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# Topological and limit-space subcategories of countably-based equilogical spaces

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There are two main approaches to obtaining ‘topological’ cartesian-closed categories. Under one approach, one restricts to a full subcategory of topological spaces that happens to be cartesian closed – for example, the category of *sequential* spaces. Under the other, one generalises the notion of space – for example, to Scott’s notion of *equilogical* space. In this paper, we show that the two approaches are equivalent for a large class of objects. We first observe that the category of countably based equilogical spaces has, in a precisely defined sense, a largest full subcategory that can be simultaneously viewed as a full subcategory of topological spaces. In fact, this category turns out to be equivalent to the category of all quotient spaces of countably based topological spaces. We show that the category is bicartesian closed with its structure inherited, on the one hand, from the category of sequential spaces, and, on the other, from the category of equilogical spaces.

We also show that the category of countably based equilogical spaces has a larger full subcategory that can be simultaneously viewed as a full subcategory of *limit spaces*. This full subcategory is locally cartesian closed and the embeddings into limit spaces and countably based equilogical spaces preserve this structure. We observe that it seems essential to go beyond the realm of topological spaces to achieve this result.

## 1. Introduction

It is important in computer science to reconcile topological and type-theoretic structure. On the one hand, as has often been stressed, see, for example, Smyth (1992), topological structure accounts for an abstract notion of observable property, and continuity provides a mathematical alternative to computability, emphasising the finitary aspect of computation whilst avoiding the technicalities of recursion theory. On the other hand, type constructors, such as function space, arise fundamentally in both the syntax and semantics of programming languages. The challenge for reconciling them is provided by the well-known mathematical anomaly: the category, **Top**, of topological spaces is *not* cartesian closed.

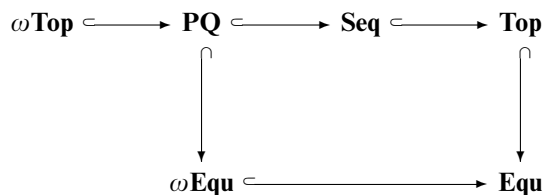
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Of course very many reconciliations of this situation have been proposed. One possibility is to cut down the category of topological spaces to a full subcategory that is cartesian closed. Some well-known examples are: Steenrod’s category of compactly-generated Hausdorff spaces (Mac Lane 1971); the category, **Seq**, of *sequential* spaces (which contains many computationally important non-Hausdorff spaces) (Hyland 1979b); or the even larger category of quotients of exponentiable spaces considered in Day (1972). However, the received wisdom about such categories is that their function spaces are topologically hard to understand. It is much quoted that the exponential  $\mathbb{N}^{\mathbb{B}}$ , where  $\mathbb{B}$  is Baire space, can never be first-countable (Hyland 1979b), whereas an ideal approach from a computational viewpoint would allow effectivity issues to be addressed, and the stricter requirement of second-countability is often claimed to be necessary for such, see, for example, Smyth (1992).

A second alternative is to expand the category **Top** by adding new objects and hence new potential exponentials. Again there are many ways of doing this. A very elegant construction is to take the *regular completion* of **Top** (as a left-exact category) or the related *exact completion* (Birkedal *et al.* 1998; Carboni and Rosolini 2000; Rosolini 2000). The regular completion has a straightforward description as a category of equivalence relations on topological spaces, whose importance (in the case of  $T_0$  spaces) was first recognised by Dana Scott (Bauer *et al.* 1998). Following Scott, we call such structures, consisting of spaces together with equivalence relations, *equilogical spaces* (although we do not make the restriction to  $T_0$  spaces), and we call the associated category **Equ**. Not only is **Equ** cartesian closed, but recent investigations have shown that other approaches to expanding **Top** to a cartesian-closed category (such as Hyland’s *filter space* approach (Hyland 1979b)) can be naturally embedded within **Equ** (Hyland 1979a; Heckmann 1998; Rosolini 2000). A further important feature of **Equ** is that its full subcategory,  $\omega$ **Equ**, of countably based equilogical spaces is also a cartesian-closed category (with its structure inherited from **Equ**). This fact allows equilogical spaces to support an analysis of effectivity at higher types. It is also the basis of an interesting connection with realizability semantics. The category  $\omega$ **Equ** is equivalent to the category of assemblies over the combinatory algebra  $\mathcal{P}\omega$  defined by Scott in Scott (1976).

In this paper we demonstrate an interesting connection between the subcategory and supercategory approaches to achieving cartesian closure. We first show that the categories **Top** and  $\omega$ **Equ** share, in a precisely defined sense, a largest common full subcategory. This category, **PQ**, turns out to be none other than the full subcategory of **Top** consisting of all quotient spaces of countably based topological spaces. This includes, of course, all the countably based spaces themselves. The following diagram depicts the relationship between the categories mentioned above (the square does *not* commute).



The remarkable fact is that **PQ** is also bicartesian closed (with finite limits). As a category of topological spaces, **PQ** inherits its bicartesian-closed structure from **Seq** (which contains all quotients of countably based spaces). Similarly, as a category of equilogical spaces, **PQ** inherits its bicartesian-closed structure from **Equ**. Thus one may conclude that, at least for (iterated) exponentials over countably based spaces, the subcategory approach, as exemplified by **Seq**, and the supercategory approach, as exemplified by **Equ**, give equivalent ways of modelling continuity at higher types.

On the other hand,  $\omega$ **Equ** supports a still richer type structure: it is *locally cartesian closed*. It seems that no non-trivial topological subcategory can share this richer structure (we give a partial result to this effect in Section 9.1). However, we can, nonetheless, obtain an extensional account of local cartesian closure using the category **Lim** of Kuratowski *limit spaces*, into which **Seq** fully embeds. By analogy with the earlier results, we show that:

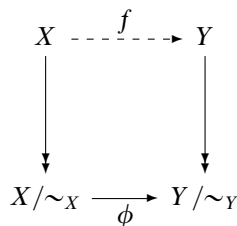
- **Lim** and  $\omega$ **Equ** share a largest common full subcategory, **PQ<sub>L</sub>**;
- **PQ<sub>L</sub>** is locally cartesian closed;
- the embeddings of **PQ<sub>L</sub>** into **Lim** and  $\omega$ **Equ** preserve the locally cartesian closed structure.

**2. Topological subcategories of equilogical spaces**

D.S. Scott introduced the category of *Equilogical spaces* as a simple extension with very good properties of the category of  $T_0$  topological spaces. The idea generalises immediately from  $T_0$  spaces to arbitrary spaces and in the present paper we use the term *equilogical space* to mean this natural generalisation.

**Definition 2.1.**

- 1 An *equilogical space* is a pair  $(X, \sim)$  where  $X$  is a topological space and  $\sim$  is an arbitrary equivalence relation on the underlying set of  $X$ .
- 2 An *equivariant map*<sup>†</sup>  $\phi : (X, \sim_X) \rightarrow (Y, \sim_Y)$  is a function  $\phi$  from the quotient set  $X/\sim_X$  to the quotient set  $Y/\sim_Y$  that is *realized* by some continuous  $f : X \rightarrow Y$  that preserves the equivalence relations (that is, the diagram below commutes).



We write **Equ** for the category of equilogical spaces and equivariant maps.

Scott’s interesting insight was that the category of equilogical spaces is cartesian closed (Bauer *et al.* 1998). The proof made use of his old result that the injective objects in the

<sup>†</sup> In Bauer *et al.* (1998) the equivariant maps are defined as equivalence classes of equivalence-relation preserving continuous functions, rather than as functions between quotient sets.

category of  $T_0$  spaces, the *continuous lattices*, themselves form a cartesian-closed category (Scott 1972). Subsequently, Carboni and Rosolini realised that the construction is an example of a *regular completion* of a left-exact category, and that cartesian closure (and even *local* cartesian closure) are obtained for very general reasons (Birkedal *et al.* 1998; Carboni and Rosolini 2000; Rosolini 2000). The original  $T_0$  version of equilogical spaces is just the regular completion of  $\mathbf{Top}_0$  (the category of  $T_0$  spaces). Similarly, the category  $\mathbf{Equ}$  defined above is the regular completion of  $\mathbf{Top}$ . In Section 8 we sketch a direct proof of the cartesian closure of  $\mathbf{Equ}$  using constructions from Bauer *et al.* (1998) and Rosolini (2000). Yet another proof is presented in Rosický (1999).

The evident functor  $I : \mathbf{Top} \rightarrow \mathbf{Equ}$ , mapping a topological space  $X$  to the equilogical space  $(X, =)$ , exhibits  $\mathbf{Top}$  as a full subcategory of  $\mathbf{Equ}$ . We call the objects (isomorphic to those) in its image the *topological* objects of  $\mathbf{Equ}$ . As  $\mathbf{Top}$  is not cartesian closed, it is clear that  $\mathbf{Equ}$  also contains many non-topological objects, and some such objects can be obtained by exponentiation from topological objects. One example is the object  $\mathbb{N}^{\mathbb{B}}$  (Hyland 1979b).

The inclusion functor  $I$  has a left-adjoint  $Q : \mathbf{Equ} \rightarrow \mathbf{Top}$  that maps an equilogical space  $(X, \sim)$  to the topological quotient  $X/\sim$ . Thus  $\mathbf{Top}$  is a full reflective subcategory of  $\mathbf{Equ}$ . The topological quotient functor  $Q$  has another important property: it is faithful. This fact motivates the following definition of when a full subcategory of  $\mathbf{Equ}$  can be viewed as a ‘topological’ category (that is, as a category of topological spaces and all continuous functions between them).

**Definition 2.2.** We say that a full subcategory  $\mathbf{C}$  of  $\mathbf{Equ}$  is *topological* if the (faithful) composite functor  $\mathbf{C} \hookrightarrow \mathbf{Equ} \xrightarrow{Q} \mathbf{Top}$  is full.

In other words,  $\mathbf{C}$  is topological if  $Q : \mathbf{Equ} \rightarrow \mathbf{Top}$  cuts down to an equivalence between  $\mathbf{C}$  and a full subcategory of  $\mathbf{Top}$ .

It is easily seen that the full subcategory of topological objects of  $\mathbf{Equ}$  gives one topological subcategory of  $\mathbf{Equ}$ . Moreover, this category can be shown to be *a maximal* (but not *the maximum* – see below!) topological subcategory of  $\mathbf{Equ}$ : any strictly larger full subcategory of  $\mathbf{Equ}$  is not topological.

These remarks are hardly surprising. However, what is interesting about the notion of topological subcategory is that there exist other topological subcategories of  $\mathbf{Equ}$  that contain non-topological equilogical spaces amongst their objects (and are hence incomparable with the maximal topological subcategory identified above). We shall see that one such subcategory arises in a very natural way.

Let us consider what happens when equilogical spaces are restricted to equivalence relations over countably based spaces. We say that a topological space is *countably based* if there exists some countable base for its topology (Smyth 1992). Such spaces are also known as second-countable spaces. We write  $\omega\mathbf{Top}$  for the category of countably based topological spaces and  $\omega\mathbf{Equ}$  for the category of those equilogical spaces  $(X, \sim)$  where  $X$  is countably based. As mentioned in the introduction,  $\omega\mathbf{Equ}$  is cartesian closed with its cartesian-closed structure inherited from  $\mathbf{Equ}$  (see Section 8). From a computer science viewpoint, the restriction to countably based spaces is natural, allowing  $\omega\mathbf{Equ}$  to be used to formalise issues of effectivity at higher types.

Clearly, the functor  $I : \mathbf{Top} \rightarrow \mathbf{Equ}$  cuts down to a functor  $I : \omega\mathbf{Top} \rightarrow \omega\mathbf{Equ}$ , identifying (up to isomorphism) the topological objects in  $\omega\mathbf{Equ}$ . We also have the topological quotient functor  $Q : \omega\mathbf{Equ} \rightarrow \mathbf{Top}$ . Note that the image of  $Q$  does not land in  $\omega\mathbf{Top}$  as topological quotients of countably based spaces are not in general countably based.

As with  $\mathbf{Equ}$ , the topological objects of  $\omega\mathbf{Equ}$  form a topological subcategory of  $\omega\mathbf{Equ}$ . The difference this time is that the topological objects do *not* form a maximal topological subcategory. Instead, there is a unique maximal topological subcategory of  $\omega\mathbf{Equ}$ , including all topological objects, but also containing many non-topological equilogical spaces.

**Definition 2.3.** We say that a full subcategory  $\mathbf{C}$  of  $\omega\mathbf{Equ}$  contains  $\omega\mathbf{Top}$  if the functor  $I : \omega\mathbf{Top} \rightarrow \omega\mathbf{Equ}$  factors through the inclusion  $\mathbf{C} \hookrightarrow \omega\mathbf{Equ}$ .

**Theorem 1.** There exists a unique largest topological full subcategory,  $\mathbf{C}$ , of  $\omega\mathbf{Equ}$  containing  $\omega\mathbf{Top}$ . (That is, for any other topological full subcategory,  $\mathbf{C}'$ , of  $\omega\mathbf{Equ}$  also containing  $\omega\mathbf{Top}$ , the inclusion  $\mathbf{C}' \hookrightarrow \omega\mathbf{Equ}$  factors through the inclusion  $\mathbf{C} \hookrightarrow \omega\mathbf{Equ}$ .)

In order to prove the theorem, we define the largest topological subcategory explicitly.

