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# On the motive of the group of units of a division algebra 

Evgeny Shinder

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#### Abstract

We consider the algebraic group $G L_{1}(A)$, where $A$ is a division algebra of prime degree over a field $F$, and the associated motive in the Voevodsky category of motivic complexes $D M_{-}^{\text {eff }}(F)$. We relate the motive of $G L_{1}(A)$ to the motive of the Čech simplicial scheme $\mathcal{X}$, associated to the Severi-Brauer variety of $A$, and compute the second differential in the resulting spectral sequence for motivic cohomology.


Keywords: division algebra, Severi-Brauer variety, motivic cohomology
Mathematics Subject Classication 2010: 17A35, 11E57, 14F42, 19E15

## 1 Introduction

In this paper we consider motives and motivic cohomology of algebraic groups $G L_{1}(A)$ for a division algebra $A$ of prime degree $n$ over a perfect field $F$. Motivation to study these groups, as well as more complicated groups $S L_{1}(A)$ comes from the problems arising in algebraic $K$-theory, in particular non-triviality of $S K_{1}(A)$ [S91b], [Me].

It is proved by Biglari [B] that motives of split reductive algebraic groups such as $G L_{n}(F)$ and $S L_{n}(F)$ are Tate motives. Furthermore, using higher Chern classes in motivic cohomology constructed by Pushin $[\mathrm{Pu}]$ one can write down explicit direct sum decompositions for the motives of these two groups with integral coefficients. Proposition 4.2 in the present paper deals with the case of $G L_{n}(F)$, and the case of $S L_{n}(F)$ can be treated similarly. Non-split algebraic groups such as $G L_{1}(A)$ and $S L_{1}(A)$ are more intricate. We note however that all the complications lie in $n$-torsion effects $(n=\operatorname{deg}(A))$ : we are back in the split case if we consider motives with coefficients in $\mathbb{Z}[1 / n]$.

The motive of $G L_{1}(A)$ is closely related to the motive of the Severi-Brauer variety $S B(A)$. We follow an idea of Suslin to break up the motive $M\left(G L_{1}(A)\right)$ into two pieces: the first piece is a very simple Tate motive, whereas the second piece is a twisted Tate motive $M$ over $\mathcal{X}$, where $\mathcal{X}$ is the Čech simplicial scheme associated to the Severi-Brauer variety $S B(A)$ (Theorem 4.7). We investigate the structure of the latter motive $M$ using the twisted slice filtration, and compute the second differential in the arising spectral sequence (Theorem 4.9). Using the spectral sequence we compute some lower weight motivic cohomology groups of $G L_{1}(A)$ (Corollary 4.16) when $A$ is
given by a symbol $\theta=(\chi, a)$. We also consider the case of degree 2 algebra where one can write explicit decomposition for $M\left(G L_{1}(A)\right)$ (Proposition 4.5).

We now describe the structure of the paper in some detail.
In section 2 we recall the basic facts on central simple algebras, Severi-Brauer varieties and the groups $G L_{1}(A)$. We formulate and prove Proposition 2.8 , which is one of the key geometric tools we use. Some classical references on Severi-Brauer varieties include $[\mathrm{A}]$ and $[\mathrm{Q}]$.

In section 3 we recall some constructions and results due to Voevodsky [V00], [V03a], [V10a], [V10b], and formulate Propositions 3.5 and 3.6, which constitute the second geometric tool we need and whose proofs are rather straightforward modulo Voevodsky's general machinery. We include a version of the Rost nilpotence theorem (Corollary 3.10), which will not be used in the main body of the text, but fits naturally in the context of motives over $\mathcal{X}$ and the slice filtration and whose proof is this context is also rather straightforward.

In section 4 we consider the motive and motivic cohomology of $G L_{1}(A)$ by first looking at the split case, then the case of $n=2$ and finally the general case of prime $n \geq 3$.

Notation: Everywhere in the paper $F$ stands for a perfect field and $A$ is a central simple algebra over $F$ of degree $n$ which is assumed to be prime in Section 4. Throughout the text we keep track of a simple explicit example of a quaternion algebra $(n=2)$ in which case we assume $\operatorname{char}(F) \neq 2$. We often use the equality sign to indicate a canonical isomorphism between algebraic varieties or motives.

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## 2 Varieties associated to central simple algebras

A central simple algebra $A$ of degree $n$ over a field $F$ is an associative unital algebra of dimension $n^{2}$ over $F$ that has no nontrivial two-sided ideals and such that the center of $A$ coincides with $F$.

According to the Wedderburn theorem, $A$ is isomorphic to the matrix algebra $M_{n}(D)$ over a central division algebra $D$ over $F$. $A$ is called split if it is isomorphic to $M_{n}(F)$. It is well known that any central simple algebra splits in some finite separable extension of scalars $E / F$ :

$$
A_{E}=A \otimes_{F} E \cong M_{n}(E)
$$

Galois descent implies that det : $M_{n}\left(F^{\text {sep }}\right) \rightarrow F^{\text {sep }}$ and $t r: M_{n}\left(F^{\text {sep }}\right) \rightarrow F^{\text {sep }}$ descend to define the so called reduced norm map $N r d: A \rightarrow F$ and the reduced trace map $\operatorname{Trd}: A \rightarrow F$.
Example 2.1. Let $\operatorname{char}(F) \neq 2$. A quaternion algebra $\binom{a, b}{F}$ is defined for $a, b \in F^{*}$ to be an $F$ vector space of dimension 4 with the basis $1, i, j, k$ and multiplication $i^{2}=a, j^{2}=b, i j=-j i=k$. It follows from the Wedderburn theorem, that $\binom{a, b}{F}$ either splits or is a division algebra. Trd and $N r d$ are the usual trace and norm: $\operatorname{Tr} d(x+y i+z j+w k)=2 x, N r d(x+y i+z j+w k)=$ $x^{2}-a y^{2}-b z^{2}+a b w^{2}$.

Any central simple algebra of degree two is in fact isomorphic to a quaternion algebra.

Recall that the Severi-Brauer variety $S B(A)$ is a closed subvariety in $\operatorname{Gr}(n, A)$ representing the functor which associates to a commutative algebra $R$ over $F$ the set

$$
S B(A)(R)=\{\text { right ideals of } A \otimes R \text { which are projective of rank } n \text { over } R\} .
$$

Remark 2.2. Let $V$ be a vector space of dimension $n$ over $F$, and let $A$ be a split central simple algebra $A=\operatorname{End}(V)$. In this case we have a canonical identification

$$
S B(\operatorname{End}(V))=\mathbb{P}(V)
$$

where a one-dimensional subspace $U \subset V$ corresponds to a right ideal of operators on $V$ whose image is contained in $U$. In general we have such a description only over a splitting field of $A$, so that an arbitrary Severi-Brauer variety $S B(A)$ is a twisted form of the projective space $\mathbb{P}^{n-1}$.
Remark 2.3. If $S B(A)$ has a rational point that is to say $A$ has a right ideal $I$ of rank $n$, then $A$ has to be split. Indeed, the right multiplication action $R_{\alpha}: I \rightarrow I, a \in A$ satisfies $R_{\alpha \beta}=R_{\beta} R_{\alpha}$, and the homomorphism

$$
R: A \rightarrow \operatorname{End}(I)^{o p}=\operatorname{End}\left(I^{*}\right)
$$

is an isomorphism by the Schur lemma.
Example 2.4. In the case $A=\binom{a, b}{F}, S B(A)$ is isomorphic to a conic in $\mathbb{P}^{2}$ defined by the equation $x^{2}=a y^{2}+b z^{2}$.

By definition, $S B(A)$ being a subvariety in a Grassmannian is endowed with a locally free sheaf $\mathcal{J}$ of rank $n$ with a right $A$ action. $\mathcal{J}$ is a subsheaf of $\mathcal{O}_{S B(A)} \otimes A$. We write $\mathcal{J}^{*}$ for the dual of $\mathcal{J}$.
Remark 2.5. In the split case $A=\operatorname{End}(V), \mathcal{J}$ is identified with $V^{*} \otimes \mathcal{O}(-1)=\mathcal{H o m}(V, \mathcal{O}(-1))$ over $\mathbb{P}(V)$.
Lemma 2.6. The sheaf of algebras $\mathcal{O}_{S B(A)} \otimes A$ is isomorphic to $\mathcal{E} n d\left(\mathcal{J}^{*}\right)$.
Proof. The isomorphism is given by the right action of $A$ on $\mathcal{J}$, which is fiberwise given in Remark 2.3.

We now define the linear algebraic group $G L_{1}(A)$. For any $R$ is a commutative algebra over $F$ the $R$-points of this groups are:

$$
G L_{1}(A)(R)=\left(A \otimes_{F} R\right)^{*}=\left\{g \in A \otimes_{F} R: N r d(g) \neq 0\right\}
$$

One can consider $G L_{1}(A)$ either an open subscheme in $\mathbb{A}^{n^{2}}$ or as a form of $G L_{n}(F)$ twisted by the cocycle defining $A$.
Example 2.7. For the quaternion algebra $A=\binom{a, b}{F}, G L_{1}(A)$ is an open subscheme in $\mathbb{A}^{4}$ defined by $x^{2}-a y^{2}-b z^{2}+a b w^{2} \neq 0$.

Let $E \rightarrow T$ be a vector bundle of rank $n$ and consider the associated group scheme $\mathbf{G L}_{T}(E)$ of local automorphisms of $E$ over $T$. Let $\alpha_{E}$ be the tautological automorphism of $p^{*}(E)=\mathbf{G L}_{T}(E) \times_{T}$ $E\left(p: \mathbf{G L}_{T}(E) \rightarrow T\right.$ is the projection) which maps $(g, v)$ to $(g, g \cdot v)$. Via explicit description of $K_{1}$ by Gillet and Grayson [GG], $\alpha_{E}$ defines an element $\left[\alpha_{E}\right] \in K_{1}\left(\mathbf{G L}_{T}(E)\right)$.

