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AN EXAMPLE OF A NON-FOURIER-MUKAI FUNCTOR BETWEEN DERIVED CATEGORIES OF COHERENT SHEAVES

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WITH AN APPENDIX BY AMNON NEEMAN

ABSTRACT. Orlov's famous representability theorem asserts that any fully faithful exact functor between the bounded derived categories of coherent sheaves on smooth projective varieties is a Fourier-Mukai functor. In this paper we show that this result is false without the fully faithfulness hypothesis. We also show that our functor does not lift to the homotopy category of spectral categories if the ground field is $\mathbb{Q}$.

CONTENTS

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1. Introduction

Throughout k is a field of characteristic zero. All objects and constructions are assumed to be k-linear.

Recall the following seminal theorem proved by Orlov almost 20 years ago:

Theorem 1.1. [41, Thm 2.2] *Let X/k, Y/k be smooth projective schemes. Then every fully faithful exact functor $\Psi : D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(Y))$ is isomorphic to a Fourier-Mukai functor associated to an object in $D^b(\mathrm{coh}(X \times_k Y))$.*

Although this theorem was generally believed to be false without the hypothesis that Ψ is fully faithful, no counterexamples were known. *In the current paper we fill this gap by constructing such a counterexample.*

Another example of a functor that is not Fourier-Mukai was obtained by Vologodsky [54] shortly after this paper appeared on the arXiv. While our base field has characteristic zero, the example in loc. cit. is over $\mathbb{F}_p$. Vologodsky's functor is in fact a composition of derived functors and while it is not a Fourier-Mukai functor over $\mathbb{F}_p$, it is still represented by a morphism in the homotopy category of $\mathbb{Z}$-linear DG-categories (see Lemma B.3.1 below). In contrast we will show in Appendix B that if $k = \mathbb{Q}$ then our functor does not even have a lift to the homotopy category of spectral categories.

Our counterexample is presented in Theorem 1.4 below, but we will first discuss the underlying ideas on which it is based. Some initial progress toward the construction of non-Fourier-Mukai functors had already been made in [46] where we systematically analyzed functors whose source category is the derived category of a field. Leveraging this theory, we were able to construct a non-Fourier-Mukai functor $D^b(\mathrm{coh}(X)) \to D^b(\mathrm{Qch}(Y))$ by factoring through the localization at the generic point of X. Unfortunately, such methods do not allow one to replace $\mathrm{Qch}(Y)$ by $\mathrm{coh}(Y)$.

A highly nontrivial topological example of an exact functor which is not Fourier-Mukai in an appropriate sense is given in the beautiful paper [37]. Let $\mathrm{Ho}(\mathrm{Sp})$ be the homotopy category of spectra. In [37], Neeman constructs an exact functor[1]

$$D^b(\mathbb{Z}[1/2]) \to \mathrm{Ho}(\mathrm{Sp})[1/2] \tag{1.1}$$

which sends $\mathbb{Z}[1/2]$ to the sphere spectrum S^0.

Now note that $\mathrm{Ho}(\mathrm{Sp})^{-1}(S^0, S^0) = \pi_1(S^0) = \mathbb{Z}/2\mathbb{Z}$. So

$$\mathrm{Ho}(\mathrm{Sp})[1/2]^{-1}(S^0, S^0) = 0$$

and Neeman's proof strongly suggests that it is precisely this gap in the negative Ext's that makes this example work.

In the first version of this paper which appeared on the arXiv the authors proved a generalization of this result, valid for more complicated categories, at the cost of requiring the vanishing of more negative Ext-groups. Our proof was for triangulated categories of algebraic nature so it did not recover Neeman's original result.

[1]The actual result proved in loc. cit. is for $D(\mathbb{Z}[1/2])$.

However Amnon Neeman succeeded in finding a yet more general argument which is valid for triangulated categories satisfying Neeman's more restrictive axioms [7] (i.e. all those that occur in nature) so it encompasses both our algebraic result and Neeman's result for Ho(Sp)[1/2]. Amnon Neeman's proof is included as Appendix E in the current paper. Its main results are summarized in §5 and in particular the now simple construction of the functor (1.1) is presented in Example 5.5.

For the purpose of this introduction we state a simple corollary of these results:

Theorem 1.2. *(see Appendix C)*

(1) *Let B be a DG-algebra and let $R \to H^0(B)$ be a k-algebra morphism. Assume that $\operatorname{gl\,dim} R = m$ and $H^i(B) = 0$ for $i = -1, \dots, -m$. Then the natural functor*
$$R \to D(B) : R \mapsto B$$
(R considered as a one-object category) extends to an exact functor
$$L : D^b(R) \to D(B). \tag{1.2}$$
(2) *If L is isomorphic to a functor of the form $U \overset{L}{\otimes}_R -$, for U a complex of k-central $B-R$-bimodules, then the graded k-algebra morphism $R \to H^*(B)$ may be lifted to an A_∞-morphism $R \to B$.*

As expected, if R is a field Theorem 1.2(1) imposes no conditions on B and hence this theorem may be regarded as an extension of the basic principle underlying the constructions in [46].

In order do be able to apply part (2) of Theorem 1.2 we note that there are very concrete obstructions against the lifting of a morphism of graded rings to an A_∞-morphism (see §8.2 below). In the setting of Theorem 1.2, these obstructions take values in the Hochschild cohomology groups
$$\mathrm{HH}^i(R, H^{2-i}(B)) \tag{1.3}$$
for $i \geq m + 3$. Since such obstructions are easily controlled, Theorem 1.2(1) immediately gives a supply of functors which are non-Fourier-Mukai in an appropriate sense. The most basic case is the following.

Proposition 1.3. *(see Appendix D.2). Let R be a k-algebra of global dimension m and let M be a k-central R-bimodule. Let $\eta \in \mathrm{HH}^n(R, M)$ be a non-zero class in Hochschild cohomology, with $n \geq m + 3$. Let R_η be the A_∞-algebra $R \oplus \Sigma^{n-2} M\epsilon$, $\epsilon^2 = 0$, whose multiplication is twisted by η (i.e. $m_{R_\eta,n}$ is given by $\eta : R^{\otimes n} \to M$). Finally, let R_η^{dg} be the DG-hull of R_η (see Appendix D.1 below). Then the functor L in Theorem 1.2(1), with $B = R_\eta^{\mathrm{dg}}$ is not isomorphic to a functor of the form $U \overset{L}{\otimes}_R -$ for U a $R_\eta^{\mathrm{dg}}-R$-DG-bimodule.*

In this proposition we have introduced R_η^{dg} to stay within the world of DG-algebras but in fact the distinction between R_η and R_η^{dg} is only of technical relevance and we will ignore it in the rest of this introduction.

The functors constructed via Proposition 1.3 may be called "non-Fourier-Mukai" in a generalized sense. Unfortunately they are essentially non-geometric because:

(1) They involve DG-algebras.

(2) If R is commutative and finitely generated over an algebraically closed field, the Hochschild dimension of R is equal to the global dimension. This implies for example that the inequality $n \geq m+3$ cannot be satisfied if we want $\operatorname{Spec} R$ to be an affine variety. Note however that the equality can be satisfied with R being *essentially* of finite type (e.g. take $R = M = k(x,y,z)$).

To get rid of the first problem in the ring case, one may consider the situation where there is some k-algebra morphism $f : S \to R$ such that[2] $0 = f_*(\eta) \in \mathrm{HH}^n(S,M)$. In that case one may show that there is a commutative diagram of A_∞-algebras (see Proposition 8.2.6 below)

$$\begin{tikzcd} R_\eta \arrow[dr] & & S \arrow[ll, dotted, "\tilde{f}"'] \arrow[dl, "f"] \\ & R & \end{tikzcd} \tag{1.4}$$

Starting from (1.4) one could hope that in some cases the composition

$$\Psi : D^b(R) \xrightarrow{(1.2)} D(R_\eta) \xrightarrow{\tilde{f}_*} D(S) \tag{1.5}$$

is not given by tensoring with a complex of bimodules. We have not really investigated how well this method works for contructing non-Fourier-Mukai functors between derived categories of rings but the underlying idea is used in the geometric case we consider below. By a stroke of luck the geometric construction will also be seen to yield an example of a non-Fourier-Mukai functor between derived categories of finite dimensional algebras. See Corollary 1.5 below.

To deal with the second problem note that, if X is a smooth projective variety of dimension m, then $\operatorname{gl\,dim} \operatorname{Qch}(X) = m$, whereas the Hochschild dimension of X is $2m$ (see e.g. §10.6 below). So in that case there is no problem satisfying the inequality $n \geq m+3$ when $m \geq 3$.

As a thought experiment we will consider the following situation: X is as in the previous paragraph, $n \geq m+3$, M is an $\mathcal{O}_X$-module. Let $0 \neq \eta \in \mathrm{HH}^n(X,M) \overset{\mathrm{def}}{=} \operatorname{Ext}^n_{X\times_k X}(i_{\Delta,*}\mathcal{O}_X, i_{\Delta,*}M)$ ($i_\Delta : X \to X \times X$ being the diagonal). Then we expect there to be some kind of derived deformation (or infinitesimal thickening) $X \to X_\eta$ corresponding to η. We will discuss this further below, but for now assume X_η exists. Assume furthermore there is some morphism $f : X \to Y$ with Y smooth such that $f_*(\eta) = 0$. Then, as in the ring case, we *expect* there to be a diagram of the type

$$\begin{tikzcd} X_\eta \arrow[rr, dotted, "\tilde{f}"] & & Y \\ & X \arrow[ul] \arrow[ur, "f"'] & \end{tikzcd} \tag{1.6}$$

(the arrows are reversed with respect to (1.4) by the usual algebra-geometry duality) which should in principle allow us to define a functor as in (1.5).

[2]Here f_* is shorthand for the composition ${}_SS_S \xrightarrow{f} {}_SR_S \xrightarrow{{}_S\eta_S} {}_SM_S[n]$ in $D(S \otimes_k S^\circ)$.

Now we have to deal with the question: what is X_η? One canonical answer is to use η to deform an enhancement of $D(\mathrm{Qch}(X))$ [25, 32, 33]. But then one hits the so-called "curvature" problem: the result will in general only be a cA_∞-category [17, 29], i.e. roughly speaking it will not satisfy $d^2 = 0$. Homological algebra over cA_∞-categories is possible [17, 42] but presents rather serious technical difficulties.

Another approach is to view X_η as a kind of DG-gerbe on X. However in our examples η will be very non-local, so the "higher gluing" required to understand X_η will be necessarily subtle.

In this paper we have opted for a third approach (based on [30]) which is much cheaper but nonetheless sufficient for our purposes. The idea is to embed $\mathrm{Qch}(X)$ into a *category of presheaves* associated to an affine covering of X. Such presheaves form a module category so we can directly apply the algebraic constructions discussed above. In particular there is no curvature problem.

Let $X = \bigcup_{i=1}^n U_i$ be an affine covering. For $I \subset \{1, \ldots, n\}$ put $U_I = \bigcap_{i\in I} U_i$. Let $\mathcal{I}$ be the set $\{I \subset \{1, \ldots, n\} \mid I \neq \emptyset\}$ and let $\mathcal{X}$ be the category with objects $\mathcal{I}$ and Hom-sets

$$\mathcal{X}(I, J) = \begin{cases} \mathcal{O}_X(U_J) & I \subset J \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

In other words $\mathcal{X}$ is the subcategory of $\mathrm{Qch}(X)$ spanned by the objects $(i_{U_I,*}\mathcal{O}_{U_I}))_{I\in\mathcal{I}}$ where we only allow maps $I \to J$ when $I \subset J$. Since $\mathcal{I}$ is finite one may even think of $\mathcal{X}$ as an actual ring $\bigoplus_{I,J\in\mathcal{I}} \mathcal{X}(I, J)$.

It is easy to see that $\mathrm{Mod}(\mathcal{X})$ is the category of modules over the presheaf of rings $(I, \Gamma(U_I, \mathcal{O}_{U_I}))_{I\in\mathcal{I}}$. In particular $\mathrm{Mod}(\mathcal{X})$ contains $\mathrm{Qch}(X)$ as a full subcategory. Furthermore by the "Special Cohomology Comparison Theorem" [18, 31] one has $\mathrm{HH}^*(\mathcal{X}, \mathcal{M}) = \mathrm{HH}^*(X, M)$ (see §9.3 below) where $\mathcal{M}$ is the $\mathcal{X} - \mathcal{X}$-bimodule associated to M defined by a similar formula as (1.7). It follows that we may define and A_∞-category $\mathcal{X}_\eta$ in exactly the same way as we defined R_η.

Assume now that f, X, Y, η are as above and that in addition $f : X \to Y$ is a closed immersion, and start with an affine covering of Y. By giving X the induced covering, we may then construct a diagram of A_∞-categories and functors

$$\begin{tikzcd} \mathcal{X}_\eta \arrow[dr] & & \mathcal{Y} \arrow[ll, dotted, "\tilde{f}"'] \arrow[dl, "f"] \\ & \mathcal{X} & \end{tikzcd}$$

(the arrows are again reversed with respect to (1.6) since we are now back in an algebraic framework as in (1.4)) and we may construct an exact functor

$$\Psi : D^b(\mathrm{Qch}(X)) \xrightarrow{L} D_{\mathrm{Qch}}(\mathcal{X}_\eta) \xrightarrow{\tilde{f}_*} D_{\mathrm{Qch}}(\mathcal{Y}) \cong D(\mathrm{Qch}(Y))\,, \tag{1.8}$$

where the first functor is a geometric version of (1.2) and where $D_{\mathrm{Qch}}(-)$ means that we only consider complexes with quasi-coherent cohomology, through the embedding $\mathrm{Qch}(X) \subset \mathrm{Mod}(\mathcal{X})$. In fact, since $\mathrm{Qch}(X)$ has enough injectives but not projectives, the first functor is given by a construction dual to the one presented in Theorem 1.2(1). See Remark 11.2 below.

Now we may state our main theorem.

Theorem 1.4. *(see §12 below) Let X be a smooth quadric in $Y = \mathbb{P}^4$ whose defining equation has maximal isotropy index*[3] *and let $f : X \to Y$ be the inclusion. Let $M = \omega_X^{\otimes 2}$ and let $0 \neq \eta \in \mathrm{HH}^6(X, \omega_X^{\otimes 2}) \cong k$. Then $f_*\eta \in \mathrm{HH}^6(Y, f_*(\omega_X^{\otimes 2}))$ is zero. The functor Ψ in (1.8) restricts to an exact functor*

$$\Psi : D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(Y))$$

which is not a Fourier-Mukai functor.

Recall [16, 44] that even for a non-Fourier-Mukai functor one may still define sheaves $\mathcal{H}^i$ on $X \times_k Y$ which would be the cohomology of the kernel - if the latter existed. In our case we have (see (A.1) below)

$$\mathcal{H}^i = \begin{cases} \mathcal{O}_{\Gamma_f} & \text{if } i = 0 \\ \omega_{\Gamma_f}^{\otimes -2} & \text{if } i = 4 \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

where $\Gamma_f \subset X \times_k Y$ is the graph of f.

To conclude this introduction, let us indicate how we prove that Ψ in Theorem 1.4 is not Fourier-Mukai (see §12 below). The technical details are somewhat involved but the underlying idea is the following. The basic feature of a Fourier-Mukai functor is that it is compatible with *base change*. In fact, if $\Psi : D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(Y))$ is a Fourier-Mukai functor and Z is a smooth proper scheme over k, then by extending the kernel of Ψ we obtain a Fourier-Mukai functor $D^b(\mathrm{coh}(X \times Z)) \to D^b(\mathrm{coh}(Y \times Z))$. If $Z = X$ then the kernel of Ψ is the image via this functor of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset X \times X$.

Because of this last fact we do not expect base change to hold for non-Fourier-Mukai functors suggesting a possible method to identify them. Unfortunately it seems not so obvious to give a workable definition of base change in this more general setting. There is one situation however which is easier to handle and which applies to our example. Assume that X has a tilting bundle T and let $\Gamma = \mathrm{End}_X(T)$. Then $D^b(\mathrm{coh}(X \times X))$ is equivalent to $D^b(\mathrm{coh}(X)_\Gamma)$ where $\mathrm{coh}(X)$ is the abelian category of coherent sheaves on X equipped with a Γ-action and under this equivalence $\mathcal{O}_\Delta$ correponds to T (which is indeed a coherent sheaf on X, naturally equipped with a Γ-action). So it is a natural idea to replace $D^b(\mathrm{coh}(X \times X))$ by the more algebraic $D^b(\mathrm{coh}(X)_\Gamma)$.

If Ψ is a Fourier Mukai functor then since the standard constructions of derived pullback, tensor product and pushforward are compatible with the action of Γ, the kernel for Ψ defines at the same time a "Γ-equivariant lift" $\Psi_\Gamma : D^b(\mathrm{coh}(X)_\Gamma) \to D^b(\mathrm{coh}(Y)_\Gamma)$ of Ψ, i.e. a functor which behaves as Ψ if we ignore the Γ-action. So we should try to prove that such a Γ-equivariant lift of Ψ does not exist. The above discussion suggests we should only consider the object $\Psi_\Gamma(T)$ since it represents the (would be) kernel of Ψ under the equivalence $D^b(\mathrm{coh}(X \times Y)) \cong D^b(\mathrm{coh}(Y)_\Gamma)$.

We are now ready to describe the key argument. By functoriality $\Psi(T)$ is an object in $D(\mathrm{coh}(Y))_\Gamma$ (that is: the category of objects in $D(\mathrm{coh}(Y))$ equipped with a Γ-action) and as explained above, if Ψ is Fourier-Mukai then $\Psi(T)$ lifts to the

[3]This condition is only relevant for a non-algebraically closed base field. One may take $x_0^2 + x_1x_2 + x_3x_4 = 0$.

object $\Psi_\Gamma(T)$ in $D^b(\mathrm{coh}(Y)_\Gamma)$. It turns out that in our setting there is a homological obstruction against lifting $\Psi(T)$ under the obvious functor

$$D(\mathrm{coh}(Y)_\Gamma) \to D(\mathrm{coh}(Y))_\Gamma \tag{1.10}$$

and we have to prove it is non-zero. Now it is more or less a tautology that $L(T) \in D^b(\mathcal{X}_\eta)_\Gamma$ (with L as in (1.8)) does not lift to $D(\mathcal{X}_\eta \otimes \Gamma)$ (as L is a non-Fourier-Mukai functor). We show in §8 that by general principles the obstruction to the latter lift is sent by a suitable incarnation of f_* to the corresponding obstruction against the lift (1.10). Now all we have to do is to show in our example that f_* induces an isomorphism between the cohomology groups which are the targets for the obstruction. This is carried out §12.