**Definition 2.4.** We say that an object  $A$  (in any category) is *projective with respect to* a map  $r : B \rightarrow R$  if for every  $f : A \rightarrow R$  there exists some  $\bar{f} : A \rightarrow B$  such that  $r.\bar{f} = f$ .

**Definition 2.5.** We say that a morphism  $r : B \rightarrow R$  in  $\mathbf{Top}$  is  $\omega$ -*projecting* if every countably based space is projective with respect to it.

**Definition 2.6.** We write  $\mathbf{EPQ}$  for the full subcategory of  $\omega\mathbf{Equ}$  consisting of those objects  $(A, \sim)$  for which the induced quotient  $A \rightarrow (A/\sim)$  in  $\mathbf{Top}$  is  $\omega$ -projecting.

The acronym EPQ stands for Equilogical  $\omega$ -Projecting Quotient.

*Proof of Theorem 1.* We show that  $\mathbf{EPQ}$  is the category characterised by the theorem. First,  $\omega\mathbf{Top}$  is trivially contained in  $\mathbf{EPQ}$ . For the fullness of  $Q : \mathbf{EPQ} \rightarrow \mathbf{Top}$ , suppose we have  $f : (A/\sim_A) \rightarrow (B/\sim_B)$  in  $\mathbf{Top}$  where  $(B, \sim_B)$  is in  $\mathbf{EPQ}$ . Then, as the quotient  $q_B : B \rightarrow (B/\sim_B)$  is  $\omega$ -projecting, there exists  $g : A \rightarrow B$  such that  $q_B.g = f.q_A$ . Then  $g$  realizes the equivariant map  $f : (A, \sim_A) \rightarrow (B, \sim_B)$ . Thus  $\mathbf{EPQ}$  is a topological subcategory containing  $\omega\mathbf{Top}$ .

It remains to show that  $\mathbf{EPQ}$  is the largest such subcategory. Suppose that an object  $(B, \sim)$  of  $\omega\mathbf{Equ}$  lies in some other such category  $\mathbf{C}'$ . To show the quotient  $q : B \rightarrow (B/\sim)$  is  $\omega$ -projecting, suppose  $A$  is countably based and take any  $f : A \rightarrow (B/\sim)$  in  $\mathbf{Top}$ . As  $\mathbf{C}'$  contains  $\omega\mathbf{Top}$ , the object  $(A, =)$  is in  $\mathbf{C}'$ . As  $\mathbf{C}'$  is a topological subcategory, the continuous function  $f : A \rightarrow (B/\sim)$  gives an equivariant map  $\phi_f : (A, =) \rightarrow (B, \sim)$  in  $\omega\mathbf{Equ}$ . Then any realizer  $\bar{f} : A \rightarrow B$  for  $\phi_f$  will satisfy  $q_B.\bar{f} = f$ . Thus, we have  $q_B : B \rightarrow (B/\sim)$  is  $\omega$ -projecting. □

It is not immediately obvious that  $\mathbf{EPQ}$  is not just the category of all topological objects of  $\omega\mathbf{Equ}$ . That this is not the case is given by the following surprising theorem, whose proof will eventually be given in Section 8. A consequence of the theorem is that the non-topological equilogical space  $\mathbb{N}^{\mathbb{B}}$  (see Section 1) is an object of  $\mathbf{EPQ}$ .

**Theorem 2.** The category  $\mathbf{EPQ}$  is bicartesian closed with finite limits. Moreover, the inclusion functor  $\mathbf{EPQ} \hookrightarrow \omega\mathbf{Equ}$  preserves this structure.

By its definition as a topological subcategory of  $\mathbf{Equ}$ , we have that  $\mathbf{EPQ}$  is equivalent to a full subcategory of  $\mathbf{Top}$ , which, because of the equivalence, must itself be cartesian closed. Thus, even though exponentiation in  $\mathbf{EPQ}$  goes outside the world of the topological objects of  $\mathbf{Equ}$ , it can nonetheless be viewed as a purely topological phenomenon. Accordingly, it is of interest to give an explicit description of the equivalent topological category.

**Definition 2.7.** We write  $\mathbf{PQ}$  for the full subcategory of  $\mathbf{Top}$  consisting of those spaces  $Q$  for which there exists a countably based space  $A$  together with an  $\omega$ -projecting topological quotient  $q : A \twoheadrightarrow Q$ .

The acronym PQ stands for  $\omega$ -Projecting Quotient spaces. It is immediate from the definitions that the functor  $Q : \mathbf{Equ} \rightarrow \mathbf{Top}$  cuts down to the claimed equivalence of categories  $Q : \mathbf{EPQ} \rightarrow \mathbf{PQ}$ . In Sections 3–7 we shall prove the bicartesian closure of  $\mathbf{PQ}$  directly, culminating in Theorem 4 of Section 7. Theorem 2 will be derived from this in Section 8.

Although the above definition of  $\mathbf{PQ}$  is the one needed for the proof of Theorem 2, the definition itself is not particularly satisfying, as it does not yield an easy method of showing that a space is in  $\mathbf{PQ}$ . The next result addresses this problem, and also demonstrates that  $\mathbf{PQ}$  is a more natural category than its definition, at first, suggests.

**Theorem 3.**  $\mathbf{PQ}$  is the full subcategory of  $\mathbf{Top}$  consisting of all quotient spaces of countably based spaces.

The proof, for which Matthias Schröder provided the key idea, is given in Section 7.

We conclude the present section with some remarks and questions. The fact that  $\mathbf{PQ}$  is a cartesian-closed category consisting entirely of quotients of countably based spaces is important as it offers a means of extending Weihrauch's 'Type 2' computability to higher-type computation. Furthermore, Scott's approach to computability in countably based algebraic lattices (Scott 1976) can be applied to  $\omega\mathbf{Equ}$ , and hence to  $\mathbf{EPQ}$ . Thus the equivalence between  $\mathbf{PQ}$  and  $\mathbf{EPQ}$  opens up the possibility of comparing Weihrauch's and Scott's approaches. In fact, recent research programmes along these lines have been carried out by Matthias Schröder (Schröder 2000b) and Andrej Bauer (Bauer 2000; Bauer 2001). See the Addendum to this paper for further discussion.

The potential relation to computability gives a computational motivation for the choice of  $\omega\mathbf{Top}$  as the basis for the identification of  $\mathbf{EPQ}$  as the category characterised by Theorem 1. However, it is interesting to consider what variation is possible in this choice of topological category. For any full subcategory  $\mathbf{T}$  of  $\mathbf{Top}$ , we can form the evident full subcategory  $\mathbf{Equ}_{\mathbf{T}}$  of equilogical spaces over  $\mathbf{T}$ . For any such category  $\mathbf{T}$ , the proof of Theorem 1 generalises to determine a largest topological subcategory  $\mathbf{LT}_{\mathbf{T}}$  of  $\mathbf{Equ}_{\mathbf{T}}$  containing  $\mathbf{T}$  itself. As we have already mentioned, in the case that  $\mathbf{T}$  is  $\mathbf{Top}$ , we have  $\mathbf{LT}_{\mathbf{Top}}$  is equivalent to  $\mathbf{Top}$  itself, hence  $\mathbf{LT}_{\mathbf{Top}}$  is not cartesian closed. Why is it then that in the case that  $\mathbf{T}$  is  $\omega\mathbf{Top}$ , we do obtain a cartesian-closed category for  $\mathbf{LT}_{\mathbf{T}}$ ? We do not know a good general answer to this question, but the choice of  $\omega\mathbf{Top}$  seems very constrained.

For example, one can show that if  $\mathbf{T}$  is the full subcategory  $\kappa$ -based topological spaces for any cardinal  $\kappa \geq 2^\omega$ , then  $\mathbf{LT}_\mathbf{T}$  is *not* a cartesian-closed subcategory of  $\mathbf{Equ}$ . The essential problem is that all such categories contain the equilogical space  $(\mathbb{N}^\mathbb{B}, =)$  with the exponential  $\mathbb{N}^\mathbb{B}$  calculated in  $\mathbf{Seq}$ , given which, the definition of topological subcategory prevents the actual exponential  $(\mathbb{N}, =)^{(\mathbb{B}, =)}$  in  $\mathbf{Equ}$  from being in  $\mathbf{LT}_\mathbf{T}$  (if it were in  $\mathbf{LT}_\mathbf{T}$ , it would have to be isomorphic in  $\mathbf{Equ}$  to  $(\mathbb{N}^\mathbb{B}, =)$ , which is not the case).

However, there may be other ways of obtaining categories  $\mathbf{T}$  such that  $\mathbf{LT}_\mathbf{T}$  is a cartesian-closed subcategory of  $\mathbf{Equ}$ . Two possibilities for  $\mathbf{T}$  that could be worth investigating are: the category of first-countable spaces, *cf.* Franklin (1965); and the category of exponentiable spaces, *cf.* Day (1972).

### 3. Sequential spaces and limit spaces

In this section we introduce the category  $\mathbf{Seq}$  of sequential spaces (Franklin 1965), which is a full subcategory of  $\mathbf{Top}$ . We also introduce the category  $\mathbf{Lim}$  of limit spaces in the sense of Kuratowski (Kuratowski 1952). Although this category is not a subcategory of  $\mathbf{Top}$ , it does embed the category of sequential spaces. It is easy to prove that  $\mathbf{Lim}$  is cartesian closed because products and exponentials have straightforward definitions. We use this to prove the known result that  $\mathbf{Seq}$  is also cartesian closed and that it inherits this structure from that in  $\mathbf{Lim}$  (Day 1972; Hyland 1979b). These properties of  $\mathbf{Seq}$  and  $\mathbf{Lim}$  will be used in Sections 4–7 to prove the cartesian closure of  $\mathbf{PQ}$ .

#### 3.1. Sequential spaces

The sequential spaces are those topological spaces whose topologies are determined by sequence convergence. Explicitly, say that a sequence  $(x_i)$  of elements of a set  $X$  is *eventually in* a subset  $O \subseteq X$  if there exists  $l$  such that, for all  $i \geq l$ ,  $x_i \in O$ . Recall that, in an arbitrary topological space  $X$ , a sequence  $(x_i)$  is said to *converge to a point*  $x$  if, for every neighbourhood of  $x$ , the sequence is eventually in the neighbourhood.

**Definition 3.1.** Let  $X$  be a topological space.

- 1 A subset  $O$  of  $X$  is *sequentially open* if every sequence converging to a point in  $O$  is eventually in  $O$ .
- 2 A subset  $O$  of  $X$  is *sequentially closed* if no sequence in  $O$  converges to a point not in  $O$ .
- 3  $X$  is *sequential* if every sequentially open subset is open or, equivalently, if every sequentially closed subset is closed.

Let  $\mathbf{Seq}$  denote the category of sequential spaces and continuous functions. For sequential spaces, the notion of continuity has a natural reformulation. In order to state it properly we define a *convergent sequence* (with limit) to be a sequence  $(x_i)$  together with a point  $x$  such that  $(x_i)$  converges to  $x$ . It is easy to check that a function  $f : X \rightarrow Y$  between sequential spaces is continuous if and only if it preserves convergent sequences.

There is another way of viewing convergent sequences. Let  $\mathbb{N}^+$  denote the one point compactification of the natural numbers. This has  $\mathbb{N} \cup \{\infty\}$  as underlying set and its



topology is given by the following base  $\{\{n\} \mid n \in \mathbb{N}\} \cup \{\{n, n+1, \dots, \infty\} \mid n \in \mathbb{N}\}$ . That is, a sequence converges to some  $n \in \mathbb{N}$  if and only if the sequence is eventually equal to  $n$ . On the other hand, a sequence converges to  $\infty$  if and only if, for all  $n$ , the sequence is eventually greater than  $n$ . It is easily verified that, for any topological space  $X$ , the convergent sequences in  $X$  are in one-to-one correspondence with the continuous functions from  $\mathbb{N}^+$  to  $X$ .

It is easy to check that every countably based space is sequential (as, indeed, is any first-countable space). Thus  $\omega\mathbf{Top}$  is a full subcategory of  $\mathbf{Seq}$ . Moreover, the embedding  $\omega\mathbf{Top} \hookrightarrow \mathbf{Seq}$  preserves countable products, countable coproducts and subspaces (equalizers).

The set of sequentially open subsets of any topological space is a sequential topology. This fact induces a functor  $\mathbf{Top} \rightarrow \mathbf{Seq}$ , which is right adjoint to the embedding in the opposite direction. That is,  $\mathbf{Seq}$  is a full coreflective subcategory of  $\mathbf{Top}$ . This shows that  $\mathbf{Seq}$  is complete and cocomplete and explains why, in  $\mathbf{Top}$ , coproducts and quotients of sequential spaces are again sequential spaces (Franklin 1965). It follows that every quotient of a countably based space is sequential. Thus, in particular,  $\mathbf{PQ}$  is a full subcategory of  $\mathbf{Seq}$ .

On the other hand, in contrast to the countably based case, subspaces and (even finite) products (in  $\mathbf{Top}$ ) of sequential spaces, need not be sequential in general. Thus, products in  $\mathbf{Seq}$  do not always coincide with topological products. Similarly, regular subobjects in  $\mathbf{Seq}$  do not, in general, have the subspace topology.

### 3.2. Limit spaces

In order to gain a better understanding of the structure of  $\mathbf{Seq}$ , we introduce the related notion of Kuratowski *limit space* (Kuratowski 1952).