This applies in particular to the case of the trivial rank $n$ bundle $E=F^{n}$ over a point, in which case we denote the corresponding element in $K_{1}\left(G L_{n}(F)\right)$ by $\left[\alpha_{0}\right]$.

Proposition 2.8. There is a canonical isomorphism of varieties over $S B(A)$

$$
S B(A) \times G L_{1}(A) \cong \mathbf{G} \mathbf{L}_{S B(A)}\left(\mathcal{J}^{*}\right)
$$

where $\mathcal{J}$ is the tautological sheaf of ideals on $S B(A)$.
Furthermore, the tautological class $\left[\alpha_{\mathcal{J}^{*}}\right] \in K_{1}\left(\mathbf{G L}\left(\mathcal{J}^{*}\right)\right)$ corresponds under this isomorphism to a class in $K_{1}\left(S B(A) \times G L_{1}(A)\right)$ which in the split case is identified with $\left[p_{1}^{*}(\mathcal{O}(1))\right] \cdot\left[p_{2}^{*}\left(\alpha_{0}\right)\right]$ where the product is the standard multiplication for algebraic $K$-groups $K_{0} \otimes K_{1} \rightarrow K_{1}$.

Proof. The first assertion follows from Lemma 2.6. Indeed we have a commutative diagram of locally free sheaves

and we simply need to pass to subvarieties of non-degenerate elements in both rows.
To prove the second assertion, consider the split case $A=\operatorname{End}(V)$, and identify $\mathcal{J}^{*}$ with $V \otimes \mathcal{O}(1)$ by Remark 2.5. Then the isomorphism in question becomes the canonical identification:

$$
\mathbb{P}(V) \times G L_{1}(E n d(V))=\mathbf{G} \mathbf{L}_{\mathbb{P}(V)}(V \otimes \mathcal{O})=\mathbf{G} \mathbf{L}_{\mathbb{P}(V)}(V \otimes \mathcal{O}(1)),
$$

and the claim follows from the following lemma.
Lemma 2.9. Let $E$ be a vector bundle and $L$ be a line bundle over the same quasiprojective base $T$. Then the tautological class $\left[\alpha_{E \otimes L}\right] \in K_{1}\left(\mathbf{G L}_{T}(E \otimes L)\right)$ corresponds to $\left[p^{*} L\right] \cdot\left[\alpha_{E}\right] \in K_{1}\left(\mathbf{G L}_{T}(E)\right)$ ( $p$ is the projection to $T$ ) under the canonical isomorphism of group schemes over $T$

$$
\mathbf{G} \mathbf{L}_{T}(E) \cong \mathbf{G} \mathbf{L}_{T}(E \otimes L)
$$

Proof. Let $\phi: \mathbf{G L}_{T}(E) \rightarrow \mathbf{G L}_{T}(E \otimes L)$ denote the isomorphism in question. $\phi$ sends each pair $\left(t \in T, g \in \operatorname{Aut}\left(E_{t}\right)\right)$, to $\left(t, g \otimes i d \in \operatorname{Aut}\left(E_{t} \otimes L_{t}\right)\right)$. Thus it follows that for $\phi^{*}\left(\alpha_{E \otimes L}\right) \in$ $\operatorname{Aut}\left(p^{*}(E) \otimes p^{*}(L)\right)$ we have

$$
\begin{equation*}
\phi^{*}\left(\alpha_{E \otimes L}\right)=\alpha_{E} \otimes i d_{p^{*}(L)} . \tag{2.1}
\end{equation*}
$$

Using the Jouanolou trick [J], we may assume that $T=\operatorname{Spec}(R)$ is affine, and then $E$ corresponds to a finitely generated projective module $M$ over $R$. In this setting $\mathbf{G L}_{T}(E)$ is also affine. Indeed if $M$ is free of $\operatorname{rank} r$, then $\mathbf{G} \mathbf{L}_{T}(E)=T \times G L_{r}(F)$, and in general $M$ is a direct summand of a trivial $R$-module, hence $\mathbf{G L}_{T}(E)$ is closed in some $T \times G L_{r}(F)$.

In the affine case the claim follows from (2.1) which is the definition of the product $K_{0}(S) \otimes$ $K_{1}(S) \rightarrow K_{1}(S)$ (see [Mi], page 27).

## 3 Motivic slice filtration

### 3.1 Generalities on Voevodsky's categories of motives

We recall some definitions and notation from [V00], [V03a], [V10a]. We work in the category $D M_{-}^{\text {eff }}(F)$ of motivic complexes over $F$ as defined in [V00] and in its full subcategory $D M_{\mathcal{X}}$ defined in [V10a] for a simplicial scheme $\mathcal{X}$ over $F$.

Recall that $D M_{-}^{e f f}(F)$ is a tensor triangulated category which admits a covariant monoidal functor from the category of smooth varieties over $F$

$$
M: S m / F \rightarrow D M_{-}^{e f f}(F)
$$

satisfying the usual properties such as Mayer-Vietoris and localization distinguished triangles.
The category of Tate motives is defined as the full subcategory $D M_{-}^{\text {eff }}(F)$ generated by Tate motives $\mathbb{Z}(q)[p], q \geq 0, p \in \mathbb{Z}$. For example $\mathbb{P}^{k}$ and $\mathbb{A}^{k}-\{0\}$ have Tate motives:

$$
\begin{align*}
M\left(\mathbb{P}^{k}\right) & =\bigoplus_{j=0}^{k} \mathbb{Z}(j)[2 j]  \tag{3.1}\\
M\left(\mathbb{A}^{k}-\{0\}\right) & =\mathbb{Z} \oplus \mathbb{Z}(k)[2 k-1] .
\end{align*}
$$

We will frequently use the Cancellation Theorem [V10b]

$$
\begin{equation*}
\operatorname{Hom}_{D M_{-}^{\text {eff }}(F)}(M(1), N(1))=\operatorname{Hom}_{D M_{-}^{\text {eff }}(F)}(M, N) \tag{3.2}
\end{equation*}
$$

where $M=M \otimes \mathbb{Z}(1)$ and by equality we mean a canonical isomorphism given by the map from the group on the right to the group on the left.

For any smooth variety $X$ the morphism $X \rightarrow \operatorname{Spec}(F)$ gives rise to a morphism of motives

$$
M(X) \rightarrow M(\operatorname{Spec}(F))=\mathbb{Z}
$$

One includes this morphism into a distinguished triangle

$$
\begin{equation*}
\widetilde{M}(X) \rightarrow M(X) \rightarrow \mathbb{Z} \rightarrow \widetilde{M}(X)[1] . \tag{3.3}
\end{equation*}
$$

A choice of rational point on $X$ (in the case a rational point exists) determines a splitting

$$
\begin{equation*}
M(X)=\widetilde{M}(X) \oplus \mathbb{Z} \tag{3.4}
\end{equation*}
$$

Taking the category $D M_{-}^{e f f}(F)$ for granted the motivic cohomology groups and the reduced motivic cohomology groups of degree $p \in \mathbb{Z}$ and weight $q \geq 0$ can be defined to be

$$
\begin{aligned}
& H^{p, q}(X):=\operatorname{Hom}_{D M_{-}^{e f f}(F)}(M(X), \mathbb{Z}(q)[p]) \\
& \widetilde{H}^{p, q}(X):=\operatorname{Hom}_{D M_{-}^{e f f}(F)}(\widetilde{M}(X), \mathbb{Z}(q)[p]),
\end{aligned}
$$

so that distinguished triangles in $D M_{-}^{e f f}(F)$ become long exact sequences in motivic cohomology of each weight. It is convenient to define motivic cohomology for $q<0$ to be identically zero.

If $Z$ is a closed subvariety in $X$, then we define the motive of $X$ with supports in $Z, M_{Z}(X)$ as

$$
M_{Z}(X):=C_{*}\left(\mathbb{Z}_{t r}(X) / \mathbb{Z}_{t r}(X-Z)\right)
$$

We have a distinguished a triangle of motives

$$
M(X \backslash Z) \rightarrow M(X) \rightarrow M_{Z}(X) \rightarrow M(X \backslash Z)[1]
$$

Recall that if $Z$ is smooth of codimension $c$ then we have the Gysin isomorphism ([SV], Theorem 4.10)

$$
\begin{equation*}
M_{Z}(X) \cong M(Z)(c)[2 c] . \tag{3.5}
\end{equation*}
$$

Lemma 3.1. If $T_{1} \subset T_{0} \subset S$ is a sequence of closed embeddings, then there is a distinguished triangle in $D M_{-}^{\text {eff }}(F)$

$$
\begin{equation*}
M_{T_{0} \backslash T_{1}}\left(S \backslash T_{1}\right) \rightarrow M_{T_{0}}(S) \rightarrow M_{T_{1}}(S) \rightarrow M_{T_{0} \backslash T_{1}}\left(S \backslash T_{1}\right)[1] . \tag{3.6}
\end{equation*}
$$

Proof. The octahedron axiom of triangulated categories ([BBD], Proposition 1.1.11) implies that the commutative square

can be completed to a $3 \times 3$ commutative square with rows and columns being distinguished triangles:


Thus cone $\left(M_{T_{0}}(S) \rightarrow M_{T_{1}}(S)\right) \cong M_{T_{0} \backslash T_{1}}\left(S \backslash T_{1}\right)[1]$ and we get the distinguished triangle (3.6).

Recall that the Čech simplicial scheme $\mathcal{X}=\check{C}(S B(A))$ (see [V03a], appendix B) is defined by Voevodsky to consist of $\mathcal{X}_{k}=S B(A)^{k+1}$ with the face and degeneracy maps taken to be partial projections and diagonals. The canonical morphism $M(\mathcal{X}) \rightarrow \mathbb{Z}$ is an isomorphism if $S B(A)$ has an $F$-point (i.e. if algebra $A$ splits). Recall that $\mathcal{X}$ is an embedded simplicial scheme, which means be definition that $M(\mathcal{X}) \otimes M(\mathcal{X})=M(\mathcal{X})$.