Valery Lunts suggested adding the following immediate corollary of Theorem 1.4 for finite dimensional algebras. Recall that Rickard proved in [43, Theorem 3.3] that if Γ and Λ are derived equivalent k-algebras, then there is an equivalence of the form

$$U \overset{L}{\otimes}_\Gamma - : D^b(\mathrm{mod}(\Gamma)) \to D^b(\mathrm{mod}(\Lambda))$$

where U is a complex of $\Lambda - \Gamma$ bimodules. However, in contrast with the geometric situation (see Theorem 1.1), it is unknown whether all derived equivalences between rings are of this form. The next corollary states that there do indeed exist exact functors between derived categories of finite dimensional algebras that are not given by tensoring with a bimodule. They are however not derived equivalences.

Corollary 1.5. *With the notation of Theorem 1.4, let $\Gamma = \mathrm{End}_X(T)$ where T is the tilting bundle on X described in Theorem 12, $\Lambda = \mathrm{End}_{\mathbb{P}^4}\left(\oplus_{i=0}^4 \mathcal{O}(i)\right)$. Then the functor Ψ induces a functor*

$$\Phi : D^b(\mathrm{mod}(\Gamma)) \to D^b(\mathrm{mod}(\Lambda)) \tag{1.11}$$

which is not of the form $U \overset{L}{\otimes}_\Gamma -$ for U a complex of $\Lambda - \Gamma$-bimodules.

Proof. We have $D^b(\mathrm{coh}(X)) \cong D^b(\mathrm{mod}(\Gamma))$, $D^b(\mathrm{coh}(Y) \cong D^b(\mathrm{mod}(\Lambda))$ so that we may indeed define the functor Φ as in (1.11). If Φ were of the form $U \overset{L}{\otimes}_\Gamma -$ then it would be induced from a DG-functor and hence the same would be true for Φ as in Theorem 1.4. But then the latter would be a Fourier-Mukai functor by [52, Thm 8.15]. $\square$

A number of extensions and variants of Orlov's theorem are known. See e.g. [3, 11, 12, 14, 16, 22, 32, 44, 45]. For excellent surveys on the current state of knowledge see [15, 13].

2. Outline

The paper consists of a number of parts which are independent of each other.

- In §6 we discuss our main technical result (a dual version of Theorem 1.2) which is at the heart of our construction of a non-Fourier-Mukai functor. The proof reduces quickly to the general result by Amnon Neeman contained in Appendix E.
- In §7,§8 we discuss the main facts concerning A_∞-categories that we will need in the rest of the paper. This culminates in §8.3 where we discuss the obstructions for lifting objects under the functor $D(\mathfrak{b} \otimes_k \Gamma) \to D(\mathfrak{b})_\Gamma$. The relevance of this has been explained in the introduction.

- In §9 we relate the properties of a quasi-compact separated scheme X to similar properties of the corresponding category $\mathcal{X}$ defined in the introduction. This material is necessary as we use $\mathcal{X}$ to deform the derived category of quasi-coherent sheaves on X.
- In §10 we discuss the behavior of Hochschild cohomology under restriction to a smooth hypersurface. This is used to verify that the quadruple (X, Y, f, η) in Theorem 1.4 indeed has the property $\eta \neq 0$, $f_*\eta = 0$.
- In §11, §12 we give the proof of Theorem 1.4.
- We include several appendices. In Appendix A we prove (1.9). In Appendix B we prove that Ψ does not lift to a functor of spectral categories (if $k = \mathbb{Q}$). In Appendices C, D we prove Theorem 1.2 and Proposition 1.3 from the introduction. Finally Appendix E is written by Amnon Neeman. In contrast to the other appendices, this one is essential for our paper!

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4. Notation and preliminaries

Throughout k is a field of characteristic zero and all constructions will be k-linear.

4.1. **Modules and bimodules over categories.** Let $\mathfrak{a}$ be a pre-additive category. A left $\mathfrak{a}$-module M is a covariant additive functor $M : \mathfrak{a} \to \mathbf{Ab}$. We view it as a collection of abelian groups $M(A)_{A\in \mathrm{Ob}(\mathfrak{a})}$ depending covariantly on A. We write $\mathrm{Mod}(\mathfrak{a})$ for the additive category of left $\mathfrak{a}$-modules.

A right $\mathfrak{a}$-module is an object in $\mathrm{Mod}(\mathfrak{a}^\circ)$. If $\mathfrak{a}$, $\mathfrak{b}$ are k-linear categories, a (k-linear) $\mathfrak{a}$-$\mathfrak{b}$-bimodule is an object in $\mathrm{Mod}(\mathfrak{a} \otimes_k \mathfrak{b}^\circ)$. We view it as a collection of k-vector spaces $M(B, A)_{B\in \mathrm{Ob}(\mathfrak{b}), A\in \mathrm{Ob}(\mathfrak{a})}$ depending contravariantly on B and covariantly on A. We will sometimes write $\mathrm{Bimod}_k(\mathfrak{a}, \mathfrak{b})$ for $\mathrm{Mod}(\mathfrak{a} \otimes_k \mathfrak{b}^\circ)$. Given

$M \in \mathrm{Mod}(\mathfrak{b})$ and a functor $\mathfrak{b} \to \mathfrak{a}$, we will write $\mathfrak{a} \otimes_{\mathfrak{b}} M$ to denote the tensor product of M with the $\mathfrak{a}$-$\mathfrak{a}$ bimodule $\mathfrak{a}(-,-)$ viewed as an $\mathfrak{a}$-$\mathfrak{b}$-bimodule.

If $\mathfrak{a}$, $\mathfrak{b}$ are DG-categories, the above notions have obvious generalizations to DG-modules, DG-bimodules, etc. We write $\underline{\mathrm{Mod}}(\mathfrak{a})$ for the category of left DG-$\mathfrak{a}$-modules.

4.2. **A_∞-notions.** If $\mathfrak{a}$ is a DG-graph, we denote by $\mathbb{B}\mathfrak{a}$ its bar-cocategory. I.e. $\mathrm{Ob}(\mathbb{B}\mathfrak{a}) = \mathrm{Ob}(\mathfrak{a})$ and

$$\mathbb{B}\mathfrak{a}(A,B) = \bigoplus_{A_1,\dots,A_{i-1}\in \mathrm{Ob}(\mathfrak{a})} \Sigma\mathfrak{a}(A_{i-1},B) \otimes \ldots \otimes \Sigma\mathfrak{a}(A,A_1)$$

An A_∞-structure on $\mathfrak{a}$ is a codifferential $b_{\mathfrak{a}}$ on $\mathbb{B}\mathfrak{a}$. Similarly, an A_∞-functor between A_∞-categories $\mathfrak{a}$, $\mathfrak{b}$ is a cofunctor $f : \mathbb{B}\mathfrak{a} \to \mathbb{B}\mathfrak{b}$ such that $b_{\mathfrak{b}} \circ f = f \circ b_{\mathfrak{a}}$.

As usual, we describe A_∞-structures and morphisms via their Taylor coefficients: $(b_{\mathfrak{a},j})_j$, $(f_j)_j$ which may be evaluated on sequences of j composable maps.

All A_∞-constructions will always be implicitly assumed to be strictly unital. Note that any reasonable A_∞-construction can be strictified, which is ultimately due to the fact that Hochschild cohomology may be computed using normalized cocycles. See [26, Ch. 3]. We routinely apply standard constructions for A_∞-algebras to A_∞-categories. This simply means that operations are only applied to composable arrows.

If $\mathfrak{b}$ is an A_∞-category, we will denote by $C^u_\infty(\mathfrak{b})$ the category of (strictly unital) A_∞-$\mathfrak{b}$-modules with A_∞-morphisms. Let $D_\infty(\mathfrak{b})$ be obtained from $C^u_\infty(\mathfrak{b})$ by identifying homotopic maps. This is one of several equivalent constructions for the derived category of an A_∞-category. See [26] for details.

5. Construction of a functor

We briefly summarize the results we will need from Appendix E written by Amnon Neeman and also present a relevant example.

If $\mathcal{H}$ is a full subcategory of a triangulated category $\mathcal{T}$ then we denote by $\mathcal{H}^*$ the *extension closure* of $\mathcal{H}$, i.e. the smallest full subcategory of $\mathcal{T}$ which contains $\mathcal{H}$ and which has the property that if there is a distinguished triangle $x \to y \to z$ with $x, z \in \mathcal{H}^*$ then $y \in \mathcal{H}^*$. We say that $\mathcal{H}$ is *extension closed* if $\mathcal{H}^* = \mathcal{H}$.

Definition 5.1. Let $H : \mathcal{R} \to \mathcal{T}$ be an exact functor between triangulated categories. The pair of full subcategories $(\mathcal{A} \subset \mathcal{R}, \mathcal{B} \subset \mathcal{R})$ is called a *good couple with respect to H* if

(1) $\Sigma^{-1}\mathcal{A} \subset \mathcal{A}$ and $\Sigma\mathcal{B} \subset \mathcal{B}$.
(2) The map $\mathcal{R}(a,b) \to \mathcal{T}(Ha,Hb)$ is an isomorphism if $a \in \mathcal{A}$ and $b \in \mathcal{B}$, and is surjective if $a \in \mathcal{A}$ and $b \in \Sigma^{-1}\mathcal{B}$.

A good couple $(\mathcal{A},\mathcal{B})$ is called *excellent* if $\mathcal{A}$, $\mathcal{B}$ are extension closed.

Remark 5.2. If $(\mathcal{A},\mathcal{B})$ is a good couple for H, then it is clear that the restriction of H to $\mathcal{A} \cap \mathcal{B} \subset \mathcal{R}$ is fully faithful.

Proposition 5.3 (See Corollaries E.7, E.9 and their proofs)**.** *If $(\mathcal{A},\mathcal{B})$ is a good couple then $(\mathcal{A}^*,\mathcal{B}^*)$ is an excellent couple. Moreover if $\mathcal{C} = H(\mathcal{A}\cap\mathcal{B})$ is the essential image of $\mathcal{A}\cap\mathcal{B}$ then $\mathcal{C}^* \subset H(\mathcal{A}^*\cap\mathcal{B}^*)$.*

The following result is a version of Theorem E.10 of Appendix E. We have slightly altered the notation so as to be more compatible with the body of the paper.

Theorem 5.4. *Let $H : \mathcal{R} \to \mathcal{T}$ be an exact functor. Assume the category $\mathcal{R}$ satisfies the axioms of the article* [36]*. Suppose further that $\mathcal{T}$ has a non-degenerate t-structure with heart $\mathcal{T}^\heartsuit$, let $\mathcal{H} : \mathcal{T} \to \mathcal{T}^\heartsuit$ be the standard homological functor from $\mathcal{T}$ to the heart, and let $\mathcal{D} \subset \mathcal{T}^\heartsuit$ be a full, abelian subcategory closed under extensions and define*

$$\mathcal{T}^b_{\mathcal{D}} = \left\{ t \in \mathcal{T} \;\middle|\; \begin{array}{c} \mathcal{H}^i(t) = 0 \textit{ for all but finitely many } i \in \mathbb{Z} \\ \mathcal{H}^i(t) \in \mathcal{D} \textit{ for every } i \in \mathbb{Z} \end{array} \right\}$$

Assume now that $(\mathcal{A}, \mathcal{B})$ is an excellent couple in $\mathcal{R}$ such that $\mathcal{D} \subset H(\mathcal{A} \cap \mathcal{B})$. Then there exists an exact functor $G : \mathcal{T}^b_{\mathcal{D}} \to \mathcal{R}$ which fits in a commutative diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[r, dotted, "G|\mathcal{D}"'] \arrow[d, hook] \arrow[rr, bend left, hook] & \mathcal{A} \cap \mathcal{B} \arrow[r, "\cong", "H|\mathcal{A}\cap\mathcal{B}"'] \arrow[d, hook] & H(\mathcal{A} \cap \mathcal{B}) \arrow[d, hook] \\
\mathcal{T}^b_{\mathcal{D}} \arrow[r, dotted, "G"] \arrow[rr, bend right, hook] & \mathcal{R} \arrow[r, "H"] & \mathcal{T}
\end{tikzcd} \tag{5.1}$$

Example 5.5 (Neeman)**.** Suppose $\mathcal{R} = \mathrm{Ho}(\mathrm{Sp})[1/2]$ is the homotopy category of spectra with 2 inverted, let $\mathcal{T} = D(\mathbb{Z}[1/2])$, and let $H : \mathcal{R} \to \mathcal{T}$ be the functor taking a spectrum to its singular chain complex. We define $\mathrm{Free}(S^0)$ to be the full subcategory of $\mathcal{R}$ whose objects are bouquets of zero-spheres. Set $\mathcal{A} = \{\Sigma^n P \mid P \in \mathrm{Free}(S^0), n \le 1\}$ and $\mathcal{B} = \{\Sigma^n P \mid P \in \mathrm{Free}(S^0), n \ge 0\}$. That is the objects of $\mathcal{A}$ and $\mathcal{B}$ are just shifts of bouquets of the zero-sphere S^0, with the shifts as prescribed. We claim that $(\mathcal{A}, \mathcal{B})$ is a good couple. This comes down to $\mathcal{R}(S^0, S^0) = \pi_0(S^0)[1/2] = \mathbb{Z}[1/2]$, $\mathcal{R}(S^0, \Sigma^{-1} S^0) = \pi_1(S^0)[1/2] = 0$.

We define $\mathrm{Free}(\mathbb{Z}[1/2])$ to be the category of free $\mathbb{Z}[1/2]$–modules, viewed as objects of $\mathcal{T}$ concentrated in degree 0. Let $\mathcal{C}$ be the essential image of $\mathcal{A} \cap \mathcal{B}$. Then $\mathcal{C}$ contains $\{\Sigma^n P \mid P \in \mathrm{Free}(\mathbb{Z}[1/2]), n = 0 \text{ or } 1\}$, and it's easy to see that $\mathcal{C}^*$ contains the heart $\mathcal{T}^\heartsuit$ of the standard t-structure on $\mathcal{T}$ (a single extension is enough as $\mathbb{Z}[1/2]$ is hereditary). Proposition 5.3 now informs us that $(\mathcal{A}^*, \mathcal{B}^*)$, is an excellent couple and that the essential image of $\mathcal{A}^* \cap \mathcal{B}^*$ contains $\mathcal{T}^\heartsuit$. Applying Theorem 5.4 with $\mathcal{D} = \mathcal{T}^\heartsuit$ we obtain a functor $G : D^b(\mathbb{Z}[1/2]) \to \mathrm{Ho}(\mathrm{Sp})$ such that the composition HG is the inclusion $D^b(\mathbb{Z}[1/2]) \to D(\mathbb{Z}[1/2])$.

Our main application of Theorem 5.4 will be Proposition 6.3.1 below. Another application will be given in Appendix C.

6. The main technical result

6.1. **Derived injectives.** This is part of ongoing work of the second author with Francesco Genovese and Wendy Lowen. Assume that $\mathcal{T}$ is a well generated triangulated category equipped with a t-structure with heart $\mathcal{T}^\heartsuit$, such that $H^0(-)$ respects coproducts. If $I \in \mathrm{Inj}\,\mathcal{T}^\heartsuit$, by Brown representability [39] the cohomological functor $\mathcal{T} \mapsto \mathbf{Ab}$ given by $\mathcal{T}^\heartsuit(H^0(-), I)$ is representable. Denote the representing object by $L(I)$. So we have that for $X \in \mathcal{T}$

$$\mathcal{T}^{\heartsuit}(H^0(X), I) = \mathcal{T}(X, L(I)). \tag{6.1}$$

We will call $L(I)$ the *derived injective* associated to I.

Remark 6.1.1. If $\mathcal{T} = \mathrm{Ho}(\mathrm{Sp})$ is the homotopy category of spectra with the standard t-structure with heart $\mathbf{Ab}$ and $I = \mathbb{Q}/\mathbb{Z}$, then $L(I)$ is the Brown-Comenetz dual of the sphere spectrum.

The following properties are easily verified:

$$L(I) \in \mathcal{T}_{\geq 0}, \tag{6.2}$$

$$H^0(L(I)) = I. \tag{6.3}$$

For $I, J \in \operatorname{Inj} \mathcal{T}^{\heartsuit}$, one has

$$\mathcal{T}(L(I), \Sigma^i L(J)) = \begin{cases} \mathcal{T}^{\heartsuit}(I, J) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \tag{6.4}$$

Thus, in particular, we have a fully faithful functor

$$L : \operatorname{Inj} \mathcal{T}^{\heartsuit} \to \mathcal{T} : I \mapsto L(I). \tag{6.5}$$

6.2. **Derived injectives in a DG-category.** Now assume that $\mathcal{D}$ is $D(\mathfrak{c})$ with $\mathfrak{c}$ a DG-category concentrated in degrees ≤ 0. Equip $\mathcal{D}$ with the standard t-structure [23] with heart $\mathcal{T}^{\heartsuit} = \mathrm{Mod}(H^0(\mathfrak{c}))$. One verifies for $I \in \operatorname{Inj} \mathcal{T}^{\heartsuit}$

$$H^*(L(I)) = \mathrm{Hom}_{H^0(\mathfrak{c})}(H^{-*}(\mathfrak{c}), I) \tag{6.6}$$

as graded $H^*(\mathfrak{c})$-modules. We also find

(6.7)

$$\mathrm{Hom}^i_{D(\mathfrak{c})}(L(I), L(J)) = \mathrm{Hom}_{H^0(\mathfrak{c})}(H^{-i}(L(I)), J) = \mathrm{Hom}_{H^0(\mathfrak{c})}(\mathrm{Hom}_{H^0(\mathfrak{c})}(H^i(\mathfrak{c}), I), J)$$

Finally note the following

Lemma 6.2.1. *There is a commutative diagram*

$$\begin{tikzcd} \operatorname{Inj} \mathrm{Mod}(H^0(\mathfrak{c})) \arrow[r, "L"] \arrow[dr, hook] & D(\mathfrak{c}) \arrow[d, "\mathrm{RHom}_{\mathfrak{c}}(H^0(\mathfrak{c}), -)"] \\ & D(H^0(\mathfrak{c})) \end{tikzcd}$$

were the horizontal map is obtained from (6.5).