#### Definition 3.2.

- 1 A *limit space* consists of a set  $X$  together with a distinguished family of functions  $(\mathbb{N} \cup \{\infty\}) \rightarrow X$ , called *convergent sequences* in  $X$ . We say that  $(x_i)$  *converges* to  $x_\infty$  in  $X$  if the induced function  $(\mathbb{N} \cup \{\infty\}) \rightarrow X$  is one of the convergent sequences in  $X$ . The convergent sequences must satisfy the following axioms:
  - (a) The constant sequence  $(x)$  converges to  $x$ .
  - (b) If  $(x_i)$  converges to  $x$ , then so does every subsequence of  $(x_i)$ .
  - (c) If  $(x_i)$  is a sequence such that every subsequence of  $(x_i)$  contains a subsequence converging to  $x$ , then  $(x_i)$  converges to  $x$ .
- 2 A function between limit spaces is said to be *continuous* if it preserves convergent sequences.

Actually, Kuratowski (Kuratowski 1952) imposed the further axiom that a sequence should have at most one limit. The notion of limit space at the level of generality above seems to have appeared first in Johnstone (1979) (where they are called *subsequential spaces*) and Hyland (1979b) (where they are called *L-spaces*).

When manipulating limit spaces, we usually write  $(x_i) \rightarrow x$  as a shorthand for  $(x_i)$  converges to  $x$ . We also write  $(x_{f_i})$  for a subsequence of  $(x_i)$ , where  $f$  is tacitly assumed to be an injective monotonic function from  $\mathbb{N}$  to  $\mathbb{N}$ .

It is easy to see that **Seq** is a full subcategory of **Lim**. The embedding assigns to each sequential space, the limit space with same underlying set and as convergent sequences those that converge topologically.

Viewed as a limit space, the one point compactification of the natural numbers,  $\mathbb{N}^+$ , acts as a generic convergent sequence in **Lim**: convergent sequences, in any limit space  $X$ , are in one-to-one correspondence with the continuous functions (in the limit space sense) from  $\mathbb{N}^+$  to  $X$ . This fact will be useful later in the proofs of Propositions 3.1 and 5.2.

Let **Lim** denote the category of limit spaces and continuous maps. In Johnstone (1979), it is shown that it arises as the full and reflective subcategory of  $\neg\neg$ -separated sheaves of a Grothendieck topos. This fact implies that **Lim** is a quasitopos. Although we shall mainly use properties of the categorical structure of **Lim** that are true in any quasitopos, it is instructive to give an explicit description of finite limits, finite colimits and exponentials.

There is an evident forgetful functor **Lim**  $\rightarrow$  **Set**. It has a ‘chaotic’ right adjoint  $\nabla$  that assigns to each set, the limit space with this underlying set and where every sequence converges to every point. It also has a ‘discrete’ left adjoint that assigns to each set, the limit space with this underlying set but where a sequence converges to a point if and only if the sequence is eventually the constant sequence of that point.

The existence of these adjoints implies that the forgetful functor preserves limits and colimits. This gives us the underlying sets of many constructions among limit spaces. The corresponding convergent sequences are as follows.

Let  $X$  and  $Y$  be limit spaces. A sequence  $((x_i, y_i))$  of pairs converges to  $(x, y)$  in  $X \times Y$  iff  $(x_i) \rightarrow x$  in  $X$  and  $(y_i) \rightarrow y$  in  $Y$ .

A sequence  $(z_i)$  converges in  $X + Y$  to an  $x \in X$  if there exists a  $k$  such that for each  $j \geq k$ ,  $z_j \in X$  (that is,  $(z_i)$  is eventually in  $X$ ) and  $(z_j)_{j \geq k}$  converges to  $x$  in  $X$ , and similarly for  $y \in Y$ .

The underlying set of  $Y^X$  is the set of continuous functions from  $X$  to  $Y$  and  $(f_i) \rightarrow f$  if for each  $(x_i) \rightarrow x$  in  $X$ ,  $(f_i x_i) \rightarrow f x$  in  $Y$ .

Monos are exactly those morphisms with injective underlying functions, and epis are exactly those morphisms with surjective underlying functions.

A mono  $m : A \rightarrow X$  is regular if and only if  $(ma_i) \rightarrow ma$  in  $X$  implies  $(a_i) \rightarrow a$  in  $A$ .

An epi  $q : X \rightarrow Q$  is regular if and only if for each  $(z_i) \rightarrow z$  in  $Q$  it holds that for every subsequence  $(z_{\alpha i})$  there exists a subsequence  $(z_{\alpha\beta i})$  and a sequence  $(x_i) \rightarrow x$  in  $X$  such that, for each  $i$ ,  $qx_i = z_{\alpha\beta i}$  and  $qx = z$ .

### 3.3. Seq as a reflective subcategory of Lim

We say that a limit space is *topological* if it lies in the image of the embedding of **Seq** in **Lim**. Such limit spaces are easily characterised explicitly. We say that a subset  $U$  of the underlying set of a limit space  $X$  is *sequentially open* if every sequence in  $X$  converging to a point in  $U$  is eventually in  $U$ . We say that a sequence  $(x_i)$  *topologically converges* to a point  $x$  in  $X$  if, for every sequentially open subset  $U$  containing  $x$ , the sequence  $(x_i)$

is eventually in  $U$ . Clearly,  $(x_i) \rightarrow x$  implies  $(x_i)$  topologically converges to  $x$ . The limit space  $X$  is topological if and only if the converse holds, that is,  $X$  is topological if and only if convergence agrees with topological convergence.

Underlying the above characterisation is a reflection functor from **Lim** to **Seq**. The family of sequentially open subsets of a limit space forms a topology and the resulting topological space is sequential. This operation determines a functor  $F : \mathbf{Lim} \rightarrow \mathbf{Seq}$  that is left adjoint to the embedding in the opposite direction (Johnstone 1979; Hyland 1979b). An immediate consequence of this is that the embedding preserves products and equalizers. Also, using the explicit description of coproducts in **Lim**, it is easy to see that the embedding also preserves coproducts. We shall use these facts later.

In the proof of Corollary 10.2 of Hyland (1979b), the following property of the reflection is stated as obvious. We thought it worth giving a proof.

**Proposition 3.1.** The left adjoint  $F : \mathbf{Lim} \rightarrow \mathbf{Seq}$  preserves finite products.

*Proof.* It is clear that  $F(X \times_{\mathbf{Lim}} Y)$  and  $FX \times_{\mathbf{Seq}} FY$  have the same underlying set and that the identity function  $F(X \times_{\mathbf{Lim}} Y) \rightarrow FX \times_{\mathbf{Seq}} FY$  is continuous. So we need only prove that every open in  $F(X \times_{\mathbf{Lim}} Y)$  is open in  $FX \times_{\mathbf{Seq}} FY$ , that is, that every sequentially open subset of  $X \times_{\mathbf{Lim}} Y$  is a sequentially open subset of  $FX \times_{\mathbf{Lim}} FY$  (as the inclusion from **Seq** to **Lim** preserves products).

By the symmetry of product, it suffices to prove that if a subset  $W \subseteq X \times_{\mathbf{Lim}} Y$  is sequentially open, then  $W$  is sequentially open in  $X \times_{\mathbf{Lim}} FY$ . Suppose then that  $((a_i, b_i)) \rightarrow (a, b)$  in  $X \times_{\mathbf{Lim}} FY$  where  $(a, b) \in W$ . As  $\{x \in X \mid (x, b) \in W\}$  is sequentially open in  $X$ , there exists an  $m$  such that, for all  $i \geq m, (a_i, b) \in W$ . Write  $a_\infty$  for  $a$  and define

$$V = \{y \in Y \mid \text{for all } j \text{ with } m \leq j \leq \infty, (a_j, y) \in W\}.$$

We now prove that  $V \subseteq Y$  is sequentially open. Suppose for contradiction that, in  $Y$ ,  $(y_i) \rightarrow y \in V$  but  $(y_i)$  is not eventually in  $V$ . Then, there exists a subsequence  $y_{g_i} \rightarrow y$  in  $Y$  with each  $y_{g_i}$  not in  $V$ . So for each  $i$  there exists  $f_i$  with  $m \leq f_i \leq \infty$  such that  $(a_{f_i}, y_{g_i})$  is not in  $W$ . The sequence  $(f_i)$  is an arbitrary sequence of elements of  $\mathbb{N}^+$ . By the compactness of  $\mathbb{N}^+$ ,  $(f_i)$  has a converging subsequence in  $\mathbb{N}^+$ ,  $(f_{h_i}) \rightarrow j$  for some  $j$  with  $m \leq j \leq \infty$ . But then we have that  $(a_{f_{h_i}}) \rightarrow a_j$  in  $X$  and that  $(y_{g_{h_i}}) \rightarrow y$  in  $Y$ . So  $((a_{f_{h_i}}, y_{g_{h_i}})) \rightarrow (a_j, y)$  in  $X \times_{\mathbf{Lim}} Y$ .

But  $(a_j, y) \in W$ , as  $y \in V$ . Yet for no  $i$  is  $(a_{f_{h_i}}, y_{g_{h_i}})$  in  $W$ . This contradicts the assumption that  $W$  is sequentially open in  $X \times_{\mathbf{Lim}} Y$ . So  $V$  is sequentially open.

Then  $(b_i)$  is eventually in  $V$ . Hence,  $((a_i, b_i))$  is indeed eventually in  $W$ , proving that  $W$  is sequentially open.  $\square$

By an elementary categorical argument (Freyd and Scedrov 1990, 1.857), it follows that **Seq** is an exponential ideal of **Lim** (that is, if  $X$  is a sequential space and  $Y$  is a limit space, the object  $X^Y$  of **Lim** is topological). This means, in particular, that **Seq** is a cartesian-closed category, and that the embedding  $\mathbf{Seq} \hookrightarrow \mathbf{Lim}$  preserves the cartesian-closed structure.

**4. Pre-embeddings and pre-extensional spaces**

In this section we introduce the notion of a pre-embedding and use it to give an abstract characterisation of sequential spaces as a subcategory of **Lim**. Pre-embeddings will also be important later for obtaining injectivity results.

A continuous  $f : X \rightarrow Y$  between topological spaces is a (topological) *pre-embedding* if for every open  $U$  in  $X$  there exists an open  $V$  in  $Y$  such that  $f^{-1}V = U$ . Notice that if  $f : X \rightarrow Y$  is a pre-embedding and  $Y$  is countably based, then  $X$  is countably based. Also, consider the following fact whose easy proof we omit.

**Proposition 4.1.** Let  $f : X \rightarrow Y$  be a pre-embedding between topological spaces. If  $(fx_i)$  converges to  $fx$  in  $Y$ , then  $(x_i)$  converges to  $x$  in  $X$ .

This proposition suggests how to formulate the notion of pre-embedding between limit spaces.

We say that a map  $f : X \rightarrow Y$  in **Lim** is a **Lim-pre-embedding** if  $(fx_i) \rightarrow fx$  in  $Y$  implies  $(x_i) \rightarrow x$  in  $X$ . Note that a map in **Lim** is a regular mono if and only if it is both mono and a **Lim-pre-embedding**. In fact, **Lim-pre-embeddings** in general share many of the properties of regular monos.

**Proposition 4.2.** Let  $f : X \rightarrow Y$  be a **Lim-pre-embedding**.

- 1 If  $g : Y \rightarrow Z$  is also a **Lim-pre-embedding**, the composition  $g.f$  is too.
- 2 For an arbitrary  $h : Z \rightarrow Y$ , the pullback  $h^*f$  of  $f$  along  $h$  is a **Lim-pre-embedding**.
- 3 If  $f' : X' \rightarrow Y'$  is a **Lim-pre-embedding**, the product  $f \times f'$  is also.
- 4 For any object  $Z$ ,  $f^Z : X^Z \rightarrow Y^Z$  is a **Lim-pre-embedding**.

*Proof.* The first two are easy calculations, and the third follows from them. The last is also easy, but we will give it explicitly as an example. Let  $(f^Z h_i) \rightarrow f^Z h$  in  $Y^Z$ . We want to prove that  $(h_i) \rightarrow h$  in  $X^Z$ . To do this, let  $(z_i) \rightarrow z$  in  $Z$ . Then  $((f^Z h_i)z_i) \rightarrow (f^Z h)z$ . That is,  $(f(h_i z_i)) \rightarrow f(hz)$ . As  $f$  is a **Lim-pre-embedding**,  $(h_i z_i) \rightarrow hz$ . So, in fact,  $(h_i) \rightarrow h$ . Hence, as required,  $f^Z$  is also a **Lim-pre-embedding**. □

It is worth noting that **Lim-pre-embeddings** have a nice categorical characterisation from which the above properties follow. Recall the ‘chaotic’ inclusion  $\nabla : \mathbf{Set} \rightarrow \mathbf{Lim}$  and for any limit space  $X$ , let  $\nabla X$  be the corresponding chaotic limit space; also let  $X \rightarrow \nabla X$  be the unit of the adjunction and  $\nabla f : \nabla X \rightarrow \nabla Y$  be the reflection of  $f$ . A map  $f : X \rightarrow Y$  is a **Lim-pre-embedding** if and only if the following square is a pullback.

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow & & \downarrow \\
 \nabla X & \xrightarrow{\nabla f} & \nabla Y
 \end{array}$$

As we have already said, in **Top**, subspaces of sequential spaces need not be sequential. The following may then come as a surprise.

**Proposition 4.3.** Let  $X$  be a sequential space and let  $f : A \rightarrow X$  be a **Lim**-pre-embedding. Then:

- 1  $A$  is topological.
- 2 If  $X$  is countably based, then  $f$  is a topological pre-embedding.