In [V10a], Voevodsky introduces a tensor triangulated category $D M_{-}^{\text {eff }}(\mathcal{X})$ of motives over $\mathcal{X}$ and its close relative $D M_{\mathcal{X}}$, a full subcategory of $D M_{-}^{\text {eff }}(F)$, consisting of objects $M$ satisfying the property that the canonical morphism

$$
M \otimes M(\mathcal{X}) \rightarrow M \otimes \mathbb{Z}=M
$$

is an isomorphism. Note that $M(\mathcal{X})$ is an object in $D M_{\mathcal{X}}$ and we will occasionally write $\mathbb{Z}_{\mathcal{X}}$ for $M(\mathcal{X})$ to emphasize that in the split case $\mathbb{Z}_{\mathcal{X}}$ is canonically isomorphic to $\mathbb{Z}$.

The full embedding $D M_{\mathcal{X}} \subset D M_{-}^{e f f}(F)$ admits a right adjoint functor

$$
\Phi: D M_{-}^{e f f}(F) \rightarrow D M_{\mathcal{X}}
$$

which on objects is defined to be

$$
\Phi(M)=M \otimes M(\mathcal{X})
$$

(see Lemma 6.10 in [V10a].)
Remark 3.2. It follows from the adjunction property that for any motive $M$ in $D M_{\mathcal{X}}, q \geq 0$, $p \in \mathbb{Z}$

$$
H^{p, q}(M, \mathbb{Z})=\operatorname{Hom}_{D M_{-}^{e f f}(F)}(M, \mathbb{Z}(q)[p]) \cong \operatorname{Hom}_{D M_{\mathcal{X}}}\left(M, \mathbb{Z}_{\mathcal{X}}(q)[p]\right)
$$

Let $D T(\mathcal{X}) \subset D M_{-}^{e f f}(\mathcal{X})$ denote the subcategory of effective Tate motives over $\mathcal{X}$.

### 3.2 Twisted motivic slice filtration

We need a version of a slice filtration on the categories of motivic complexes (see [V10a] and [HK]).
Let $M$ be an object in $D M_{\mathcal{X}}$. For each $q \geq 0$ we define the $q$-th term of the slice filtration of $M$ to be:

$$
\nu_{\mathcal{X}}^{\geq q} M=\underline{H o m}_{\left.D M_{-}^{e f f}(F)\right)}(\mathbb{Z}(q), M)(q) \otimes \mathbb{Z}_{\mathcal{X}} .
$$

The internal Hom-object above exists by [V00], Proposition 3.2.8.
Remark 3.3. It is easy to see using the adjunction property that

$$
{\underline{\operatorname{Hom}_{D M_{-}^{e f f}(F)}}}^{(\mathbb{Z}(q), M)(q) \otimes M(\mathcal{X})}
$$

is in fact isomorphic to

$$
\underline{\operatorname{Hom}}_{D M_{\mathcal{X}}}\left(\mathbb{Z}_{\mathcal{X}}(q), M\right)(q) .
$$

It is also easy to see that for Tate motives our slice filtration coincides with the one from [V10a].
We define $\nu_{\mathcal{X}}^{q}$ as the cone in the distinguished triangle

$$
\nu_{\mathcal{X}}^{\geq q+1}(M) \rightarrow \nu_{\mathcal{X}}^{\geq q}(M) \rightarrow \nu_{\mathcal{X}}^{q}(M) \rightarrow \nu_{\mathcal{X}}^{\geq q+1}(M)[1] .
$$

The triangulated functors $\left\{\nu_{\mathcal{X}}^{\geq q}\right\}$ commute with extension of scalars and for each $k, j \geq 0$ satisfy

$$
\nu_{\overline{\mathcal{X}}}^{\geq k+j}(M(j))=\nu_{\overline{\mathcal{X}}}^{\geq k}(M)(j) .
$$

Remark 3.4. For a split Tate motive $M=\oplus_{p, q} \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p, q}}$ we have

$$
\nu_{\overline{\mathcal{X}}}^{\geq k}(M)=\oplus_{p \geq k, q} \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p, q}}
$$

and

$$
\nu_{\mathcal{X}}^{k}(M)=\oplus_{q} \mathbb{Z}_{\mathcal{X}}(k)[q]^{\oplus a_{k, q}} .
$$

The following two propositions provide geometric criteria for motives to lie in $D M_{\mathcal{X}}$ and $D T(\mathcal{X})$ respectively.

Proposition 3.5. Let $T$ be a variety over $F$.

1. If $T$ is smooth and for each generic point $\eta$ of $T A_{F(\eta)}$ is a split algebra then $M(T)$ lies in $D M_{\mathcal{X}}$.
2. Let $T \subset S$ be a closed embedding of $T$ into a smooth variety $S$. If for each scheme-theoretic point $z \in T A_{F(z)}$ is a split algebra then $M_{T}(S)$ lies in $D M_{\mathcal{X}}$.

Proof. (1) We need to show that $M(T) \otimes C=0$ where $C=\operatorname{cone}(M(\mathcal{X}) \rightarrow \mathbb{Z})$. This follows from [V03a], Lemma 4.5.
(2) We filter $T$ by closed subvarieties

$$
T_{N} \subset T_{N-1} \subset \cdots \subset T_{1} \subset T_{0}=T \subset S
$$

where $T_{k} \backslash T_{k+1}$ are nonsingular. We prove by the descending induction on $k$ that $M_{T_{k}}(S)$ is an object in $D M_{\mathcal{X}}$. The base case $k=N$ follows from (1) and the Gysin isomorphism (3.5): since $T_{N}$ is smooth,

$$
M_{T_{N}}(S) \cong M\left(T_{N}\right)(c)[2 c] \in D M_{\mathcal{X}} .
$$

For the induction step, we use the distinguished triangle of Lemma 3.1:

$$
M_{T_{k} \backslash T_{k+1}}\left(S \backslash T_{k+1}\right) \rightarrow M_{T_{k}}(S) \rightarrow M_{T_{k+1}}(S) \rightarrow M_{T_{k} \backslash T_{k+1}}\left(S \backslash T_{k+1}\right)[1]
$$

Since by induction hypothesis and by applying the first claim of the Lemma again, $M_{T_{k+1}}(S)$ and $M_{T_{k} \backslash T_{k+1}}\left(S \backslash T_{k+1}\right)$ lie in $D M_{\mathcal{X}}, M_{T_{k}}(S)$ also lies in $D M_{\mathcal{X}}$.

Proposition 3.6. Let $M$ be an object in $D M_{\mathcal{X}}$. Assume that $M_{F(S B(A))}$ is a split Tate motive of the form $\oplus_{p, q} \mathbb{Z}(p)[q]^{\oplus a_{p, q}}$. Then the slice filtration of $M$ in $D M_{\mathcal{X}}$ has successive cones which are split Tate motives

$$
\nu_{\mathcal{X}}^{p}(M)=\oplus_{q} \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p, q}} .
$$

In particular, $M$ is a mixed Tate motive over $\mathcal{X}$.
For the proof we need the following lemma, which we borrow from $[\mathrm{S}]$.
Lemma 3.7. For any $M$ from $D M_{-}^{\text {eff }}(F)$ and $p \in \mathbb{Z}$ the extension of scalars $H^{p, 0}(M) \rightarrow$ $H^{p, 0}\left(M_{F(S B(A))}\right)$ is an isomorphism.

Proof. It is sufficient to prove the statement in the case $M=M(S)[j]$ where $S$ is a smooth connected scheme over $F$. In this case the homomorphism in question takes the form:

$$
H^{p-j, 0}(S) \rightarrow H^{p-j, 0}\left(S_{F(S B(A))}\right),
$$

and both groups are equal 0 for $p \neq j$.
$S$ is connected, and $S B(A)$ being geometrically irreducible has separably generated function field $F\left(S B(A)\right.$ ), hence $S_{F(S B(A))}$ is connected as well. Therefore if $p=j$ both cohomology groups in question are isomorphic to $\mathbb{Z}$ with the map being the identity.

Proof of Proposition 3.6. Let $\nu_{\mathcal{X}}^{p} M=c_{p}(M)(p)$. Then

$$
\begin{aligned}
& \quad \operatorname{Hom}\left(\nu_{\mathcal{X}}^{p} M, \mathbb{Z}_{\mathcal{X}}(p)[q]\right) \\
& =\operatorname{Hom}\left(c_{p}(M), \mathbb{Z}_{\mathcal{X}}[q]\right) \text { by the Cancellation Theorem }(3.2) \\
& \quad=H^{q, 0}\left(c_{p}(M), \mathbb{Z}\right) \text { by Remark } 3.2 \\
& =H^{q, 0}\left(c_{p}\left(M_{F(S B(A))}\right), \mathbb{Z}\right) \text { by Lemma 3.7 } \\
& =H^{q, 0}\left(\oplus_{r} \mathbb{Z}[r]^{\oplus a_{p, r}}, \mathbb{Z}\right) \text { by Remark } 3.4 \\
& \quad=\mathbb{Z}^{\oplus a_{p, q}} .
\end{aligned}
$$

Therefore there exists a morphism $\phi_{p}: \nu_{\mathcal{X}}^{p} M \rightarrow \oplus_{q} \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p, q}}$ such that $\phi_{p}$ becomes an isomorphism after scalar extension to $F(S B(A))$. This implies that cone $\left(\phi_{p}\right)_{F(S B(A))}=0$, so that $\operatorname{cone}\left(\phi_{p}\right)=\operatorname{cone}\left(\phi_{p}\right) \otimes M(\mathcal{X})=0$ by [V03a], Lemma 4.5, and thus $\phi_{p}$ is an isomorphism.

Remark 3.8. As the example of $M=M(S B(A))$ shows, $M$ itself is not always a split Tate motive. Indeed it is a result of Karpenko $[\mathrm{K}]$ that for a division algebra $A, M(S B(A))$ is indecomposable ${ }^{1}$.