Proof. Let $I \in \operatorname{Inj} \mathrm{Mod}(H^0(\mathfrak{c}))$. We have to contruct a natural isomorphism

$$\mathrm{RHom}_{\mathfrak{c}}(H^0(\mathfrak{c}), L(I)) \cong I \tag{6.8}$$

in $D(H^0(\mathfrak{c}))$. Let $Y \in D(H^0(\mathfrak{c}))$. We have

$$\begin{aligned} \mathrm{Hom}_{H^0(\mathfrak{c})}(Y, \mathrm{RHom}_{\mathfrak{c}}(H^0(\mathfrak{c}), L(I))) &= \mathrm{Hom}_{\mathfrak{c}}(Y, L(I)) \\ &= \mathrm{Hom}_{H^0(\mathfrak{c})}(H^0(Y), I) = \mathrm{Hom}_{H^0(\mathfrak{c})}(Y, I) \end{aligned}$$

The first equality is "change of rings", the second equality is (6.1) and the last equality is because I is injective. The isomorphism (6.8) now follows by Yoneda's lemma. $\square$

6.3. **The main technical result.**

Proposition 6.3.1. *Let $\mathfrak{c}$ be a DG-category concentrated in degree ≤ 0, satisfying in addition*

$$H^i(\mathfrak{c}) = 0 \qquad \text{for } i = -1, \ldots, -m$$

and assume that there is a full abelian subcategory $\mathcal{D} \subset \operatorname{Mod}(H^0(\mathfrak{c}))$ such that

- *$\mathcal{D}$ has enough injectives and $\operatorname{gl}\dim \mathcal{D} \leq m$.*
- *$\operatorname{Inj} \mathcal{D} \subset \operatorname{Inj} \operatorname{Mod} H^0(\mathfrak{c})$ (and hence $D^b(\mathcal{D}) = D^b_{\mathcal{D}}(H^0(\mathfrak{c})) \subset D^b(H^0(\mathfrak{c}))$, in particular $\mathcal{D}$ is extension closed).*

Then there is a commutative diagram of additive functors where the top arrow is the functor L introduced in (6.5) *(with $\mathcal{D} = D(\mathfrak{c})$).*

(6.9)
$$\begin{tikzcd}
\operatorname{Inj}\mathcal{D} \arrow[r, hook, "L"] \arrow[d, hook] & D(\mathfrak{c}) \arrow[d, equal] & \\
D^b(\mathcal{D}) \arrow[r, dotted, "L", "exact"'] \arrow[rr, hook, bend right] & D(\mathfrak{c}) \arrow[r, "\operatorname{RHom}_{\mathfrak{c}}(H^0(\mathfrak{c}),-)"] & D(H^0(\mathfrak{c}))
\end{tikzcd}$$

Proof. We will use the notation of §5. Let $\mathcal{R} = D(\mathfrak{c})$, $\mathcal{T} = D(H^0(\mathfrak{c}))$ and let $H : \mathcal{R} \to \mathcal{T}$ be given by $\operatorname{RHom}_{\mathfrak{c}}(H^0(\mathfrak{c}), -)$. Let

$$\begin{aligned}
\mathcal{A} &= \{\Sigma^n I | I \in L(\operatorname{Inj}\mathcal{D}), n \leq 0\} \quad \subset \mathcal{R} \\
\mathcal{B} &= \{\Sigma^n I | I \in L(\operatorname{Inj}\mathcal{D}), n \geq -m\} \subset \mathcal{R}
\end{aligned}$$

Then we claim that $(\mathcal{A}, \mathcal{B})$ is a good couple with respect to H. In fact by (6.8) we have $\operatorname{RHom}_{\mathfrak{c}}(H^0(\mathfrak{c}), L(I)) \cong I$ and for $I, J \in \operatorname{Inj}\mathcal{D}$ we have vanishing $\operatorname{Hom}_{D^b(H^0(\mathfrak{c}))}(\Sigma^n I, \Sigma^{n'} J) = 0$ for $n \neq n'$ and moreover

- $\operatorname{Hom}_{D(\mathfrak{c})}(\Sigma^n L(I), \Sigma^{n'} L(J)) = 0$ for $n \leq 0, n' \geq -m, n \neq n'$: in fact
  (i) $\operatorname{Hom}_{D(\mathfrak{c})}(\Sigma^n L(I), \Sigma^{n'} L(J)) = 0$ if $n' - n > 0$ by (6.4);
  (ii) $\operatorname{Hom}_{D(\mathfrak{c})}(\Sigma^n L(I), \Sigma^{n'} L(J)) = 0$ if $-m \leq n' - n < 0$ by (6.7) since $H^i(\mathfrak{c}) = 0$ for $i = -1, \ldots, -m$.
- $\operatorname{Hom}_{D(\mathfrak{c})}(\Sigma^n L(I), \Sigma^n L(J)) = \operatorname{Hom}_{D^b(H^0(\mathfrak{c}))}(\Sigma^n I, \Sigma^n J)$ by (6.4).

The rest of the proof follows by observing that the essential image $\mathcal{C}$ of $\mathcal{A} \cap \mathcal{B}$ under the functor $\operatorname{RHom}_{\mathfrak{c}}(H^0(\mathfrak{c}), -)$ contains all the objects $\{I[n] \mid I \in \operatorname{Inj}\mathcal{D}, -m \leq n \leq 0\}$. Since $\operatorname{Inj}\dim \mathcal{D} \leq m$, it is clear that $\mathcal{C}^*$ contains all of the category $\mathcal{D} \subset \operatorname{Mod} H^0(\mathfrak{c})$. Proposition 5.3 now informs us that $(\mathcal{A}^*, \mathcal{B}^*)$ is an excellent couple such that the essential image of $\mathcal{A}^* \cap \mathcal{B}^*$ contains $\mathcal{D} \subset \operatorname{Mod} H^0(\mathfrak{c}) = \mathcal{T}^\heartsuit$. Applying Theorem 5.4 we find an exact functor $G : D^b_{\mathcal{D}}(H^0(\mathfrak{c})) \to D(\mathfrak{c})$ such that the composition HG is the inclusion $D^b_{\mathcal{D}}(H^0(\mathfrak{c})) \to D(\mathfrak{c})$. We now put $L = G$. $\square$

We will need the following technical property of L later.

Corollary 6.3.2. *Let L be as in the lower row of* (6.9). *If $X \in D^b(\mathcal{D})$ and $N \in D^b(H^0(\mathfrak{c}) \otimes H^0(\mathfrak{c})^\circ)$ then*

$$\operatorname{RHom}_{\mathfrak{c}}(N, L(X)) \cong \operatorname{RHom}_{H^0(\mathfrak{c})}(N, X) \tag{6.10}$$

in $D^b(H^0(\mathfrak{c}))$.

Proof. By the standard "change of rings" identity we have

$$\mathrm{RHom}_{\mathfrak{c}}(N, L(X)) = \mathrm{RHom}_{H^0(\mathfrak{c})}(N, \mathrm{RHom}_{\mathfrak{c}}(H^0(\mathfrak{c}), L(X))).$$

Using the lower row in (6.9), we obtain

$$\mathrm{RHom}_{\mathfrak{c}}(H^0(\mathfrak{c}), L(X)) \cong X.$$

This finishes the proof. □

7. Deformations

7.1. **A_∞-deformations of linear categories.** If $\mathfrak{a}$ is a k-linear category and M is a k-central $\mathfrak{a}$-bimodule, we write $\mathbf{C}^\bullet(\mathfrak{a}, M)$ for the Hochschild complex of M and $\bar{\mathbf{C}}^\bullet(\mathfrak{a}, M)$ for its subcomplex of normalized cochains (i.e. those cochains vanishing on identity morphisms). The inclusion $\bar{\mathbf{C}}^\bullet(\mathfrak{a}, M) \to \mathbf{C}^\bullet(\mathfrak{a}, M)$ is a quasi-isomorphism by [27, §1.5.7]. We write $\mathrm{HH}^\bullet(\mathfrak{a}, M)$ for the corresponding cohomology. Note that we have

$$\mathrm{HH}^n(\mathfrak{a}, M) = \mathrm{Ext}^n_{\mathfrak{a}\otimes_k \mathfrak{a}^\circ}(\mathfrak{a}, M).$$

Now let $\eta \in Z^n\bar{\mathbf{C}}^\bullet(\mathfrak{a}, M)$. Let $\tilde{\mathfrak{a}}$ be the DG-category $\mathfrak{a} \oplus \Sigma^{n-2}M$: its objects are the objects of $\mathfrak{a}$, morphisms are given by $\mathfrak{a}(-,-) \oplus \Sigma^{n-2}M(-,-)$ and composition is coming from the composition in $\mathfrak{a}$ and the action of $\mathfrak{a}$ on M.

We denote by $\mathfrak{a}_\eta$ the A_∞-category $\tilde{\mathfrak{a}}$, with deformed A_∞-structure given by

$$b_{\mathfrak{a}_\eta} = b_{\tilde{\mathfrak{a}}} + \eta,$$

where we view η as a map of degree one $(\Sigma\mathfrak{a})^{\otimes n} \to \Sigma(\Sigma^{n-2}M)$ and extend it to a map $\eta : (\Sigma\mathfrak{a}_\eta)^{\otimes n} \to \Sigma\mathfrak{a}_\eta$ of degree one by making the unspecified components zero. Clearly we have $H^*(\mathfrak{a}_\eta) = \tilde{\mathfrak{a}}$. Furthermore, since η is normalized, it is clear that $\mathfrak{a}_\eta$ is strictly unital.

Lemma 7.1.1. *If η, η' represent the same element of $\mathrm{HH}^n(\mathfrak{a}, M)$ then there is an A_∞-isomorphism $f : \mathfrak{a}_\eta \to \mathfrak{a}_{\eta'}$ whose only non-trivial component is of the form*

$$f_{n-1} : (\Sigma\mathfrak{a})^{\otimes n-1} \to \Sigma(\Sigma^{n-2}M)$$

Proof. This is an easy and standard verification. □

Because the construction of $\mathfrak{a}_\eta$ only depends on the cohomology class of η, we will often write $\mathfrak{a}_{\bar\eta}$ with $\bar\eta \in \mathrm{HH}^n(\mathfrak{a}, M)$ to denote $\mathfrak{a}_\eta$, where η is a lift of $\bar\eta$ to $Z^n\bar{\mathbf{C}}^\bullet(\mathfrak{a}, M)$.

7.2. **Tensoring with a DG-category.** Let $\mathfrak{a}$, $\mathfrak{b}$ be k-linear DG-categories, and let M be a k-central $\mathfrak{a}$-bimodule. Then then there is a morphism of complexes

$$\mathbf{C}(\mathfrak{a}, M) \to \mathbf{C}(\mathfrak{a} \otimes_k \mathfrak{b}, M \otimes_k \mathfrak{b}) : \eta \mapsto \eta \cup 1$$

where $\eta \cup 1$ is defined by (for suitable composable arrows)

$$(\eta \cup 1)(a_1 \otimes b_1 \otimes \cdots \otimes a_n \otimes b_n) = \pm\eta(a_1, \ldots, a_n) \otimes b_1 \cdots b_n$$

with the sign given by the Koszul convention. It is easy to see that on the level of cohomology $\eta \cup 1$ has the usual interpretation as a map

$$\mathrm{Ext}^*_{\mathfrak{a}\otimes_k \mathfrak{b}^\circ}(\mathfrak{a}, M) \to \mathrm{Ext}^*_{\mathfrak{a}\otimes_k \mathfrak{b}\otimes_k \mathfrak{a}^\circ \otimes_k \mathfrak{b}^\circ}(\mathfrak{a} \otimes_k \mathfrak{b}, M \otimes_k \mathfrak{b})$$

where 1 now refers to the identity element of $\mathrm{Hom}_{\mathfrak{b}\otimes_k \mathfrak{b}^\circ}(\mathfrak{b}, \mathfrak{b})$.

It is nontrivial to construct the tensor product of two A_∞-categories. However, no difficulty arises when one of the A_∞-categories is a DG-category: this is a special case of tensoring an algebra over an asymmetric operad with a DG-algebra, where again only suitable composable arrows should be considered. Specifically, assume that $\mathfrak{a}$ is a k-A_∞-category and $\mathfrak{b}$ is a k-DG-category. Then we define

$$\begin{aligned} b^n_{\mathfrak{a}\otimes_k\mathfrak{b}}(s(a_1\otimes b_1),\ldots,s(a_n\otimes b_n)) &= \pm b^n_{\mathfrak{a}}(sa_1,\ldots,sa_n)\otimes b_1\cdots b_n \\ b^1_{\mathfrak{a}\otimes_k\mathfrak{b}}(s(a\otimes b)) &= b^1(sa)\otimes b+(-1)^{|sa|}sa\otimes d(b) \end{aligned}$$

(for suitably composable arrows) where the sign is given by the Koszul convention after making the identification $s(a_i\otimes_k b_i)=(sa_i)\otimes_k b_i$.

With this definition is easy to see that, if $\eta\in Z^n\bar{\mathbf{C}}(\mathfrak{a},M)$ and $\eta\cup 1\in\bar{\mathbf{C}}(\mathfrak{a}\otimes_k\mathfrak{b},M\otimes_k\mathfrak{b})$ is the extended cocycle, then

$$\mathfrak{a}_\eta\otimes\mathfrak{b}=(\mathfrak{a}\otimes\mathfrak{b})_{\eta\cup 1}.$$

7.3. **The characteristic morphism.** Assume again that $\mathfrak{a}$ is a k-linear category and let N be an $\mathfrak{a}$-module. Then there is a so-called characteristic map [28]

$$c_N:\mathrm{HH}^n(\mathfrak{a},M)\to\mathrm{Ext}^n_{\mathfrak{a}}(N,M\overset{L}{\otimes}_{\mathfrak{a}}N),$$

which may be constructed by interpreting $\eta\in\mathrm{HH}^n(\mathfrak{a},M)$ as a map $\mathfrak{a}\to\Sigma^nM$ in $D(\mathfrak{a}\otimes_k\mathfrak{a}^\circ)$. Applying the functor $-\overset{L}{\otimes}_{\mathfrak{a}}N$ to η we obtain a map $N\to\Sigma^nM\overset{L}{\otimes}_{\mathfrak{a}}N$ which is $c_N(\eta)$.

There is a dual characteristic map

$$c_N:\mathrm{HH}^n(\mathfrak{a},M)\to\mathrm{Ext}^n_{\mathfrak{a}}(\mathrm{RHom}_{\mathfrak{a}}(M,N),N),$$

obtained by applying $\mathrm{RHom}_{\mathfrak{a}}(-,N)$ to η. For the sequel we note the following obvious fact.

Lemma 7.3.1. *Assume that M is an invertible $\mathfrak{a}$-bimodule. In that case we have a commutative diagram*

$$\begin{tikzcd}
 & \mathrm{Ext}^n_{\mathfrak{a}}(N,M\overset{L}{\otimes}_{\mathfrak{a}}N) \arrow[dd, "\cong"', "M^*\overset{L}{\otimes}_{\mathfrak{a}}-"] \\
\mathrm{HH}^n(\mathfrak{a},M) \arrow[ur, "c_N(\eta)"] \arrow[dr, "c_N^*(\eta)"'] & \\
 & \mathrm{Ext}^n_{\mathfrak{a}}(\mathrm{RHom}_{\mathfrak{a}}(M,N),N)
\end{tikzcd}$$

In other words in the context of Lemma 7.3.1 we do not have to make a distinction between the two characteristic maps.

Now assume that M is right flat over $\mathfrak{a}$. It is well-known that in that case c_N can be constructed directly on the level of complexes. One starts with the identification

$$\mathrm{Ext}^n_{\mathfrak{a}}(N,M\otimes_{\mathfrak{a}}N)=\mathrm{HH}^n(\mathfrak{a},\underline{\mathrm{Hom}}_k(N,M\otimes_{\mathfrak{a}}N)).$$

With this identification c_N is obtained by passing to cohomology from the map of complexes

$$c_N:\mathbf{C}^\bullet(\mathfrak{a},M)\to\mathbf{C}^\bullet(\mathfrak{a},\underline{\mathrm{Hom}}_k(N,M\otimes_{\mathfrak{a}}N)),$$

which is obtained from the obvious map of $\mathfrak{a}$-bimodules

$$M\to\underline{\mathrm{Hom}}_k(N,M\otimes_{\mathfrak{a}}N). \tag{7.1}$$

A similar statement holds for c_N^*. In this case (7.1) is replaced by the equally obvious map of $\mathfrak{a}$-bimodules

$$M \to \underline{\mathrm{Hom}}_k(\mathrm{Hom}(M,N),N).$$

7.4. **Deformation of objects.** Let the notation be as in §7.1, but to simplify things we will restrict to the case $n \geq 3$.

Assume that M is right flat over $\mathfrak{a}$. Let $U \in \mathrm{Mod}(\mathfrak{a})$. A lift of U to $\mathfrak{a}_\eta$ is a pair (V,ϕ) where V is in $D_\infty(\mathfrak{a}_\eta)$ and ϕ is an isomorphism of graded $H^*(\mathfrak{a}_\eta)$-modules $H^*(V) \cong H^*(\mathfrak{a}_\eta) \otimes_{\mathfrak{a}} U$.

Similarly, if M is left projective, then a colift of U to $\mathfrak{a}_\eta$ is a pair (V,ϕ), where V is in $D_\infty(\mathfrak{a}_\eta)$ and ϕ is an isomorphism of graded $H^*(\mathfrak{a}_\eta)$-modules $H^*(V) \cong \mathrm{Hom}_{\mathfrak{a}}(H^*(\mathfrak{a}_\eta),U)$.

We recall the following well-known fact.

