,$$

and suppose that $\sum_{i=0}^{n} a_i > d$. We call I the index of the Fano variety X. Note that I should not be confused with the Fano index of X (see Remark 1.8).

Definition 1.4. The global log canonical threshold of the Fano variety X is the number

$$\mathrm{lct}\big(X\big) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } \big(X, \lambda D\big) \text{ is log canonical} \\ \text{for every effective } \mathbb{Q}\text{-divisor } D \equiv -K_X \end{array}\right\} \in \mathbb{R}.$$

The number $\mathrm{lct}(X)$ is an algebraic counterpart of the α-invariant introduced in [18]. In particular, the global log canonical threshold and the α-invariant are known to coincide in the nonosingular case (see e. g. [7, Theorem A.3]). One of the important applications of (either of) these invariants is the problem of existence of an orbifold Kähler–Einstein metric on the variety X.

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Theorem 1.5 ([18], [9])**.** The variety X admits an orbifold Kähler–Einstein metric if

$$\mathrm{lct}\big(X\big) > \frac{\dim\big(X\big)}{\dim\big(X\big)+1}.$$

There are Fano orbifolds that do not admit orbifold Kähler–Einstein metrics (see [16], [11],[12]).

Theorem 1.6 ([12])**.** The variety X admits no Kähler–Einstein metrics if either $I > na_0$ or

$$dI^n > n^n \prod_{i=0}^{n} a_i.$$

The two inequalities mentioned in Theorem 1.6 are known as Lichnerowicz and Bishop obstructions, respectively. A remarkable fact is that in our case they are not independent. Namely, we prove the following result in Section 3.

Theorem 1.7. Let $\bar{a}_0 \leqslant \bar{a}_1 \leqslant \ldots \leqslant \bar{a}_n$ and $\bar{d}$ be positive real numbers such that

$$\bar{d}\Bigg(\sum_{i=0}^{n} \bar{a}_i - \bar{d}\Bigg)^{n} > n^n \prod_{i=0}^{n} \bar{a}_i,$$

and $\bar{d} < \sum_{i=0}^{n} \bar{a}_i$. Then $\sum_{i=0}^{n} \bar{a}_i - \bar{d} > n\bar{a}_0$.

It is well-known that $I \leqslant n = \dim(X) + 1$ if X is smooth. On the other hand, we know that

$$dI^n > n^n \prod_{i=0}^{n} a_i \iff I\Big(-K_X\Big)^{n-1} > \Big(\dim\big(X\big)+1\Big)^{n}.$$

Remark 1.8. Let U be a smooth Fano variety of dimension m. Define the Fano index $\daleth$ of U to be the maximal integer such that $-K_U \sim \daleth H$ for some $H \in \mathrm{Pic}(U)$. Then the inequality

$$\daleth\Big(-K_U\Big)^{m} \leqslant \Big(\dim\big(U\big)+1\Big)^{m+1}$$

fails in general if $m \gg 1$ (see [8, Proposition 5.22]). But we always have $\daleth \leqslant m+1$.

Suppose that $n = 3$. Then X is a del Pezzo surface with at most quotient singularities, which is an interesting object of study, in particular from the point of the question of existence of orbifold Kähler–Einstein metrics and Sasakian–Einstein structures (see e. g. [14], [1], [3], [4]) and some others (see e. g. [10]). The classification of such surfaces X with $I = 1$ is known due to [14].

Theorem 1.9 ([14, Theorem 8])**.** Suppose that $I = 1$. Then

- either $(a_0, a_1, a_2, a_3, d) = (2, 2m+1, 2m+1, 4m+1, 8m+4)$, where m is a positive integer,
- or the quintuple (a_0, a_1, a_2, a_3, d) lies in the sporadic set

$$\left\{\begin{aligned}
&(1,1,1,1,3),(1,1,1,2,4),(1,1,2,3,6),(1,2,3,5,10),\\
&(1,3,5,7,15),(1,3,5,8,16),(2,3,5,9,18),(3,3,5,5,15),(3,5,7,11,25),\\
&(3,5,7,14,28),(3,5,11,18,36),(5,14,17,21,56),(5,19,27,31,81),\\
&(5,19,27,50,100),(7,11,27,37,81),(7,11,27,44,88),(9,15,17,20,60),\\
&(9,15,23,23,69),(11,29,39,49,127),(11,49,69,128,256),\\
&(13,23,35,57,127),(13,35,81,128,256)
\end{aligned}\right\}.$$

Note that we can not apply Theorem 1.5 to the surface X if $I \geqslant 3a_0/2$, because $\mathrm{lct}(X) \leqslant a_0/I$.

The authors of [4] went further to classify the cases with $2 \leqslant I \leqslant 10$ and suggest that I cannot attain larger values.

Theorem 1.10 (cf. [4, Theorem 4.5]). Suppose that $2 \leqslant I \leqslant 10$ and $I < 3a_0/2$. Then

- either there exist a non-negative integer $k < I$ and a positive integer $a \geqslant I+k$ such that
$$\big(a_0, a_1, a_2, a_3, d\big) = \big(I-k, I+k, a, a+k, 2a+k+I\big),$$
- or the quintuple (a_0, a_1, a_2, a_3, d) belongs to one of the following infinite series:
  - $(3, 3m, 3m+1, 3m+1, 9m+3)$,
  - $(3, 3m+1, 3m+2, 3m+2, 9m+6)$,
  - $(3, 3m+1, 3m+2, 6m+1, 12m+5)$,
  - $(3, 3m+1, 6m+1, 9m, 18m+3)$,
  - $(3, 3m+1, 6m+1, 9m+3, 18m+6)$,
  - $(4, 2m+3, 2m+3, 4m+4, 8m+12)$,
  - $(4, 2m+3, 4m+6, 6m+7, 12m+18)$,
  - $(6, 6m+3, 6m+5, 6m+5, 18m+15)$,
  - $(6, 6m+5, 12m+8, 18m+9, 36m+24)$,
  - $(6, 6m+5, 12m+8, 18m+15, 36m+30)$,
  - $(8, 4m+5, 4m+7, 4m+9, 12m+23)$,
  - $(9, 3m+8, 3m+11, 6m+13, 12m+35)$,

  where m is a positive integer,
- or the quintuple (a_0, a_1, a_2, a_3, d) lies in the sporadic set[1]

$$\left\{\begin{array}{l}
(2,3,4,7,14), (3,4,5,10,20), (3,4,6,7,18), (3,4,10,15,30), (5,13,19,22,57),\\
(5,13,19,35,70), (6,9,10,13,36), (7,8,19,25,57), (7,8,19,32,64),\\
(9,12,13,16,48), (9,12,19,19,57), (9,19,24,31,81), (10,19,35,43,105),\\
(11,21,28,47,105), (11,25,32,41,107), (11,25,34,43,111), (11,43,61,113,226),\\
(13,18,45,61,135), (13,20,29,47,107), (13,20,31,49,111), (13,31,71,113,226),\\
(14,17,29,41,99), (5,7,11,13,33), (5,7,11,20,40), (11,21,29,37,95),\\
(11,37,53,98,196), (13,17,27,41,95), (13,27,61,98,196), (15,19,43,74,148),\\
(9,11,12,17,45), (10,13,25,31,75), (11,17,20,27,71), (11,17,24,31,79),\\
(11,31,45,83,166), (13,14,19,29,71), (13,14,23,33,79), (13,23,51,83,166),\\
(11,13,19,25,63), (11,25,37,68,136), (13,19,41,68,136), (11,19,29,53,106),\\
(13,15,31,53,106), (11,13,21,38,76), (3,7,8,13,29), (3,10,11,19,41),\\
(5,6,8,9,24), (5,6,8,15,30), (2,3,4,5,12), (7,10,15,19,45),\\
(7,18,27,37,81), (7,15,19,32,64), (7,19,25,41,82), (7,26,39,55,117).
\end{array}\right\}.$$

Note that Theorem 1.10 differs from [4, Theorem 4.5] in the following way.