*Proof.* To prove 1 we are going to show that if  $(a_i)$  is eventually in every sequentially open neighbourhood of  $a$ , then  $(a_i) \rightarrow a$  in  $A$ . In order to do this, let  $U$  be an open neighbourhood of  $fa$ . As  $f^{-1}U$  is sequentially open,  $(a_i)$  is eventually in  $f^{-1}U$ . Then,  $(fa_i)$  is eventually in  $U$ . As  $X$  is topological, this means that  $(fa_i) \rightarrow fa$  in  $X$ . As  $f$  is a pre-embedding,  $(a_i) \rightarrow a$  in  $A$ .

To prove 2 we are going to use the following property of countably based spaces: the closure of any subset is obtained by adding the limits of all convergent sequences in the subset. Moreover, we are going to use the characterisation of sequential spaces in terms of closed sets.

By 1, we know that  $A$  is topological. We now show that if  $U \subseteq A$  is sequentially closed, then there exists a sequentially closed  $V \subseteq X$  such that  $f^{-1}V = U$ .

Suppose  $U$  is sequentially closed. Now take the closure  $\overline{fU}$  of  $fU$ , the image of  $U$  under  $f$ . We are going to prove that  $U = f^{-1}\overline{fU}$ . Trivially  $U \subseteq f^{-1}\overline{fU}$ . For the other inclusion, let  $fa \in \overline{fU}$ . As  $X$  is countably based, there exists a sequence  $(fa_i)$  in  $fU$  such that  $(fa_i) \rightarrow fa$ . As  $f$  is a pre-embedding,  $(a_i) \rightarrow a$ . As  $U$  is closed,  $a \in U$ . So  $U = f^{-1}\overline{fU}$ . □

Actually, property 2 holds for every space that satisfies the condition mentioned in the proof. Such spaces are known as *Fréchet* spaces (Franklin 1965).

By Propositions 4.1 and 4.3, it follows that it is irrelevant to distinguish between topological and **Lim**-pre-embeddings into countably based spaces.

**Corollary 4.1.** In **Lim**:

- 1 Regular subobjects of topological objects are topological (though they need not have the subspace topology).
- 2 However, regular subobjects of countably based spaces are in one-to-one correspondence with topological subspaces.

We conclude this section with an application of pre-embeddings in order to obtain an abstract characterisation of the topological objects in **Lim**. This characterisation will play a surprising role in the proof of Theorem 3 in Section 7.

Let  $\Sigma$  be Sierpinski space (that is, the two element space  $\{\perp, \top\}$  with the singleton  $\{\top\}$  as the only non-trivial open). It is an easy fact in topology that the continuous functions from any topological space  $X$  to  $\Sigma$  are in one-to-one correspondence with the open subsets of  $X$ . Similarly,  $\Sigma$  is also a limit space and the maps from any limit space  $X$  to  $\Sigma$  are in one-to-one correspondence with the sequentially open subsets of  $X$ .

By the last observation,  $\Sigma^X$  in **Lim** is an object of sequentially open subsets of a limit space  $X$ . Moreover, as **Seq** is an exponential ideal of **Lim**, the object  $\Sigma^X$  is topological. (Warning – in general, its topology is *not* the Scott topology!) For any limit space  $X$  let  $\Omega : X \rightarrow \Sigma^{\Sigma^X}$  denote the transpose of the evaluation map. If  $X$  is topological, it is easy

to check that  $\Omega$  is mono if and only if  $X$  is a  $T_0$  space. It is useful to consider a stronger property of  $\Omega$ .

**Definition 4.1.** A limit space  $X$  is *extensional* if  $\Omega : X \rightarrow \Sigma^X$  is a regular mono. It is *pre-extensional* if the map is a **Lim**-pre-embedding.

The terminology is taken from Hyland (1991).

As  $\Sigma^X$  is topological, by Proposition 4.3, it follows that so is any pre-extensional object. Moreover, if  $X$  is extensional, then  $\Omega : X \rightarrow \Sigma^X$  is also mono, so  $X$  is  $T_0$ .

Recall that  $F : \mathbf{Lim} \rightarrow \mathbf{Seq}$  is the reflection functor.

**Proposition 4.4.** If  $(\Omega x_i) \rightarrow \Omega x$  in  $\Sigma^X$  then  $(x_i) \rightarrow x$  in  $FX$ .

*Proof.* Let  $O$  be sequentially open in  $X$  and  $x \in O$ . It is clear that  $(O) \rightarrow O$  in  $\Sigma^X$ . Then, as  $(\Omega x_i) \rightarrow \Omega x$ ,  $((\Omega x_i)O) \rightarrow (\Omega x)O$ . That is,  $((\Omega x_i)O)$  must be eventually  $\top$ . In other words,  $(x_i)$  must be eventually in  $O$ . So  $(x_i) \rightarrow x$  in  $FX$ . □

So, if  $X$  is a sequential space,  $\Omega : X \rightarrow \Sigma^X$  is a **Lim**-pre-embedding.

**Corollary 4.2.** In **Lim**:

- 1 The full subcategory of pre-extensional objects is equivalent to **Seq**.
- 2 The full subcategory of extensional objects is equivalent to the category of  $T_0$  sequential spaces.

### 5. Projectivity

Recall the notion of  $\omega$ -projecting map used to define **PQ** in Section 2. As  $\omega\mathbf{Top}$  is a full subcategory of **Seq** and hence also of **Lim**, it is clear that we can also define the  $\omega$ -projecting maps in any of these categories. We shall be mainly interested in the  $\omega$ -projecting maps in **Lim**, and their relationship to  $\omega$ -projecting quotients in **Top**.

We first prove some closure properties of  $\omega$ -projecting maps.

**Proposition 5.1.** Let  $f : X \rightarrow Y$  be an  $\omega$ -projecting map in **Lim**.

- 1 If  $g : Y \rightarrow Z$  is also  $\omega$ -projecting, the composition  $g.f$  is also.
- 2 For an arbitrary  $h : Y' \rightarrow Y$ , the pullback  $h^*f$  of  $f$  along  $h$  is  $\omega$ -projecting.
- 3 If  $f : X' \rightarrow Y'$  is  $\omega$ -projecting, the product  $f \times f'$  is also.
- 4 If  $B$  is a countably based space,  $f^B : X^B \rightarrow Y^B$  is  $\omega$ -projecting.

*Proof.* Statement 1 is straightforward.

To prove 2, let  $f' : X' \rightarrow Y'$  be the pullback of  $f$  along any  $h : Y' \rightarrow Y$ . For any  $g : A \rightarrow Y'$  where  $A$  is countably based, let  $g' : A \rightarrow X'$  be such that  $f.g' = h.g$  (as given by  $f$  being  $\omega$ -projecting). Let  $\bar{g} : A \rightarrow X'$  be given by the universal property of the pullback. Then  $f'.\bar{g} = g$ , as required.

Statement 3 is a consequence of 1 and 2.

To prove 4, consider any map  $g : A \rightarrow Y^B$  where  $A$  is countably based. Take the exponential transpose  $h : A \times B \rightarrow Y$  and extend to  $\bar{h} : A \times B \rightarrow X$  such that  $f.\bar{h} = h$  (as

$f$  is  $\omega$ -projecting). Defining  $\bar{g} : A \rightarrow X^B$  as the exponential transpose of  $\bar{h}$ , we have that  $f^B \cdot \bar{g} = g$ , as required.  $\square$

Now observe that, by our explicit description of regular epis in **Lim** (given in Section 3.2), if  $\mathbb{N}^+$  is projective with respect to a map  $h$ , then  $h$  is a regular epi. As  $\mathbb{N}^+$  is countably based, we obtain the following proposition.

**Proposition 5.2.** If  $q : X \rightarrow Y$  is  $\omega$ -projecting, it is a regular epi.

By the previous two propositions, if  $q : X \rightarrow Y$  is  $\omega$ -projecting in **Lim**, then, for every countably based space  $A$ , the map  $q^A : X^A \rightarrow Y^A$  is a regular epi. Note that the converse holds trivially. Thus we have that  $q : X \rightarrow Y$  is  $\omega$ -projecting if and only if, for every countably based space  $A$ , the following property holds in the internal logic of **Lim**<sup>†</sup>:

$$\mathbf{Lim} \models (\forall f \in Y^A)(\exists \bar{f} \in X^A)(f = q \cdot \bar{f})$$

Thus the original *external* notion of being  $\omega$ -projecting is equivalent to its natural *internal* analogue.

In section 7 we are going to prove the cartesian closure of **PQ**, by working inside **Lim** and using the closure properties of  $\omega$ -projecting maps. In order to do this, we need to study what projecting quotients in **Top** look like from the perspective of **Lim**.

**Proposition 5.3.** Let  $r : B \rightarrow R$  be a continuous function between sequential spaces. The following are equivalent:

- 1  $r : B \rightarrow R$  is  $\omega$ -projecting in **Top**.
- 2  $r : B \rightarrow R$  is an  $\omega$ -projecting quotient in **Top**.
- 3  $r : B \rightarrow R$  is  $\omega$ -projecting in **Lim**.

*Proof.* As **Seq** is a full subcategory of both **Top** and **Lim**, it is clear that 1 and 3 are equivalent and that 2 implies both of them.

We now prove that 3 implies 2. By the previous proposition,  $r$  is a regular epi in **Lim**. But the functor **Lim**  $\rightarrow$  **Seq**  $\rightarrow$  **Top** has a right adjoint and so preserves regular epis. As  $B$  and  $R$  are sequential spaces, the functor maps  $r$  to the continuous function  $r : B \rightarrow R$  in **Top**. Therefore  $r$  is a regular epi in **Top**, that is, it is a topological quotient.  $\square$

Beware, in **Top** (unlike in **Seq**), there exist  $\omega$ -projecting maps, which are not necessarily between sequential spaces, that are not topological quotients.

## 6. Injectivity

In order to prove the cartesian closure of **PQ**, we need to investigate injectivity, the dual notion to projectivity.

**Definition 6.1.** In any category, we say that an object  $X$  is *injective with respect to a map*  $g : Y \rightarrow Z$  if for every  $f : Y \rightarrow X$  there exists  $\bar{f} : Z \rightarrow X$  such that  $f = \bar{f} \cdot g$ .

<sup>†</sup> As **Lim** is a quasitopos it has a full first-order intuitionistic internal logic. However, the property in question can be interpreted more generally in any cartesian-closed regular category.

We shall be interested, in particular, in objects that are injective with respect to all pre-embeddings between countably based spaces. (Recall from Section 4 that topological pre-embeddings and **Lim**-pre-embeddings agree between countably based spaces.) In **Lim**, such injective objects are related to  $\omega$ -projecting maps as follows.

**Proposition 6.1.** In **Lim**,  $E$  is injective with respect to pre-embeddings between countably based spaces if and only if, for every pre-embedding  $a : A \rightarrow B$  between countably based spaces,  $E^a : E^B \rightarrow E^A$  is  $\omega$ -projecting.

*Proof.* For the ‘if’ direction, suppose  $E^a$  is  $\omega$ -projecting. Then, given any  $f : A \rightarrow E$ , we obtain  $g : \mathbf{1} \rightarrow E^A$  by exponential transpose, then  $\bar{g} : \mathbf{1} \rightarrow E^B$  because  $E^a$  is  $\omega$ -projecting, and then  $\bar{f} : B \rightarrow E$  again by exponential transpose. The equation  $\bar{f}.a = f$  is easily verified.

For the ‘only if’ direction, suppose  $E$  is injective with respect to pre-embeddings between countably based spaces, and let  $a : A \rightarrow B$  be a pre-embedding between two countably based spaces. Take any  $f : C \rightarrow E^A$  where  $C$  is countably based. We then obtain  $g : A \times C \rightarrow E$  (by exponential transpose), whence  $\bar{g} : B \times C \rightarrow A$  (because  $a \times id_C : A \times C \rightarrow B \times C$  is a pre-embedding between countably based spaces by Proposition 4.2), whence  $\bar{f} : C \rightarrow E^B$  (again by exponential transpose). The equation  $E^a.\bar{f} = f$  is easily verified. □

In Scott (1972), Dana Scott introduced the *continuous lattices*, and characterised these as the injective objects with respect to subspace embeddings in the category of  $T_0$  topological spaces. Martín Escardó pointed out to us that, in **Top** itself, the continuous lattices are, more generally, injective with respect to topological pre-embeddings. (Note that the topological pre-embeddings between  $T_0$  spaces are exactly the subspace embeddings.)

For our purposes, we require only a convenient collection of injective objects in  $\omega$ **Top**. Although we could work with countably based continuous lattices, it suffices to restrict attention to the (even more manageable) algebraic lattices. We assume that the reader is familiar with the definition of these (Davey and Priestly 1990; Gierz *et al.* 1980). We shall only sketch the various constructions on algebraic lattices that we shall require.

**Proposition 6.2.** Every algebraic lattice is injective with respect to every topological pre-embedding.

*Proof.* Let  $a : X \rightarrow Y$  be any topological pre-embedding. Suppose  $D$  is an algebraic lattice. Consider any  $f : X \rightarrow D$ . Then the extension  $\bar{f} : Y \rightarrow D$  is defined by

$$\bar{f}(y) = \bigsqcup \left\{ \bigcap f(a^{-1}U) \mid U \text{ is an open neighbourhood of } y \right\}.$$

The proof that this is a continuous extension of  $f$  is identical to the standard proof of the injectivity of continuous lattices with respect to subspace embeddings between  $T_0$  spaces (Scott 1972). □

**Proposition 6.3.** Every topological space can be topologically pre-embedded into an algebraic lattice. Moreover, every countably based space can be pre-embedded in a countably based algebraic lattice.