Lemma 7.4.1. *The object $U \in \mathrm{Mod}(\mathfrak{a})$ has a lift to $\mathfrak{a}_\eta$ if and only if $c_U(\bar\eta) = 0$. It has a colift if and only if $c_U^*(\bar\eta) = 0$.*

Proof. Both cases are similar, so we will consider the case of a lift. Thus in that case we assume that M is right flat. Let U' be the graded $H^*(\mathfrak{a}_\eta)$-module $U \oplus \Sigma^{n-2}M \otimes_{\mathfrak{a}} U$. If V is an $\mathfrak{a}_\eta$-lifting of U then we may assume that V is represented by a "minimal model" object $V = (U', b_V)$ with $b_{V,1} = 0$. We now have a graded functor between graded categories

$$\mathfrak{a} \oplus \Sigma^{n-2}M \xrightarrow{f} \Lambda := \begin{pmatrix} \mathrm{End}_k(U) & 0 \\ \mathrm{Hom}_k(U, \Sigma^{n-2}M \otimes_{\mathfrak{a}} U) & \mathrm{End}_k(U) \end{pmatrix}$$

representing the action of $H^*(\mathfrak{a}_\eta)$ on U', and we have to change it to an A_∞-morphism

$$(\mathfrak{a} \oplus \Sigma^{n-2}M, b_{\mathfrak{a}_\eta}) \xrightarrow{f+\xi} \begin{pmatrix} \mathrm{End}_k(U) & 0 \\ \mathrm{Hom}_k(U, \Sigma^{n-2}M \otimes_{\mathfrak{a}} U) & \mathrm{End}_k(U) \end{pmatrix}$$

with $\xi : (\Sigma\mathfrak{a})^{\otimes n-1} \to \Sigma\,\mathrm{Hom}_k(U, \Sigma^{n-2}M \otimes_{\mathfrak{a}} U)$ and $b_{\mathfrak{a}_\eta} = b_{\mathfrak{a}} + \eta$. The required compatibility between cofunctors and codifferentials may be expressed as

$$(f+\xi) \circ (b_{\mathfrak{a}} + \eta) - b_\Lambda \circ (f+\xi) = 0,$$

with b_Λ being the codifferential on Λ. As usual we only have to check this after performing the projection $\mathbb{B}\Lambda \to \Lambda$. So the only possible non-trivial evaluation is on $\Sigma\mathfrak{a}^{\otimes n}$ and we get

$$\xi \circ b_{\mathfrak{a}} - b_\Lambda \circ \xi + f \circ \eta = 0,$$

which may be rewritten as

$$d_{\mathrm{Hoch}}(\xi) = f \circ \eta = c_U(\eta).$$

This proves what we want. □

8. Obstruction theory

8.1. **Preliminaries on A_n-categories and A_n-functors.** A_n categories and functors are defined by replacing $\mathbb{B}\mathfrak{b}$ with the n-truncated $(\mathbb{B}\mathfrak{b})_{\leq n}$ bar-cocategory. I.e. $\mathrm{Ob}(\mathbb{B}\mathfrak{b})_{\leq n} = \mathrm{Ob}(\mathfrak{b})$ and

$$\mathbb{B}\mathfrak{b}_{\leq n}(A,B) = \bigoplus_{i \leq n, A_1,\ldots,A_{i-1} \in \mathrm{Ob}(\mathfrak{b})} \Sigma\mathfrak{b}(A_{i-1},B) \otimes \ldots \otimes \Sigma\mathfrak{b}(A,A_1)$$

As usual, we describe A_n-structures and morphisms via their Taylor coefficients, which may be evaluated on sequences of $i \le n$ composable maps.

Given an A_n category $\mathfrak{b}$, we write $\mathfrak{b}_{\le m}$ for the corresponding category viewed as an A_m-category for $m \le n$. A similar convention applies to functors.

Like for the A_∞-case, all A_n-notions will be assumed to be strictly unital.

8.2. **Obstructions for A_∞-morphisms.** We will use [46, §10] as a convenient reference. The following lemma is a more precise version of [46, Lemma 10.3.1] (see [5] for a related result).

Lemma 8.2.1. *Let $f_i : \mathfrak{c} \to \tilde{\mathfrak{c}}$ be an A_i-functor between A_∞-categories. Then there is an "obstruction" $o_{i+1}(f_i) \in \mathrm{HH}^{i+1}(H^*(\mathfrak{c}), H^*(\tilde{\mathfrak{c}}))^{-i+1}$ with the following property: $o_{i+1}(f_i)$ vanishes if and only if there exists $\delta_i : (\Sigma\mathfrak{c})^{\otimes i} \to \Sigma\tilde{\mathfrak{c}}$ such that $b_{\tilde{\mathfrak{c}},1} \circ \delta_i - \delta_i \circ b_{\mathfrak{c},1} = 0$ and such that $f_i + \delta_i$ extends to an A_{i+1}-functor. The obstruction $o_{i+1}(f_i)$ is natural in the following sense: for A_∞-functors $G : \mathfrak{c}' \to \mathfrak{c}$, $g : \tilde{\mathfrak{c}} \to \tilde{\mathfrak{c}}'$ we have*

$$o_{i+1}(g \circ f_i \circ G) = H^*(g) \circ o_{i+1}(f_i) \circ H^*(G). \tag{8.1}$$

Proof. First note that in [46] we worked with non-strictly unital A_∞-functors (in fact: morphisms). We may however equally well perform the construction in the strictly unital context by working with the normalized Hochschild complex. Here we will follow this approach.

We view f_i as a cofunctor $f_i : \mathbb{B}\mathfrak{c}_{\le i+1} \to \mathbb{B}\tilde{\mathfrak{c}}_{\le i+1}$ by making its $i{+}1$'th Taylor coefficient zero. Define the f_i-coderivation $D : \mathbb{B}\mathfrak{c}_{\le i+1} \to \mathbb{B}\tilde{\mathfrak{c}}_{\le i+1}$ as follows:

$$D = b_{\tilde{\mathfrak{c}}} \circ f_i - f_i \circ b_{\mathfrak{c}}\,.$$

It is clear that

$$b_{\tilde{\mathfrak{c}}} \circ D + D \circ b_{\mathfrak{c}} = 0. \tag{8.2}$$

Moreover, by construction we have that the Taylor coefficients D_n of D satisfy

$$D_n = 0 \qquad \text{for } n = 1, \dots, i \tag{8.3}$$

so that the only data in D is D_{i+1}, which by (8.2) descends to a linear map $\bar{D}_{i+1} : H^*(\Sigma\mathfrak{c})^{\otimes i+1} \to H^*(\Sigma\tilde{\mathfrak{c}})$. As in the proof of [46, Lemma 10.3.1], from (8.2) one computes

$$0 = d_{\mathrm{Hoch}}(\bar{D}_{i+1}) \tag{8.4}$$

where d_{Hoch} represent the Hochschild differential. Computing degrees one sees that $\bar{D}_{i+1}$ represents an element of $\mathrm{HH}^{i+1}(H^*(\mathfrak{c}), H^*(\tilde{\mathfrak{c}}))^{-i+1}$, which we will call $o_{i+1}(f_i)$.

Furthermore - again as in the proof of [46, Lemma 10.3.1] - one sees that, if $o_{i+1}(f_i)$ vanishes, then f_i extends to f_{i+1} in the way described in the statement of the lemma.

To show that the implication goes in both directions, let us repeat the argument. The data f_{i+1}, δ_i as in the statement of the lemma will exist if and only the following equation has a solution in $(f_{i+1})_{i+1}$, δ_i:

$$\begin{aligned}
0 &= (b_{\tilde{\mathfrak{c}}} \circ f_{i+1} - f_{i+1} \circ b_{\mathfrak{c}})_{i+1} \\
&= D_{i+1} + (b_{\tilde{\mathfrak{c}}} \circ (f_{i+1} - f_i) - (f_{i+1} - f_i) \circ b_{\mathfrak{c}})_{i+1} \\
&= D_{i+1} + b_{\tilde{\mathfrak{c}},1} \circ (f_{i+1})_{i+1} - (f_{i+1})_{i+1} \circ b_{\mathfrak{c},1} \\
&\qquad + b_{\tilde{\mathfrak{c}},2} \circ (\delta_i \otimes f_1 + f_1 \otimes \delta_i) - \sum_{a+b+2=i+1} (\delta_i \circ (\mathrm{id}^{\otimes a} \otimes b_{\mathfrak{c},2} \otimes \mathrm{id}^{\otimes b})
\end{aligned} \tag{8.5}$$

It is clear that this has a solution if and only if the corresponding equation in cohomology $0 = \bar{D}_{i+1} + d_{\mathrm{Hoch}}\bar{\delta}_i = 0$ has a solution, i.e. if and only the obstruction $o_{i+1}(f_i)$ vanishes.

Naturality: let us write $D(f_i)$ for D as introduced above. Then it follows from the definition (8.2) that for $f_i' = g \circ f_i \circ G$

$$D(f_i') = g \circ D(f_i) \circ G.$$

By (8.3) this yields

$$D(f_i')_{i+1} = g_0 \circ D(f_i)_{i+1} \circ G_0,$$

and passing to cohomology

$$\overline{D(f_i')}_{i+1} = \overline{g}_0 \circ \overline{D(f_i)}_{i+1} \circ \overline{G}_0$$

which is (8.1). □

If $f : H^*(\mathfrak{c}) \to H^*(\tilde{\mathfrak{c}})$ is a graded functor, then it can always be completed to an A_2-functor $f_2 : \mathfrak{c} \to \tilde{\mathfrak{c}}$ (this is essentially choosing homotopies). So the first non-trivial obstruction is $o_3(f_2) \in \mathrm{HH}^3(H^*(\mathfrak{c}), H^*(\tilde{\mathfrak{c}}))^{-1}$. It is easy to see that it only depends on f. Indeed, two choices of f_2 only differ by a $\delta_2 : (\Sigma\mathfrak{c})^{\otimes 2} \to \Sigma\tilde{\mathfrak{c}}$ commuting with differentials, and this δ_2 disappears in the obstruction. See Corollary 8.2.4 for a variation on this fact.

In general, if we start from $f : H^*(\mathfrak{c}) \to H^*(\tilde{\mathfrak{c}})$ we may compute obstructions

$$o_3(f_2), o_4(f_3), o_5(f_4), \ldots$$

We will informally write these as

$$o_3(f), o_4(f), o_5(f), \ldots$$

with the proviso that $o_{i+1}(f)$ only exists when $o_3(f), \ldots, o_i(f)$ vanish, and furthermore $o_{i+1}(f)$ depends on prior choices. So it may be zero for one such choice and non-zero for another.

We will always apply Lemma 8.2.1 with $\mathfrak{c}$ being a k-linear category (i.e. concentrated in degree zero). In that case we have

$$o_i(f) \in \mathrm{HH}^i(\mathfrak{c}, H^{-i+2}(\tilde{\mathfrak{c}})). \tag{8.6}$$

Corollary 8.2.2. *Consider a commutative diagram of graded functors*

$$\begin{tikzcd} H^*(\mathfrak{c}_1) \arrow[r, "f_1"] & H^*(\tilde{\mathfrak{c}}_1) \arrow[d, "H^*(g)"] \\ H^*(\mathfrak{c}_2) \arrow[u, "H^*(G)"] \arrow[r, "f_2"'] & H^*(\tilde{\mathfrak{c}}_2) \end{tikzcd} \tag{8.7}$$

with g, G being A_∞-quasi-isomorphisms. Let $\{i_1, i_2\} = \{1, 2\}$. When making choices for computing the obstructions for f_{i_1}, we may make corresponding choices for computing the obstructions for f_{i_2} such that $o_*(f_2) = H^*(g) \circ o_*(f_1) \circ H^*(G)$.

Proof. When $i_1 = 1$, $i_2 = 2$ this is follows from naturality (see Lemma 8.2.1). For the case $i_1 = 2$, $i_2 = 1$ we use the fact that g, G have inverses up to homotopy in the A_∞-category. Such inverses are true inverses on cohomology. Now use again the naturality for A_∞-morphisms. □

Remark 8.2.3. It follows from Corollary 8.2.2 that in order to calculate obstructions for $f : \mathfrak{c} \to H^*(\tilde{\mathfrak{c}})$ with $\mathfrak{c}$ a k-linear category we may replace $\tilde{\mathfrak{c}}$ with a (strictly unital) minimal model $\bar{\mathfrak{c}}$. By definition the underlying complex of $\bar{\mathfrak{c}}$ is $H^*(\tilde{\mathfrak{c}})$, with zero differential and there is an A_∞-quasi-isomorphism $g : \bar{\mathfrak{c}} \to \tilde{\mathfrak{c}}$ such that g_1 induces the identity on cohomology. Let $f_1 : \mathfrak{c} \to H^*(\bar{\mathfrak{c}})$ be such that $H^*(g_1) \circ f_1 = f$. By naturality we have $o_*^{\tilde{\mathfrak{c}}}(f) = o_*^{\tilde{\mathfrak{c}}}(H^*(g_1) \circ f_1) = H^*(g_1) \circ o_*^{\bar{\mathfrak{c}}}(f_1)$. How we use that $H^*(g_1)$ is the identity to obtain $o_*^{\tilde{\mathfrak{c}}}(f) = o_*^{\bar{\mathfrak{c}}}(f_1)$.

Corollary 8.2.4. *Assume that $\mathfrak{c}$ is a k-linear category and let $f : \mathfrak{c} \to H^*(\tilde{\mathfrak{c}})$ be a k-linear functor. If $-n < 0$ is maximal with the property that $H^{-n}(\tilde{\mathfrak{c}}) \neq 0$ then*

$$o_3(f) = \cdots = o_{n+1}(f) = 0 \tag{8.8}$$

and $o_{n+2}(f)$ does not depend on any choices.

Proof. Since by (8.6) $o_j(f) \in \mathrm{HH}^j(\mathfrak{c}, H^{-j+2}(\tilde{\mathfrak{c}}))$ we have $o_j(f) = 0$ for $-n+1 \leq -j+2 \leq -1$ which yields (8.8).

To prove the statement about $o_{n+2}(f)$ we may as in Remark 8.2.3 replace $\tilde{\mathfrak{c}}$ with a minimal model $\bar{\mathfrak{c}}$. But then $\bar{\mathfrak{c}}^{-n+1} = \cdots = \bar{\mathfrak{c}}^{-1} = 0$. Following the proof of Lemma 8.2.1, we have

$$f = f_1 = f_2 = f_3 = \cdots = f_n$$

To compute f_{n+1} we first have to compute

$$D(f_n)_{n+1} = (b_{\bar{\mathfrak{c}}} \circ f - f \circ b_{\mathfrak{c}})_{n+1}$$

If $n = 1$ then this is zero since f respects the multiplication. If $n > 1$ then using the fact that the lowest $b_{\bar{\mathfrak{c}},j}$ for $j > 2$ which can be non-zero is $b_{\bar{\mathfrak{c}},n+2}$ we also get zero which is compatible with the fact that we already know $o_{n+1}(f_n) = 0$.

To compute the possible lifts of f_n we have to solve (see (8.5))

$$0 = [d, (f_{n+1})_{n+1}] + d_{\mathrm{Hoch}}(\delta_n)$$

For degree reasons we must have $\delta_n = 0$. Since both $\mathfrak{c}$ and $\bar{\mathfrak{c}}$ have zero differential, it follows that we may choose $(f_{n+1})_{n+1} : (\Sigma\mathfrak{c})^{\otimes n+1} \to \Sigma(\mathfrak{c})^{-n}$ freely.

By definition $o_{n+2}(f)$ is the class of

$$(b_{\bar{\mathfrak{c}}} \circ f_{n+1} - f_{n+1} \circ b_{\mathfrak{c}})_{n+2} = b_{\bar{\mathfrak{c}},n+2} \circ (f \otimes \cdots \otimes f) + d_{\mathrm{Hoch}}((f_{n+1})_{n+1})$$

One easily checks that $b_{\bar{\mathfrak{c}},n+2}$ is a Hochschild cocycle, and moreover if we replace $\bar{\mathfrak{c}}$ by another A_∞-isomorphic minimal model (see Remark 8.2.3) then $b_{\bar{\mathfrak{c}},n+2}$ changes only by a Hochschild boundary. From this we deduce that $o_{n+2}(f)$ is well defined. $\square$

Remark 8.2.5. If we write $p_{n+2}(\tilde{\mathfrak{c}}) \overset{\text{def}}{=} \bar{b}_{\bar{\mathfrak{c}},n+2} \in \mathrm{HH}^{n+2}(H^0(\mathfrak{c}), H^{-n}(\tilde{\mathfrak{c}}))$ then it was shown in the previous proof that $p_{n+2}(\tilde{\mathfrak{c}})$ is well defined and moreover

$$o_{n+2}(f) = p_{n+2}(\tilde{\mathfrak{c}}) \circ f$$

Note that we may also think of $p_{n+2}(\tilde{\mathfrak{c}})$ as $o_{n+2}(j)$ where $j : H^0(\tilde{\mathfrak{c}}) \to H^*(\tilde{\mathfrak{c}})$ is the inclusion.

If $\mathfrak{c}$ is a k-linear category, $\eta \in \mathrm{HH}^{n+2}(\mathfrak{c}, M)$ for $n \geq 1$ and $j : \mathfrak{c} \to H^*(\mathfrak{c}_\eta) = \mathfrak{c} \oplus \Sigma^i M$ is the inclusion then we obtain $o_3(j) = \cdots = o_{n+1}(j) = 0$ and $o_{n+2}(j) = \eta$.

We will use the following application.

Proposition 8.2.6. *Let $\mathfrak{a}$, $\mathfrak{c}$ be k-linear categories, M an $\mathfrak{a}$-bimodule, $\eta \in \mathrm{HH}^n(\mathfrak{a}, M)$. Assume $f : \mathfrak{c} \to \mathfrak{a}$ is an additive functor such that $f_*(\eta) \overset{def}{=} \eta \circ f = 0$. Then there is a commutative diagram of A_∞-categories*

(8.9)
$$\begin{tikzcd} \mathfrak{a}_\eta \arrow[dr] & & \mathfrak{c} \arrow[ll, dotted, "\tilde{f}"'] \arrow[dl, "f"] \\ & \mathfrak{a} & \end{tikzcd}$$

Proof. According to Corollary 8.2.4 and Remark 8.2.5 $f_*(\eta)$ is the single obstruction against the existence of the diagram (8.9). □

8.3. **Scalar extensions of derived categories.** Let Γ a k-algebra. For a k-linear category $\mathcal{A}$ we write $\mathcal{A}_\Gamma$ for the category Γ-objects in $\mathcal{A}$, i.e. pairs (M, ρ) where $M \in \mathrm{Ob}(\mathcal{A})$ and $\rho : \Gamma \to \mathcal{A}(M, M)$ is a k-algebra morphism.

There is a forgetful functor [45]

$$F : D(\mathfrak{b} \otimes_k \Gamma) \to D(\mathfrak{b})_\Gamma$$

forgetting the action of Γ on the level of complexes but remembering it on the level of the derived category. There is a similar result as Lemma 8.2.1:

Lemma 8.3.1. *(1) Let $T \in D(\mathfrak{b})_\Gamma$. Then there is a sequence of obstructions*

$$o_{i+2}(T) \in \mathrm{HH}^{i+2}(\Gamma, \mathrm{Ext}_{\mathfrak{b}}^{-i}(T, T))$$

for $i \geq 1$ such that T lifts to an object in $D(\mathfrak{b} \otimes_k \Gamma)$ if and only if all obstructions vanish. More precisely $o_{i+1}(T)$ is only defined if $o_3(T), \ldots, o_i(T)$ vanish and it depends on choices.