- The series $(3, 3m+1, 3m+2, 6m+1, 12m+5)$ is omitted in [4, Theorem 4.5].
- We have removed the quintuple $(5, 7, 8, 9, 23)$ from the list of sporadic cases since $(5, 7, 8, 9, 23) = (I-k, I+k, a, a+k, 2a+k+I)$ for $I = 6$, $k = 1$ and $a = 8$.
- The infinite series in [4, Theorem 4.5] corresponding to our series $(4, 2m+3, 4m+6, 6m+7, 12m+18)$ starts from $m = 0$; we have shifted it and extracted the sporadic case $(3, 4, 6, 7, 18)$ corresponding to $m = 0$.
- The infinite series in [4, Theorem 4.5] corresponding to our series $(8, 4m+5, 4m+7, 4m+9, 12m+23)$ in [4, Theorem 4.5] starts with $m = 0$; we have shifted it and extracted the sporadic case $(5, 7, 8, 9, 23)$ corresponding to $m = 0$.

[1] We group these quintuples according to the value of I, and in each group the quintuples are ordered lexicographically.

- The infinite series in [4, Theorem 4.5] corresponding to our series $(9, 3m+8, 3m+11, 6m+13, 12m+35)$ starts with $m=-1$; we have shifted it and extracted the sporadic case $(8, 9, 11, 13, 35)$ corresponding to $m=0$ (note that the quintuple $(5, 7, 8, 9, 23)$ corresponding to $m=-1$ has already appeared from the previous series).

Remark 1.11. Arguing as in the proof of [4, Lemma 5.2], one can show that

$$\mathrm{lct}\big(X\big) \geqslant 2/3 \iff \big(a_0, a_1, a_2, a_3, d\big) \in \Big\{\big(1,1,1,1,3\big), \big(1,1,2,3,6\big)\Big\}$$

in the case when $(a_0, a_1, a_2, a_3, d) = (I-k, I+k, a, a+k, 2a+k+I)$ for some non-negative integer $k < I$ and some positive integer $a \geqslant I+k$ (cf. [5, Theorem 1.7]). These two cases are exactly ones when X is smooth.

The main purpose of this paper is to prove a technical result (see Theorem 2.2 in Section 2), which we derive from the classification of isolated quasi-homogeneous rational three-dimensional hypersurface singularities obtained in [20]. Being not very attractive on its own, Theorem 2.2 easily implies the following.

Theorem 1.12. The assertion of Theorem 1.10 holds without the assumption $I \leqslant 10$.

Therefore, we obtain a proof of the (corrected version of the) half-experimental result of [4] (i. e. Theorem 1.10) modulo [20].

As the second application of Theorem 2.2 we derive from it a classificational result in the style of [14] which is more explicit than the corresponding result of [4]. Namely, we list the cases with $I=2$. Note that obtaining the list of the cases with any bounded index requires just a bit of elementary computation modulo Theorem 2.2.

Corollary 1.13. Suppose that $I=2$. Then

- either $(a_0, a_1, a_2, a_3, d) = (1, 1, s, r, s+r)$, where $s \leqslant r$ are positive integers,
- or the quintuple (a_0, a_1, a_2, a_3, d) belongs to one of the following infinite series:
  - $(1, 2, m+1, m+2, 2m+4)$,
  - $(1, 3, 3m, 3m+1, 6m+3)$,
  - $(1, 3, 3m+1, 3m+2, 6m+5)$,
  - $(3, 3m, 3m+1, 3m+1, 9m+3)$,
  - $(3, 3m+1, 3m+2, 3m+2, 9m+6)$,
  - $(3, 3m+4, 3m+5, 6m+7, 12m+17)$,
  - $(3, 3m+1, 6m+1, 9m, 18m+3)$,
  - $(3, 3m+1, 6m+1, 9m+3, 18m+6)$,
  - $(4, 2m+3, 2m+3, 4m+4, 8m+12)$,
  - $(4, 2m+3, 4m+6, 6m+7, 12m+18)$,

  where m is a positive integer,
- or the quintuple (a_0, a_1, a_2, a_3, d) lies in the sporadic set

$$\left\{\begin{array}{c}(1,1,2,2,4),(1,4,5,7,15),(1,4,5,8,16),(1,5,7,11,22),(1,6,9,13,27),\\(1,7,12,18,36),(1,8,13,20,40),(1,9,15,22,45),(1,3,4,6,12),(1,4,6,9,18),\\(1,6,10,15,30),(2,3,4,5,12),(2,3,4,7,14),(3,4,5,10,20),(3,4,6,7,18),(3,4,10,15,30),\\(3,4,6,7,18),(5,13,19,22,57),(5,13,19,35,70),(6,9,10,13,36),(7,8,19,25,57),\\(7,8,19,32,64),(9,12,13,16,48),(9,12,19,19,57),(9,19,24,31,81),\\(10,19,35,43,105),(11,21,28,47,105),(11,25,32,41,107),(11,25,34,43,111),\\(11,43,61,113,226),(13,18,45,61,135),(13,20,29,47,107),\\(13,20,31,49,111),(13,31,71,113,226),(14,17,29,41,99)\end{array}\right\}.$$

As was already mentioned above, an interesting question about a surface X is whether X admits an orbifold Kähler–Einstein metric or not. Some obstructions are provided by Theorem 1.6, and the main instrument to prove the existence is the sufficient condition given by Theorem 1.5. Most of the examples mentioned in Theorems 1.9 and 1.10 have already been studied from this point of view. As for the series omitted in [4], we have the following.

Theorem 1.14. Suppose that

$$\big(a_0, a_1, a_2, a_3, d\big) = \big(3, 3m + 1, 3m + 2, 6m + 1, 12m + 5\big),$$

where $m \in \mathbb{Z}_{>0}$. Then $\mathrm{lct}(X) = 1$.

Theorem 1.14 can be proved along the same lines as the results of [6].

The results of [19], [14], [1], [3], [4], [6] together with Theorem 1.14 imply the following result concerning orbifold Kähler–Einstein metrics on the Del Pezzo hypersurfaces X.

Corollary 1.15. Suppose that $I < 3a_0/2$. Then

- either X admits an orbifold Kähler–Einstein metric,
- or one of the following possible exceptions occur:
  - there exist a non-negative integer $k < I$ and a positive integer $a \geqslant I + k$ such that
    $$\big(a_0, a_1, a_2, a_3, d\big) = \big(I - k, I + k, a, a + k, 2a + k + I\big),$$
  - the quintuple (a_0, a_1, a_2, a_3, d) lies in the set
    $$\left\{\begin{array}{l}(2, 3, 4, 7, 14), (7, 10, 15, 19, 45), (7, 18, 27, 37, 81),\\ (7, 15, 19, 32, 64), (7, 19, 25, 41, 82), (7, 26, 39, 55, 117)\end{array}\right\},$$
  - $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and $\phi(x_0, x_1, x_2, x_3)$ does not contain $x_1x_2x_3$,
  - $(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12)$ and $\phi(x_0, x_1, x_2, x_3)$ does not contain $x_1x_2x_3$.

Remark 1.16. One can show that there are infinitely many quintuples

$$(i - k, i + k, a, a + k, 2a + k + i)$$

such that there exists a quasismooth well-formed hypersurface in $\mathbb{P}(i - k, i + k, a, a + k)$ of degree $2a + k + i$, where k, a, i are non-negative integers such that $0 \leqslant k < i$ and $a \geqslant i + k$.