Example 3.9. Let $A=\binom{a, b}{F}$, and let $M_{a, b}=M(S B(A))$ be the Rost motive. In this case the slice filtration is the distinguished triangle

$$
\mathbb{Z}_{\mathcal{X}}(1)[2] \rightarrow M_{a, b} \rightarrow \mathbb{Z}_{\mathcal{X}} \rightarrow \mathbb{Z}_{\mathcal{X}}(1)[3]
$$

from [V03a], Theorem 4.4.
As a corollary of Proposition 3.7 and the existence of the slice filtration we easily deduce the following version of the Rost nilpotence theorem (cf [CGM], Cor. 8.4 and [R], Cor. 10).

Proposition 3.10. Let $M$ be a Tate motive of the form $M=\bigoplus_{k=0}^{n} \mathbb{Z}\left(i_{k}\right)\left[2 i_{k}\right]$. Let

$$
f: M(S B(A)) \otimes M \rightarrow M(S B(A)) \otimes M
$$

be a morphism of motives. If $f_{F(S B(A))}$ is an isomorphism then $f$ is an isomorphism.
Proof. Consider the slice filtration on $M(S B(A)) \otimes M$. By Lemma 3.6 the slices $\nu_{\mathcal{X}}^{p}(M(S B(A)) \otimes$ $M)$ are equal to $\mathbb{Z}_{\mathcal{X}}(p)[2 p]^{\oplus a_{p}}$, for some $a_{p} \geq 0$. The morphisms induced on the slices are given by matrices with coefficients in $\operatorname{Hom}\left(\mathbb{Z}_{\mathcal{X}}, \mathbb{Z}_{\mathcal{X}}\right)$, and this group is identified with $\mathbb{Z}$ using Remark 3.2 and Lemma 3.7.

The slice filtration gives rise to an exact couple for each weight $j$

$$
\begin{gathered}
E^{p, q}=H^{p+q}\left(\nu_{\mathcal{X}}^{q}(M), \mathbb{Z}(j)\right), \\
D^{p, q}=H^{p+q}\left(M, \nu_{\mathcal{X}}^{\geq q+1}(M), \mathbb{Z}(j)\right),
\end{gathered}
$$

[^0]$$
\cdots \rightarrow D^{p+1, q-1} \rightarrow D^{p, q} \rightarrow E^{p, q} \rightarrow D^{p+2, q-1} \rightarrow \ldots
$$
and the corresponding spectral sequence
\[

$$
\begin{equation*}
E_{2}^{p, q}=H^{p+q}\left(\nu_{\mathcal{X}}^{q}(M), \mathbb{Z}(j)\right) \Rightarrow H^{p+q}(M, \mathbb{Z}(j)) \tag{3.7}
\end{equation*}
$$

\]

with the differential $d_{2}: H^{p+q-1}\left(\nu_{\mathcal{X}}^{q+1} M, \mathbb{Z}(j)\right) \rightarrow H^{p+q}\left(\nu_{\mathcal{X}}^{q} M, \mathbb{Z}(j)\right)$ induced by the $q$-th connecting morphism $\partial_{q, M}$ given by the composition of morphisms forming the slice filtration:

$$
\begin{equation*}
\partial_{q, M}: \nu_{\mathcal{X}}^{q}(M) \rightarrow \nu_{\mathcal{X}}^{\geq q+1}(M)[1] \rightarrow \nu_{\mathcal{X}}^{q+1}(M)[1] . \tag{3.8}
\end{equation*}
$$

## 4 The motive of $G L_{1}(A)$

### 4.1 The split case

We consider the group variety $G L_{n}(F)$ over a field $F$. To give an explicit description of $M\left(G L_{n}(F)\right)$ we use the higher Chern classes $c_{j, i}$ for motivic cohomology

$$
\begin{equation*}
c_{j, i}: K_{j}(X) \rightarrow H^{2 i-j, i}(X), i, j \geq 0 \tag{4.1}
\end{equation*}
$$

Note that the ordinary Chern classes are $c_{i}=c_{0, i}$. In the computations in this section we use $c_{1, i}$.
We recall the construction of the higher Chern classes using $\mathbb{A}^{1}$-motivic homotopy category $\mathcal{H}_{\bullet}(F)$ of Morel and Voevodsky. The construction we give is essentially the same as in $[\mathrm{Pu}]$ but we follow the approach of $[\mathrm{Ri}]$. The basic references for $\mathbb{A}^{1}$-homotopy is [MV], see [V98] for a short introduction.

In the homotopy category of pointed spaces $\mathcal{H}_{\bullet}(F)$ both higher algebraic $K$-theory and motivic cohomology are representable: if $X$ is a smooth variety over $F$, then

$$
\begin{aligned}
K_{j}(X) & =\operatorname{Hom}_{\mathcal{H}(F)}\left(\Sigma^{j} X_{+}, \mathbb{Z} \times \mathbf{G r}\right) \\
H^{2 i-j, i}(X) & =\operatorname{Hom}_{\mathcal{H}(F)}\left(\Sigma^{j} X_{+}, \mathbf{H}(\mathbb{Z}(i), 2 i)\right)
\end{aligned}
$$

in analogy with the situation in topology. If in addition we define

$$
\widetilde{K}_{j}(X)=\operatorname{Hom}_{\mathcal{H} \cdot(F)}\left(\Sigma^{j} X_{+}, \mathbf{G r}\right)
$$

then

$$
\begin{aligned}
& K_{0}(X)=\widetilde{K}_{0}(X) \oplus \mathbb{Z} \\
& K_{j}(X)=\widetilde{K}_{j}(X), j>0 .
\end{aligned}
$$

The Chern classes (4.1) are induced by a morphism of pointed spaces

$$
\mathbf{c}_{i}: \mathbb{Z} \times \mathbf{G r} \rightarrow \mathbf{H}(\mathbb{Z}(i), 2 i)
$$

(cf [Ri], Theorem 6.2.1.2). It follows from this definition that $c_{j, i}$ are natrual transformations of functors. We will need the following product formula.

Proposition 4.1. Let $X$ be a smooth variety. If $\lambda \in \operatorname{Pic}(X)=H^{2,1}(X)$ and $\alpha \in K_{j}(X), j>0$ or $\alpha \in \widetilde{K}_{0}(X)$, then

$$
\begin{align*}
& c_{j, i}(\lambda \cdot \alpha)=\sum_{l=0}^{k-1}(-1)^{l}\binom{i-1}{l} \lambda^{l} c_{j, i-l}(\alpha)  \tag{4.2}\\
& =c_{j, i}(\alpha)-(i-1) \lambda c_{j, i-1}(\alpha)+\frac{(i-1)(i-2)}{2} \lambda^{2} c_{j, i-2}(\alpha)+\cdots+(-1)^{i-1} \lambda^{i-1} c_{j, 1}(\alpha)
\end{align*}
$$

(the formula is independent of $j$ ).
Proof. Assume that $\alpha$ is an element in $K_{0}(X)$ of virtual rank $r$. Using the splitting principle it is easy to see that

$$
\begin{equation*}
c_{i}(\lambda \cdot \alpha)=\sum_{l=0}^{k}\binom{r-i+l}{i} \lambda^{l} c_{i-l}(\alpha) \in H^{2 i, i}(X, \mathbb{Z}) \tag{4.3}
\end{equation*}
$$

In particular, if $\alpha \in \widetilde{K}_{0}(X)$, so that $r=0$, then $\binom{-i+l}{l}=(-1)^{l}\binom{i-1}{l}$ and

$$
\begin{equation*}
c_{i}(\lambda \cdot \alpha)=\sum_{l=0}^{k-1}(-1)^{l}\binom{i-1}{l} \lambda^{l} c_{i-l}(\alpha) . \tag{4.4}
\end{equation*}
$$

To extend the formula (4.4) to $K_{j}$ we use the method of [Ri]: we consider two natural transformations of presheaves on the category $S m / X$ of smooth schemes over $X$ :

$$
\theta_{j}, \theta_{j}^{\prime}: K_{j}(-) \rightarrow H^{2 i-j, i}(-)
$$

given for $p: Y \rightarrow X$ by

$$
\alpha \in K_{j}(Y) \mapsto c_{j, i}\left(p^{*}(\lambda) \cdot \alpha\right)
$$

and

$$
\alpha \in K_{j}(Y) \mapsto \sum_{l=0}^{k-1}(-1)^{l}\binom{r(\alpha)-i+l}{l} p^{*}(\lambda)^{l} c_{j, i-l}(\alpha)
$$

respectively. Note that the virtual rank $r(\alpha)$ can be non-zero only for $\alpha \in K_{0}(X)$.
By construction $\theta_{j}, \theta_{j}^{\prime}$ are induced by two morphisms

$$
\Theta, \Theta^{\prime}: \mathbb{Z} \times \mathbf{G r} \rightarrow \mathbf{H}(\mathbb{Z}(i), 2 i)
$$

(independent of $j \geq 0$ ). By [Ri] Theorem 1.1.6 to check that $\Theta=\Theta^{\prime}$ is suffices to show that $\theta_{0}=\theta_{0}^{\prime}: K_{0}(-) \rightarrow H^{2 i, i}(-)$. This holds by (4.3).