(2) If $f : \mathfrak{c} \to \mathfrak{b}$ is a DG-functor and $f_* : D(\mathfrak{b}) \to D(\mathfrak{c})$ is the corresponding change of rings functor then after having made choices for T we may make corresponding choices for $f_*(T)$ in such a way that

$$f_*(o_{i+2}(T)) = o_{i+2}(f_*(T))$$

Proof. If T has a lift, then it is represented by a cofibrant object $\tilde{T}$ in $\underline{\mathrm{Mod}}(\mathfrak{b} \otimes_k \Gamma)$ for the standard projective model structure, which is in particular a cofibrant object in $\underline{\mathrm{Mod}}(\mathfrak{b})$ equipped with an A_∞-Γ-action. Conversely, an object in $\underline{\mathrm{Mod}}(\mathfrak{b})$ with an A_∞-Γ-action may be regarded as an object in $D(\mathfrak{b} \otimes_k \Gamma)$. See [46, Lemma 10.2.1] for an analogous statement which is proved in the same way.

Let $\tilde{T}$ be a cofibrant object in $\underline{\mathrm{Mod}}(\mathfrak{b})$ representing T. Put $\tilde{\Gamma} = \mathrm{Hom}_{\mathfrak{b}}(\tilde{T}, \tilde{T})$. We have a graded k-algebra morphism $f : \Gamma \to H^*(\tilde{\Gamma})$, and the question is when can we lift it to an A_∞-morphism $\Gamma \to \tilde{\Gamma}$. This is controlled by the obstructions $o_i(T) \overset{\mathrm{def}}{=} o_i(f)$. We still have to show, however, that $o_i(T)$ is independent of the choice of $\tilde{T}$. Suppose that $\tilde{T}_1$, $\tilde{T}_2$ are cofibrant objects both representing T, i.e. that there is a quasi-isomorphism $u : \tilde{T}_1 \to \tilde{T}_2$ inducing the identity on T. Put $\tilde{\Gamma}_i = \mathrm{End}_{\mathfrak{b}}(\tilde{T}_i)$ and let $f_i : \Gamma \to H^i(\tilde{\Gamma}_i)$ be the corresponding actions. Let $\tilde{X} = \mathrm{cone}\, u$. Note that $\tilde{X}$ is cofibrant and acyclic. Let $\tilde{\Gamma}$ be the sub-DG-algebra of $\mathrm{End}_{\mathfrak{b}}(\tilde{X})$ given by

$$\tilde{\Gamma} = \begin{pmatrix} \tilde{\Gamma}_1 & 0 \\ \mathrm{Hom}_{\mathfrak{b}}(\Sigma\tilde{T}_1, \tilde{T}_2) & \tilde{\Gamma}_2 \end{pmatrix}.$$

The kernels of the projections $\mathrm{pr}_1 : \tilde{\Gamma} \to \tilde{\Gamma}_1$, $\mathrm{pr}_2 : \tilde{\Gamma} \to \tilde{\Gamma}_2$ are respectively given by $\mathrm{Hom}_{\mathfrak{b}}(\tilde{X}, \tilde{T}_2)$ and $\mathrm{Hom}_{\mathfrak{b}}(\Sigma\tilde{T}_1, \tilde{X})$ and hence they are acyclic. It follows that pr_1, pr_2 are quasi-isomorphisms.

We obtain a morphism of graded rings $f : \Gamma \to H^*(\tilde{\Gamma})$ given by

$$f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}.$$

Since $\mathrm{pr}_i : \tilde{\Gamma} \to \tilde{\Gamma}_i$ are quasi-isomorphisms of DG-algebras, and the resulting identification of $H^*(\tilde{\Gamma}_1)$ and $H^*(\tilde{\Gamma}_2)$ is conjugation by $H^*(u)$, we conclude by Corollary 8.2.2 that the obstructions against lifting f, f_1 and f_2 all coincide, and those of f_1, f_2 are naturally identified. This finishes the proof that $o_i(T)$ is well defined.

The verification that the obstructions are natural is similar. Let $\tilde{T}_2 \to T$ be a cofibrant replacement of T in $\underline{\mathrm{Mod}}(\mathfrak{b})$ and let $u : \tilde{T}_1 \to f_*\tilde{T}_2$ be a cofibrant replacement of $f_*\tilde{T}_2$ in $\underline{\mathrm{Mod}}(\mathfrak{c})$. Put $\tilde{\Gamma}_1 = \mathrm{End}_{\mathfrak{c}}(\tilde{T}_1)$, $\tilde{\Gamma}_2 = \mathrm{End}_{\mathfrak{b}}(\tilde{T}_2)$ and consider $\mathrm{Hom}_{\mathfrak{c}}(\Sigma\tilde{T}_1, f_*\tilde{T}_2)$ as $\tilde{\Gamma}_2 - \tilde{\Gamma}_1$-module. Put

$$\tilde{\Gamma} = \begin{pmatrix} \tilde{\Gamma}_1 & 0 \\ \mathrm{Hom}_{\mathfrak{c}}(\Sigma\tilde{T}_1, f_*\tilde{T}_2) & \tilde{\Gamma}_2 \end{pmatrix}.$$

with the differential being the sum of the natural one given by the differentials on $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$ and the commutator with $\left(\begin{smallmatrix} 0 & 0 \\ u & 0 \end{smallmatrix}\right)$. We now have projection maps $p_i : \tilde{\Gamma} \to \tilde{\Gamma}_i$ with p_2 being a quasi-isomorphism (since its kernel is given by $\mathrm{Hom}_{\mathfrak{c}}(\Sigma\tilde{T}_1, \operatorname{cone} u)$, $\Sigma\tilde{T}_1$ is cofibrant and $\operatorname{cone} u$ is acyclic). Moreover we find that $H^*(p_1) \circ H^*(p_2)^{-1} : H^*(\tilde{\Gamma}_2) \to H^*(\tilde{\Gamma}_1)$ is the map $f_* : \mathrm{Ext}^*_{\mathfrak{b}}(T,T) \to \mathrm{Ext}^*_{\mathfrak{c}}(f_*T, f_*T)$ given by functoriality. The naturally of the obstructions now follows by applying Corollary 8.2.2. □

From Corollary 8.2.4 and the proof of Lemma 8.3.1 we also deduce:

Corollary 8.3.2. *Let $T \in D(\mathfrak{b})_\Gamma$. If $-n < 0$ is maximal with the property that $\mathrm{Ext}^{-n}_{\mathfrak{b}}(T,T) \neq 0$ then $o_3(T) = \cdots = o_{n+1}(T) = 0$, and $o_{n+2}(T)$ does not depend on any choices.*

9. Sheaves and presheaves

9.1. **Introduction.** This technical section is mainly concerned with relating properties of quasi-compact separated schemes X to similar properties of the corresponding categories $\mathcal{X}$ defined in the introduction. This material is necessary as we will use $\mathcal{X}$ to deform the derived category of quasi-coherent sheaves on X. We discuss:

- the relation on the level of modules and bimodules (the functors w, W in §9.3);
- compatibility with Hochschild cohomology (see (9.5));
- compatibility with certain Fourier-Mukai functors (Lemma 9.4.1, (9.13));
- compatibility with the characteristic morphism (see (9.14));
- functoriality of $X \mapsto \mathcal{X}$ for closed immersions (see §9.7);
- vector bundles (see §9.8).

The last section §9.9 discusses an auxilliary result which will be needed later. The reader may be willing to skip this section on first reading.

9.2. **Presheaves.** We discuss some notions introduced in [18, 31]. We follow more or less [31], but as we use left instead of right modules the conventions will be slightly different.

Let $(I, \leq)$ be a poset and let $\mathcal{O}$ be a presheaf of k-algebras on I. For $j \leq i$, we denote the corresponding restriction morphism $\mathcal{O}(i) \to \mathcal{O}(j)$ by ρ_{ij}. In [31, §2.2] (following a similar construction in [18]) a k-linear category, which we will denote by $\tilde{\mathcal{O}}$, is associated to $\mathcal{O}$ as follows: $\mathrm{Ob}(\tilde{\mathcal{O}}) = I$ and

$$\tilde{\mathcal{O}}(i,j) = \begin{cases} \mathcal{O}(j) & \text{if } j \leq i \\ 0 & \text{otherwise.} \end{cases}$$

The non-trivial compositions for $k \leq j \leq i$

$$\tilde{\mathcal{O}}(j,k) \otimes_k \tilde{\mathcal{O}}(i,j) \to \tilde{\mathcal{O}}(i,k)$$

are given by

$$\mathcal{O}(k) \otimes_k \mathcal{O}(j) \xrightarrow{\text{restriction}} \mathcal{O}(k) \otimes_k \mathcal{O}(k) \xrightarrow{\text{multiplication}} \mathcal{O}(k).$$

Let $\mathrm{Mod}(\mathcal{O})$ be the category of presheaves of $\mathcal{O}$-modules. There is an equivalence of categories

$$\pi^* : \mathrm{Mod}(\mathcal{O}) \to \mathrm{Mod}(\tilde{\mathcal{O}})$$

such that

$$\pi^*(M)(i) = M(i),$$

and the multiplication map for $j \leq i$

$$\tilde{\mathcal{O}}(i,j) \otimes_k (\pi^* M)(i) \to (\pi^* M)(j)$$

is given by

$$\mathcal{O}(j) \otimes_k M(i) \xrightarrow{\text{restriction}} \mathcal{O}(j) \otimes_k M(j) \xrightarrow{\text{action}} M(j). \tag{9.1}$$

Conversely, we may recover the restriction map $\rho_{ij} : M(i) \to M(j)$ as the action of the element $1_{\mathcal{O}(j)} \in \tilde{\mathcal{O}}(i,j)$.

Assume that $\mathcal{O}'$ is a second presheaf of rings on I. Write $\mathrm{Bimod}_k(\mathcal{O}, \mathcal{O}') = \mathrm{Mod}(\mathcal{O} \otimes_k \mathcal{O}'^{\circ})$. Then there is a functor (see [18][31, §3.4, Lemma 5.2])

$$\Pi^* : \mathrm{Bimod}_k(\mathcal{O}, \mathcal{O}') \to \mathrm{Bimod}(\tilde{\mathcal{O}}, \tilde{\mathcal{O}}')$$

defined as follows

$$\Pi^*(M)(i,j) = \begin{cases} M(j) & \text{if } j \leq i \\ 0 & \text{otherwise.} \end{cases}$$

The bimodule structure on $\Pi^*(M)(i,j)$ is defined using a similar formula as (9.1).

The functor Π^* is obviously exact and by [18][31, Thm 4.1, Lemma 5.2] the corresponding derived functor

$$\Pi^* : D(\mathrm{Bimod}_k(\mathcal{O}, \mathcal{O}')) \to D(\mathrm{Bimod}(\tilde{\mathcal{O}}, \tilde{\mathcal{O}}'))$$

is fully faithful.

Lemma 9.2.1. *Let $U \in D(\mathcal{O}')$ and $M \in D(\mathrm{Bimod}_k(\mathcal{O}, \mathcal{O}'))$. Then there is a natural isomorphism*

$$\pi^*(M \overset{L}{\otimes}_{\mathcal{O}'} U) = \Pi^*(M) \overset{L}{\otimes}_{\tilde{\mathcal{O}}'} \pi^* U \tag{9.2}$$

as objects in $D(\tilde{\mathcal{O}})$.

Proof. It suffices to prove the non-derived statement for U a projective object in $\mathrm{Mod}(\mathcal{O}')$, as π^*U is then also projective in $\mathrm{Mod}(\tilde{\mathcal{O}}')$ since π^* is an equivalence of categories. Equivalently, we may assume $U = \pi_* V$, with V a projective object in $\mathrm{Mod}(\tilde{\mathcal{O}}')$ where $\pi_* = (\pi^*)^{-1}$. Furthermore, we may assume that V is of the form $\tilde{\mathcal{O}}'(i, -)$.

Then we find

$$\begin{aligned}\pi^*(M \otimes_{\mathcal{O}'} \pi_* \tilde{\mathcal{O}}'(i,-))(j) &= M(j) \otimes_{\mathcal{O}'(j)} \tilde{\mathcal{O}}'(i,j) \\ &= \begin{cases} M(j) \otimes_{\mathcal{O}'(j)} \mathcal{O}'(j) = M(j) & \text{if } j \le i \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Similarly, by the general property of tensor products

$$\Pi^*(M) \otimes_{\tilde{\mathcal{O}}'} \tilde{\mathcal{O}}'(i,-) = \Pi^*(M)(i,-),$$

and hence, by the definition of $\Pi^*(-)$,

$$(\Pi^*(M) \otimes_{\tilde{\mathcal{O}}'} \tilde{\mathcal{O}}'(i,-))(j) = \begin{cases} M(j) & \text{if } j \le i \\ 0 & \text{otherwise.}\end{cases}$$

In other words, for all i, j

$$\pi^*(M \otimes_{\mathcal{O}'} \pi_* \tilde{\mathcal{O}}'(i,-))(j) = (\Pi^*(M) \otimes_{\tilde{\mathcal{O}}'} \tilde{\mathcal{O}}'(i,-))(j)$$

It remains to show that this identification is natural in i, j. This is a routine verification, which we omit. $\square$

9.3. **Sheaves.** Here we recall some results from [30, §7.5ff]. Unless otherwise specified, in the rest of this section X will be a quasi-compact separated k-scheme. The separatedness hypothesis ensures that $D(\mathrm{Qch}(X)) \cong D_{\mathrm{Qch}}(\mathrm{Mod}(X))$ [7]. Hence we will make no distinction between those two categories. Note, in particular, that $D(\mathrm{Qch}(X))$ has enough homotopically flat objects (see [1]), so the derived tensor product may be computed entirely on the quasi-coherent level.

Let $X = \bigcup_{i=1}^n U_i$ be an affine covering. For $I \subset \{1, \dots, n\}$ put $U_I = \bigcap_{i \in I} U_i$. Let $\mathcal{I}$ be the poset $\{I \subset \{1, \dots, n\} \mid I \neq \emptyset\}$, ordered in such a way that $I \le J$ if $J \subset I$ (the strange ordering is motivated by the fact that $J \subset I$ implies $U_I \subset U_J$).

Let $\widehat{\mathcal{O}}_X$ be the presheaf of rings on $\mathcal{I}$ associated to $\mathcal{O}_X$, and, for a quasi-coherent sheaf M on X, let $\epsilon^* M$ be the corresponding presheaf of $\widehat{\mathcal{O}}_X$-modules. The corresponding derived functor

$$\epsilon^* : D(\mathrm{Qch}(X)) \to D(\mathrm{Mod}(\widehat{\mathcal{O}}_X))$$

has a right adjoint [30], which we will denote by $R\epsilon_*$. It may be computed using a version of the Čech complex. More precisely

$$R\epsilon_*(M) = \left(\bigoplus_{I \in \mathcal{I}} j_{U_I,*} \Sigma^{-|I|+1} \widetilde{M(I)}, d \right), \tag{9.3}$$

where $\tilde{?}$ is the quasi-coherent sheaf associated to a module over a commutative ring, and $j_{U_I} : U_I \to X$ is the inclusion map. The differential d is the usual alternating sum of restriction morphisms. Recall the following[4]:

[4]This is stated in somewhat greater generality than in loc. cit. However, it can be proved in the same way.

Lemma 9.3.1. [30, Theorem 7.6.6] *The functor $\epsilon^* : D(\mathrm{Qch}(X)) \to D(\widehat{\mathcal{O}}_X)$ is fully faithful and a left inverse is given by $R\epsilon_*$. Furthermore, the essential image of ϵ^* is $D_{\epsilon^*\,\mathrm{Qch}(X)}(\widehat{\mathcal{O}}_X)$.*

Below, if X is a quasi-compact separated scheme, we will denote by the corresponding curly letter $\mathcal{X}$ the category $\widetilde{\widehat{\mathcal{O}}}_X$, as introduced in this section and the previous one. There is now a fully faithful embedding

$$w : D(\mathrm{Qch}(X)) \to D(\mathcal{X}), \tag{9.4}$$

given by the composition

$$D(\mathrm{Qch}(X)) \xrightarrow{\epsilon^*} D(\mathrm{Mod}(\widehat{\mathcal{O}}_X)) \xrightarrow[\cong]{\pi^*} D(\mathcal{X}).$$

We have a similar statement for bimodules. Let $D^\delta(\mathrm{Qch}(X))$ be the category whose objects are the same as those of $D(\mathrm{Qch}(X))$, but whose Hom-sets are given by

$$\mathrm{Hom}_{D^\delta(\mathrm{Qch}(X))}(M,N) = \mathrm{Hom}_{D(\mathrm{Qch}(X\times_k X))}(i_{\Delta,*}M, i_{\Delta,*}N),$$

where $i_\Delta : X \to X \times_k X$ is the diagonal. Let $Z = \bigcup_{i=1}^n U_i \times_k U_i$. Note that $\widehat{\mathcal{O}}_Z = \widehat{\mathcal{O}}_X \otimes_k \widehat{\mathcal{O}}_X$. Then the map

$$D^\delta(\mathrm{Qch}(X)) \xrightarrow{M\mapsto i_{\Delta,*}M|Z} D(\mathrm{Qch}(Z))$$

is fully faithful, since the support of i_*M is closed in $X \times_k X$ and contained in the open set Z. By the above discussion, we obtain a fully faithful embedding

$$W : D^\delta(\mathrm{Qch}(X)) \to D(\mathcal{X} \otimes_k \mathcal{X}^\circ)$$

given as the composition

$$D^\delta(\mathrm{Qch}(X)) \xrightarrow{M\mapsto i_*M|Z} D(\mathrm{Qch}(Z)) \xrightarrow{\epsilon^*} D(\widehat{\mathcal{O}}_X \otimes_k \widehat{\mathcal{O}}_X) \xrightarrow{\Pi^*} D(\mathcal{X} \otimes_k \mathcal{X}^\circ).$$

If M is a quasi-coherent $\mathcal{O}_X$-module, then following [50] its Hochschild cohomology is defined as

$$\mathrm{HH}^*(X,M) \overset{\mathrm{def}}{=} \mathrm{Ext}^*_{X\times_k X}(i_{\Delta,*}\mathcal{O}_X, i_{\Delta,*}M).$$

Hence by the full faithfulness of W we have a canonical isomorphism [30]

$$\mathrm{HH}^*(X,M) \cong \mathrm{HH}^*(\mathcal{X}, W(M)). \tag{9.5}$$

9.4. **Actions of bimodules on modules.** Consider the following bifunctor,

$$\mathcal{F} : D^\delta(\mathrm{Qch}(X))\times D(\mathrm{Qch}(X)) \to D(\mathrm{Qch}(X)) : (M,U) \mapsto R\,\mathrm{pr}_{1*}(i_{\Delta,*}M \overset{L}{\otimes}_{\mathcal{O}_{X\times X}} \mathrm{pr}_2^* U) \tag{9.6}$$

Lemma 9.4.1. *The following diagram is commutative:*

$$\begin{tikzcd}
D^\delta(\mathrm{Qch}(X)) \arrow[d, "W"'] & \times & D(\mathrm{Qch}(X)) \arrow[r, "\mathcal{F}"] \arrow[d, "w"] & D(\mathrm{Qch}(X)) \arrow[d, "w"] \\
D(\mathcal{X}\otimes_k \mathcal{X}^\circ) & \times & D(\mathcal{X}) \arrow[r, "-\overset{L}{\otimes}_{\mathcal{X}}-"'] & D(\mathcal{X})
\end{tikzcd}$$

Proof. Let $\alpha, \beta : Z \to X$ be the first and the second projection respectively. Then we have

$$R\,\mathrm{pr}_{1*}(i_{\Delta,*}M \overset{L}{\otimes}_{\mathcal{O}_{X\times X}} \mathrm{pr}_2^* U) = R\alpha_*(i_{\Delta,*}M|_Z \overset{L}{\otimes}_{\mathcal{O}_Z} \beta^* U).$$

So, by the definition of w and W, we have to prove that for $N = i_{\Delta,*}M \in D(\mathrm{Qch}(Z))$ there is an isomorphism

$$\pi^*\epsilon^*(R\alpha_*(N \overset{L}{\otimes}_{\mathcal{O}_Z} \beta^* U)) = \Pi^*\epsilon^* N \overset{L}{\otimes}_{\mathcal{X}} \pi^*\epsilon^* U,$$

which is natural in N considered as an object in $D(\mathrm{Qch}(Z))$.