Example 1.17. A general hypersurface in $\mathbb{P}(1, 2n-1, 2n-1, 3n-2)$ of degree $6n-3$ is a quasismooth well-formed del Pezzo surface for every positive integer n. This series corresponds to the values $k = n - 1$, $a = 2n - 1$ and $i = n$ of Remark 1.16.

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2. Technical result

Let X be a quasismooth hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d (throughout this section we will not assume that the numbers a_i are ordered). The hypersurface X is given by

$$\phi\big(x, y, z, w\big) = 0 \subset \mathbb{P}\big(a_0, a_1, a_2, a_3\big) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, w\big]\Big),$$

where $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(w) = a_3$, and $\phi(x, y, z, w)$ is a quasihomogeneous polynomial of degree d.

Definition 2.1. We say that X is degenerate if $d = a_i$ for some i (cf. [13, Definition 6.5]).

The purpose of this section is to prove the following result.

Theorem 2.2. Suppose that $a_0 \leqslant \ldots \leqslant a_3$, and the hypersurface $X \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ is a well-formed non-degenerate del Pezzo surface. Then

- either there exist a non-negative integer $k < I$ and a positive integer $a \geqslant I + k$ such that
$$\big(a_0, a_1, a_2, a_3, d\big) = \big(I - k, I + k, a, a + k, 2a + k + I\big),$$
- or $I = a_i + a_j$ for some distinct i and j,
- or $I = a_i + \frac{a_j}{2}$ for some distinct i and j,
- or $(a_0, a_1, a_2, a_3, d, I)$ belongs to one of the infinite series listed in Table 1,
- or $(a_0, a_1, a_2, a_3, d, I)$ lies in the sporadic set listed in Table 2.

Remark 2.3. Note that the first three cases of Theorem 2.2 are not mutually exclusive. On the other hand, since the most interesting cases (say, from the point of view of Kähler–Einstein metrics) appear in the last two cases of Theorem 2.2, we designed the tables in Appendix A so that the cases listed there are mutually exclusive, and none of them is contained in any of the first three cases of Theorem 2.2. One can check that for each sixtuple $(a_0, a_1, a_2, a_3, d, I)$ listed in Table 1 and 2, there exists a well-formed quasismooth hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d (apparently, this is not the case with the first three cases of Theorem 2.2).

Remark 2.4. If $I = a_i + a_j$ or $I = a_i + a_j/2$ for some i and j, then $\mathrm{lct}(X) \leqslant 2/3$. Unfortunately, we do not know how to handle the problem of existence of Kähler–Einstein metrics in these cases. Neither we know this for the first case of Theorem 2.2. Note that the Bishop and Lichnerowicz obstructions (see Theorem 1.6) are not enough to settle this question.

The proof of Theorem 2.2 is based on the classification of isolated three-dimensional quasihomogeneous rational hypersurface singularities. Consider a singularity (V, O) defined by the equation
$$\phi\big(x, y, z, w\big) = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z, w\big]\Big),$$
where O is the origin of $\mathbb{C}^4$. Suppose that (V, O) is an isolated singularity (this happens if and only if the corresponding hypersurface $X \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ is quasismooth). Suppose also that V is indeed singular at the point O, i. e. $\mathrm{mult}_O(V) \geqslant 2$ (this happens if and only if the corresponding hypersurface $X \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ is non-degenerate). The following classificational result may be obtained by studying Newton diagrams of the corresponding polynomials.

Theorem 2.5 ([20, Theorem 2.1])**.** One has
$$\phi\big(x, y, z, w\big) = \xi\big(x, y, z, w\big) + \chi\big(x, y, z, w\big)$$
where $\xi(x, y, z, w)$ and $\chi(x, y, z, w)$ are quasihomogeneous polynomials of degree d with respect to the weights $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$ such that the quasihomogeneous polynomials $\xi(x, y, z, w)$ and $\chi(x, y, z, w)$ do not have common monomials, the equation
$$\xi\big(x, y, z, w\big) = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z, w\big]\Big),$$
defines an isolated singularity, and $\xi(x, y, z, w)$ is one of the following polynomials:

I $Ax^\alpha + By^\beta + Cz^\gamma + Dw^\delta$,
II $Ax^\alpha + By^\beta + Cz^\gamma + Dzw^\delta$,
III $Ax^\alpha + By^\beta + Cz^\gamma w + Dzw^\delta$,
IV $Ax^\alpha + Bxy^\beta + Cz^\gamma + Dzw^d$,
V $Ax^\alpha y + Bxy^\beta + Cz^\gamma + Dzw^\delta$,
VI $Ax^\alpha y + Bxy^\beta + Cz^\gamma w + Dzw^\delta$,
VII $Ax^\alpha + By^\beta + Cyz^\gamma + Dzw^\delta$,
VIII $Ax^\alpha + By^\beta + Cyz^\gamma + Dyw^\delta + Ez^\epsilon w^\zeta$,
IX $Ax^\alpha + By^\beta w + Cz^\gamma w + Dyw^\delta + Ey^\epsilon z^\zeta$,

X $Ax^\alpha + By^\beta z + Cz^\gamma w + Dyw^\delta$,
XI $Ax^\alpha + Bxy^\beta + Cyz^\gamma + Dzw^\delta$,
XII $Ax^\alpha + Bxy^\beta + Cxz^\gamma + Dyw^\delta + Ey^\epsilon z^\zeta$,
XIII $Ax^\alpha + Bxy^\beta + Cyz^\gamma + Dyw^\delta + Ez^\epsilon w^\zeta$,
XIV $Ax^\alpha + Bxy^\beta + Cxz^\gamma + Dxw^\delta + Ey^\epsilon z^\zeta + Fz^\eta w^\theta$,
XV $Ax^\alpha y + Bxy^\beta + Cxz^\gamma + Dzw^\delta + Ey^\epsilon z^\zeta$,
XVI $Ax^\alpha y + Bxy^\beta + Cxz^\gamma + Dxw^\delta + Ey^\epsilon z^\zeta + Fz^\eta w^\theta$,
XVII $Ax^\alpha y + Bxy^\beta + Cyz^\gamma + Dxw^\delta + Ey^\epsilon w^\zeta + Fx^\eta z^\theta$,
XVIII $Ax^\alpha z + Bxy^\beta + Cyz^\gamma + Dyw^\delta + Ez^\epsilon w^\zeta$,
XIX $Ax^\alpha z + Bxy^\beta + Cz^\gamma w + Dyw^\delta$,

where $\alpha, \beta, \gamma, \delta$ are positive integers, $\epsilon, \zeta, \eta, \theta$ are non-negative integers, and A, B, C, D, E, F are complex numbers.

We will refer to the latter polynomials according to case labelling in Theorem 2.5. For simplicity of notations, we suppose that $A = B = C = D = E = F = 1$ in the rest of the paper[2].