From now on in this section we only work with Chern classes

$$
c_{i}:=c_{1, i}: K_{1}(-) \rightarrow H^{2 i-1, i}(-) .
$$

If $\alpha \in K_{1}(X)$ and $I$ is a multi-index

$$
I=\left\{1 \leq i_{1}<\cdots<i_{r} \leq n\right\}
$$

we let

$$
\begin{gathered}
|I|=i_{1}+\cdots+i_{r} \\
l(I)=r
\end{gathered}
$$

and

$$
c_{I}(\alpha)=c_{i_{1}}(\alpha) \cdots \cdots c_{i_{r}}(\alpha) \in H^{2|I|-l(I),|I|}(X) .
$$

Proposition 4.2. The motive $M\left(G L_{n}(F)\right)$ admits the following direct sum decomposition:

$$
M\left(G L_{n}(F)\right) \cong \bigoplus_{I} \mathbb{Z}(|I|)[2|I|-l(I)]
$$

where the morphism

$$
M\left(G L_{n}(F)\right) \rightarrow \mathbb{Z}(|I|)[2|I|-l(I)]
$$

corresponds to the class

$$
c_{I}(\alpha) \in H^{2|I|-l(I),|I|}\left(G L_{n}(F)\right),
$$

$[\alpha]$ is the tautological class in $K_{1}\left(G L_{n}(F)\right)$ defined in the paragraph preceding Proposition 2.8.
Proof. We define the morphism

$$
\phi: M\left(G L_{n}(F)\right) \rightarrow \bigoplus_{I} \mathbb{Z}(|I|)[2|I|-l(I)]
$$

using the classes $c_{I}$. We claim that $\phi$ is an isomorphism.
First note, that for any reductive split group $G$ over $F$ the motive $M(G)$ is a Tate motive [B], Proposition 4.2. Therefore by the Yoneda lemma it is sufficient to check that $\phi$ induces isomorphism on the motivic cohomology groups.

According to $[\mathrm{Pu}]$, Lemma 13, motivic cohomology of $G L_{n}(F)$ is generated freely by the classes $c_{I}(\alpha)$ and the statement follows.

We also need a relative version of Proposition 4.2.
Proposition 4.3. Let $E \rightarrow T$ be a vector bundle of rank $n$, and let $\alpha_{E}$ be the tautological class in $K_{1}(\mathbf{G} \mathbf{L}(E))$. The motive $M(\mathbf{G L}(E))$ admits the following decomposition:

$$
M(\mathbf{G L}(E))=\bigoplus_{I} M(T)(|I|)[2|I|-l(I)]
$$

where the morphism

$$
M(\mathbf{G L}(E)) \rightarrow M(T)(|I|)[2|I|-l(I)]
$$

is the composition

$$
\begin{gathered}
M(\mathbf{G L}(E)) \rightarrow M(\mathbf{G L}(E)) \otimes M(\mathbf{G L}(E)) \rightarrow M(\mathbf{G L}(E))(|I|)[2|I|-l(I)] \rightarrow \\
M(T)(|I|)[2|I|-l(I)]
\end{gathered}
$$

of multiplication by the class

$$
c_{I}\left(\alpha_{E}\right) \in H^{2|I|-l(I),|I|}(\mathbf{G} \mathbf{L}(E)) .
$$

followed by the canonical projection.
Proof. The statement follows from Proposition 4.2 and the Mayer-Vietoris distinguished triangle.

### 4.2 The case $n=2$

Let $A=\binom{a, b}{F}$, and let $C=S B(A)$ be the norm conic. In this case $G L_{1}(A)$ is the complement to $Q \subset \mathbb{A}^{4}-\{0\}$ in $\mathbb{A}^{4}-\{0\}$, where

$$
Q=\left\{(x, y, z, w) \in \mathbb{A}^{4}-\{0\}: x^{2}-a y^{2}-b z^{2}+a b w^{2}=0\right\}
$$

Lemma 4.4. $M(Q)=M(C) \oplus M(C)(2)[3]$.
Proof. First note that the projective quadric $\left\{x^{2}-a y^{2}-b z^{2}+a b w^{2}=0\right\} \subset \mathbb{P}^{3}$ is isomorphic to $C \times C$. Indeed we have the Segre embedding

$$
C \times C=S B(A) \times S B(A) \cong S B(A) \times S B\left(A^{\vee}\right) \rightarrow S B\left(A \otimes A^{\vee}\right) \cong S B\left(\operatorname{End}_{F}(A)\right) \cong \mathbb{P}(A) \cong \mathbb{P}^{3}
$$

and the image consists of elements of rank 1 and thus the image is given by one homogeneous equation $\operatorname{Nrd}(\alpha)=x^{2}-a y^{2}-b z^{2}+a b w^{2}=0$.

It can be proved analogously to Proposition 2.8 that $C \times C$ is a projective line bundle over $C$, therefore

$$
M(C \times C)=M(C) \oplus M(C)(1)[2]
$$

$Q$ over $C \times C$ is the complement to the zero section in the line bundle $\mathcal{O}(-1)$. We have a distinguished triangle

$$
M(C)(1)[1] \oplus M(C)(2)[3] \rightarrow M(Q) \rightarrow M(C) \oplus M(C)(1)[2] \rightarrow M(C)(1)[2] \oplus M(C)(2)[4]
$$

with the third morphism being the natural one and the claim follows since after separating the summand $M(C)(1)[2]$ the resulting distinguished triangle is split.

Proposition 4.5. There is a decomposition

$$
M\left(G L_{1}(A)\right)=\mathbb{Z} \oplus M(C)(1)[1] \oplus \mathbb{Z}_{a, b}(3)[4]
$$

where we temporarily use the notation $\mathbb{Z}_{a, b}$ for the cone of the canonical morphism $\mathbb{Z}(1)[2] \rightarrow M(C)$ corresponding to the fundamental class $[C] \in C H^{0}(C)=C H_{1}(C)$.

Proof. Consider the distinguished triangle corresponding to the open embedding

$$
\begin{gather*}
G L_{1}(A) \subset \mathbb{A}^{4}-\{0\}: \\
M_{Q}\left(\mathbb{A}^{4}-\{0\}\right)[-1] \rightarrow \widetilde{M}\left(G L_{1}(A)\right) \rightarrow \widetilde{M}\left(\mathbb{A}^{4}-\{0\}\right) \rightarrow M_{Q}\left(\mathbb{A}^{4}-\{0\}\right) \tag{4.5}
\end{gather*}
$$

We have $\widetilde{M}\left(\mathbb{A}^{4}-\{0\}\right)=\mathbb{Z}(4)[7]$ and also

$$
M_{Q}\left(\mathbb{A}^{4}-\{0\}\right)=M(Q)(1)[2]=M(C)(1)[2] \oplus M(C)(3)[5],
$$

with the first equality being Gysin isomorphism and the second one comes from Lemma 4.4.
The distinguished triangle (4.5) now can be rewritten as:

$$
M(C)(1)[1] \oplus M(C)(3)[4] \rightarrow \widetilde{M}\left(G L_{1}(A)\right) \rightarrow \mathbb{Z}(4)[7] \rightarrow M(C)(1)[2] \oplus M(C)(3)[5]
$$

By dimension reasons $\operatorname{Hom}(\mathbb{Z}(4)[7], M(C)(1)[2])=0$, therefore

$$
\widetilde{M}\left(G L_{1}(A)\right)=M(C)(1)[1] \oplus \operatorname{cone}(\mathbb{Z}(4)[7] \rightarrow M(C)(3)[5])[-1] .
$$

The morphism $\mathbb{Z}(4)[7] \rightarrow M(C)(3)[5]$ corresponds to a class in $C H_{1}(C)=C H^{0}(C)$ which can be computed after passing to a splitting field by Lemma 3.7. In the split case we can verify that the morphism in question corresponds via the Cancellation Theorem (3.2) to the fundamental class [C].

Remark 4.6. Note that in the split case $C=\mathbb{P}^{1}$ and $\mathbb{Z}_{a, b}=\mathbb{Z}$ so that the we have

$$
M\left(G L_{2}(F)\right)=\mathbb{Z} \oplus \mathbb{Z}(1)[1] \oplus \mathbb{Z}(2)[3] \oplus \mathbb{Z}(3)[4]
$$

in agreement with Proposition 4.2.

### 4.3 The general case

We assume $n \geq 3$ is a prime. Let $Z$ be the complement of $G L_{1}(A)$ in $\mathbb{A}^{n^{2}}-\{0\}$, i.e. the subvariety in $\mathbb{A}^{n^{2}}-\{0\}$ given by equation $N r d_{A}=0$. Let $M=M_{Z}\left(\mathbb{A}^{n^{2}}-\{0\}\right)[-1]$ be a motive with supports which is determined by the distinguished triangle

$$
\begin{equation*}
M \rightarrow M\left(G L_{1}(A)\right) \rightarrow M\left(\mathbb{A}^{n^{2}}-\{0\}\right) \rightarrow M[1] \tag{4.6}
\end{equation*}
$$

We concentrate on studying the motive $M$.
Theorem 4.7. 1. For $j<n^{2}$ and $p \in \mathbb{Z}$ we have a canonical isomorphism

$$
\widetilde{H}^{p, j}\left(G L_{1}(A)\right) \rightarrow H^{p, j}(M) .
$$

2. If $A$ splits, then we have a decomposition

$$
M=\widetilde{M}\left(G L_{1}(A)\right) \oplus \mathbb{Z}\left(n^{2}\right)\left[2 n^{2}-2\right]=\bigoplus_{I \neq \emptyset} \mathbb{Z}_{\mathcal{X}}(|I|)[2|I|-l(I)] \oplus \mathbb{Z}\left(n^{2}\right)\left[2 n^{2}-2\right]
$$

3. $M$ is an object in $D T(\mathcal{X})$ and the slices of the slice filtration are given by:

$$
\nu_{\mathcal{X}}^{q}(M)= \begin{cases}\bigoplus_{|I|=q} \mathbb{Z}_{\mathcal{X}}(q)[2 q-l(I)], & 1 \leq q \leq \frac{n(n+1)}{2} \\ \mathbb{Z}_{\mathcal{X}}\left(n^{2}\right)\left[2 n^{2}-2\right], & q=n^{2} \\ 0, & \text { otherwise }\end{cases}
$$

Proof. Motivic cohomology of $G L_{1}(A)$ and that of $M$ are related via the long exact sequence

$$
\widetilde{H}^{p, j}\left(\mathbb{A}^{n^{2}}-\{0\}\right) \rightarrow \widetilde{H}^{p, j}\left(G L_{1}(A)\right) \rightarrow H^{p, j}(M) \rightarrow \widetilde{H}^{p+1, j}\left(\mathbb{A}^{n^{2}}-\{0\}\right),
$$

and the first claim follows since using (3.1) we see that

$$
\widetilde{H}^{p, j}\left(\mathbb{A}^{n^{2}}-\{0\}\right)=H^{p, j}\left(\mathbb{Z}\left(n^{2}\right)\left[2 n^{2}-1\right]\right)=0
$$

for $j<n^{2}$ and any $p \in \mathbb{Z}$.
If the algebra $A$ is split, then in the distinguished triangle

$$
M \rightarrow \widetilde{M}\left(G L_{n}(F)\right) \rightarrow \widetilde{M}\left(\mathbb{A}^{n^{2}}-\{0\}\right) \rightarrow M[1]
$$

the second morphism is zero, since as a simple computation using Proposition 4.2 shows, $\operatorname{Hom}\left(\widetilde{M}\left(G L_{n}(F)\right), \widetilde{M}\left(\mathbb{A}^{n^{2}}-\{0\}\right)\right)=0$. The triangle splits yielding the first equality in the second claim. The second equality follows from Proposition 4.2.