It follows from (9.11) and Lemma 9.4.5 below that for any $N \in D(\mathrm{Qch}(Z))$ there is a canonical morphism

$$\epsilon^* R\alpha_*(N \overset{L}{\otimes}_{\mathcal{O}_Z} \beta^* U) \to \epsilon^* N \overset{L}{\otimes}_{\widehat{\mathcal{O}}_X} \epsilon^* U,$$

which is moreover an isomorphism if $N = i_{\Delta,*}M$.

Applying π^* and using Lemma 9.2.1 we get a canonical morphism

$$\pi^*\epsilon^* R\alpha_*(N \overset{L}{\otimes}_{\mathcal{O}_Z} \beta^* U) \to \Pi^*\epsilon^* N \overset{L}{\otimes}_{\mathcal{X}} \pi^*\epsilon^* U$$

having the same property. This finishes the proof. $\square$

We now give the lemmas on which the previous proof was based.

Lemma 9.4.2. *Let $N, U \in D(\mathrm{Qch}(X))$. Then*

$$\epsilon^*(N \overset{L}{\otimes}_{\mathcal{O}_X} U) \cong \epsilon^* N \overset{L}{\otimes}_{\widehat{\mathcal{O}}_X} \epsilon^* U \tag{9.7}$$

Proof. We may assume that U is homotopically flat [1] and it is easy to see that this implies that $\epsilon^* U$ is also homotopically flat. Hence we have to prove (9.7) for quasi-coherent sheaves, which is obvious. $\square$

Lemma 9.4.3. *Let P be in $D(\mathrm{Qch}(Z))$. Then*

$$R\alpha_* P = R\epsilon_*^{\mathrm{left}}(\epsilon^* P), \tag{9.8}$$

where ϵ^{left} refers to the fact that we only consider the left $\widehat{\mathcal{O}}_X$-structure on $\epsilon^* P$.

Proof. Since P is a complex of quasi-coherent sheaves P is quasi-isomorphic to its Čech complex. In other words it is isomorphic to

$$\left(\bigoplus_{I\in\mathcal{I}} j_{U_I\times_k U_I,*}\Sigma^{-|I|+1} P(U_I \times_k U_I)\tilde{\ }, d\right) \tag{9.9}$$

(9.9) consists of modules which are acyclic for α_*. In fact, $j_{U_I\times_k U_I}$ and $\alpha \circ j_{U_I\times_k U_I}$ are affine and hence have no higher direct images for quasi-coherent sheaves. It follows by the Leray spectral sequence that the same is true for α. Moreover, α_* has finite cohomological dimension. Hence we have

$$\begin{aligned} R\alpha_* P &= \left(\bigoplus_{I\in\mathcal{I}} \alpha_* j_{U_I\times_k U_I,*}\Sigma^{-|I|+1} P(U_I \times_k U_I)\tilde{\ }, d\right) \\ &= \left(\bigoplus_{I\in\mathcal{I}} j_{U_I *}\,\mathrm{pr}_{U_I\times_k U_I, U_I *}\,\Sigma^{-|I|+1} P(U_I \times_k U_I)\tilde{\ }, d\right) \end{aligned}$$

where $\mathrm{pr}_{U_I\times_k U_I, U_I *}$ is the projection map. This is precisely $R\epsilon_*^{\mathrm{left}}$ applied to the presheaf on $\mathcal{I}$ given by $I \mapsto P(U_I \times_k U_I)$, and the latter is of course $\epsilon^* P$. $\square$

Lemma 9.4.4. *Let N be in $D(\mathrm{Qch}(Z))$ and $U \in D(\mathrm{Qch}(X))$. Then we have*

$$R\alpha_*(N \overset{L}{\otimes}_{\mathcal{O}_Z} \beta^* U) = R\epsilon_*(\epsilon^* N \overset{L}{\otimes}_{\widehat{\mathcal{O}}_X} \epsilon^* U) \tag{9.10}$$

Proof. We have

$$\begin{aligned} R\alpha_*(N \overset{L}{\otimes}_{\mathcal{O}_Z} \beta^* U) &= R\epsilon_*^{\mathrm{left}}(\epsilon^*(N \overset{L}{\otimes}_{\mathcal{O}_Z} \beta^* U)) \\ &= R\epsilon_*(\epsilon^* N \overset{L}{\otimes}_{\widehat{\mathcal{O}}_Z} \epsilon^* \beta^* U) \\ &= R\epsilon_*(\epsilon^* N \overset{L}{\otimes}_{\widehat{\mathcal{O}}_Z} (\widehat{\mathcal{O}}_Z \otimes_{\widehat{\mathcal{O}}_X} \epsilon^* U)) \\ &= R\epsilon_*(\epsilon^* N \overset{L}{\otimes}_{\widehat{\mathcal{O}}_X} \epsilon^* U) \end{aligned}$$

where in the first equality we use (9.8), and in the second equality we use (9.7). □

By adjointness, we obtain from (9.10) a canonical morphism

$$\epsilon^* R\alpha_*(N \overset{L}{\otimes}_{\mathcal{O}_Z} \beta^* U) \xrightarrow{\cong} \epsilon^* R\epsilon_*(\epsilon^* N \overset{L}{\otimes}_{\widehat{\mathcal{O}}_X} \epsilon^* U) \xrightarrow{\text{counit}} \epsilon^* N \overset{L}{\otimes}_{\widehat{\mathcal{O}}_X} \epsilon^* U. \tag{9.11}$$

Lemma 9.4.5. (9.11) *is an isomorphism if N is of the form $i_{\Delta,*}M$.*

Proof. In that case, one easily checks that

$$\epsilon^* i_{\Delta,*} M \overset{L}{\otimes}_{\widehat{\mathcal{O}}_X} \epsilon^* U = \epsilon^*(M \overset{L}{\otimes}_{\mathcal{O}_X} U).$$

Since $M \overset{L}{\otimes}_{\mathcal{O}_X} U$ is quasi-coherent, we obtain that the counit morphism in (9.11) is an isomorphism. □

9.5. **Equivariant version.** Assume now that Γ is a k-algebra (non necessarily commutative). Let $\mathrm{Qch}(X)_\Gamma$ be the category of quasi-coherent sheaves on X equipped with a left Γ-action. Let $\mathcal{O}_{X,\Gamma} = \mathcal{O}_X \otimes_k \Gamma$ so that $\mathrm{Qch}(X)_\Gamma \cong \mathrm{Qch}(\mathcal{O}_{X,\Gamma})$. Furthermore put $\mathcal{X}_\Gamma = \mathcal{X} \otimes_k \Gamma$. Then $\mathcal{X}_\Gamma$ is obtained from $\mathcal{O}_{X,\Gamma}$ in the same way as $\mathcal{X}$ is obtained from $\mathcal{O}_X$. Then using [30, Theorem 7.6.6] we obtain as in (9.4) a full faithful embedding

$$w : D(\mathrm{Qch}(X)_\Gamma) \to D(\mathcal{X}_\Gamma) \tag{9.12}$$

and furthermore we have a commutative diagram with the same proof as Lemma 9.4.1

$$\begin{tikzcd} D^\delta(\mathrm{Qch}(X)) \arrow[d, "W"'] & \times & D(\mathrm{Qch}(X)_\Gamma) \arrow[r, "\mathcal{F}"] \arrow[d, "w"] & D(\mathrm{Qch}(X)_\Gamma) \arrow[d, "w"] \\ D(\mathcal{X} \otimes_k \mathcal{X}^\circ) & \times & D(\mathcal{X}_\Gamma) \arrow[r, "- \overset{L}{\otimes}_{\mathcal{X}} -"'] & D(\mathcal{X}_\Gamma) \end{tikzcd} \tag{9.13}$$

9.6. **The characteristic morphism.** In §7.3 we introduced the characteristic morphism for DG-categories (following [28]). A similar definition works for schemes. We present a restricted version which is sufficient for our applications. Let X be as above and let $M, U \in D(\mathrm{Qch}(X))$. Then the characteristic morphism

$$c_U : \mathrm{HH}^*(X, M) \to \mathrm{Ext}^*_X(U, M \overset{L}{\otimes}_X U)$$

is defined as follows. Let $\eta \in \mathrm{HH}^n(X, M)$, and view it as a map $\mathcal{O}_X \to \Sigma^n M$ in the category $D^\delta(\mathrm{Qch}(X))$. Then

$$c_U(\eta) = \mathcal{F}(\eta, \mathrm{id}),$$

with $\mathcal{F}$ as in (9.6). From Lemma 9.4.1 we immediately obtain the following commutative diagram:

$$\begin{tikzcd} \mathrm{HH}^*(X,M) \arrow[r, "c_U"] \arrow[d, "W"', "\cong"] & \mathrm{Ext}^*_X(U, M \overset{L}{\otimes}_X U) \arrow[d, "\cong"', "w"] \\ \mathrm{HH}^*(\mathcal{X}, W(M)) \arrow[r, "c_{w(U)}"'] & \mathrm{Ext}^*_{\mathcal{X}}(w(U), W(M) \overset{L}{\otimes}_{\mathcal{X}} w(U)) \end{tikzcd} \tag{9.14}$$

(the vertical maps are isomorphism because of (9.5), Lemma 9.4.1, and the fact that w is fully faithful (see §9.3). If U is an object in $D(\mathrm{Qch}(X)_\Gamma)$, then there is a characteristic map

$$c_{U,\Gamma} : \mathrm{HH}^*(X, M) \to \mathrm{Ext}^*_{\mathrm{Qch}(X)_\Gamma}(U, M \overset{L}{\otimes}_X U) \tag{9.15}$$

which fits in a similar commutative diagram as (9.14).

9.7. **Functoriality.** Now assume that X, Y are quasi-compact separated k-schemes, and let $f : X \to Y$ be a closed immersion. Let $Y = \bigcup_i^n V_i$ be an affine covering, and let $U_i = f^{-1}(V_i)$ be the induced covering on X.

The map f induces a dual functor

$$f : \mathcal{Y} \to \mathcal{X}$$

and hence a "change of rings" functor

$$f_* : D(\mathcal{X}) \to D(\mathcal{Y}).$$

Lemma 9.7.1. *The following diagram is commutative:*

$$\begin{tikzcd} D(\mathrm{Qch}(X)) \arrow[r, "f_*"] \arrow[d, "w"'] & D(\mathrm{Qch}(Y)) \arrow[d, "w"] \\ D(\mathcal{X}) \arrow[r, "f_*"'] & D(\mathcal{Y}) \end{tikzcd}$$

Proof. All functors are induced from exact functors on the level of abelian categories. Hence it suffices to check the commutativity on the level of sheaves, which is obvious. □

The functor $(f, f)_* : D(\mathrm{Qch}(X \times_k X)) \to D(\mathrm{Qch}(Y \times_k Y))$ descends to a functor

$$f_* : D^\delta(\mathrm{Qch}(X)) \to D^\delta(\mathrm{Qch}(Y)).$$

Lemma 9.7.2. *The following diagram is commutative:*

$$\begin{tikzcd} D^\delta(\mathrm{Qch}(X)) \arrow[r, "f_*"] \arrow[d, "W"'] & D^\delta(\mathrm{Qch}(Y)) \arrow[d, "W"] \\ D(\mathcal{X} \otimes_k \mathcal{X}^\circ) \arrow[r, "(f,f)_*"'] & D(\mathcal{Y} \otimes_k \mathcal{Y}^\circ) \end{tikzcd} \tag{9.16}$$

Proof. All functors are induced from exact functors on the level of abelian categories. Hence it suffices to check the commutativity on the level of sheaves, which is obvious. □

Applying (9.16) to morphisms $\mathcal{O}_X \to \Sigma^n M$ in $D^\delta(\mathrm{Qch}(X))$, we get a commutative diagram

$$\begin{tikzcd}
D^\delta(\mathrm{Qch}(X))(\mathcal{O}_X, \Sigma^n M) \arrow[r, "f_*"] \arrow[d, "W"', "\cong"] & D^\delta(\mathrm{Qch}(Y))(f_*\mathcal{O}_X, \Sigma^n f_*M) \arrow[r] \arrow[d, "\cong"', "W"] & D^\delta(\mathrm{Qch}(Y))(\mathcal{O}_Y, \Sigma^n f_*M) \arrow[d, "\cong"', "W"] \\
\mathrm{Ext}^n_{\mathcal{X}\otimes_k\mathcal{X}^\circ}(\mathcal{X}, W(M)) \arrow[r, "(f,f)_*"'] & \mathrm{Ext}^n_{\mathcal{Y}\otimes_k\mathcal{Y}^\circ}(\mathcal{X}, (f,f)_*(W(M))) \arrow[r] & \mathrm{Ext}^n_{\mathcal{Y}\otimes_k\mathcal{Y}^\circ}(\mathcal{Y}, (f,f)_*(W(M)))
\end{tikzcd}$$

where the rightmost square is obtained by precomposing with $\mathcal{O}_Y \to f_*\mathcal{O}_X$ and $\mathcal{Y} \to \mathcal{X}$ respectively.

We obtain a commutative diagram on the level of Hochschild cohomology

$$\begin{tikzcd}
\mathrm{HH}^*(X, M) \arrow[r, "f_*"] \arrow[d, "W"', "\cong"] & \mathrm{HH}^*(Y, f_*M) \arrow[d, "\cong"', "W"] \\
\mathrm{HH}^*(\mathcal{X}, W(M)) \arrow[r, "(f,f)_*"'] & \mathrm{HH}^*(\mathcal{Y}, (f,f)_*(W(M)))
\end{tikzcd} \tag{9.17}$$

9.8. **Vector bundles and projectives.**

Lemma 9.8.1. *(1) Let M, N we quasi-coherent sheaves on X and put $\mathcal{M} = W(M)$, $\mathcal{N} = W(N)$. Then $W(M \otimes_{\mathcal{O}_X} N) = \mathcal{M} \otimes_{\mathcal{X}} \mathcal{N}$.*

(2) Assume that M is a vector bundle on X. Then $\mathcal{M}$ is projective on the left and on the right. That is, for every $I \in \mathcal{I}$ we have that $\mathcal{M}(I,-)$ and $\mathcal{M}(-,I)$ are respectively projective left and right $\mathcal{X}$-modules.

Proof. (1) It is an immediate verification that

$$\begin{aligned}
\mathcal{M}(I,-) &= \mathcal{X}(I,-) \otimes_{\mathcal{O}(U_I)} M(U_I) \\
\mathcal{M}(-,I) &= M(U_I) \otimes_{\mathcal{O}(U_I)} \mathcal{X}(-,I)
\end{aligned} \tag{9.18}$$

We compute

$$\begin{aligned}
(\mathcal{M} \otimes_{\mathcal{X}} \mathcal{N})(I_1, I_2) &= \mathcal{M}(-, I_2) \otimes_{\mathcal{X}} \mathcal{N}(I_1, -) \\
&= M(U_{I_2}) \otimes_{\mathcal{O}(U_{I_2})} \mathcal{X}(-, I_2) \otimes_{\mathcal{X}} \mathcal{X}(I_1, -) \otimes_{\mathcal{O}(U_{I_1})} N(U_{I_1}) \\
&= M(U_{I_2}) \otimes_{\mathcal{O}(U_{I_2})} \mathcal{X}(I_1, I_2) \otimes_{\mathcal{O}(U_{I_1})} N(U_{I_1}).
\end{aligned}$$

Assume now $I_2 \subset I_1$ (for otherwise there is nothing to prove). Then we have

$$\begin{aligned}
M(U_{I_2}) \otimes_{\mathcal{O}(U_{I_2})} \mathcal{X}(I_1, I_2) \otimes_{\mathcal{O}(U_{I_1})} N(U_{I_1}) &= M(U_{I_2}) \otimes_{\mathcal{O}(U_{I_2})} \mathcal{O}_X(U_{I_2}) \otimes_{\mathcal{O}(U_{I_1})} N(U_{I_1}) \\
&= M(U_{I_2}) \otimes_{\mathcal{O}(U_{I_2})} N(U_{I_2}) \\
&= (M \otimes_{\mathcal{O}_X} N)(U_{I_2}) \\
&= W(M \otimes_{\mathcal{O}_X} N)(I_1, I_2)
\end{aligned}$$

(2) Now $M(U_I)$ is a finitely generated "projective $\mathcal{O}(U_I)$-module, and hence a summand of a free module. By (9.18) this implies that $\mathcal{M}(I,-)$ is a summand of $\mathcal{X}(I,-)^{\oplus n}$ for some n, and similarly for $\mathcal{M}(-,I)$. This means both are projective.