In order to prove Theorem 2.2 we will suppose that $d < \sum_{i=0}^{3} a_i$ (this happens if and only if X is a del Pezzo surface, provided that X is well-formed). Then the singularity (V, O) is canonical (see Remark 1.2), and thus $\mathrm{mult}_O(V) \leqslant 3$. Moreover, the singularity (V, O) is rational (see Remark 1.2). The main result of [20] is a classification of (the deformation families of) the quasihomogeneous polynomials that define isolated three-dimensional quasihomogeneous *rational* hypersurface singularities up to an analytical change of coordinates (in some sense it is a refinement of Theorem 2.5). To give a classification of quasismooth del Pezzo hypersurfaces in the weighted projective spaces we actually need the classification of such polynomials up to the change of coordinates that is compatible with the corresponding $\mathbb{C}^*$-action (i. e., the change of coordinates that respects the weights).[3] Indeed, while the weights of the variables are not fixed even if one fixes a polynomial $\xi(x, y, z, w)$ from Theorem 2.5 that is homogeneous with respect to these weights (since one can multiply all of them by some constant), the corresponding *well-formed* weighted projective space and thus the family of the corresponding well-formed hypersurfaces becomes fixed in this case. Fortunately, these two classifications are not very far from each other. To recover the latter from the former is not a difficult task, but still it requires some additional work. Luckily, to prove Theorem 2.2 we don't need to do it in full generality, since we can disregard polynomials whose degree d (and thus the index I either) equals a sum of two of the weights. The latter are included in one of the types of our resulting classification (see Theorem 2.2). If there is a *unique* choice of weights $\mathrm{wt}(x)$, $\mathrm{wt}(y)$, $\mathrm{wt}(z)$, $\mathrm{wt}(w)$ that makes some of the polynomials obtained from the polynomial $\xi(x, y, z, w)$ by an analytical change of coordinates quasihomogeneous, then one trivially obtains that any change of coordinates that turns ξ into another quasihomogeneous polynomial must agree with the corresponding $\mathbb{C}^*$-action. Furthermore, this is the case if we restrict ourselves to the weights that are at most $d/2$, where d is the total weight of a corresponding polynomial (see [17, Lemma 4.3]). Therefore, the polynomials that we need to recover must be homogeneous with respect to the weights such that one of the weights, say $\mathrm{wt}(x)$, is strictly larger than $d/2$. In this case we have

[2]The singularity defined by $\xi(x, y, z, w)$ is not necessary isolated if $A = \ldots = F = 1$ (this happens, for instance, in the case XIX if $\alpha = \beta = \gamma = \delta = 1$). We hope that such abuse of notation will not lead to a confusion.

[3] Note that these two classifications indeed differ. Say, if one denotes by $\upsilon(x, y, z, w)$ the $(\alpha, \beta, \gamma, \delta)$-part of the polynomial $\xi(x, y, z, w)$, one sees that the cases when υ has less than 4 different monomials are absent from the list of [20]. These are $\xi(x, y, z, w) = x^\alpha + y^\beta + z^\gamma w + zw^\delta$ with $\gamma = \delta = 1$ (cf. [20, Case III]), $\xi(x, y, z, w) = x^\alpha y + xy^\beta + z^\gamma + zw^\delta$ with $\alpha = \beta = 1$ (cf. [20, Case V]), and $\xi(x, y, z, w) = x^\alpha y + xy^\beta + z^\gamma w + zw^\delta$ with $\alpha = \beta = 1$ or/and $\gamma = \delta = 1$ (cf. [20, Case VI]). It is easy to check that the listed cases are equivalent up to an analytical change of coordinates to some other cases that are present in the list of [20], but one can choose the weights of variables so that there does not exist such change of coordinates that respects the weights.

$\xi = xg + h$, where g and h are polynomials that do not depend on x. By quasismoothness at least one other variable occurs linearly in g, so by a $\mathbb{C}^*$-equivariant coordinate transformation we may assume that g is a coordinate, say y. Now collect all terms divisible by y and absorb them in xy by a ($\mathbb{C}^*$-equivariant) coordinate change in x. Still we have to take care of all polynomials that are obtained from ξ by an analytical change of coordinates (note that these may not contain a monomial that is a product of two variables even if ξ does). The rank of the hypersurface singularity in question is at least 2. The latter is preserved under the analytical change of coordinates, so it is enough for our purposes to describe all possible quasihomogeneous polynomials f (say, in variables x_0, x_1, x_2 and x_4) giving a singularity of rank r equal to 2, 3 or 4, and not containing monomials x_ix_j for $i \neq j$. The latter condition implies that (up to $\mathbb{C}^*$-equivariant coordinate change) $f = x_0^2 + \ldots + x_r^2 + g(x_{r+1}, \ldots, x_4)$, where g is a polynomial in $4-r$ variables of rank 0 (i. e. corank $0 \leqslant 4-r \leqslant 2$). If $r = 4$, then $g = 0$, and if $r = 3$, then $g = x_3^n$, so that in both of these cases f is found in [20, Case I]. If $r = 2$, applying [2, §13.1] (and keeping in mind [17, Lemma 4.3]), we again see that f is contained in the list of [20] (cases I.1, II.1 and III.1).[4]

To summarize, for every $\xi(x, y, z, w)$ the possible values (up to a $\mathbb{C}^*$-equivariant change of coordinates) of the quadruple $(\alpha, \beta, \gamma, \delta)$ are listed in [20] up to the polynomials that contain a monomial which is a product of two variables. Unfortunately, as it usually happens with long lists, in the list of [20] there are some omissions. Namely, apart from minor misprints (see Examples 2.8 and 2.14 below) the following cases are omitted[5]

XI $\xi(x, y, z, w) = x^\alpha + xy^\beta + yz^\gamma + zw^\delta$ and $(\alpha, \beta, \gamma, \delta) = (2, 4, 13, 3)$,
XII $\xi(x, y, z, w) = x^\alpha + xy^\beta + xz^\gamma + yw^\delta + y^\epsilon z^\zeta$ and

$$(\alpha, \beta, \gamma, \delta, \epsilon, \zeta) \in \Big\{ \big(5, 4, 3, 2, 1, 3\big), \big(7, 4, 3, 2, 2, 2\big), \big(6, 5, 3, 2, 1, 3\big) \Big\}.$$

Remark 2.6. Note that different cases in the list of [20] are not mutually exclusive. For example, for [20, Case I.1] with $r = s = 2$ and [20, Case XIII.1(7)] with $r = 2$ there is a $\mathbb{C}^*$-action and a change of coordinates equivariant with respect to this action such that the two (deformation families of) the singularities are the same (actually, such coincidences are numerous in [20]). A side effect of this is that sometimes one has to make a ($\mathbb{C}^*$-equivariant) coordinate change to find a given polynomial in the list of [20]. For example, the polynomial $\xi = x^4 + xy^4 + xz^3 + yw^2 + y^4z$ is not found in [20, Case XII] as one could possibly expect, but in the new coordinates $x' = x$, $y' = z - x$, $z' = y$ and $w' = w$ it gives the same deformation family as [20, Case XI.3(16)] for $r = s = 4$.

Therefore, given a list of [20], to prove Theorem 2.2, we must find all singularities in this list that correspond to the well-formed hypersurfaces $X \subset \mathbb{P}(a_0, a_1, a_2, a_3)$. This means that we need to find all possible values of the quadruple $(\alpha, \beta, \gamma, \delta)$ such that

$$\gcd\big(a_i, a_j, a_k\big) = 1$$

and d is divisible by $\gcd(a_i, a_j)$ for all $i \neq j \neq k \neq i$. Let us show how to do this in the few typical cases.

Example 2.7. Suppose that the hypersurface X is well-formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in the third part of [20, Case X.3(1)]. Then

$$\xi\big(x, y, z, w\big) = x^2 + y^3z + z^5w + yw^u,$$

[4]We are grateful to J. Stevens who explained this argument to us.
[5] We are grateful to L. Morris who checked the computations of [20] and found these omissions.

where $5 \leqslant u \leqslant 18$. Hence $2a_0 = 3a_1 + a_2 = 5a_2 + a_3 = a_1 + ua_3$. Put $a_3 = a$. Then

$$\big(a_0, a_1, a_2, a_3, d\big) = \left(\frac{(15u+1)a}{22}, \frac{(4u+1)a}{11}, \frac{(3u-2)a}{11}, a, \frac{(15u+1)a}{11}\right),$$

where either $a = 1$ or $a = 11$, because $\gcd(a_1, a_2, a_3) = 1$.