To prove the third claim note that any point of $z \in Z$ splits $A: A_{F(z)}$ has a non-zero noninvertible element (given by $z$ ) therefore $A_{F(z)}$ is not a division algebra, and since we assume that the degree $n$ of $A$ is prime, $A_{F(z)}$ splits. The third claim now follows from Propositions 3.5 and 3.6.

We investigate the slice spectral sequence (3.7) for the motive $M$. If we consider the weights $j<n^{2}$, then by Theorem 4.7 the spectral sequence in question actually converges to $\widetilde{H}^{*, j}\left(G L_{1}(A)\right)$. It also follows from Theorem 4.7 that the second page $E_{2}$ of the spectral sequence will be formed from the motivic cohomology groups of $\mathbb{Z}_{\mathcal{X}}$. The second differential will be naturally given in terms of cohomology classes in $H^{3,1}\left(\mathbb{Z}_{\mathcal{X}}\right)$.

Lemma 4.8. If $A$ is non-split, then there is a canonical isomorphism

$$
H^{3,1}\left(\mathbb{Z}_{\mathcal{X}}\right)=\mathbb{Z} / n
$$

and if $A$ is split, $H^{3,1}\left(\mathbb{Z}_{\mathcal{X}}\right)=0$.
Proof. Assume first that $A$ is non-split. The isomorphism

$$
H^{3,1}\left(\mathbb{Z}_{\mathcal{X}}\right) \cong \operatorname{Ker}\left(\text { res }: H_{e t}^{2}\left(F, \mu_{n}\right) \rightarrow H_{e t}^{2}\left(F(S B(A)), \mu_{n}\right)\right) .
$$

is established in [MS], Proposition 1.4 (the assumption made in [MS] that the class $[A] \in{ }_{n} B r(F)$ is a symbol does not play a role in the proof).

On the other hand for any field $H_{e t}^{2}\left(F, \mu_{n}\right)$ is canonically isomorphic to the $n$-torsion of the Brauer group $\operatorname{Br}(F)$, and the kernel of the restriction map $\operatorname{Br}(F) \rightarrow \operatorname{Br}(F(X))$ is generated by the class of algebra $A$ by the classical theorem of Amitsur. Since the period of $A$ is equal to $n$ the statement of the Lemma follows.

If $A$ is split, then $\mathbb{Z}_{\mathcal{X}}=M(\operatorname{Spec}(F))$ and we have $H^{3,1}\left(\mathbb{Z}_{\mathcal{X}}\right)=H^{3,1}(\operatorname{Spec}(F))=0$ by standard vanishing theorems for motivic cohomology.

We denote the generator of $H^{3,1}\left(\mathbb{Z}_{\mathcal{X}}\right)=\mathbb{Z} / n$ corresponding to $[A] \in \operatorname{Br}(F)$ in the proof of Lemma 4.8 by $\delta$. This notation is consistent with [MS], 1.5.

Let $1 \leq q<\frac{n(n+1)}{2}$. The second differential $d_{2}$ in the slice spectral sequence for $M$ is induced by the morphism of motives (3.8)

with components

$$
\begin{equation*}
\partial_{I, J}: \mathbb{Z}_{\mathcal{X}}(q)[2 q-l(I)] \rightarrow \mathbb{Z}_{\mathcal{X}}(q+1)[2 q+3-l(J)] \tag{4.7}
\end{equation*}
$$

corresponding to multi-indices $I,|I|=q$ and $J,|J|=q+1$. Each morphism $\partial_{I, J}$ determines a class

$$
\partial_{I, J} \in H^{3-l(J)+l(I), 1}\left(\mathbb{Z}_{\mathcal{X}}\right) .
$$

Theorem 4.9. Let $A$ be a division algebra of prime degree $n \geq 3$.

1. The morphism $\partial_{I, J}$ in (4.7) is zero unless $l(I)=l(J)$ and the sequence $J$ is obtained from the sequence I by increasing exactly one index by one.
2. If $A$ is a division algebra, then there exists $c=c(A) \in \mathbb{Z} / n, c \neq 0$ with the following property: if the sequence $J$ is obtained from the sequence $I$ by increasing an index $i_{t}$ by one, then

$$
\partial_{I, J}=i_{t} \cdot c \cdot \delta \in H^{3,1}\left(\mathbb{Z}_{\mathcal{X}}\right)
$$

Finally, if $A$ is a split algebra, then all $\partial_{I, J}=0$.
Proof. The idea of the proof is to compare the slice filtration of $M$ with that of the motive of the Severi-Brauer variety $M(S B(A))$. More precisely we will express all potentially non-vanishing $\partial_{I, J}$ in terms of the 0 -th connecting morphism $\partial^{\prime}:=\partial_{0, M(S B(A))}$ (3.8) in the slice filtration of $M(S B(A))$.

We fix a weight $q$ and a multi-index

$$
I=\left\{i_{1}, \ldots, i_{r}\right\}
$$

such that

$$
|I|=\sum_{t=1}^{r} i_{t}=q .
$$

Consider the motive $M\left(S B(A) \times G L_{1}(A)\right)$. According to Proposition 2.8

$$
S B(A) \times G L_{1}(A)=\mathbf{G} \mathbf{L}_{S B(A)}\left(\mathcal{J}^{*}\right)
$$

and Proposition 4.3 implies that $M\left(S B(A) \times G L_{1}(A)\right)$ admits a direct summand

$$
M(S B(A))(q)[2 q-r] \subset M\left(S B(A) \times G L_{1}(A)\right)
$$

corresponding to the class $c_{I}\left(\alpha_{E}\right)$. We denote this embedding by $\iota$ and consider the diagram


Lemma 4.10. There exists a unique morphism $\phi$ which fits in the diagram:


Proof. From the distinguished triangle (4.6) defining $M$ we see that it is sufficient to show that

$$
\operatorname{Hom}\left(M(S B(A))(q)[2 q-r], M\left(\mathbb{A}^{n^{2}}-\{0\}\right)[\epsilon]\right)=0
$$

for $\epsilon=0,-1$. We have $M\left(\mathbb{A}^{n^{2}}-\{0\}\right)[\epsilon]=\mathbb{Z}[\epsilon] \oplus \mathbb{Z}\left(n^{2}\right)\left[2 n^{2}-1+\epsilon\right]$ so that $\operatorname{Hom}\left(M(S B(A))(q)[2 q-r], M\left(\mathbb{A}^{n^{2}}-\{0\}\right)[\epsilon]\right)=H^{\epsilon-(2 q-r),-q}(S B(A)) \oplus H^{2 n^{2}-1+\epsilon-(2 q-r), n^{2}-q}(S B(A))$.

Now both cohomology groups are zero: the first one because it is of strictly negative weight, and second one because the degree is greater than weight plus dimension:

$$
2 n^{2}-1+\epsilon-(2 q-r)-\left(n^{2}-q\right)=n^{2}-q+r-1+\epsilon>\operatorname{dim}(S B(A))=n-1
$$

under the assumptions $n \geq 3$ and $q<\frac{n(n+1)}{2}$.
The morphism

$$
\phi: M(S B(A))(q)[2 q-r] \rightarrow M
$$

that we have just defined induces a morphism on the slice filtrations of the source and target motives. For each $q \leq k \leq q+n-1$ we get a commutative diagram

where the equality on the left follows from Proposition 3.6 and the equality on the right is established by Theorem 4.7.

Each $\nu_{\mathcal{X}}^{k}(\phi)_{J},|J|=k$ is an element in the group

$$
\operatorname{Hom}\left(\mathbb{Z}_{\mathcal{X}}(k)[2 k-r] \mathbb{Z}_{\mathcal{X}}(k)[2 k-l(J)]\right)=\operatorname{Hom}\left(\mathbb{Z}_{\mathcal{X}}, \mathbb{Z}_{\mathcal{X}}[r-l(J)]\right)=H^{r-l(J), 0}\left(\mathbb{Z}_{\mathcal{X}}\right)
$$

(the second isomorphism comes from Remark 3.2). By Lemma 3.7 the latter cohomology group is isomorphic to $\mathbb{Z}$ when $l(J)=r$ and is zero otherwise. Thus in what follows each symbol $\nu_{\mathcal{X}}^{k}(\phi)_{J}$ will be considered as an integer or zero.
Lemma 4.11. 1. For a sequence $J$ with $|J|=q$, we have

$$
\nu_{\mathcal{X}}^{q}(\phi)_{J}= \begin{cases}1, & J=I=\left\{i_{1}, \ldots, i_{r}\right\} \\ 0, & \text { otherwise }\end{cases}
$$

2. For a sequence $J$ with $|J|=q+1$,

$$
\nu_{\mathcal{X}}^{q+1}(\phi)_{J}= \begin{cases}i_{t}, & J=\left\{i_{1}, \ldots, i_{t}+1, \ldots, i_{r}\right\}, t=1 \ldots r \\ 0, & \text { otherwise }\end{cases}
$$

Proof. According to Lemma 3.7, integers $\nu_{\mathcal{X}}^{q}(\phi)_{J}$ and $\nu_{\mathcal{X}}^{q+1}(\phi)_{J}$ do not change under the extension of scalars to the field $F(S B(A))$. Therefore we may assume that $A$ is split.