*Proof.* For any topological space  $X$ , construct the algebraic lattice  $D$  as the set of all filters of opens ordered by inclusion. The function mapping  $x$  to its neighbourhood filter is a topological pre-embedding (with respect to the Scott topology on  $D$ ).

For a countably based space, choose a countable base containing the empty set and the whole set. Construct  $D$  as the set of filters of basic opens ordered by inclusion. The pre-embedding is given by the function mapping  $x$  to its filter of basic open neighbourhoods. □

### 7. Bicartesian closure

In this section we finally prove, as Theorem 4, that **PQ** is a full bicartesian-closed subcategory of **Seq**, and we also prove Theorem 3.

We write  $\omega\mathbf{Alg}$  for the category of countably based algebraic lattices. It is well known that  $\omega\mathbf{Alg}$  is cartesian closed (Davey and Priestly 1990; Gierz *et al.* 1980). We assume that the reader is familiar with the construction of exponentials in this category. In particular, for compact elements  $a \in D$  and  $b \in E$  of any two objects  $D, E$  in  $\omega\mathbf{Alg}$ , we write  $(a \searrow b) : D \rightarrow E$  for the related step function. Explicitly,

$$(a \searrow b)d = \begin{cases} b & \text{if } a \leq d \\ \perp & \text{otherwise.} \end{cases}$$

**Lemma 7.1.** The embedding  $S : \omega\mathbf{Alg} \rightarrow \mathbf{Lim}$  is a cartesian closed functor.

*Proof.* The embedding  $S$  assigns to each countably based algebraic lattice the corresponding space with the Scott topology. It is easy to see that it preserves products. Now, for  $D, E$  countably based algebraic lattices, it is also clear that  $S(E^D)$  and  $SE^{SD}$  have the same underlying set, so we need only prove that they have the same convergent sequences. So, let  $(f_i) \rightarrow f$  in  $S(E^D)$  and let  $(x_i) \rightarrow x$  in  $SD$ . We must show that  $(f_i x_i) \rightarrow fx$  in  $SE$ . In order to do this, given any compact  $e \leq fx$ , we will prove that  $(f_i x_i)$  is eventually above  $e$ .

So, let  $(a_i)$  be an ascending sequence of compact elements such that  $\bigsqcup a_i = x$ . Then,  $f(\bigsqcup a_i) = \bigsqcup f a_i = fx$ . So there exists an  $m$  such that  $e \leq f a_m$ . That is,  $(a_m \searrow e) \leq f$ . As  $a_m$  is compact, there exists  $L$  such that for all  $j \geq L$ , we have  $x_j \geq a_m$ . On the other hand, as  $(a_m \searrow e)$  is compact, there exists  $L'$  such that for all  $j \geq L'$ ,  $f_j \geq (a_m \searrow e)$ . Now let  $M = \text{Max}\{L, L'\}$ . For all  $j \geq M$ ,  $f_j \geq (a_m \searrow e)$ , so  $e \leq f_j a_m$ . Also,  $a_m \leq x_j$ , and then  $f_j a_m \leq f_j x_j$ . So  $e \leq f_j x_j$ . That is,  $(f_i x_i)$  is eventually above  $e$ .

We now prove the converse, so assume  $(f_i) \rightarrow f$  in  $SE^{SD}$ . For any compact  $c \leq f$  we will show that  $(f_i)$  is eventually above  $c$ . Actually, as it is known that the compact elements are finite joins of step functions, it is enough to prove that  $(f_i)$  is eventually above  $(a \searrow b)$  for compact elements  $a, b$  in  $D$  and  $E$ , respectively, such that  $(a \searrow b) \leq f$ . To see this, consider the sequence that is constantly  $a$ . By hypothesis,  $(f_i a) \rightarrow fa$ . As  $(a \searrow b) \leq f$  if and only if  $b \leq fa$ , it follows that  $(f_i a)$  is eventually above  $b$ . That is, there exists  $L$  such that for all  $j \geq L$ ,  $f_j a \geq b$ . Then, for all  $j \geq L$ ,  $(a \searrow b) \leq f_j$ . □

**Theorem 4.** The category **PQ** is bicartesian closed with finite limits. Moreover, the inclusion  $\mathbf{PQ} \hookrightarrow \mathbf{Seq}$  preserves this structure.

*Proof.* As the inclusion from **Seq** to **Lim** preserves finite limits, exponentials and coproducts, it suffices to show that **PQ** inherits all the specified structure from **Lim**.

Let  $Q$  and  $R$  be in **PQ**. Then, there exist  $\omega$ -projecting maps  $q : A \rightarrow Q$  and  $r : B \rightarrow R$  in **Lim** with  $A$  and  $B$  countably based.

To prove that  $Q \times R$  is in **PQ**, just recall that  $Q \times R$  is topological, that  $A \times B$  is countably based and that  $\omega$ -projecting maps are closed under products in **Lim**, by Proposition 5.1. Thus, by Proposition 5.3,  $Q \times R$  is an  $\omega$ -projecting quotient of  $A \times B$  in **Top**.

For equalizers, we show that any regular subobject  $m : Q' \rightarrowtail Q$  (in **Lim**) of  $Q$  is in **PQ**. Construct the pullback

$$\begin{array}{ccc}
 A' & \xrightarrow{\quad} & A \\
 q' \downarrow & \lrcorner & \downarrow q \\
 Q' & \xrightarrow{m} & Q
 \end{array}$$

Then  $A'$  is countably based and  $Q'$  is topological, both by Corollary 4.1, and  $q'$  is  $\omega$ -projecting, by Proposition 5.1. Again, by Proposition 5.3,  $Q'$  is an  $\omega$ -projecting quotient of  $A'$  in **Top**.

For coproducts,  $Q + R$  is topological and  $A + B$  is countably based, so, by Proposition 5.1, we need only prove that  $(q + r) : A + B \rightarrow Q + R$  is  $\omega$ -projecting. So, let  $C$  be countably based and take any  $h : C \rightarrow Q + R$ .

As coproducts are stable, we get that  $C$  is isomorphic to  $F + G$  and  $h$  is isomorphic to  $f + g$  in the following diagram:

$$\begin{array}{ccccc}
 F & \xrightarrow{in_F} & C & \xleftarrow{in_G} & G \\
 f \downarrow & \lrcorner & \downarrow h & \lrcorner & \downarrow g \\
 Q & \xrightarrow{in_Q} & Q + R & \xleftarrow{in_R} & R
 \end{array}$$

As  $C$  is countably based and the injections are regular monos,  $F$  and  $G$  are countably based. Then, as  $q$  and  $r$  are projecting, there exist  $\bar{f} : F \rightarrow A$  and  $\bar{g} : G \rightarrow B$  such that  $q \cdot \bar{f} = f$  and  $r \cdot \bar{g} = g$ . So we have  $\bar{h} = \bar{f} + \bar{g} : C \cong F + G \rightarrow A + B$  such that  $(q + r) \cdot \bar{h} = h$ .

Now we consider exponentials. Let  $q : A \rightarrow Q$  and  $r : B \rightarrow R$  be as before. As **Seq** is an exponential ideal of **Lim**,  $R^Q$  is topological. So, by Proposition 5.3, it suffices to construct an  $\omega$ -projecting map  $e : [A, B] \rightarrow R^Q$  from a countably based space  $[A, B]$ .

Using Proposition 6.3, let  $A$  and  $B$  arise as domains of pre-embeddings  $a : A \rightarrow D$  and  $b : B \rightarrow E$  into  $\omega$ -algebraic lattices  $D$  and  $E$ .

We define  $[A, B]$  by taking pullbacks as follows:

$$\begin{array}{ccccc}
 [A, B] & \xleftarrow{\chi} & \langle A, B \rangle & \xrightarrow{\psi} & E^D \\
 \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow E^a \\
 & & B^A & \xrightarrow{b^A} & E^A \\
 \downarrow e & & \downarrow r^A & & \\
 R^Q & \xleftarrow{R^q} & R^A & & 
 \end{array}$$

As  $b$  is a pre-embedding between countably based spaces, it is a **Lim**-pre-embedding by Proposition 4.1. By Proposition 4.2,  $b^A$  is also, and so  $\psi$  is too. By Lemma 7.1,  $E^D$  is countably based, so  $\langle A, B \rangle$  is countably based too.

As  $E$  is injective with respect to pre-embeddings between countably based spaces, we have by Proposition 6.1 that  $E^a$  is  $\omega$ -projecting. It then follows by Proposition 5.1, that  $d, r^A, r^A.d$  and finally  $e$  are  $\omega$ -projecting. So, in order to prove that  $R^Q$  is in **PQ**, we need only prove that  $[A, B]$  is countably based. To see this, notice that as the functor  $R^{(-)}$  carries colimits to limits, and  $q$  is a regular epi (by Proposition 5.2), then  $R^q : R^Q \rightarrow R^A$  is a regular mono. Then  $\chi$  is also, and as  $\langle A, B \rangle$  is countably based,  $[A, B]$  is countably based too. □

As **EPQ** is equivalent to **PQ**, we obtain the following.

**Corollary 7.1.** **EPQ** is bicartesian closed with finite limits.

We conclude this section with the proof of Theorem 3. The proof is a minor adaptation of the proof of a closely related result by Matthias Schröder (private communication). Theorem 3 is an immediate consequence of Proposition 7.1 below, which says that every quotient of a countably based space has an  $\omega$ -projecting countably based ‘cover’.

**Lemma 7.2.** Suppose  $Q$  is a quotient of a countably based space and  $R$  is in **PQ**. Then  $R^Q$  (calculated in **Seq**) is in **PQ**.

*Proof.* The assumptions give a quotient  $q : A \rightarrow Q$  and an  $\omega$ -projecting quotient  $r : B \rightarrow R$  with  $A, B$  countably based. Construct the map  $e : [A, B] \rightarrow R^Q$  as in the proof of closure under exponentials for Theorem 4. As  $q$  is a quotient in **Top**, it is the coequaliser of its kernel pair, which, because  $A$  is countably based, has countably based domain. Thus  $q$  is the coequaliser in **Top** of maps between countably based (hence sequential) spaces. Then, as **Seq** is a full coreflective subcategory of **Top**,  $q$  also coequalises these maps in **Seq**, so  $q$  is a regular epi in **Seq**. This allows the above proof that  $e : [A, B] \rightarrow R^Q$  is  $\omega$ -projecting to go through (without the assumption that  $q$  is  $\omega$ -projecting). □

**Proposition 7.1.** If  $Q$  is a quotient of a countably based space, there exists an  $\omega$ -projecting quotient  $q : A \rightarrow Q$  with  $A$  countably based.

*Proof.* As in Section 4, let  $\Sigma$  be Sierpinski space. By the above lemma (twice),  $\Sigma^{\Sigma^0}$  is in **PQ**. Thus there exists an  $\omega$ -projecting quotient  $q : B \rightarrow \Sigma^{\Sigma^0}$  with  $B$  countably based. By Corollary 4.2,  $\Omega : Q \rightarrow \Sigma^{\Sigma^0}$  is a **Lim**-pre-embedding. Then the pullback  $\Omega^*q : A \rightarrow Q$  is an  $\omega$ -projecting quotient, by Proposition 5.1. Moreover,  $q^*\Omega : A \rightarrow B$  is a **Lim**-pre-embedding, by Proposition 4.2, and thus  $A$  is indeed countably based, by Proposition 4.3.  $\square$

### 8. Relating to equilogical spaces

To complete the proof of Theorem 2, it remains to show that the embedding of **EPQ** in  $\omega$ **Equ** preserves all the identified structure. By the description of finite limits and coproducts in  $\omega$ **Equ**, and the fact that countably based spaces are closed under these operations, it follows that **EPQ** inherits this structure from  $\omega$ **Equ**. It remains to prove that the embedding **EPQ**  $\rightarrow$   $\omega$ **Equ** preserves exponentials. For this, we need explicitly to introduce the cartesian-closed structure on  $\omega$ **Equ**. This is most easily done by considering an equivalent category, introduced in Bauer *et al.* (1998).

#### Definition 8.1.

- 1 An *assembly* (over an algebraic lattice)  $M$  is a triple  $M = (|M|, \delta_M, D_M)$  such that  $|M|$  is a set,  $D_M$  is an algebraic lattice and  $\delta_M$  is a function from  $|M|$  to the set of non-empty subsets of  $D_M$ .
- 2 A morphism between assemblies  $f : M \rightarrow N$  is a function  $f : |M| \rightarrow |N|$  such that there exists a continuous  $\bar{f} : D_M \rightarrow D_N$  *realizing*  $f$  in the sense that for all  $m \in |M|$  and  $d \in \delta_M(m)$ , we have  $\bar{f}d \in \delta_N(fm)$ .

Let **Ass** be the category of assemblies over algebraic lattices and morphisms between them. The proposition below appears in Remark 3.1 of Rosolini (2000).