Suppose that $a = 1$. Then $3u - 2$ and $4u + 1$ are divisible by 11. We see that $u = 8$. Then

$$a_0 = \frac{(15u+1)a}{22} = \frac{121}{22} \notin \mathbb{Z},$$

which is a contradiction.

We see that $a = 11$. Then u must be odd for a_0 to be integer. Thus, we obtain 7 solutions:

- $(a_0, a_1, a_2, a_3, d, I) = (38, 21, 13, 11, 76, 7)$,
- $(a_0, a_1, a_2, a_3, d, I) = (53, 29, 19, 11, 106, 6)$,
- $(a_0, a_1, a_2, a_3, d, I) = (68, 37, 25, 11, 136, 5)$,
- $(a_0, a_1, a_2, a_3, d, I) = (83, 45, 31, 11, 166, 4)$,
- $(a_0, a_1, a_2, a_3, d, I) = (98, 53, 37, 11, 196, 3)$,
- $(a_0, a_1, a_2, a_3, d, I) = (113, 61, 43, 11, 226, 2)$,
- $(a_0, a_1, a_2, a_3, d, I) = (128, 69, 49, 11, 256, 1)$.

Example 2.8. Suppose that the hypersurface X is well-formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in the second part of [20, Case XII.3(16)]. Then[6]

$$\xi\big(x, y, z, w\big) = x^3 + xy^5 + xz^2 + yw^4 + y^\epsilon z^\zeta,$$

which gives $3a_0 = a_0 + 5a_1 = a_0 + 2a_2 = a_1 + 4a_3$, which contradicts the well-formedness of X.

Example 2.9. Suppose that the hypersurface X is well-formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in [20, Case I.2]. Then

$$\xi\big(x, y, z, w\big) = x^2 + y^3 + z^3 + w^r,$$

where $r \in \mathbb{Z}_{\geqslant 3}$. Hence $2a_0 = 3a_1 = 3a_2 = ra_3$. Thus $a_0 = 3$ and $a_1 = a_2 = 2$, because

$$\gcd\big(a_0, a_1, a_2\big) = 1.$$

We see that $a_3 = 6/r$. Since $r \geqslant 3$, we have $a_3 = 1$, because $\gcd(a_1, a_2, a_3) = 1$.

Example 2.10. Suppose that the hypersurface X is well-formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in the fourth part of [20, Case IX.3(3)]. Then

$$\xi\big(x, y, z, w\big) = x^3 + y^2 w + z^6 w + yw^s + y^\epsilon z^\zeta,$$

where $s \in \mathbb{Z}_{\geqslant 6}$. Hence $3a_0 = 2a_1 + a_3 = 6a_2 + a_3 = a_1 + sa_3$. Put $a_3 = a$. Then

$$\big(a_0, a_1, a_2, a_3, d\big) = \left(\frac{(2s-1)a}{3}, (s-1)a, \frac{(s-1)a}{3}, a, (2s-1)a\right),$$

where d is divisible by $\gcd(a_1, a_2) = (s-1)a/3$. Thus, we have

$$s - 1 \mid 3\big(2s - 1\big),$$

which is possible only if 3 is divisible by $s - 1$, which contradicts the assumption $s \geqslant 6$.

Example 2.11. Suppose that the hypersurface X is well-formed, and suppose that the quasihomogeneous polynomial $\xi(x, y, z, w)$ is found in the first part of [20, Case VIII.3(5)]. Then

$$\xi\big(x, y, z, w\big) = x^2 + y^s + yz^3 + yw^3 + z^\epsilon w^\zeta,$$

[6]Note that there is a misprint in [20, Case XII.3(16)], and one should read $(5, 4)$ instead of $(4, 5)$.

where $s \in \mathbb{Z}_{\geqslant 4}$. Hence $2a_0 = sa_1 = a_1 + 3a_2 = a_1 + 3a_3$. Put $a_1 = a$. Then

$$\big(a_0, a_1, a_2, a_3, d\big) = \left(\frac{sa}{2}, a, \frac{(s-1)a}{3}, \frac{(s-1)a}{3}, sa\right),$$

where $d = sa$ is divisible by $\gcd(a_2, a_3) = (s-1)a/3$, because X is well-formed. Thus

$$s - 1 \mid 3s,$$

which implies that $s = 4$, because $s \geqslant 4$. Hence, we have

$$\big(a_0, a_1, a_2, a_3, d\big) = \big(2a, a, a, a, 4a\big),$$

which gives $a = 1$. Then X is a smooth del Pezzo surface X such that $K_X^2 = 2$.

Example 2.12. Suppose that the hypersurface X is well-formed, and suppose that the quasi-homogeneous polynomial $\xi(x, y, z, w)$ is found in the second part of [20, Case XVIII.2(2)]. Then

$$\xi\big(x, y, z, w\big) = x^2z + xy^2 + yz^s + yw^3 + z^\epsilon w^\zeta,$$

where $s \in \mathbb{Z}_{\geqslant 4}$. Hence $2a_0 + a_2 = a_0 + 2a_1 = a_1 + sa_2 = a_1 + 3a_3$. Put $a_2 = a$. Then

$$\big(a_0, a_1, a_2, a_3, d\big) = \left(\frac{(2s-1)a}{3}, \frac{(s+1)a}{3}, a, \frac{sa}{3}, \frac{(4s+1)a}{3}\right).$$

Since either s or $s+1$ is not divisible by 3, we see that a is divisible by 3. But

$$\gcd\big(a_0, a_1, a_2\big) = 1,$$

because X is well-formed. Then $a = 3$. Thus, we have

$$\big(a_0, a_1, a_2, a_3, d\big) = \big(2s-1, s+1, 3, s, 4s+1\big),$$

where $s \in \mathbb{Z}_{\geqslant 4}$. Note that if $s = 2 \mod 3$, then

$$\gcd\big(a_0, a_1, a_2\big) = 3,$$

which is impossible. Then either $s = 0 \mod 3$ or $s = 1 \mod 3$.

Suppose that $s = 0 \mod 3$. Then $s = 3n$ for some $n \in \mathbb{Z}_{\geqslant 2}$. We have

$$\big(a_0, a_1, a_2, a_3, d\big) = \big(6n-1, 3n+1, 3, 3n, 12n+1\big),$$

and d is not divisible by $\gcd(a_2, a_3) = 3$, which contradicts the well-formedness of X.

We see that $s = 1 \mod 3$. Then $s = 3n+1$ for some $n \in \mathbb{Z}_{\geqslant 2}$. We have

$$\big(a_0, a_1, a_2, a_3, d, I\big) = \big(6n+1, 3n+2, 3, 3n+1, 12n+5, 2\big).$$

Example 2.13. Suppose that the hypersurface X is well-formed, and suppose that the quasi-homogeneous polynomial $\xi(x, y, z, w)$ is found in [20, Case IX.2(1)]. Then

$$\xi\big(x, y, z, w\big) = x^2 + y^2w + z^rw + yw^s + y^\epsilon z^\zeta,$$

where $r \in \mathbb{Z}_{\geqslant 2} \ni s$. Hence $2a_0 = 2a_1 + a_3 = ra_2 + a_3 = a_1 + sa_3$. Put $a_2 = a$. Then

$$\big(a_0, a_1, a_2, a_3, d\big) = \left(\frac{(2s-1)ra}{4(s-1)}, \frac{ra}{2}, a, \frac{ra}{2(s-1)}, \frac{(2s-1)ra}{2(s-1)}\right).$$

Note that $\gcd(2s-1, 4(s-1)) = 1$. Thus ra is divisible by $4(s-1)$. But

$$\gcd\big(a_0, a_1, a_3\big) = 1,$$

because the hypersurface X is well-formed. Then $ra = 4(s-1)$. Hence, we have

$$\big(a_0, a_1, a_2, a_3, d\big) = \left(2s-1, 2s-2, \frac{4s-4}{r}, 2, 4s-2\right),$$

where d is divisible by $\gcd(a_1, a_2)$. Hence $r(4s-2)$ is divisible by $s-1$. Then

$$r = k(s-1)$$

for some $k\in\mathbb{Z}_{\geqslant 1}$. Since $4/k=a_2\in\mathbb{Z}_{>0}$, one obtains that $k\in\{1,2,4\}$.