9.9. **Compact generators.** For a perfect complex P in $D(\mathrm{Qch}(X))$, put $P^D = \mathrm{R}\mathcal{H}om_X(P, \mathcal{O}_X)$. Recall the following

Lemma 9.9.1. *Let P be perfect object in $D(\mathrm{Qch}(X))$. Then P generates $D(\mathrm{Qch}(X))$ if and only if P^D generates $D(\mathrm{Qch}(X))$.*

Proof. By [38, 35] P generates $D(\mathrm{Qch}(X))$ if and only if it classically generates the category $\mathrm{Perf}(X)$ of perfect complexes in $D(\mathrm{Qch}(X))$. The fact that $(-)^D$ is a duality on $\mathrm{Perf}(X)$ proves what we want. $\square$

Proposition 9.9.2. *Let $T \in \mathrm{Qch}(X)$ be a tilting bundle, i.e. a vector bundle generating $D(\mathrm{Qch}(X))$ such that $\mathrm{Ext}^i_X(T,T) = 0$ for $i > 0$. Set $\Gamma = \mathrm{End}_X(T)$. Then*

$$c_{T,\Gamma} : \mathrm{HH}^*(X, M) \to \mathrm{Ext}^*_{\mathrm{Qch}(X)_\Gamma}(T, M \overset{L}{\otimes}_X T)$$

is an isomorphism.

Proof. We claim that the functor

$$H : D(\mathrm{Qch}(X \times_k X)) \to D(\mathrm{Qch}(X)_\Gamma) : N \mapsto R\,\mathrm{pr}_{1*}(N \overset{L}{\otimes}_{X\times X} \mathrm{pr}_2^* T)$$

is an equivalence of categories. This implies what we want.

By Lemma 9.9.1 and [10, §3.4], $T \boxtimes T^D$ is a compact generator for $D(\mathrm{Qch}(X \times_k X))$ and it is also clear that $T \otimes_k \Gamma$ is a compact generator for $D(\mathrm{Qch}(X)_\Gamma)$. We compute

$$\begin{aligned} H(T \boxtimes T^D) &= R\,\mathrm{pr}_{1*}((T \boxtimes T^D) \overset{L}{\otimes}_{X\times X} \mathrm{pr}_2^* T) \\ &= R\,\mathrm{pr}_{1*}(T \boxtimes \mathrm{R}\mathcal{H}om_X(T,T)) \\ &= T \otimes_k \Gamma \end{aligned}$$

and is clear that in this way H yields an isomorphism between

$$\mathrm{REnd}_{X\times_k X}(T \boxtimes T^D) = \Gamma \otimes_k \Gamma^\circ$$

and

$$\mathrm{REnd}_{\mathrm{Qch}(X)_\Gamma}(T \otimes_k \Gamma) = \Gamma \otimes_k \Gamma^\circ$$

This implies that H is an equivalence in the usual way. $\square$

10. Some properties of divisors

10.1. **Preliminaries.** Let X be a quasi-compact separated scheme. If $\mathcal{A}$, $\mathcal{B}$ are sheaves of k-algebras on X then an $\mathcal{A}-\mathcal{B}$-bimodule $\mathcal{F}$ is defined to be a sheaf of $\mathcal{A}\otimes_k \mathcal{B}^\circ$-modules. Note that even if $\mathcal{A}$, $\mathcal{B}$ are quasi-coherent this will usually not be the case for $\mathcal{A}\otimes_k \mathcal{B}^\circ$. To compute things like $\mathcal{F} \overset{L}{\otimes}_{\mathcal{B}} -$ we may take a flat resolution of $\mathcal{F}$ as sheaf of $\mathcal{A}\otimes_k \mathcal{B}^\circ$-modules. This is then automatically also a flat resolution as right $\mathcal{B}$-modules which can be used to compute the derived tensor product.

Define $D^\delta(\mathcal{O}_X)$ as the full subcategory of $D_{\mathrm{Qch}}(\mathcal{O}_X \otimes_k \mathcal{O}_X)$ whose objects are obtained from complexes of quasi-coherent $\mathcal{O}_X$-modules with $\mathcal{O}_X \otimes \mathcal{O}_X$ acting via the multiplication map $\mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X$. We will need the following lemma

Lemma 10.1.1. *There is an equivalence of categories*

$$D^\delta(\mathcal{O}_X) \cong D^\delta(\mathrm{Qch}(X))$$

which is the identity on objects where $D^\delta(\mathrm{Qch}(X))$ was introduced in §9.3.

Proof. As usual let $i_\Delta : X \to X \times X$ be the diagonal. Put $\mathcal{A} = i_\Delta^{-1}(\mathcal{O}_{X\times X})$. Then there is an obvious morphism of sheaves of algebras $\mathcal{O}_X \otimes_k \mathcal{O}_X \to \mathcal{A}$ and we claim it is flat. Indeed the stalk of $\mathcal{A}$ at $x \in X$ is equal to the stalk at $\mathcal{O}_{X\times X}$ at $\Delta(x)$. So this stalk is equal to the localization of $\mathcal{O}_{X,x} \otimes \mathcal{O}_{X,x}$ at the kernel of the map $\mathcal{O}_{X,x} \otimes \mathcal{O}_{X,x} \to k(x) \otimes k(x) \to k(x)$. Note that $\mathcal{A}$ also maps to $\mathcal{O}_X$, so we may define $D^\delta(\mathcal{A})$ as the full subcategory of $D(\mathcal{A})$ spanned by objects which are obtained from complexes of quasi-coherent $\mathcal{O}_X$-modules.

We have pairs of adjoint functors

$$\begin{tikzcd}
D(\mathcal{O}_X \otimes \mathcal{O}_X) \arrow[rr, bend left, "\mathcal{A}\otimes_{\mathcal{O}_X\otimes\mathcal{O}_X} -"] & & D(\mathcal{A}) \arrow[ll, bend left, "-|_{\mathcal{O}_X\otimes\mathcal{O}_X}"] \arrow[r, bend right, "i_{\Delta,*}"'] & D(\mathcal{O}_{X\times X}) \arrow[l, bend right, "i_\Delta^{-1}"']
\end{tikzcd} \tag{10.1}$$

whose unit/counit maps are isomorphisms in the categories $D^\delta(-)$. From this it follows immediately that the functors in (10.1) define inverse equivalences between the $D^\delta(-)$. $\square$

Corollary 10.1.2. *If $M \in D(\mathrm{Qch}(X))$ then*

$$\mathrm{HH}^*(X, M) = \mathrm{Ext}^*_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{O}_X, M).$$

10.2. **The characteristic class of a divisor.** Unless otherwise specified, in the rest of this section X will be a closed subscheme of a quasi-compact separated k-scheme Y defined by an invertible ideal I. With a slight abuse of notation, we will write $\mathcal{O}_X = \mathcal{O}_Y/I$ and we consider $\mathcal{O}_X$ as a sheaf of k-algebras on Y.

We prove a technical result (Lemma 10.2.1 below) which will be used to show that certain Hochschild cohomology classes are non-trivial (see Proposition 10.3.1 below). The result is probably known in some form to experts. For example Andrei Căldăraru tells us that it would also follow from his work with Arinkin [2] on derived self-intersections, modulo some technical verifications. Nonetheless, since we were unable to find a written proof in the literature, we provide one here.

We consider the complex of $\mathcal{O}_X$-bimodules

$$C(X/Y) \overset{\mathrm{def}}{=} \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{O}_X$$

To compute the cohomology of $C(X/Y)$ we may view $C(X/Y)$ as a complex of $\mathcal{O}_Y - \mathcal{O}_Y$-bimodules. Using the obvious $\mathcal{O}_Y$-flat resolution of $\mathcal{O}_X$

$$0 \to I \to \mathcal{O}_Y \to \mathcal{O}_X \to 0,$$

we find

$$H^*(C(X/Y)) = \begin{cases} \mathcal{O}_X & \text{if } i = 0 \\ I/I^2 & \text{if } i = -1 \\ 0 & \text{otherwise.} \end{cases} \tag{10.2}$$

In particular, we have a distinguished triangle of complexes of $\mathcal{O}_X$-bimodules

$$C(X/Y) \to \mathcal{O}_X \xrightarrow{\xi_{X/Y}} \Sigma^2 I/I^2 \to \tag{10.3}$$

with $\xi_{X/Y} \in \mathrm{Ext}^2_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{O}_X, I/I^2) = \mathrm{HH}^2(X, I/I^2)$, where we have used Corollary 10.1.2.

We will now give a classical avatar of $\xi_{X/Y}$ (see Lemma 10.2.1). By change of rings we have

$$\mathrm{HH}^2(X, I/I^2) = \mathrm{Ext}^2_{\mathcal{O}_X\otimes\mathcal{O}_X}(\mathcal{O}_X, I/I^2) = \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_X\otimes\mathcal{O}_X} \mathcal{O}_X, I/I^2) \tag{10.4}$$

and if X, Y are smooth there is the Hochschild-Kostant-Rosenberg isomorphism in the derived category of $\mathcal{O}_X$-modules

$$\mathrm{HKR}_* : \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_{X\times X}} \mathcal{O}_X \to \wedge^\bullet \Omega_X \tag{10.5}$$

If we represent $\mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_X\otimes\mathcal{O}_X} \mathcal{O}_X$ by the usual Hochschild complex

$$\mathbf{C}_\bullet(X) := \cdots \to \mathcal{O}_X \otimes_k \mathcal{O}_X \otimes_k \mathcal{O}_X \to \mathcal{O}_X \otimes_k \mathcal{O}_X \to \mathcal{O}_X,$$

then HKR_* is given by sending a local section $f_0 \otimes f_1 \otimes \cdots \otimes f_n$ of $\mathcal{O}_X^{\otimes n+1}$ to $f_0 df_1 \cdots df_n$.

In particular combining (10.5) with (10.4) we get a split injective map

$$\mathrm{HKR} : \mathrm{Ext}^1_X(\Omega_X, I/I^2) \to \mathrm{HH}^2(X, I/I^2)$$

Lemma 10.2.1. *Assume that X, Y are smooth, and let $\xi'_{X/Y} \in \mathrm{Ext}^1_X(\Omega_X, I/I^2)$ correspond to the conormal sequence* [20, Prop. 8.4A]

$$0 \to I/I^2 \to \mathcal{O}_X \otimes_{\mathcal{O}_Y} \Omega_Y \to \Omega_X \to 0.$$

Then

$$\xi_{X/Y} = \mathrm{HKR}(\xi'_{X/Y}). \tag{10.6}$$

In particular, if $\xi'_{X/Y}$ is non-zero then so is $\xi_{X/Y}$.

Proof. To prove this lemma we have to understand better the distinguished triangle

$$\Sigma I/I^2 \to \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{O}_X \to \mathcal{O}_X \to \tag{10.7}$$

corresponding to $\xi_{X/Y}$. We represent $\mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{O}_X$ by the bar complex $B_\bullet(X/Y)$

$$\cdots \to \mathcal{O}_X \otimes_k \mathcal{O}_Y \otimes_k \mathcal{O}_Y \otimes_k \mathcal{O}_X \to \mathcal{O}_X \otimes_k \mathcal{O}_Y \otimes_k \mathcal{O}_X \to \mathcal{O}_X \otimes_k \mathcal{O}_X$$

with the usual bar-differential. The analogous bar-complex $B_\bullet(X/X)$ is quasi-isomorphic to $\mathcal{O}_X$ and the map $\mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{O}_X \to \mathcal{O}_X$ in (10.7) is represented by the map of complexes $B_\bullet(X/Y) \to B_\bullet(X/X)$. So we obtain an exact sequence of complexes of $\mathcal{O}_X$-bimodules

$$0 \to J_\bullet \to B_\bullet(X/Y) \to B_\bullet(X/X) \to 0 \tag{10.8}$$

where $J_\bullet$ is a complex concentrated in degrees ≤ -1 of the form

$$\cdots \to \underbrace{\mathcal{O}_X \otimes (I \otimes \mathcal{O}_Y + \mathcal{O}_Y \otimes I) \otimes \mathcal{O}_X}_{J_2} \to \underbrace{\mathcal{O}_X \otimes I \otimes \mathcal{O}_X}_{J_1} \to \underbrace{0}_{J_0} \to 0$$

Now one computes locally that $H^{-1}(J^\bullet) = I/I^2$ and since $J^\bullet$ is acyclic in other degrees by (10.2) we obtain a quasi-isomorphism

$$J_\bullet \to \Sigma(I/I^2) \tag{10.9}$$

which sends local sections $f \otimes g \otimes h$ of J_1 to $f\bar{g}h$. Hence (10.8)(10.9) define a distinguished triangle *isomorphic* to (10.7). It takes some more straightforward verifications to show that the two distinguished triangles are actually the same, which we leave to the reader. If one does not want to do this then one may take

(10.8)(10.9) as defining $\xi_{X/Y}$. It then differs from the prior definition by at most a global unit (as I/I^2 is a line bundle on X, this is the ambiguity in the choise of (10.9)). Since we will be only interested in when $\xi_{X/Y} \neq 0$, this makes no difference.

Now we tensor (10.8) on the left by $\mathcal{O}_X \otimes_{\mathcal{O}_X \otimes \mathcal{O}_X} -$. Since all complexes in (10.8) are flat over $\mathcal{O}_X \otimes \mathcal{O}_X$ this is in fact the derived tensor product. Furthermore one has obvious identifications of complexes of $\mathcal{O}_X$-modules

$$\begin{aligned}\mathcal{O}_X \otimes_{\mathcal{O}_X \otimes \mathcal{O}_X} B_\bullet(X/Y) &= \mathcal{O}_X \otimes_{\mathcal{O}_Y} C_\bullet(Y)\\ \mathcal{O}_X \otimes_{\mathcal{O}_X \otimes \mathcal{O}_X} B_\bullet(X/X) &= C_\bullet(X)\end{aligned}$$

Moreover $\mathcal{O}_X \otimes_{\mathcal{O}_X \otimes \mathcal{O}_X} J_\bullet$ is a complex which ends with the term $\mathcal{O}_X \otimes I$ in degree -1. We now immediately verify that we have a commutative diagram of complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X \otimes_{\mathcal{O}_X \otimes \mathcal{O}_X} J_\bullet \arrow[r] \arrow[d, "\mathrm{can}"'] & \mathcal{O}_X \otimes_{\mathcal{O}_Y} C_\bullet(Y) \arrow[r] \arrow[d] & C_\bullet(X) \arrow[r] \arrow[d, "\mathrm{HKR}"] & 0 \\ 0 \arrow[r] & \Sigma(I/I^2) \arrow[r] & \Sigma(\mathcal{O}_X \otimes_{\mathcal{O}_Y} \Omega_Y) \arrow[r] & \Sigma\Omega_X \arrow[r] & 0 \end{tikzcd} \tag{10.10}$$

where the middle vertical map sends $f \otimes g \otimes h$ (in degree -1) to $f\bar{g}dh$. Then (10.10) give a map between distinguished triangles in the derived category of $\mathcal{O}_X$-modules. (10.11)

$$\begin{tikzcd} \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_X \otimes \mathcal{O}_X} \Sigma(I/I^2) \arrow[r] \arrow[d, "\mathrm{can}"'] & \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_X \otimes \mathcal{O}_X} C(X/Y) \arrow[r] \arrow[d] & \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_X \otimes_k \mathcal{O}_X} \mathcal{O}_X \arrow[r, "1\otimes\xi_{X/Y}"] \arrow[d, "\mathrm{HKR}"] & {} \\ \Sigma(I/I^2) \arrow[r] & \Sigma(\mathcal{O}_X \otimes_{\mathcal{O}_Y} \Omega_Y) \arrow[r] & \Sigma\Omega_X \arrow[r, "\Sigma\xi'_{X/Y}"'] & {} \end{tikzcd}$$

Completing (10.11) with a third commutative square we get

$$\begin{tikzcd} \mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_X \otimes_k \mathcal{O}_X} \mathcal{O}_X \arrow[r, "\mathrm{id}\otimes\xi_{X/Y}"] \arrow[d, "\mathrm{HKR}"'] & \Sigma^2(\mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_X \otimes \mathcal{O}_X} I/I^2) \arrow[d, "\mathrm{can}"] \\ \Sigma\Omega_X \arrow[r, "\Sigma\xi'_{X/Y}"'] & \Sigma^2(I/I^2) \end{tikzcd} \tag{10.12}$$

Under the isomorphism (10.4), $\xi_{X/Y}$ corresponds to the diagonal composition can $\circ$ $(\xi_{X/Y} \otimes \mathrm{id})$, whereas $\mathrm{HKR}(\xi'_{X/Y})$ corresponds (by construction) to the other diagonal composition. Hence we are done. □

10.3. **A concrete example.** Here is a concrete example of a situation where $\xi_{X/Y} \neq 0$.