#### Proposition 8.1. Ass and Equ are equivalent.

*Proof.* First define a functor  $E' : \mathbf{Equ} \rightarrow \mathbf{Ass}$ . For any space  $X$ , let  $\eta_X : X \rightarrow \widehat{X}$  be its representation as a chosen pre-embedding into an algebraic lattice. To each  $(X, \sim_X)$  in **Equ**, assign  $(X/\sim_X, \delta_X, \widehat{X})$ , where  $\delta_X$  assigns to each  $[x]$  in  $X/\sim_X$  the non-empty subset  $\{\eta_{x'} \mid x' \sim x\}$  of  $\widehat{X}$ .

The action on maps is the identity (using Proposition 6.2 to see that this produces a morphism between assemblies). It is easy to see that this functor is full and faithful.

The functor  $E : \mathbf{Ass} \rightarrow \mathbf{Equ}$  is defined as follows. For an assembly  $M = (|M|, \delta_M, D_M)$ , let  $E_M$  be the topological space with underlying set  $\{(m, d) \in |M| \times D_M \mid d \in \delta_M m\}$  and with the unique topology that makes the projection  $E_M \rightarrow D_M$  into a pre-embedding. Let  $\sim_{E_M}$  be the equivalence relation defined by

$$(m, d) \sim_{E_M} (m', d') \text{ iff } m = m'.$$

Define  $E(M) = (E_M, \sim_{E_M})$ .

To define the action on arrows, note that  $E_M/\sim_{E_M}$  is isomorphic to  $M$ . So the action of  $E$  on arrows is the identity up to the evident isomorphism.

It is straightforward to check that this functor is also full and faithful and that together with  $E'$  they give an equivalence between **Ass** and **Equ**.  $\square$

The advantage of **Ass** over **Equ** is that its exponentials have an easy description. For assemblies  $M, N$  let  $|N^M|$  be the set of morphisms from  $M$  to  $N$ . Then, the exponential is defined by  $N^M = (|N^M|, \delta_{NM}, D_N^{D_M})$  where  $D_N^{D_M}$  is the exponential of algebraic lattices and  $\delta_{NM}(f : M \rightarrow N) = \{g : D_M \rightarrow D_N \mid g \text{ realizes } f\}$ .

Let  $\omega\mathbf{Ass}$  denote the category of assemblies between countably based algebraic lattices. It is not difficult to see that the equivalence of the Proposition 8.1 cuts down to one between  $\omega\mathbf{Ass}$  and  $\omega\mathbf{Equ}$  so long as the choice of pre-embedding in the definition of  $E'$  is chosen so as to preserve the countable base. Also, the description of exponentials in  $\omega\mathbf{Ass}$  is identical to that in **Ass**.

We can now prove that the embedding of **EPQ** in  $\omega\mathbf{Equ}$  preserves exponentials. To calculate the exponential in **EPQ**, we use its equivalence with **PQ**.

Given objects  $(A, \sim_A)$  and  $(B, \sim_B)$  in **EPQ**, we write  $q : A \rightarrow Q$  and  $r : B \rightarrow R$  for the induced  $\omega$ -projecting regular epis in **Lim**. In Section 7, we constructed the  $\omega$ -projecting regular epi  $e : [A, B] \rightarrow R^Q$  and a pre-embedding  $c : [A, B] \rightarrow \widehat{B}^A$ . Writing  $\sim$  for the induced equivalence relation on the countably based space  $[A, B]$ , we have that the quotient  $[A, B]/\sim$  is isomorphic to the exponential  $R^Q$ . As the equivalence **EPQ**  $\rightarrow$  **PQ** reflects exponentials, we obtain the following.

**Proposition 8.2.** In **EPQ**,  $(B, \sim_B)^{(A, \sim_A)}$  is isomorphic to  $([A, B], \sim)$ .

So we must prove the proposition below.

**Proposition 8.3.** In  $\omega\mathbf{Equ}$ ,  $(B, \sim_B)^{(A, \sim_A)}$  is isomorphic to  $([A, B], \sim)$ .

*Proof.* We use the equivalence between  $\omega\mathbf{Ass}$  and  $\omega\mathbf{Equ}$ . Calculate the exponential  $E'(B, \sim_B)^{E'(A, \sim_A)} = (R, \delta_B, \widehat{B})^{(Q, \delta_A, \widehat{A})} = (R^Q, \delta, \widehat{B}^A)$  where, for  $k \in R^Q$ , we have  $\delta(k) = \{c(f) \mid f \in [A, B] \text{ and } e(f) = k\}$ .

But  $E_{R^Q}$  is iso to  $[A, B]$  and the projection  $E_{R^Q} \rightarrow \widehat{B}^A$  is a pre-embedding. Moreover,  $E_{R^Q}/\sim_{E_{R^Q}} \cong R^Q$ , so the image of the exponential assembly above is isomorphic to  $([A, B], \sim)$ . As the functor  $E$  is part of an equivalence, it preserves exponentials. So  $([A, B], \sim)$  is indeed the exponential of  $(A, \sim_A)$  and  $(B, \sim_B)$  in  $\omega\mathbf{Equ}$ .  $\square$

**Corollary 8.1.** The embedding **EPQ**  $\rightarrow$   $\omega\mathbf{Equ}$  is a bicartesian-closed functor preserving finite limits.

### 9. Lim-subcategories of $\omega\mathbf{Equ}$

The category **PQ** was characterised as the largest topological category (containing  $\omega\mathbf{Top}$ ) induced by the topological quotient functor  $Q : \omega\mathbf{Equ} \rightarrow \mathbf{Top}$ . It was shown to be a bicartesian-closed category inheriting its structure from both **Seq** and  $\omega\mathbf{Equ}$ . However,  $\omega\mathbf{Equ}$  is also *locally* cartesian closed, but, as we shall see in this section, **PQ** is not. In fact, in order to achieve an extensional account of local cartesian closure, it seems essential to go beyond the realm of topological spaces. Although **PQ** is the largest common full subcategory of **Seq** and  $\omega\mathbf{Equ}$ , it turns out that  $\omega\mathbf{Equ}$  shares an even larger full subcategory

with **Lim**. This larger category is locally cartesian closed and the embeddings into **Lim** and  $\omega\mathbf{Equ}$  preserve this structure. Thus, via the use of limit spaces, this category offers an extensional approach to understanding local cartesian closure within  $\omega\mathbf{Equ}$ .

For an equivalence relation  $\sim$  on a limit space  $X$  we define the **Lim-quotient** to be the limit space on the set-theoretic quotient  $X/\sim$  determined by the requirement that the quotient function  $X \rightarrow (X/\sim)$  be regular epi in **Lim**. We can then define a functor  $Q_L : \omega\mathbf{Equ} \rightarrow \mathbf{Lim}$  that takes an object  $(A, \sim)$  to its **Lim-quotient**. As with the topological quotient functor, the functor  $Q_L$  is faithful. Thus, by analogy with Definition 2.2, we say that a full subcategory **C** of  $\omega\mathbf{Equ}$  is a **Lim-subcategory** if the composite functor  $\mathbf{C} \hookrightarrow \omega\mathbf{Equ} \xrightarrow{Q_L} \mathbf{Lim}$  is full.

Let  $\mathbf{PQ}_L$  be the full subcategory of **Lim** given by those limit spaces  $X$  for which there exists a countably based  $A$  and an  $\omega$ -projecting map  $A \rightarrow X$  in **Lim**. Also, let  $\mathbf{EPQ}_L$  be the full subcategory of  $\omega\mathbf{Equ}$  given by those  $(A, \sim)$  such that the **Lim-quotient**  $A \rightarrow (A/\sim)$  is  $\omega$ -projecting.

**Theorem 5.**

- 1  $\mathbf{EPQ}_L$  is the largest **Lim-subcategory** of  $\omega\mathbf{Equ}$  containing  $\omega\mathbf{Top}$ .
- 2  $\mathbf{PQ}_L$  and  $\mathbf{EPQ}_L$  are equivalent.
- 3  $\mathbf{PQ}_L$  is bicartesian closed with finite limits, and the embedding into **Lim** preserves this structure.
- 4 The embedding of  $\mathbf{EPQ}_L$  in  $\omega\mathbf{Equ}$  also preserves the above structure.

The proof of Theorem 5 follows exactly the lines of the proofs for **PQ** and **EPQ** (except that the category **Seq** can be avoided altogether). Indeed, because of the correspondence between  $\omega$ -projectivity in **Top** and **Lim** for sequential spaces (Proposition 5.3) we obtain the following corollary.

**Corollary 9.1.** **PQ** is a full subcategory of  $\mathbf{PQ}_L$ . Moreover, the embedding preserves the bicartesian-closed structure and finite limits.

The benefit of **PQ** is that it consists entirely of topological spaces, which are familiar mathematical objects. However, the benefit of  $\mathbf{PQ}_L$  over **PQ** is that the following theorem holds, as we shall prove in this section.

**Theorem 6.**

- 1  $\mathbf{PQ}_L$  is locally cartesian closed, and the embedding into **Lim** preserves this structure.
- 2 The embedding of  $\mathbf{EPQ}_L$  in  $\omega\mathbf{Equ}$  also preserves this structure.

In the next section (see discussion below Proposition 9.3) we show that achieving local cartesian closure necessitates considering non-topological subcategories of  $\omega\mathbf{Equ}$ . This remark relates to the observation of Normann and Waagbø (Normann and Waagbø 1998), who found that non-topological limit spaces are necessary for modelling dependent types.

It is worth mentioning that it is possible to fully embed the *whole* of **Lim** in **Equ** by composing the inclusion functors  $\mathbf{Lim} \longrightarrow \mathbf{Fil}$  (the category of *filter spaces* (Hyland

1979b, Theorem 9.2)), and **Fil**  $\longrightarrow$  **Equ** (Rosolini 2000; Heckmann 1998). However, this embedding is not cartesian closed, cf. Hyland (1979b).

9.1. Strong partial map classifiers in **Lim**

In this subsection we define strong partial map classifiers. It is a standard result in category theory that cartesian closure and the existence of strong partial maps together imply local cartesian closure (Proposition 9.1 below). Also in this subsection we describe the strong partial map classifiers in **Lim**. These results will be used in the next subsection to prove that **PQ<sub>L</sub>** is locally cartesian closed.

**Definition 9.1.** A mono  $m : Y \rightarrow Z$  is strong if for every epi  $e : X \twoheadrightarrow W$  and maps  $g : X \rightarrow Y$  and  $g' : W \rightarrow Z$  such that  $g'.e = m.g$ , we have that there exists a (necessarily unique)  $h : W \rightarrow Y$  such that  $m.h = g'$  and  $h.e = g$ .

A strong partial map  $\langle m, f \rangle : Y \rightarrow X$  is a pair consisting of a strong mono  $m : Y' \rightarrow Y$  and a map  $f : Y' \rightarrow X$ . (Normally, strong partial maps are equivalence classes of such pairs, but we shall not be concerned with the equivalence of partial maps.)

Notice that in a category with epi/regular-mono factorizations, strong monos are equalizers – for example, in **Top**.

**Definition 9.2.** A classifier for strong partial maps with codomain  $X$  is an object  $\tilde{X}$  together with a strong mono  $\tau : X \twoheadrightarrow \tilde{X}$  such that for every strong partial map  $\langle m, f \rangle : Y \rightarrow X$  there exists a unique map  $\chi_f : Y \rightarrow \tilde{X}$  such that the following square is a pullback.

$$\begin{array}{ccc}
 Y' & \xrightarrow{f} & X \\
 \downarrow m & & \downarrow \tau \\
 Y & \xrightarrow{\chi_f} & \tilde{X}
 \end{array}$$

We say that a category has strong partial map classifiers if for every  $X$  it has a classifier for strong partial maps with codomain  $X$ .

**Proposition 9.1.** If **E** is cartesian closed and has strong partial map classifiers, then **E** is locally cartesian closed.

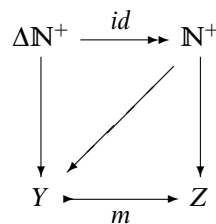
*Proof.* See, for example, Wyler (1991, paragraph 19.3). □

Then, in order to prove that **PQ<sub>L</sub>** is locally cartesian closed, it is enough to prove that it has strong partial map classifiers. To do this, we first show that the strong monos in **PQ<sub>L</sub>** are exactly the regular monos in **Lim** between objects in **PQ<sub>L</sub>**. We then describe the strong partial map classifiers in **Lim**. In the next subsection we will prove that this description also works in **PQ<sub>L</sub>**.

**Proposition 9.2.** In any of the categories **Lim**, **PQ** and **PQ<sub>L</sub>**, a mono  $m : Y \rightarrow Z$  is strong if and only if  $(my_i) \rightarrow my$  in  $Z$  implies  $(y_i) \rightarrow y$  in  $Y$ .

*Proof.* In all the categories in the statement, monos are exactly the maps with an underlying injective function. One then proves that the strong monos are exactly the regular ones as follows. Let  $m : Y \rightarrow Z$  be a strong mono. Now, assume that  $(my_i) \rightarrow my$  in  $Z$ ; we need to prove that  $(y_i) \rightarrow y$  in  $Y$ .