If $k\in\{1,2\}$, then $\gcd(a_1,a_2,a_3)=2$, which is impossible. We see that $k=4$. Then

$$\big(a_0,a_1,a_2,a_3,d\big)=\big(2s-1,2s-2,1,2,4s-2\big).$$

Example 2.14. Suppose that the hypersurface X is well-formed, and suppose that the quasi-homogeneous polynomial $\xi(x,y,z,w)$ is found in the first part of [20, Case V.3(4)]. Then[7]

$$\xi\big(x,y,z,w\big)\in\Big\{yx^3+xy^3+z^2+zw^s,\ yx^3+xy^3+z^s+zw^2\Big\},$$

where $s\in\mathbb{Z}_{\geqslant 3}$. If $\xi(x,y,z,w)=yx^3+xy^3+z^2+zw^s$, then

$$3a_0+a_1=a_0+3a_1=2a_2=a_2+sa_3$$

which contradicts the well-formedness of the hypersurface X.

We have $\xi(x,y,z,w)=yx^3+xy^3+z^s+zw^2$. Then $3a_0+a_1=a_0+3a_1=sa_2=a_2+2a_3$ and

$$\big(a_0,a_1,a_2,a_3,d\big)=\left(\frac{sa}{4},\frac{sa}{4},a,\frac{(s-1)a}{2},sa\right),$$

where $a_2=a$. Since $\gcd(a_0,a_1,a_2)=1$, we see that $4\mid a$. Then $a\in\{2,4\}$.

Suppose that $a=2$. Then

$$\big(a_0,a_1,a_2,a_3,d\big)=\left(\frac{s}{2},\frac{s}{2},2,s-1,2s\right),$$

where s is divisible by 2 and not divisible by 4. Then $s=4n+2$, where $n\in\mathbb{Z}_{\geqslant 1}$. We have

$$\big(a_0,a_1,a_2,a_3,d,I\big)=\big(2n+1,2n+1,2,4n+1,8n+4,1\big).$$

Suppose that $a=4$. Then $(a_0,a_1,a_2,a_3,d)=(s,s,4,2s-2,4s)$. Then

$$\big(a_0,a_1,a_2,a_3,d,I\big)=\big(2n+1,2n+1,4,4n,8n+4,2\big)$$

for some $n\in\mathbb{Z}_{\geqslant 1}$, because s must be odd.

Example 2.15. Suppose that the hypersurface X is well-formed, and suppose that the quasi-homogeneous polynomial $\xi(x,y,z,w)$ is found in the first part of [20, Case XI.3(14)]. Then

$$\xi\big(x,y,z,w\big)=x^3+xy^3+yz^s+zw^2,$$

where $s\in\mathbb{Z}_{\geqslant 3}$. Hence $3a_0=a_0+3a_1=a_1+sa_2=a_2+2a_3$. Put $a_2=a$. Then

$$\big(a_0,a_1,a_2,a_3,d\big)=\left(\frac{3as}{7},\frac{2as}{7},a,\frac{a(9s-7)}{14},\frac{9as}{7}\right),$$

and $\gcd(a_0,a_1,a_2)=1$, because X is well-formed. Thus either $a=1$ or $a=7$.

Suppose that $a=1$. Then

$$\big(a_0,a_1,a_2,a_3,d\big)=\left(\frac{3s}{7},\frac{2s}{7},1,\frac{9s-7}{14},\frac{9s}{7}\right),$$

which implies that $s=7k$ for some $k\in\mathbb{Z}_{\geqslant 1}$. Hence, we have

$$\big(a_0,a_1,a_2,a_3,d\big)=\left(3k,2k,1,\frac{9k-1}{2},9k\right),$$

which implies that $k=2n-1$ for some $n\in\mathbb{Z}_{\geqslant 1}$. We have

$$\big(a_0,a_1,a_2,a_3,d,I\big)=\big(6n-3,4n-2,1,9n-5,18n-9,n\big).$$

Suppose that $a=7$. Then

$$\big(a_0,a_1,a_2,a_3,d\big)=\left(3s,2s,7,\frac{9s-7}{2},9s\right),$$

[7]Note that there is a misprint in [20, Case V.3(4)] and one should read $(r,s)=(3,s)$ instead of $(s,r)=(3,s)$, and the same correction should be made in the second and the third part of this subcase.

which implies that $s=2k+1$ for some $k\in\mathbb{Z}_{\geqslant 1}$. Hence, we have

$$\big(a_0,a_1,a_2,a_3,d\big)=\big(6k+3,4k+2,7,9k+1,18k+9\big),$$

but $\gcd(a_0,a_1,a_2)=1$. Then $k\neq 3 \mod 7$. Thus, we have the following solutions:[8]

- $(a_0,a_1,a_2,a_3,d,I)=(28n-22,42n-33,7,63n-53,126n-99,7n-2)$,
- $(a_0,a_1,a_2,a_3,d,I)=(28n-18,42n-27,7,63n-44,126n-81,7n-1)$,
- $(a_0,a_1,a_2,a_3,d,I)=(28n-10,42n-15,7,63n-26,126n-45,7n+1)$,
- $(a_0,a_1,a_2,a_3,d,I)=(28n-6,42n-9,7,63n-17,126n-27,7n+2)$,
- $(a_0,a_1,a_2,a_3,d,I)=(28n-2,42n-3,7,63n-8,126n-9,7n+3)$,
- $(a_0,a_1,a_2,a_3,d,I)=(28n+2,42n+3,7,63n+1,126n+9,7n+4)$,

where $n\in\mathbb{Z}_{\geqslant 1}$.

Example 2.16. Suppose that the hypersurface X is well-formed, and suppose that the quasi-homogeneous polynomial $\xi(x,y,z,w)$ is found in [20, Case VIII.2(1)]. Then

$$\xi\big(x,y,z,w\big)=x^2+y^2+yz^r+yw^s+z^\epsilon w^\zeta,$$

where $r\in\mathbb{Z}_{\geqslant 2}\ni s$. Hence $2a_0=2a_1=a_1+ra_2=a_1+sa_3$. Put $a_2=a$. Then

$$\big(a_0,a_1,a_2,a_3,d\big)=\left(ra,ra,a,\frac{ra}{s},2ra\right),$$

where $a=\gcd(a_0,a_1,a_2)=1$, because X is well-formed. Thus, we have

$$\big(a_0,a_1,a_2,a_3,d\big)=\left(r,r,1,\frac{r}{s},2r\right),$$

where $r/s=\gcd(a_0,a_1,a_3)=1$. Then $(a_0,a_1,a_2,a_3,d)=(r,r,1,1,2r)$.