The diagram (4.8) takes the form

and for each $q \leq k \leq q+n-1$ the morphism $\psi$ gives rise to a morphism of slices

$$
\nu_{\mathcal{X}}^{k}(\psi): \mathbb{Z}(k)[2 k-r] \rightarrow \bigoplus_{|J|=k} \mathbb{Z}(k)[2 k-l(J)]
$$

We have $\nu_{\mathcal{X}}^{k}(M)=\nu_{\mathcal{X}}^{k}\left(G L_{n}(F)\right)$ by Theorem 4.7(2) and also $\nu_{\mathcal{X}}^{k}(\phi)$ is equal to $\nu_{\mathcal{X}}^{k}(\psi)$. The component $\nu_{\mathcal{X}}^{k}(\psi)_{J}$ can be non-zero only for $J$ with $l(J)=l(I)=r$, and for such $J$ it can be computed as follows. Consider the induced morphism on motivic cohomology:

$$
\psi^{*}: H^{*, *}\left(G L_{n}(F)\right) \rightarrow H^{*-(2 q-r), *-q}(\mathbb{P}(V)) .
$$

Let $h=c_{1}(\mathcal{O}(1)) \in C H^{1}(\mathbb{P}(V))$; then

$$
\begin{equation*}
\psi^{*}\left(c_{J}\left(\alpha_{0}\right)\right)=\sum_{k \geq q} \nu_{\mathcal{X}}^{k}(\psi)_{J} \cdot h^{k-q} \in C H^{*}(\mathbb{P}(V)) . \tag{4.9}
\end{equation*}
$$

By Proposition 4.3 motivic cohomology $H^{*, *}\left(\mathbf{G} \mathbf{L}_{\mathbb{P}(V)}\left(\mathcal{J}^{*}\right)\right)$ considered as a module over $H^{*, *}(\mathbb{P}(V))$ is free and both

$$
\left\{c_{J}\left(\alpha_{\mathcal{J}^{*}}\right)\right\}_{J}
$$

and

$$
\left\{c_{J}\left(p_{2}^{*} \alpha_{0}\right)\right\}_{J}
$$

are bases for this module. Note that by Proposition 2.8 we have $\left[p_{2}^{*}\left(\alpha_{0}\right)\right]=\left[p_{1}^{*}(\mathcal{O}(-1))\right] \cdot\left[\alpha_{\mathcal{J}^{*}}\right]$ and multiplicativity formula of the higher Chern classes (4.2) will give the transformation matrix between the two bases above. In particular from

$$
c_{j_{t}}\left(p_{2}^{*}\left(\alpha_{0}\right)\right) \equiv c_{j_{t}}\left(\alpha_{\mathcal{J}^{*}}\right)+\left(j_{t}-1\right) h c_{j_{t}-1}\left(\alpha_{\mathcal{J}^{*}}\right)\left(\bmod h^{2}\right)
$$

we see that for $J=\left\{j_{1}, \ldots, j_{r}\right\}$ we have

$$
\begin{aligned}
c_{J}\left(p_{2}^{*}\left(\alpha_{0}\right)\right) & =\prod_{t=1}^{r} c_{j_{t}}\left(p_{2}^{*}\left(\alpha_{0}\right)\right) \equiv \\
& \equiv \prod_{t=1}^{r}\left(c_{j_{t}}\left(\alpha_{\mathcal{J}^{*}}\right)+\left(j_{t}-1\right) h c_{j_{t}-1}\left(\alpha_{\mathcal{J}^{*}}\right)\right)\left(\bmod h^{2}\right) \equiv \\
& \equiv c_{J}\left(\alpha_{\mathcal{J}^{*}}\right)+\sum_{t=1}^{r}\left(j_{t}-1\right) h c_{j_{1}, \ldots, j_{t}-1, \ldots, j_{r}}\left(\alpha_{\mathcal{J}^{*}}\right)\left(\bmod h^{2}\right) .
\end{aligned}
$$

Therefore

$$
\psi^{*}\left(c_{J}\left(\alpha_{0}\right)\right)=\iota^{*}\left(c_{J}\left(p_{2}^{*}\left(\alpha_{0}\right)\right)\right) \equiv\left\{\begin{array}{ll}
1, & J=I \\
i_{t} h, & J=\left\{i_{1}, \ldots, i_{t}+1, \ldots, i_{r}\right\}, t=1 \ldots r \\
0, & \text { otherwise }
\end{array}\right\}\left(\bmod h^{2}\right)
$$

which together with (4.9) gives the desired result.
We consider the commutative diagram of the connecting morphisms (3.8) in the slice filtrations:

$$
\begin{gathered}
\mathbb{Z}_{\mathcal{X}}(q)[2 q-r] \longrightarrow \mathbb{Z}_{\mathcal{X}}(q+1)[2 q+3-r] \\
\nu_{\mathcal{X}}^{\nu_{\mathcal{X}}^{\prime}(\phi)} \downarrow \\
\bigoplus_{|J|=q} \mathbb{Z}_{\mathcal{X}}(q)[2 q-l(J)] \underset{\partial_{q}}{\longrightarrow} \bigoplus_{|J|=q+1} \mathbb{Z}_{\mathcal{X}}(q+1)[2 q+3-l(J)]
\end{gathered}
$$

From the first claim of Lemma 4.11 it follows that the left vertical map is the canonical embedding corresponding to $J=I$. Now we find that

$$
\begin{equation*}
\partial_{I, J}=\nu_{\mathcal{X}}^{q+1}(\phi)_{J} \circ \partial_{q}^{\prime}, \tag{4.10}
\end{equation*}
$$

where $\nu_{\mathcal{X}}^{q+1}(\phi)_{J}$ is determined in the second claim of Lemma 4.11. The class $\partial_{q}^{\prime}$ sits in $\operatorname{Hom}\left(\mathbb{Z}_{\mathcal{X}}(q)[2 q-r], \mathbb{Z}_{\mathcal{X}}(q+1)[2 q+3-r]\right)=H^{3,1}\left(\mathbb{Z}_{\mathcal{X}}\right)$. If $A$ splits, then the latter group is zero by Lemma 4.8 and therefore (4.10) implies that $\partial_{I, J}=0$.

If $A$ does not split, then by Lemma 4.8 the class $\partial_{q}^{\prime}$ must be of the form

$$
\partial_{q}^{\prime}=c_{q} \cdot \delta, c_{q} \in \mathbb{Z} / n
$$

The arrow $\partial_{q}^{\prime}$ which is $q$-th connecting morphism (3.8) in the slice filtration of $M(S B(A))(q)[2 q-r]$ is equal to $\partial^{\prime}(q)[2 q-r]$ where $\partial^{\prime}$ is the 0 -th connecting morphism for the slice filtration of $M(S B(A))$. Both morphisms $\partial_{q}^{\prime}$ and $\partial^{\prime}$ define the same element $c \cdot \delta \in H^{3,1}\left(\mathbb{Z}_{\mathcal{X}}\right)$ which shows that in fact $c_{q}=c$ is independent of $q$.
Lemma 4.12 ([S]). $c \in \mathbb{Z} / n$ is non-zero if $A$ is not split.
Proof. We exploit the slice spectral sequence (3.7) for $M(S B(A))$ and weight $j=1$ which has the $E_{2}$ term of the following form:


The connecting morphism $\partial^{\prime}=c \cdot \delta$ is responsible for the second differential $d_{2}$. If $c=0$, then the spectral sequence degenerates implying that the extension of scalars map

$$
C H^{1}(S B(A)) \rightarrow C H^{1}\left(\mathbb{P}^{n-1}\right)=\mathbb{Z}
$$

to a splitting field of $A$ is an isomorphism. The Picard-Brauer exact sequence shows that this can not happen unless $A$ is split (see [S84], Theorem 10.12 for a more general result).

Putting together (4.10), the second claim of Lemma 4.11 and Lemma 4.12 we obtain the desired description of the differential.

We would like to use the slice spectral sequence to compute motivic cohomology of $G L_{1}(A)$ for small weights. In order to do so we need to know the corresponding motivic cohomology groups of $\mathcal{X}$. These have been computed by Merkurjev and Suslin $[\mathrm{MS}]$ for the Čech simplicial scheme for any Rost variety $X_{\theta}$. We apply the results of $[\mathrm{MS}]$ when $\theta=(\chi, a)=\chi \cup(a) \in{ }_{n} B r(F)=H_{e t}^{2}\left(F, \mu_{n}\right)$, $\chi \in H_{e t}^{1}(F, \mathbb{Z} / n)=\operatorname{Hom}\left(\operatorname{Gal}\left(F^{\text {sep }} / F\right), \mathbb{Z} / n\right), a \in H_{e t}^{1}\left(F, \mu_{n}\right)=F^{*} /\left(F^{*}\right)^{n}$. In what follows we assume that $A$ is non-trivial a cyclic algebra ( $\chi, a$ )

We follow [MS] in using the notation

$$
\begin{aligned}
H^{*, *}(\mathcal{X})^{\geq 0} & :=\bigoplus_{p-q-1 \geq 0} H^{p, q}(\mathcal{X}) \\
H^{*, *}(\mathcal{X})^{\leq 0} & :=\bigoplus_{p-q-1 \leq 0} H^{p, q}(\mathcal{X})
\end{aligned}
$$

and

$$
K_{j}^{\theta}(F)=\operatorname{coker}\left(\bigoplus_{E} K_{j}^{M}(E) \rightarrow K_{j}^{M}(F)\right)
$$

where $K_{j}^{M}$ is the Milnor $K$-theory functor and the direct sum is taken over all finite extensions $E / F$ that split $A$. For example we have

$$
\begin{aligned}
& K_{0}^{\theta}(F)=\mathbb{Z} / n \mathbb{Z} \\
& K_{1}^{\theta}(F)=F^{*} / \operatorname{Nrd}\left(A^{*}\right)
\end{aligned}
$$

(for the second statement see [GS], Proposition 2.6.4 and Exercise 2.8).
There is a natural $K_{*}^{\theta}(F)$-module structure on $H^{*, *}(\mathcal{X})^{\geq 0}([\mathrm{MS}]$, Proposition 1.2). The Proposition below is a reformulation of [MS], Theorem 1.15 in the case $\left(X_{\theta}, n, l\right)=(S B(A), 2, n)$.