Let  $\Delta\mathbb{N}^+$  be the topological space with underlying set  $\mathbb{N} \cup \{\infty\}$  and the discrete topology. The identity function is obviously an epi map  $\Delta\mathbb{N}^+ \twoheadrightarrow \mathbb{N}^+$ . Then define a map  $\Delta\mathbb{N}^+ \rightarrow Y$  by sending  $n$  to  $y_n$  and  $\infty$  to  $y$ . Define also a map  $\mathbb{N}^+ \rightarrow Z$  by sending  $n$  to  $my_n$  and  $\infty$  to  $my$ . Then we have a square as below, which we can complete because  $m$  is strong:



But this means that  $(y_i)$  converges to  $y$ . □

For every limit space  $X$ , we define  $\tilde{X}$  to have underlying set  $|X| \cup \perp$  and the following convergent sequences. First, every  $(z_i)$  converges to  $\perp$ . Then, for every  $z$  in  $X$ ,  $(z_i)$  converges to  $z$  in  $\tilde{X}$  if and only if for every subsequence  $(z_{\alpha i})$  there exists a subsequence  $(z_{\alpha\beta i})$  such that one of the following holds:

- 1  $(z_{\alpha\beta i})$  is constantly  $\perp$  or
- 2  $(z_{\alpha\beta i})$  is inside  $X$  and it converges to  $z$  in  $X$ .

**Proposition 9.3.** For every  $X$ ,  $\tilde{X}$  is a limit space.

*Proof.* The first two axioms of Definition 3.2 are easy to prove.

For the third, let  $(z_i)$  be such that for every subsequence  $(z_{\alpha i})$  there exists a subsequence  $(z_{\alpha\beta i})$  that converges to  $z$ . If  $z = \perp$ , the axiom holds trivially because everything converges to  $\perp$ . So, let  $z \in X$ . By the definition of  $\tilde{X}$ , there exists a subsequence  $(z_{\alpha\gamma\delta i})$  satisfying one of the conditions above. Then put  $\beta = \gamma.\delta$ , and this proves that  $(z_i)$  converges to  $z$ . □

It is worth pointing out that the limit space  $\tilde{X}$  is almost never topological, even when  $X$  is. Indeed, if  $\tilde{X}$  were topological, the only closed inhabited set would be the entirety of  $X$ , as, for any  $x$ , the constant  $x$  sequence converges to  $\perp$  and the constant  $\perp$  sequence converges to  $x$ . Thus the topology on  $\tilde{X}$  would have to be the chaotic topology. But this cannot be the case whenever  $X$  contains two distinct elements  $x, y$  with  $x$  not in the closure of  $y$ , because then the constant  $y$  sequence does not converge to  $x$  in  $\tilde{X}$  although it does in the chaotic topology.

This example also shows that it is essential to go beyond topological spaces to achieve local cartesian closure. The reason is that  $\tilde{X}$  can be defined from  $X$  and  $\nabla(1 + 1)$  (the strong subobject classifier) using the local cartesian closed structure of **Lim**. Thus, any locally cartesian closed subcategory of **Lim** containing  $1 + 1$  and  $\nabla(1 + 1)$  must contain a non-topological limit space.



Let  $\tau_X : X \rightarrow \tilde{X}$  be the evident regular mono embedding  $X$  into  $\tilde{X}$ .

**Proposition 9.4.** For every  $X$  in **Lim**,  $\tau_X : X \rightarrow \tilde{X}$  is a strong partial map classifier in **Lim**.

*Proof.* Let  $\langle m, f \rangle : Y \rightarrow X$  be a strong partial map with  $m : Y' \rightarrow Y$ . Define  $\chi_f : Y \rightarrow \tilde{X}$  by

$$\chi_f y = \begin{cases} \tau(fy) & \text{if } y \in Y' \\ \perp & \text{otherwise.} \end{cases}$$

To prove that  $\chi_f$  is continuous, let  $(y_i) \rightarrow y$  in  $Y$ . If  $y \notin Y'$ , then  $\chi_f y = \perp$ , and hence  $(\chi_f y_i)$  converges to  $\chi_f y$  in  $\tilde{X}$ . So, let  $y \in Y'$  and consider a subsequence  $(y_{\alpha\beta i})$ .

If  $(y_{\alpha i})$  is eventually in  $Y'$ , it has a subsequence  $(y_{\alpha\beta i})$  that is completely inside  $Y'$ . As  $m$  is strong,  $(y_{\alpha\beta i})$  converges to  $y$  in  $Y'$ . Then  $(\tau(fy_{\alpha\beta i}))$  converges to  $\tau(fy)$  in  $\tilde{X}$ . That is,  $(\chi_f y_{\alpha\beta i})$  converges to  $\chi_f y$ .

If  $(y_{\alpha i})$  is not eventually in  $Y'$ , there exists a subsequence  $(y_{\alpha\beta i})$  that is completely outside  $Y'$ . So  $(\chi_f y_{\alpha\beta i})$  is a constant sequence of  $\perp$ 's and hence converges to  $\chi_f y$ .

This completes the proof that  $\chi_f$  is continuous. It is not difficult to see that  $\chi_f.m = \tau.f$ .

To prove that the diagram in Definition 9.2 is a pullback square, let  $h : Z \rightarrow Y$  and  $g : Z \rightarrow X$  be such that  $\chi_f.h = \tau.g$ . By the definitions of  $\chi_f$  and  $\tau$ , it follows that the image of  $h$  is included in the image of  $m$ . As  $m$  is a regular mono, it follows that  $h$  factors as  $h = m.h'$  for a unique  $h' : Z \rightarrow Y'$ .

On the other hand,  $\tau.g = \chi_f.h = \chi_f.m.h' = \tau.f.h'$ . As  $\tau$  is mono,  $g = f.h'$ , and hence the square is a pullback. In order to see that  $\chi_f$  is the unique map that allows us to prove this, note that there is no room for another definition. This is because the value on  $y \in Y'$  is determined by the partial map and the value on  $y \notin Y'$  has to go to  $\perp$ . This finishes the proof that  $\tau$  is a partial map classifier. □

### 9.2. **PQ<sub>L</sub>** is locally cartesian closed

In this subsection we prove that **PQ<sub>L</sub>** is locally cartesian closed. In order to do this we prove that it is closed under the formation of strong partial map classifiers. That is, if  $q : A \rightarrow Q$  is  $\omega$ -projecting in **Lim** with  $A \in \omega\mathbf{Top}$ , there exists an  $\omega$ -projecting  $r : \tilde{A} \rightarrow \tilde{Q}$  with  $\tilde{A}$  in  $\omega\mathbf{Top}$ .

An important part of the construction, though, does not depend on the topological spaces involved being countably based. In fact, the essential parts of this construction will be used later to describe the strong partial map classifiers in **Equ**.

For any topological space  $A$ , let  $a : A \rightarrow \hat{A}$  be the usual pre-embedding into an algebraic lattice  $\hat{A}$ . Also, let  $|\tilde{A}| = |\hat{A}| + |A|$  and let  $\tilde{A}$  be the topological space with underlying set  $|\tilde{A}|$  and topology given by the open sets of the form  $U \cup \{x \mid ax \in U\}$  where  $U$  is open in  $\hat{A}$ . The idea is to add to  $\hat{A}$  a copy of  $A$  in such a way that if  $x \in A$  and  $ax \in \hat{A}$ , then  $x$  and  $ax$  have the same open neighbourhoods. Notice that if  $A$  is a countably based space, we can find a countably based  $\hat{A}$ , and hence a countably based  $\tilde{A}$ .

In spite of  $\tilde{A}$  not being a coproduct, we still have continuous injections  $in_A : A \rightarrow \tilde{A}$  and

$in_{\hat{A}} : \hat{A} \rightarrow \check{A}$  that are, in fact, regular monos in **Top**. This remark restricts to countably based spaces.

**Lemma 9.1.** For any topological space  $C$  and strong partial map  $\langle m, f \rangle : C \rightarrow A$  there exists a (not necessarily unique)  $v_f : C \rightarrow \check{A}$  such that the following square is a pullback:

$$\begin{array}{ccc}
 C_0 & \xrightarrow{f} & A \\
 m \downarrow & & \downarrow in_A \\
 C & \xrightarrow{v_f} & \check{A}
 \end{array}$$

*Proof.* Consider the map  $a.f : C_0 \rightarrow \hat{A}$ . As  $\hat{A}$  is injective with respect to subspace embeddings, there exists an  $\bar{f} : C \rightarrow \hat{A}$  such that  $a.f = \bar{f}.m$ . Now define  $v_f : C \rightarrow \check{A}$  by

$$v_f c = \begin{cases} in_A(f c) & \text{if } c \in C_0 \\ in_{\hat{A}}(\bar{f} c) & \text{otherwise.} \end{cases}$$

We now prove that  $v_f$  is continuous. So, let  $V$  be open in  $\check{A}$ . Then  $V = U \cup \{x \mid ax \in U\}$  with  $U$  open in  $\hat{A}$ . So  $v_f^{-1}V = \{c \notin C_0 \mid \bar{f}c \in U\} \cup \{c \in C_0 \mid a f c = \bar{f} m c \in U\} = \bar{f}^{-1}U$ , which is open because  $\bar{f}$  is continuous.

To see that the square is a pullback, let  $j : D \rightarrow C$  and  $k : D \rightarrow A$  be such that  $v_f.j = in_A.k$ . As the image of  $v_f.j$  has to be included in the image of  $in_A$ , it follows that the image of  $f$  is included in the image of  $m$ . As  $m$  is a subspace embedding,  $j$  factors through  $m$  via a (necessarily unique)  $j' : D \rightarrow C_0$ . Using the fact that  $in_A$  is mono, one proves that  $f.j' = k$ . □

**Proposition 9.5.** If  $Q$  is in **PQ<sub>L</sub>**, then so is  $\check{Q}$ .

*Proof.* Let  $q : A \rightarrow Q$  be  $\omega$ -projecting in **Lim** with  $A \in \omega\mathbf{Top}$ . First notice that as  $in_A$  is a regular mono,  $\check{A} \xleftarrow{in_A} A \xrightarrow{q} Q$  is a strong partial map. Then we have a unique  $r : \check{A} \rightarrow \check{Q}$  making the right-hand square in the second diagram below a pullback. We now prove that this  $r$  is  $\omega$ -projecting.

Let  $C$  be countably based, let  $g : C \rightarrow \check{Q}$  and take the following pullback:

$$\begin{array}{ccc}
 C_0 & \xrightarrow{h} & Q \\
 m \downarrow & & \downarrow \tau \\
 C & \xrightarrow{g} & \check{Q}
 \end{array}$$

As  $q$  is  $\omega$ -projecting, the map  $h : C_0 \rightarrow Q$  factors as  $q.f = h$  for some  $f : C_0 \rightarrow A$ .

Then, Lemma 9.1 gives us a map  $v_f$  and the left-hand pullback in the diagram below:

$$\begin{array}{ccccc}
 C_0 & \xrightarrow{f} & A & \xrightarrow{q} & Q \\
 \downarrow m & & \downarrow in_A & & \downarrow \tau \\
 C & \xrightarrow{v_f} & \tilde{A} & \xrightarrow{r} & \tilde{Q}
 \end{array}$$

Both squares are pullbacks so the rectangle is. As  $\tau$  is a strong partial map classifier, the map  $r.v_f$  is the unique one making the rectangle a pullback. But  $q.f = h$ , so  $r.v_f = g$ . Hence  $r$  is  $\omega$ -projecting. □

**Corollary 9.2.**  $\mathbf{PQ}_L$  is locally cartesian closed and the embedding in  $\mathbf{Lim}$  preserves this structure.

*Proof.* The proof is by Proposition 9.1 and Proposition 9.5. □

### 9.3. The preservation of the local structure

We now prove that the embedding of  $\mathbf{PQ}_L$  into  $\omega\mathbf{Equ}$  preserves the locally cartesian closed structure. In order to do this, we describe the strong partial map classifiers in  $\mathbf{Equ}$ . We indicate that the description restricts to  $\omega\mathbf{Equ}$  and show that the embedding  $\mathbf{PQ}_L \rightarrow \omega\mathbf{Equ}$  preserves this structure. Then, as the cartesian closed structure is also preserved, the construction of the exponentials in the slices coincides.

Using the equivalence between  $\mathbf{PQ}_L$  and  $\widehat{\mathbf{EPQ}_L}$ , it is easy to see that for any object  $(A, \sim_A)$  in  $\mathbf{EPQ}_L$ , the partial map classifier  $(A, \sim_A)$  is  $(\tilde{A}, \sim_{\tilde{A}})$ , where  $\tilde{A}$  is the topological space associated to  $A$  as described before Lemma 9.1 and  $\sim_{\tilde{A}}$  is the equivalence relation given by:

- 1 For every  $x, x' \in A, in_A x \sim_{\tilde{A}} in_A x'$  if and only if  $x \sim_A x'$ .
- 2 For every  $z, z' \in \hat{A}, in_{\hat{A}} z \sim_{\tilde{A}} in_{\hat{A}} z'$ .

Moreover, the classifying map  $\tau : (A, \sim_A) \rightarrow (\tilde{A}, \sim_{\tilde{A}})$  is just the induced quotient of  $in_A : A \rightarrow \hat{A}$ , which clearly preserves the equivalence relations.

It is clear that we can construct such a  $\tau : (A, \sim_A) \rightarrow (\tilde{A}, \sim_{\tilde{A}})$  for any equilogical space  $(A, \sim_A)$ . We now prove that these maps are strong partial map classifiers in  $\mathbf{Equ}$ . As we mentioned before, if  $A$  is countably based, we can find a countably based  $\tilde{A}$ . It will then follow that the embedding  $\mathbf{PQ}_L \rightarrow \omega\mathbf{Equ}$  preserves strong partial map classifiers.