Example 2.17. Suppose that the hypersurface X is well-formed, and suppose that the quasi-homogeneous polynomial $\xi(x,y,z,w)$ is found in [20, Case XIV.1(1)]. Then

$$\xi\big(x,y,z,w\big)=x^r+xy+xz^s+xw^t+y^\epsilon z^\zeta+z^\eta w^\theta,$$

where $r,s,t\in\mathbb{Z}_{\geqslant 2}$. Hence $ra_0=a_0+a_1=a_0+sa_2=a_0+ta_3$. Put $a_0=a$. Then

$$\big(a_0,a_1,a_2,a_3,d\big)=\left(a,(r-1)a,\frac{(r-1)a}{s},\frac{(r-1)a}{t},ra\right),$$

It follows from the well-formedness of the hypersurface X that

$$\gcd\big(a_0,a_1,a_2\big)=\gcd\big(a_0,a_1,a_3\big)=1,$$

so that a divides s and t. Put $s=ap$ and $t=aq$ for some $q\in\mathbb{Z}_{\geqslant 1}\ni p$. Then

$$\gcd\left(\frac{r-1}{p},\frac{r-1}{q}\right)=1,$$

because $\gcd(a_1,a_2,a_3)=1$, where $r-1$ is divisible by p and q. Thus, we see that

$$p=mk,\ q=ml,\ r-1=mkl,$$

where m, k and l are positive integers such that $\gcd(k,l)=1$. Then

$$\big(a_0,a_1,a_2,a_3,d\big)=\big(a,mkla,l,k,(mkl+1)a\big)$$

By well-formedness one obtains that d is divisible by $\gcd(a_1,a_2)=l$. Then $l\mid a$ and

$$l\mid\gcd\big(a_0,a_1,a_2\big)$$

so that by well-formedness $l=1$. In a similar way we get $k=1$. Then

$$\big(a_0,a_1,a_2,a_3,d,I\big)=\big(a,ma,1,1,(m+1)a,2\big),$$

[8]Note that in the resulting tables of Appendix A we split the first of the obtained series into a sporadic case corresponding to $n=1$ and a shifted series starting from $n=2$. This is done to ensure that $a_0\leqslant\ldots\leqslant a_3$.

where m and a are arbitrary positive integers.

The proof of Theorem 2.2 is similar in the remaining cases.

3. Bishop vs Lichnerowicz

In this section, we prove Theorem 1.7. Let $\bar{a}_0, \ldots, \bar{a}_n, \bar{d}$ be positive real numbers such that

$$0 < \sum_{i=0}^{n} \bar{a}_i - \bar{d} \leqslant n\bar{a}_0$$

and $\bar{a}_0 \leqslant \bar{a}_1 \leqslant \ldots \leqslant \bar{a}_n$, where $n \geqslant 1$. To prove Theorem 1.7, we must show that

$$\bar{d}\left(\sum_{i=0}^{n} \bar{a}_i - \bar{d}\right)^n \leqslant n^n \prod_{i=0}^{n} \bar{a}_i.$$

Put $\bar{I} = \sum_{i=0}^{n} \bar{a}_i - \bar{d}$. Then $I = \alpha n \bar{a}_0$, where $\alpha \in \mathbb{R}$ such that $0 < \alpha \leqslant 1$. We must prove that

$$\left(\sum_{i=1}^{n} \bar{a}_i + \big(1 - \alpha n\big)\bar{a}_0\right)\bar{a}_0^{n-1}\alpha^n - \prod_{i=1}^{n} \bar{a}_i \leqslant 0. \tag{3.1}$$

Put $a_i = \bar{a}_i / \bar{a}_0$ for every $i \in \{1, \ldots, n\}$. Then (3.1) is equivalent to

$$\left(\sum_{i=1}^{n} a_i + 1 - \alpha n\right)\alpha^n - \prod_{i=1}^{n} a_i \leqslant 0, \tag{3.2}$$

where $a_1 \geqslant 1, a_2 \geqslant 1, \ldots, a_n \geqslant 1$. But to prove (3.2) is enough to prove that

$$\sum_{i=1}^{n} a_i + 1 - n - \prod_{i=1}^{n} a_i \leqslant 0, \tag{3.3}$$

because the derivative of the left hand side of (3.2) with respect to α equals

$$n\alpha^{n-1}\left(\sum_{i=1}^{n} a_i + 1 - \alpha\big(n+1\big)\right) \geqslant n\alpha^{n-1}\left(\sum_{i=1}^{n} a_i - n\right) \geqslant 0,$$

since $\alpha \leqslant 1$ and $a_i \geqslant 1$ every $i \in \{1, \ldots, n\}$. Let us prove (3.3) by induction on n.

We may assume that $n \geqslant 2$, and $a_i \neq 1$ for every $i \in \{1, \ldots, n\}$ by the induction assumption.

Lemma 3.4. Suppose that $a_i \geqslant n$ for some $i \in \{1, \ldots, n\}$. Then the inequality 3.3 holds.

Proof. Without loss of generality, we may assume that $a_n \geqslant n$. Then

$$\sum_{i=1}^{n} a_i + 1 - n - \prod_{i=1}^{n} a_i = \sum_{i=1}^{n-1}\left(a_i - \prod_{i=1}^{n-1} a_i\right) + \big(a_n - n + 1\big)\left(1 - \prod_{i=1}^{n-1} a_i\right) \geqslant 0$$

which completes the proof. $\square$

Put $F(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i + 1 - n - \prod_{i=1}^{n} x_i$. Let $U \subset \mathbb{R}^n$ be an open set given by

$$1 < a_1 < n,\ 1 < a_2 < n, \ldots, 1 < a_n < n,$$

and suppose that (3.3) fails. Then $F(a_1, \ldots, a_n) > 0$. But

$$\big(x_1, \ldots, x_n\big) \in \overline{U} \setminus U \Longrightarrow F\big(x_1, \ldots, x_n\big) \leqslant 0,$$

which implies that F attains its maximum at some point $(A_1, \ldots, A_n) \in U$. Thus, we have

$$A_k = \prod_{i=1}^{n} A_i$$

for every $k \in \{1, \ldots, n\}$ by the first derivative test. The latter implies $A_1 = A_2 = \ldots = A_n$. Then

$$nA_1 + 1 - n - A_1^n > 0,$$

which is impossible, because $nA_1 + 1 - n - A_1^n$ is a decreasing function of A_1 vanishing at $A_1 = 1$.

The assertion of Theorem 1.7 is proved.

Appendix A. Tables

Table 1 and Table 2 contain one-parameter infinite series and sporadic cases respectively of values of $(a_0, a_1, a_2, a_3, d, I)$ in Theorem 2.2. We always assume that $a_0 \leqslant \ldots \leqslant a_3$. The last columns represent the cases in [20] from which the sixtuples $(a_0, a_1, a_2, a_3, d, I)$ originate[9]. The parameter n is any positive integer.