Proposition 4.13. We have a canonical isomorphism

$$
H^{*, *}(\mathcal{X})^{\leq 0}=H^{*, *}(F)^{\leq 0}
$$

and a direct sum decomposition

$$
H^{*, *}(\mathcal{X})^{\geq 0}=\bigoplus_{i, k \geq 0} K_{i}^{\theta}(F) \cdot \gamma^{k} \delta \oplus \bigoplus_{i, k \geq 0} K_{i}^{\theta}(F) \cdot \gamma^{k+1}
$$

where $\delta \in H^{3,1}(\mathcal{X}), \gamma \in H^{2 n+2, n}(\mathcal{X})$ are defined in [MS], 1.6. The bidegree of $K_{i}^{\theta}(F) \cdot \gamma^{k} \delta$ is $(i+2 k(n+1)+3, i+k n+1)$ and the bidegree of $K_{i}^{\theta}(F) \cdot \gamma^{k+1}$ is $(i+2(k+1)(n+1), i+(k+1) n)$.

Corollary 4.14. In weights $0,1,2$ we have

$$
H^{p, 0}(\mathcal{X})= \begin{cases}\mathbb{Z}, & p=0 \\ 0, & \text { otherwise }\end{cases}
$$

$$
\begin{gathered}
H^{p, 1}(\mathcal{X})= \begin{cases}F^{*}, & p=1 \\
\mathbb{Z} / n \cdot \delta, & p=3 \\
0, & \text { otherwise }\end{cases} \\
H^{p, 2}(\mathcal{X})= \begin{cases}H^{p, 2}(F), & p \leq 2 \\
F^{*} / N r d\left(A^{*}\right) \cdot \delta, & p=4 \\
0, & \text { otherwise }\end{cases}
\end{gathered}
$$

Proof. First note that $H^{p, q}(\mathcal{X})=H^{p, q}(F)$ for $p \leq q+1$, this gives $H^{0,0}, H^{1,0}, H^{0,1}, H^{1,1}, H^{2,1}$, $H^{0,2}, H^{1,2}, H^{2,2}, H^{3,2}$.

Let $p>q+1$.
Weight $0: H(\mathcal{X})^{\geq 0}$ does not contribute since $i+k n+1, i+(k+1) n>0$ for all $i, k \geq 0$. (Alternatively we could argue using Lemma 3.7.)

Weight 1: $K_{i}^{\theta}(F) \cdot \gamma^{k+1}$ does not contribute since $i+(k+1) n \geq n>1 . K_{i}^{\theta}(F) \cdot \gamma^{k} \delta$ has weight 1 when $i=k=0$, thus giving

$$
H^{3,1}(\mathcal{X})=K_{0}^{\theta}(F) \cdot \delta .
$$

Weight 2: $i+k n+1=2$ implies $(i, k)=(1,0)$ thus giving

$$
H^{4,2}(\mathcal{X})=K_{1}^{\theta}(F) \cdot \delta=F^{*} / N r d\left(A^{*}\right) \cdot \delta
$$

and $i+(k+1) n=2$ is not possible since $n \geq 3$.
Remark 4.15. Recall that in this section we assume that $n$ is an odd prime. If $n=2$, then in addition to cohomology groups in weight two listed in the Corollary there is also

$$
H^{6,2}(\mathcal{X})=K_{0}^{\theta}(F) \cdot \gamma=\mathbb{Z} / 2 \cdot \gamma
$$

which appears when $(i, k)=(0,0)$ so that $i+(k+1) n=2$.
Corollary 4.16. Assume that $A$ is a cyclic algebra of prime odd degree $n$ given by the symbol $\theta$. Motivic cohomology of $G L_{1}(A)$ of weights 1, 2 and 3 are given as:

$$
\begin{gathered}
\widetilde{H}^{p, 1}\left(G L_{1}(A)\right)= \begin{cases}\mathbb{Z}, & p=1 \\
0, & \text { otherwise }\end{cases} \\
\widetilde{H}^{p, 2}\left(G L_{1}(A)\right)= \begin{cases}F^{*}, & p=2 \\
n \mathbb{Z}, & p=3 \\
0, & \text { otherwise }\end{cases} \\
\widetilde{H}^{p, 3}\left(G L_{1}(A)\right)= \begin{cases}H^{0,2}(F), & p=1 \\
H^{1,2}(F), & p=2 \\
H^{2,2}(F), & p=3 \\
\mathbb{Z} \oplus N r d\left(A^{*}\right), & p=4 \\
n \mathbb{Z}, & p=5 \\
0, & \text { otherwise }\end{cases}
\end{gathered}
$$

Here by $n \mathbb{Z}$ we mean that the extension of scalars to the splitting field for the corresponding motivic cohomology group is injective and the image is $n \mathbb{Z} \subset \mathbb{Z}$.

Proof. In weight $j$ the spectral sequence has nonzero terms

$$
E_{2}^{p, q}=H^{p+q}\left(\nu_{\mathcal{X}}^{q}(M), \mathbb{Z}(j)\right)
$$

only for $0<q \leq j$. Let us consider the weights $j=1,2,3$. In these weights the spectral sequence converges to $\widetilde{H}^{*, j}\left(G L_{1}(A)\right)$ by theorem $4.7(1)$. The first three slices of the slice filtration are given by:

$$
\begin{aligned}
\nu_{\mathcal{X}}^{1}(M) & =\mathbb{Z}_{\mathcal{X}}(1)[1] \\
\nu_{\mathcal{X}}^{2}(M) & =\mathbb{Z}_{\mathcal{X}}(2)[3] \\
\nu_{\mathcal{X}}^{3}(M) & =\mathbb{Z}_{\mathcal{X}}(3)[4] \oplus \mathbb{Z}_{\mathcal{X}}(3)[5] .
\end{aligned}
$$

In weight $j=1$ the slice spectral sequence consists of one row which contains a unique non-zero term $E_{2}^{0,1}=H^{0,0}(\mathcal{X})=\mathbb{Z}$, hence we get the isomorphism

$$
\widetilde{H}^{1,1}\left(G L_{1}(A)\right)=\mathbb{Z}
$$

and the rest of the reduced cohomology groups of $G L_{1}(A)$ of weight 1 vanish.
In weight $j=2$ we have two nonzero rows:

$$
\begin{aligned}
& E_{2}^{p, 1}=H^{p+1,2}\left(\mathbb{Z}_{\mathcal{X}}(1)[1]\right)=H^{p, 1}(\mathcal{X}) \\
& E_{2}^{p, 2}=H^{p+2,2}\left(\mathbb{Z}_{\mathcal{X}}(2)[3]\right)=H^{p-1,0}(\mathcal{X})
\end{aligned}
$$


and the differential $d_{2}$ is multiplication by $c$ which is prime to $n$ by Theorem 4.9. Thus we have

$$
\begin{aligned}
& \widetilde{H}^{2,2}\left(G L_{1}(A)\right)=F^{*} \\
& \widetilde{H}^{3,2}\left(G L_{1}(A)\right)=n \mathbb{Z}
\end{aligned}
$$

and the rest of the reduced cohomology groups of $G L_{1}(A)$ of weight 2 vanish.
In weight $j=3$ we have three nonzero rows:

$$
\begin{aligned}
& E_{2}^{p, 1}=H^{p+1,3}\left(\mathbb{Z}_{\mathcal{X}}(1)[1]\right)=H^{p, 2}(\mathcal{X}) \\
& E_{2}^{p, 2}=H^{p+2,3}\left(\mathbb{Z}_{\mathcal{X}}(2)[3]\right)=H^{p-1,1}(\mathcal{X}) \\
& E_{2}^{p, 3}=H^{p+3,3}\left(\mathbb{Z}_{\mathcal{X}}(3)[4] \oplus \mathbb{Z}_{\mathcal{X}}(3)[5]\right)=H^{p-1,0}(\mathcal{X}) \oplus H^{p-2,0}(\mathcal{X})
\end{aligned}
$$



By Theorem 4.9 the differential

$$
d_{2}: \mathbb{Z}=H^{0,0}(\mathcal{X}) \rightarrow H^{3,1}(\mathcal{X})=\mathbb{Z} / n \cdot \delta
$$

maps $k \in \mathbb{Z}$ to $\overline{2 k c} \cdot \delta$, and since $2 c$ is prime to $n$, the differential is surjective and its kernel is $n \mathbb{Z} \subset \mathbb{Z}$.

Similarly the differential

$$
d_{2}: F^{*}=H^{1,1}(\mathcal{X}) \rightarrow H^{4,2}(\mathcal{X})=F^{*} / \operatorname{Nrd}\left(A^{*}\right) \cdot \delta
$$

maps $u \in F^{*}$ to $u^{c} \cdot \delta$. Since $\left(F^{*}\right)^{n} \subset \operatorname{Nrd}\left(A^{*}\right)$, and $c$ is prime to $n, d_{2}$ is surjective with kernel $\operatorname{Nrd}\left(A^{*}\right)$. There are no higher differentials by degree reasons and we get the result.

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[^0]:    ${ }^{1}$ This result is proved in $[\mathrm{K}]$ in the category of Chow motives $C H M(F)$, which is a full subcategory of $D M_{-}^{e f f}(F)$ (see [V00], Proposition 2.1.4 and Remark after Corollary 2.1.5 for the statement in characteristic zero; for an arbitrary perfect field one also needs [V03b]). $C H M(F)$ is Karoubian, therefore any direct sum decomposition of $M(S B(A))$ in $D M_{-}^{e f f}(F)$ would lead to a decomposition in $C H M(F)$.