First we need a technical lemma on pullbacks in  $\mathbf{Equ}$ . Before stating it, let us recall that regular monos in  $(W, \sim_W) \hookrightarrow (Y, \sim_Y)$  in  $\mathbf{Equ}$  can be described as subspace embeddings  $m : W \hookrightarrow Y$  where  $\sim_W$  is the restriction of  $\sim_Y$ , and, moreover,  $W$  is closed under the equivalence relation, that is, if  $mw \sim_Y y'$ , then there exists (a necessarily unique, as  $m$  is injective)  $w'$  such that  $mw' = y'$ .

**Lemma 9.2.** Consider the following commutative square in **Equ**:

$$\begin{array}{ccc}
 (W, \sim_W) & \xrightarrow{\phi} & (X, \sim_X) \\
 \downarrow \psi & & \downarrow \psi' \\
 (Y, \sim_Y) & \xrightarrow{\phi'} & (Z, \sim_Z)
 \end{array}$$

Suppose that  $\psi$  and  $\psi'$  are realized by subspace embeddings  $m$  and  $n$  that are closed under the equivalences  $\sim_Y$  and  $\sim_Z$ , respectively. Also, let  $\phi$  and  $\phi'$  be realized by  $f$  and  $g$ . Finally, assume that the following square is a pullback in **Top**:

$$\begin{array}{ccc}
 W & \xrightarrow{f} & X \\
 \downarrow m & & \downarrow n \\
 Y & \xrightarrow{g} & Z
 \end{array}$$

Then the first square is a pullback in **Equ**.

*Proof.* Let us first calculate the pullback  $(P, \sim_P)$  of  $\psi'$  and  $\phi'$ . The usual construction of pullbacks in **Equ** gives that the underlying set of  $P$  is  $\{(y, x) \in Y \times X \mid gy \sim_Z nx\}$ . But we are going to find a more suitable representation.

Suppose that  $gy \sim_Z nx$ . Then, as  $n$  is closed under  $\sim_Z$ , we have that there exists an  $x' \in X$  such that  $nx' = gy$ . Then, it must be the case that there is a (necessarily unique)  $w$  in  $W$  such that  $mw = y$ .

So the underlying set of  $P$  can be described as  $\{(w, x) \in W \times X \mid g(mw) \sim_Z nx\}$ . The topology of  $P$  is inherited from  $W \times X$  and the equivalence relation is inherited from  $\sim_W \times \sim_X$ .

Clearly, the continuous  $\langle id, f \rangle : W \rightarrow W \times X$  factors through  $P$ . On the other hand, there is an obvious projection  $\pi : P \rightarrow W$ . We will show that these maps induce an isomorphism between  $(W, \sim_W)$  and  $(P, \sim_P)$ .

Of course,  $\pi.\langle id, f \rangle = id : W \rightarrow W$ .

On the other hand,  $\langle id, f \rangle(\pi((w, x) \in P)) = (w, fw)$ . We want to show that  $fw \sim_X x$  to be able to conclude that  $(w, fw) \sim_P (w, x)$ . But as  $(w, x)$  is in  $P$ , we have that  $g(mw) \sim_Z nx$ . Then  $n(fw) \sim_Z nx$ , so, as  $\sim_X$  is the restriction of  $\sim_Z$ ,  $fw \sim_X x$ . So  $(W, \sim_W)$  is the pullback in **Equ**. □

Clearly, the proposition above restricts to  $\omega$ **Equ**.

Now, as explained in Rosolini (2000), **Equ** is a quasitopos, so every strong mono in it is an equalizer. We use this fact to prove the following proposition.

**Proposition 9.6.**  $\tau : (A, \sim_A) \twoheadrightarrow (\check{A}, \sim_{\check{A}})$  is a strong partial map classifier in **Equ**.

*Proof.* Let  $\psi : (Y', \sim_{Y'}) \rightarrow (Y, \sim_Y)$  and  $\phi : (Y', \sim_{Y'}) \rightarrow (A, \sim_A)$  be equivariant maps in **Equ** with  $\psi$  a strong mono forming together a strong partial map.

By the description of equalizers in **Equ**, we can assume that  $\psi$  is realized by a subspace embedding  $m : Y' \rightarrow Y$  that is closed under the equivalence relation  $\sim_Y$ . Moreover,  $\sim_{Y'}$  is the restriction of  $\sim_Y$  to  $Y'$ . Also, let  $f : Y' \rightarrow A$  realize  $\phi$ .

By Lemma 9.1, we have a pullback square (of topological spaces) as follows:

$$\begin{array}{ccc}
 Y' & \xrightarrow{f} & A \\
 \downarrow m & & \downarrow in_A \\
 Y & \xrightarrow{v_f} & \check{A}
 \end{array}$$

Let us check that  $v_f$  preserves  $\sim_Y$ .

- 1 If  $y \sim_Y y'$  are both in  $Y'$ , then  $v_f(my) = in_A(fy) \sim_{\check{A}} in_A(fy') = v_f(my')$ .
- 2 Now suppose that  $y \sim_Y y'$  are not in  $Y'$ . Then it must be the case that both  $v_f y$  and  $v_f y'$  are not in the image of  $in_A : A \rightarrow \check{A}$ . Then, by the definition of  $\sim_{\check{A}}$ , we have that  $v_f y$  and  $v_f y'$  are related.

So, as claimed,  $v_f$  preserves  $\sim_Y$ .

Then it realizes a map  $\chi_\phi : (Y, \sim_Y) \rightarrow (\check{A}, \sim_{\check{A}})$ . Hence, by Lemma 9.2, we have that the following square is a pullback in **Equ**:

$$\begin{array}{ccc}
 (Y', \sim_{Y'}) & \xrightarrow{\phi} & (A, \sim_A) \\
 \downarrow \psi & & \downarrow \tau \\
 (Y, \sim_Y) & \xrightarrow{\chi_\phi} & (\check{A}, \sim_{\check{A}})
 \end{array}$$

Now we must prove that  $\chi_\phi$  is unique. So, let  $\chi'$  be any other such map and let  $h : Y \rightarrow \check{A}$  realize it. Then it must be the case that for every  $y \in Y'$ ,  $h(my) \sim_{\check{A}} in_A(fy)$ . But then, as  $in_A(fy) = v_f(my)$ , we have  $h(my) \sim_{\check{A}} v_f(my)$ . Also, for  $y \notin Y'$  it must be the case that  $hy$  is in the image of  $in_{\check{A}}$ . So,  $v_f y \sim_{\check{A}} hy$  in this case also. Hence,  $h$  realizes  $\chi_\phi$ , and this implies that  $\chi_\phi = \chi'$ .

This finishes the proof that  $\tau$  is a strong partial map classifier in **Equ**. □

Again, the proposition above restricts to  $\omega\mathbf{Equ}$ .

**Corollary 9.3.** The embedding  $\mathbf{EPQ}_L \rightarrow \omega\mathbf{Equ}$  preserves the local cartesian closed structure.

*Proof.* See the discussion at the beginning of this subsection (9.3). □

## 10. Conclusions

Our results have some immediate applications. For example, one readily sees that the space of discrete natural numbers occurs as the natural numbers object in **PQ** and **PQ<sub>L</sub>**, and that the inclusions to **Seq**, **Lim** and **Equ** all preserve the natural numbers object. Thus one gets that the type hierarchies over  $\mathbb{N}$  in both **Lim** and **Equ** agree. It has long been known that the type hierarchy over  $\mathbb{N}$  in **Lim** is given by the Kleene/Kreisel continuous functionals (Scarpellini 1971). Thus we have an alternative proof of the recent result from Bauer *et al.* (1998) that the continuous functionals arise as the full type hierarchy in **Equ**. More interesting is that a similar analysis applies to the full type hierarchy over any countably based space. For example, the type hierarchy over the Euclidean reals in **Lim** (see Normann (2000) for a detailed study of this hierarchy) coincides with the hierarchy over the topological (projective) reals in **Equ**. Also, a similar analysis is available for hierarchies of dependent types, the so called *transfinite types* (Normann and Waagbø 1998) in **PQ<sub>L</sub>**. Similar results relating type hierarchies in categories of filter spaces to type hierarchies in **Equ** have appeared recently in Rosolini (2000) and Heckmann (1998).

Our results and techniques bear comparison with recent work by Berger and Normann on totality in type hierarchies, in which they relate intensional ‘totality’ structure on Scott domains to extensional structure modelled either topologically or in limit spaces (Berger 1993; Berger 1997; Normann 2000; Normann and Waagbø 1998). Our work is similar in motivation. In fact, it seems that the techniques used in our proof of Theorem 4 generalise to give a categorical approach to proving some of their results. Also, our analysis of largest common subcategories shared by the extensional and intensional approaches provides a conceptual basis for understanding the ‘lifting theorems’ of Normann and Waagbø (Normann and Waagbø 1998).

Another interesting connection is that the proof of Theorem 4 essentially gives a categorical approach to the logical relations known as partial surjective homomorphisms, which originated in Friedman’s completeness proof for the simply-typed  $\lambda$ -calculus (Friedman 1975). It seems that the notions of injectivity and projectivity form an abstract basis for understanding such special logical relations.

The functor  $Q_L : \omega\mathbf{Equ} \rightarrow \mathbf{Lim}$ , investigated in Section 9, arises in a natural way that yields connections with topos theory. The category  $\omega\mathbf{Equ}$  is the regular completion of  $\omega\mathbf{Top}$  (as a left-exact category) and **Lim** is a regular category. Therefore the left-exact inclusion  $\omega\mathbf{Top} \hookrightarrow \mathbf{Lim}$  determines a regular functor from  $\omega\mathbf{Equ}$  to **Lim**. This functor turns out to be  $Q_L$ . Interestingly, both  $\omega\mathbf{Equ}$  and **Lim** arise as the categories of double-negation separated objects within containing toposes. In the case of  $\omega\mathbf{Equ}$  the associated topos is the realizability topos  $\mathbf{RT}(\mathcal{P}\omega)$ , which is equivalent to the exact completion of  $\omega\mathbf{Top}$ . In the case of **Lim** the topos is Johnstone’s ‘topological’ (Grothendieck) topos  $\mathcal{J}$  (Johnstone 1979). The characterisation of  $\mathbf{RT}(\mathcal{P}\omega)$  as an exact completion yields an exact functor from  $\mathbf{RT}(\mathcal{P}\omega)$  to  $\mathcal{J}$  extending  $Q_L$ . Thus the functor  $Q_L$  is part of an intriguing larger relationship between two well-studied ambient toposes.

One possible application of **PQ** is to tame the ‘troublesome’ probabilistic powerdomain (Jung 1998). Using ideas from synthetic domain theory (Hyland 1991), one can

find a natural left-exact cartesian-closed full subcategory of predomains within  $\mathbf{PQ}$ . This structure can be used to give an ‘internal’ definition of a predomain of continuous valuations on any predomain, that is, a candidate probabilistic powerdomain. It seems plausible that, because of the representation of the objects of  $\mathbf{PQ}$  as quotients of countably based spaces, this powerdomain will address the problems raised in Jung (1998).

### Addendum

The work in this paper was first submitted in November 1999 after presentation at the April 1999 MFCS in New Orleans. Since the original submission of the paper, several other papers on closely related topics have appeared. In Bauer and Birkedal (2000), dependent types in  $\mathbf{Equ}$  are related to dependent types in domains with ‘totalities’ (Berger 1997). Taken together, Theorem 1 of Bauer and Birkedal (2000), Theorem 4 of Normann and Waagbø (1998) and our Theorem 6 provide a satisfying picture of dependent types, showing that the constructions coincide in many *prima facie* different models. However, it is worth noting that our Theorem 6 also applies to many spaces, such as the reals, that fall outside the scope of Bauer and Birkedal (2000) and Normann and Waagbø (1998).

Another strand of related work has been undertaken by Matthias Schröder, who has extended Weihrauch’s notion of ‘admissible representation’ (Kreitz and Weihrauch 1985; Weihrauch and Schafer 1983; Weihrauch 2000) to non-countably based spaces (Schröder 2000b). Schröder defines an *admissible representation* of a topological space  $Q$  to be a continuous map  $q : A \rightarrow Q$ , where  $A$  is a subspace of Cantor space (equivalently a countably based zero-dimensional  $T_0$  space), and  $q$  is projecting with respect to all such sub-Cantor spaces. Schröder proves many interesting results about spaces with admissible relations, including the cartesian closure of the category of sequential spaces with admissible representations (Schröder 2000b, Section 5).

The similarity between our definitions and results and those of Schröder was first observed by Andrej Bauer, who proved that the sequential spaces with admissible representations are exactly the  $T_0$   $\mathbf{PQ}$  spaces, and used this to establish connections with Weihrauch’s work (Bauer 2000; Bauer 2001). In the light of Bauer’s results, there is some overlap between results in our Sections 3–7 and results in Schröder (2000b).

Recently, Schröder proved that every  $T_0$  space that arises as a quotient of a countably based  $T_0$  space has an admissible representation (private communication). Our proof of Theorem 3, which we posed as a question in earlier versions of the paper, is a minor adaptation of Schröder’s proof.

Schröder has also extended his notion of admissibility to limit spaces, and also to a larger category of ‘weak limit spaces’ (Schröder 2000a). Here the connection with our work in Section 9 is more tenuous as, on the one hand, Schröder is working with a more general notion of limit space, and, on the other, he proves cartesian closure rather than local cartesian closure. Nonetheless, it seems likely that, fundamentally, Schröder’s techniques for representing limit spaces are essentially interchangeable with ours.

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