Table 1: Infinite series

(a_0, a_1, a_2, a_3)	d	I	Source
$(1, 3n-2, 4n-3, 6n-5)$	$12n-9$	n	VII.2(3)
$(1, 3n-2, 4n-3, 6n-4)$	$12n-8$	n	II.2(2)
$(1, 4n-3, 6n-5, 9n-7)$	$18n-14$	n	VII.3(1)
$(1, 6n-5, 10n-8, 15n-12)$	$30n-24$	n	III.1(4)
$(1, 6n-4, 10n-7, 15n-10)$	$30n-20$	n	III.2(2)
$(1, 6n-3, 10n-5, 15n-8)$	$30n-15$	n	III.2(4)
$(1, 8n-2, 12n-3, 18n-5)$	$36n-9$	$2n$	IV.3(3)
$(2, 6n-3, 8n-4, 12n-7)$	$24n-12$	$2n$	II.2(4)
$(2, 6n+1, 8n+2, 12n+3)$	$24n+6$	$2n+2$	II.2(1)
$(3, 6n+1, 6n+2, 9n+3)$	$18n+6$	$3n+3$	II.2(1)
$(7, 28n-18, 42n-27, 63n-44)$	$126n-81$	$7n-1$	XI.3(14)
$(7, 28n-17, 42n-29, 63n-40)$	$126n-80$	$7n+1$	X.3(1)
$(7, 28n-13, 42n-23, 63n-31)$	$126n-62$	$7n+2$	X.3(1)
$(7, 28n-10, 42n-15, 63n-26)$	$126n-45$	$7n+1$	XI.3(14)
$(7, 28n-9, 42n-17, 63n-22)$	$126n-44$	$7n+3$	X.3(1)
$(7, 28n-6, 42n-9, 63n-17)$	$126n-27$	$7n+2$	XI.3(14)
$(7, 28n-5, 42n-11, 63n-13)$	$126n-26$	$7n+4$	X.3(1)
$(7, 28n-2, 42n-3, 63n-8)$	$126n-9$	$7n+3$	XI.3(14)
$(7, 28n-1, 42n-5, 63n-4)$	$126n-8$	$7n+5$	X.3(1)
$(7, 28n+2, 42n+3, 63n+1)$	$126n+9$	$7n+4$	XI.3(14)
$(7, 28n+3, 42n+1, 63n+5)$	$126n+10$	$7n+6$	X.3(1)
$(7, 28n+6, 42n+9, 63n+10)$	$126n+27$	$7n+5$	XI.3(14)
$(2, 2n+1, 2n+1, 4n+1)$	$8n+4$	1	II.3(4)
$(3, 3n, 3n+1, 3n+1)$	$9n+3$	2	III.5(1)
$(3, 3n+1, 3n+2, 3n+2)$	$9n+6$	2	II.5(1)

[9]Note that sometimes a sixtuple $(a_0, a_1, a_2, a_3, d, I)$ originates from several cases in [20].

Table 1: Infinite series

(a_0, a_1, a_2, a_3)	d	I	Source
$(3, 3n+1, 3n+2, 6n+1)$	$12n+5$	2	XVIII.2(2)
$(3, 3n+1, 6n+1, 9n)$	$18n+3$	2	VII.3(2)
$(3, 3n+1, 6n+1, 9n+3)$	$18n+6$	2	II.2(2)
$(4, 2n+3, 2n+3, 4n+4)$	$8n+12$	2	V.3(4)
$(4, 2n+3, 4n+6, 6n+7)$	$12n+18$	2	XII.3(17)
$(6, 6n+3, 6n+5, 6n+5)$	$18n+15$	4	III.5(1)
$(6, 6n+5, 12n+8, 18n+9)$	$36n+24$	4	VII.3(2)
$(6, 6n+5, 12n+8, 18n+15)$	$36n+30$	4	IV.3(1)
$(8, 4n+5, 4n+7, 4n+9)$	$12n+23$	6	XIX.2(2)
$(9, 3n+8, 3n+11, 6n+13)$	$12n+35$	6	XIX.2(2)

Table 2: Sporadic cases

(a_0, a_1, a_2, a_3)	d	I	Source	(a_0, a_1, a_2, a_3)	d	I	Source
$(1, 3, 5, 8)$	16	1	VIII.3(5)	$(2, 3, 5, 9)$	18	1	II.2(3)
$(3, 3, 5, 5)$	15	1	I.19	$(3, 5, 7, 11)$	25	1	X.2(3)
$(3, 5, 7, 14)$	28	1	VII.4(4)	$(3, 5, 11, 18)$	36	1	VII.3(1)
$(5, 14, 17, 21)$	56	1	XI.3(8)	$(5, 19, 27, 31)$	81	1	X.3(3)
$(5, 19, 27, 50)$	100	1	VII.3(3)	$(7, 11, 27, 37)$	81	1	X.3(4)
$(7, 11, 27, 44)$	88	1	VII.3(5)	$(9, 15, 17, 20)$	60	1	VII.6(3)
$(9, 15, 23, 23)$	69	1	III.5(1)	$(11, 29, 39, 49)$	127	1	XIX.2(2)
$(11, 49, 69, 128)$	256	1	X.3(1)	$(13, 23, 35, 57)$	127	1	XIX.2(2)
$(13, 35, 81, 128)$	256	1	X.3(2)	$(1, 3, 4, 6)$	12	2	I.3
$(1, 4, 6, 9)$	18	2	IV.3(3)	$(1, 6, 10, 15)$	30	2	I.4
$(2, 3, 4, 7)$	14	2	IX.3(1)	$(3, 3, 4, 4)$	12	2	V.3(4)
$(3, 4, 5, 10)$	20	2	II.3(2)	$(3, 4, 6, 7)$	18	2	VII.3(10)
$(3, 4, 10, 15)$	30	2	II.2(3)	$(5, 13, 19, 22)$	57	2	X.3(3)
$(5, 13, 19, 35)$	70	2	VII.3(3)	$(6, 9, 10, 13)$	36	2	VII.3(8)
$(7, 8, 19, 25)$	57	2	X.3(4)	$(7, 8, 19, 32)$	64	2	VII.3(3)
$(9, 12, 13, 16)$	48	2	VII.6(2)	$(9, 12, 19, 19)$	57	2	III.5(1)
$(9, 19, 24, 31)$	81	2	XI.3(20)	$(10, 19, 35, 43)$	105	2	XI.3(18)
$(11, 21, 28, 47)$	105	2	XI.3(16)	$(11, 25, 32, 41)$	107	2	XIX.3(1)
$(11, 25, 34, 43)$	111	2	XIX.2(2)	$(11, 43, 61, 113)$	226	2	X.3(1)
$(13, 18, 45, 61)$	135	2	XI.3(14)	$(13, 20, 29, 47)$	107	2	XIX.3(1)
$(13, 20, 31, 49)$	111	2	XIX.2(2)	$(13, 31, 71, 113)$	226	2	X.3(2)

Table 2: Sporadic cases

(a_0, a_1, a_2, a_3)	d	I	Source	(a_0, a_1, a_2, a_3)	d	I	Source
$(14, 17, 29, 41)$	99	2	XIX.2(3)	$(5, 7, 11, 13)$	33	3	X.3(3)
$(5, 7, 11, 20)$	40	3	VII.3(3)	$(11, 21, 29, 37)$	95	3	XIX.2(2)
$(11, 37, 53, 98)$	196	3	X.3(1)	$(13, 17, 27, 41)$	95	3	XIX.2(2)
$(13, 27, 61, 98)$	196	3	X.3(2)	$(15, 19, 43, 74)$	148	3	X.3(1)
$(5, 6, 8, 9)$	24	4	VII.3(2)	$(5, 6, 8, 15)$	30	4	IV.3(1)
$(9, 11, 12, 17)$	45	4	XI.3(20)	$(10, 13, 25, 31)$	75	4	XI.3(14)
$(11, 17, 20, 27)$	71	4	XIX.3(1)	$(11, 17, 24, 31)$	79	4	XIX.2(2)
$(11, 31, 45, 83)$	166	4	X.3(1)	$(13, 14, 19, 29)$	71	4	XIX.3(1)
$(13, 14, 23, 33)$	79	4	XIX.2(2)	$(13, 23, 51, 83)$	166	4	X.3(2)
$(6, 7, 9, 10)$	27	5	XI.3(14)	$(11, 13, 19, 25)$	63	5	XIX.2(2)
$(11, 25, 37, 68)$	136	5	X.3(1)	$(13, 19, 41, 68)$	136	5	X.3(2)
$(11, 19, 29, 53)$	106	6	X.3(1)	$(13, 15, 31, 53)$	106	6	X.3(2)
$(11, 13, 21, 38)$	76	7	X.3(1)				

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