Proposition 10.3.1. *Let X be a smooth hypersurface of degree $d > 1$ in $Y = \mathbb{P}^n$, $n \geq 2$. Then $\xi_{X/Y} \neq 0$.*

Proof. According to [53], the conormal sequence on X is not split. The conclusion then follows from Lemma 10.2.1. □

10.4. **The long exact sequence associated with a divisor.** Recall the following:

Lemma 10.4.1. *Assume that M, N be complexes of $\mathcal{O}_X$-modules. Then there is a long exact sequence*

(10.13)
$$\cdots \to \operatorname{Ext}_X^{n-2}(I/I^2 \otimes_{\mathcal{O}_X} M, N) \xrightarrow{-\circ(\xi_{X/Y} \overset{L}{\otimes} \mathrm{id}_M)} \operatorname{Ext}_X^n(M,N) \xrightarrow{f_*} \operatorname{Ext}_Y^n(M,N) \to \cdots$$

Proof. Using change of rings we have

$$\begin{aligned}\operatorname{Ext}_Y^n(M,N) &= \operatorname{Ext}_X^n((\mathcal{O}_X \overset{L}{\otimes}_{\mathcal{O}_Y} \mathcal{O}_X) \overset{L}{\otimes}_{\mathcal{O}_X} M, N)\\ &= \operatorname{Ext}_X^n(C(X/Y) \overset{L}{\otimes}_{\mathcal{O}_X} M, N)\end{aligned}$$

From (10.3) we obtain a distinguished triangle

$$C(X/Y) \overset{L}{\otimes}_{\mathcal{O}_X} M \to M \xrightarrow{\xi_{X/Y} \overset{L}{\otimes} \mathrm{id}_M} \Sigma^2(I/I^2 \overset{L}{\otimes}_{\mathcal{O}_X} M) \to$$

which yields the required long exact sequence by applying $\operatorname{Hom}_X(-,N)$. □

10.5. **Application to Hochschild cohomology.**

Proposition 10.5.1. *Let M be a complex of $\mathcal{O}_X$-modules. There is a long exact sequence*

$$\cdots \to \mathrm{HH}^{n-2}(X, (I/I^2)^{-1} \otimes_X M) \xrightarrow{\xi_{X/Y}\otimes -} \mathrm{HH}^n(X,M) \xrightarrow{f_*} \mathrm{HH}^n(Y,M) \to \cdots$$

Proof. Let $\Gamma_f \subset Y \times_k X$ be the graph of f. The long exact sequence (10.13) applied to $X \times_k X \to Y \times_k X$ becomes (using the dictionary $X \mapsto X \times_k X$, $Y \mapsto Y \times_k X$, $I/I^2 \mapsto I/I^2 \boxtimes_k \mathcal{O}_X$, $M \mapsto \mathcal{O}_{\Delta_X}$, $f \mapsto (f, \mathrm{id})$, $N \mapsto i_{\Delta_X,*}M$)

(10.14)
$$\begin{aligned}\cdots \to \operatorname{Ext}^{n-2}_{X\times_k X}((I/I^2 \boxtimes \mathcal{O}_X) \otimes_{\mathcal{O}_{X\times_k X}} \mathcal{O}_{\Delta_X}, i_{\Delta_X,*}M) &\xrightarrow{-\circ((\xi_{X/Y}\boxtimes 1)\otimes \mathrm{id}_{\mathcal{O}_{\Delta_X}})}\\ \operatorname{Ext}^n_{X\times_k X}(\mathcal{O}_{\Delta_X}, i_{\Delta_X,*}M) &\xrightarrow{(f,\mathrm{id})_*} \operatorname{Ext}^n_{Y\times_k X}(\mathcal{O}_{\Gamma_f}, (f,\mathrm{id}_X)_*M) \to \cdots\end{aligned}$$

Now $(I/I^2 \boxtimes \mathcal{O}_X) \otimes_{\mathcal{O}_{X\times_k X}} \mathcal{O}_{\Delta_X} = i_{\Delta_X,*}(I/I^2)$, and it is easy to see that with this identification we have $(\xi_{X/Y} \boxtimes 1) \otimes \mathrm{id}_{\mathcal{O}_{\Delta_X}} = \xi_{X/Y}$.

We have

$$\mathrm{HH}^n(X,M) = \operatorname{Ext}^n_{X\times_k X}(\mathcal{O}_{\Delta_X}, i_{\Delta_X,*}M)$$

If we consider M as $\mathcal{O}_{Y\times_k Y}$ module, then it is in fact supported on $X \times_k X$. Hence by adjointness

$$\begin{aligned}\mathrm{HH}^n(Y,M) &= \operatorname{Ext}^n_{Y\times_k Y}(\mathcal{O}_{\Delta_Y}, i_{\Delta_Y,*}M)\\ &= \operatorname{Ext}^n_{Y\times_k X}((\mathrm{id}_Y, f)^*\mathcal{O}_{\Delta_Y}, (f, \mathrm{id}_X)_*M)\\ &= \operatorname{Ext}^n_{Y\times_k X}(\mathcal{O}_{\Gamma_f}, (f,\mathrm{id}_X)_*M)\end{aligned}$$

Finally, since I/I^2 is an invertible $\mathcal{O}_X$-module, we have

$$\operatorname{Ext}^{n-2}_{X\times_k X}(i_{\Delta,*}(I/I^2), i_{\Delta_X,*}M) = \mathrm{HH}^{n-2}(X, (I/I^2)^{-1} \otimes_X M)$$

Substituting all this in (10.14) yields what we want. □

10.6. **The smooth proper case.** Here we assume that X and Y are smooth, proper and connected, and are of dimension m, $m+1$ respectively.

Lemma 10.6.1. *One has*
$$\mathrm{HH}^{2m}(X,\omega_X^{\otimes 2}) = k.$$

Proof. We have
$$\begin{aligned}\mathrm{HH}^{2m}(X,\omega_X^{\otimes 2}) &= \mathrm{Ext}^{2m}_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta,\omega_\Delta^{\otimes 2})\\ &= \mathrm{Ext}^{2m}_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta,\mathcal{O}_\Delta\otimes_{\mathcal{O}_{X\times X}}(\omega_X\boxtimes\omega_X))\\ &= \mathrm{Hom}_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta,\mathcal{O}_\Delta)^*\\ &= k.\end{aligned}$$

In the third line we have used that $\omega_X\boxtimes\omega_X$ is the canonical sheaf on $X\times X$, together with Serre duality [8]. $\square$

Lemma 10.6.2. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The multiplication pairing*
$$\mathrm{HH}^i(X,\mathcal{L})\otimes\mathrm{HH}^{2m-i}(X,\mathcal{L}^{-1}\otimes_X\omega_X^{\otimes 2})\to\mathrm{HH}^{2m}(X,\omega_X^{\otimes 2})=k \tag{10.15}$$
is non-degenerate.

Proof. This again a straightforward application of Serre duality, which says that the following pairing by composition is non-degenerate
$$\mathrm{Ext}^i_{X\times X}(\mathcal{O}_\Delta,i_{\Delta,*}\mathcal{L})\otimes\mathrm{Ext}^{2m-i}_{X\times X}(i_{\Delta,*}\mathcal{L},\mathcal{O}_\Delta\otimes_{X\times X}(\omega_X\boxtimes\omega_X))\to\mathrm{Ext}^{2m}_{X\times X}(\mathcal{O}_\Delta,\mathcal{O}_\Delta\otimes_{X\times X}(\omega_X\boxtimes\omega_X))$$
It is easy to see that this pairing coincides with (10.15). $\square$

Proposition 10.6.3. *Assume that $\xi_{X/Y}\neq 0$. Then*
$$f_*:\mathrm{HH}^{2m}(X,\omega_X^{\otimes 2})\to\mathrm{HH}^{2m}(Y,\omega_X^{\otimes 2})$$
is the zero map.

Proof. By Lemma 10.6.1 and Proposition 10.5.1, it is sufficient to show that the map
$$\mathrm{HH}^{2m-2}(X,(I/I^2)^{-1}\otimes_X\omega_X^{\otimes 2})\xrightarrow{\xi_{X/Y}\otimes -}\mathrm{HH}^{2m}(X,\omega_X^{\otimes 2})$$
is not zero. This follows from Lemma 10.6.2. $\square$

11. Construction of potential non-Fourier-Mukai functors

We now assume that X/k, Y/k are smooth of dimension m, $m+1$, and that X is embedded as a divisor in Y. Let $f:X\to Y$ be the inclusion. Define $\mathcal{X}$, $\mathcal{Y}$ as in §9.3. We have a fully faithful embedding $w:\mathrm{Qch}(X)\to\mathrm{Mod}(\mathcal{X})$ (see §9.3). Recall the following:

Lemma 11.1. *(1) If $E\in\mathrm{Qch}(X)$ is injective then so is wE.*
(2) Every object in $\mathrm{Qch}(X)$ has injective dimension $\leq m$.

Proof. (1) In this case, $\mathrm{Mod}(\mathcal{X})$ is a locally noetherian category, so a direct sum of injectives in $\mathrm{Mod}(\mathcal{X})$ is injective. Let $E\in\mathrm{Qch}(X)$ be injective. Since w commutes with direct sums, and since X is locally noetherian, we may without loss of generality assume that E is indecomposable, and hence as in $J(x)$ for x a not necessarily closed point in X, see [19, Thm II.7.18, proof]. Since $J(x)$ only depends on the local ring $\mathcal{O}_{X,x}$ we obtain that wE satisfies

$$(wE)(I) = \begin{cases} \Gamma(X,E) & x \in U_I \\ 0 & \text{otherwise} \end{cases}$$

where $\Gamma(X,E)$ is an injective $\Gamma(X,\mathcal{O}_{U_I})$-module for all I such that $x \in U_I$. Let I be the largest subset of $\{1,\dots,n\}$ such that $x \in U_I$. We find for $M \in \mathrm{Mod}(\mathcal{X})$:

$$\mathrm{Hom}_{\mathcal{X}}(M, wI) = \mathrm{Hom}_{\mathcal{O}_X(U_I)}(M(I), \Gamma(X,E))$$

which is an exact functor.

(2) For each $U_i \subset X$ we have $\mathrm{gl}\dim \mathcal{O}_X(U_i) = m$. It now suffices to note that on a noetherian scheme the propery of being quasi-coherent injective is local [19, Prop. II.7.17]. □

Let M be a line bundle on X, and let $\mathcal{M} = W(M)$ be the corresponding $\mathcal{X}$-bimodule. By Lemma 9.8.1 we know that $\mathcal{M}$ is invertible. We will denote its two sided inverse by $\mathcal{M}^{-1}$.

Choose $n \geq m+3$ and let $\eta \in \ker(\mathrm{HH}^n(X,M) \to \mathrm{HH}^n(Y, f_*M))$. See Lemma 10.6.1 and in particular Proposition 10.6.3 for how one may choose such η in the proper case.

For simplicity, denote the corresponding element $W(\eta) \in \mathrm{HH}^n(\mathcal{X},\mathcal{M})$ by η as well. Define $\mathcal{X}_\eta$ as in §7.1. By (9.17), we have that $W(\eta) \in \ker(\mathrm{HH}^n(\mathcal{X},\mathcal{M}) \to \mathrm{HH}^n(\mathcal{Y},(f,f)_*\mathcal{M}))$. Hence, by Proposition 8.2.6, we have a commutative diagram

$$\begin{tikzcd} \mathcal{X}_\eta \arrow[dr] & & \mathcal{Y} \arrow[ll, "\tilde{f}"'] \arrow[dl, "f"] \\ & \mathcal{X} & \end{tikzcd} \tag{11.1}$$

Put $\mathcal{X}_\eta^{\mathrm{dg}} = U^u(\mathcal{X}_\eta)$ (see Appendix D.1). Then we have

$$H^*(\mathcal{X}_\eta^{\mathrm{dg}}) = \begin{cases} \mathcal{X} & \text{if } i = 0 \\ \mathcal{M} & \text{if } i = -n+2 \\ 0 & \text{otherwise.} \end{cases} \tag{11.2}$$

We define the functor

$$L : \mathrm{Inj}\,\mathrm{Qch}(X) \to D(\mathcal{X}) : E \mapsto L(wE),$$

where $L(wE)$ is the derived injective (see §6.1) in $D(\mathcal{X})$ associated to the injective wE.

Since $\mathrm{Qch}(X)$ has global dimension m, by Lemma 11.1(2) we are now in a position to apply Proposition 6.3.1 with $\mathcal{A} = \mathrm{Qch}(X)$, $\mathfrak{c} = \mathcal{X}_\eta^{\mathrm{dg}}$. This yields an exact functor

$$L : D^b(\mathrm{Qch}(X)) \to D(\mathcal{X}_\eta^{\mathrm{dg}}). \tag{11.3}$$

Remark 11.2. We cannot apply Proposition 6.3.1 with $\mathcal{A} = \mathrm{Mod}(\mathcal{X})$, since if X is proper then it is easy to see that $\mathrm{gl}\dim \mathcal{X} \geq 2m$.

Lemma 11.3. *Let $B \in D^b(\mathrm{Qch}(X))$ and $\mathcal{B} = wB$. Then there is a distinguished triangle in $D(\mathcal{X}_\eta^{\mathrm{dg}})$*

$$\mathcal{B} \to L(B) \to \Sigma^{-n+2}\mathcal{M}^{-1} \otimes_{\mathcal{X}} \mathcal{B} \to \tag{11.4}$$

where $\mathcal{B}$, $\mathcal{M}^{-1} \otimes_{\mathcal{X}} \mathcal{B}$ are viewed as $\mathcal{X}_\eta^{\mathrm{dg}}$-modules via the map

$$\mathcal{X}_\eta^{\mathrm{dg}} \to \mathcal{X}.$$

Proof. We have a distinguished triangle in $D(\mathcal{X}_\eta^{\mathrm{dg}} \otimes_k \mathcal{X}_\eta^{\mathrm{dg},\circ})$

$$\Sigma^{n-2}\mathcal{M} \to \mathcal{X}_\eta^{\mathrm{dg}} \to \mathcal{X} \to .$$

Applying $\mathrm{RHom}_{\mathcal{X}_\eta^{\mathrm{dg}}}(-, L(B))$ gives a distinguished triangle in $D(\mathcal{X}_\eta^{\mathrm{dg}})$

$$\mathrm{Hom}_{\mathcal{X}_\eta^{\mathrm{dg}}}(\mathcal{X}, L(B)) \to L(B) \to \mathrm{Hom}_{\mathcal{X}_\eta^{\mathrm{dg}}}(\Sigma^{n-2}\mathcal{M}, L(B)) \to . \tag{11.5}$$

and using (6.10) we get in $D(\mathcal{X})$

$$\begin{aligned} \mathrm{Hom}_{\mathcal{X}_\eta^{\mathrm{dg}}}(\mathcal{X}, L(B)) &\cong \mathcal{B} \\ \mathrm{Hom}_{\mathcal{X}_\eta^{\mathrm{dg}}}(\Sigma^{n-2}\mathcal{M}, L(B)) &\cong \mathrm{Hom}_{\mathcal{X}}(\Sigma^{n-2}\mathcal{M}, \mathcal{B}). \end{aligned}$$

By applying $D(\mathcal{X}) \to D(\mathcal{X}_\eta^{\mathrm{dg}})$ we see that these identities also hold in $D(\mathcal{X}_\eta^{\mathrm{dg}})$. So (11.5) becomes a distinguished triangle in $D(\mathcal{X}_\eta^{\mathrm{dg}})$:

$$\mathcal{B} \to L(B) \to \mathrm{Hom}_{\mathcal{X}}(\Sigma^{n-2}\mathcal{M}, \mathcal{B}) \to .$$

It now suffices to observe that $\mathcal{M}$ is invertible. $\square$

Corollary 11.4. *If $B \in D^b(\mathrm{coh}(X))$ then $H^*(L(B)) \in D^b_{w\,\mathrm{coh}(X)}(\mathcal{X}_\eta^{\mathrm{dg}})$.*

Proof. This follows from Lemma 11.3 and Lemma 9.4.1. $\square$

The functor we would now want to consider is the composition

$$\Psi : D^b(\mathrm{coh}(X)) \xrightarrow{L} D^b_{w\,\mathrm{coh}(X)}(\mathcal{X}_\eta^{\mathrm{dg}}) \xrightarrow{\psi_{\mathcal{X}_\eta,*}} D^b_{w\,\mathrm{coh}(X)}(\mathcal{X}_\eta) \xrightarrow{\tilde{f}_*} D^b_{w\,\mathrm{coh}(Y)}(\mathcal{Y}) \cong D^b(\mathrm{coh}(Y)) \tag{11.6}$$

($\psi_{\mathcal{X}_\eta}$ is defined in §D.1, the last isomorphism is from Lemma 9.3.1).

12. Proof of Theorem 1.4

We remind the readers of the statement of the theorem:

Theorem 12.1. *Let X be a smooth quadric in $Y = \mathbb{P}^4$ whose defining equation has maximal isotropy index*[5] *and let $f : X \to Y$ be the inclusion. Let $M = \omega_X^{\otimes 2}$ and let $0 \neq \eta \in \mathrm{HH}^6(X, \omega_X^{\otimes 2}) \cong k$. Then $f_*\eta \in \mathrm{HH}^6(Y, f_*(\omega_X^{\otimes 2}))$ is zero. The functor Ψ in* (1.8) *restricts to an exact functor*

$$\Psi : D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(Y))$$

which is not a Fourier-Mukai functor.

[5]This condition ensures that the matrix factorization in Theorem 12.2 is defined over any field. One may take $x_0^2 + x_1x_2 + x_3x_4 = 0$.

The fact that $\mathrm{HH}^6(X, M) = k$ is Lemma 10.6.1. The fact that $f_*(\eta) = 0$ follows from Proposition 10.6.3 using Proposition 10.3.1.

Let $\mathcal{O}_Y(1)$ be the tautological line bundle on $Y = \mathbb{P}^4$ and let $\mathcal{O}_X(1)$ be its restriction to X. Then we have $\omega_X = \mathcal{O}_X(-3)$. If, as usual, I is the defining ideal of X in Y, then $I \cong \mathcal{O}_Y(-2)$ and hence $I/I^2 = \mathcal{O}_X(-2)$. Recall the following:

Theorem 12.2. [21] *There is a full strong exceptional sequence on X given by*

$$\mathcal{O}_X(-2), \mathcal{O}_X(-1), \mathcal{O}_X, C \tag{12.1}$$

where C is associated to a matrix factorization of the defining equation of X in 4×4-matrices. In particular, it has a resolution on Y given by

$$0 \to \mathcal{O}_Y(-1)^4 \to \mathcal{O}_Y^4 \to C \to 0$$

If follows that all the objects occurring in the exceptional sequence (12.1) are arithmetically Cohen-Macaulay. Furthermore since odd dimensional quadrics have up to shift only a single non-trivial indecomposable Cohen-Macaulay module $\mathcal{C}$ (e.g. by Knörrer periodicity [24]), $\mathcal{C}$ must be its own syzygy up to shift. One finds that there is a short exact sequence on X

$$0 \to C(-1) \to \mathcal{O}_X^4 \to C \to 0. \tag{12.2}$$

Let T be the sum of the exceptional collection (12.1) and put $\Gamma = \mathrm{End}_X(T)$. Since Γ is given by a directed algebra with maximal compositions of length 3, we find

$$\operatorname{gl\,dim} \Gamma \leq 3. \tag{12.3}$$

For use below we record the following technical vanishing result. This lemma, (12.3) and the fact that T is a coherent sheaf are the only special properties of T that we will use.

Lemma 12.3. *One has*

$$\mathrm{Ext}^i_X(T, (I/I^2)^{-1} \otimes_X M \otimes_X T) = 0$$

for $i = 1, 2$.

Proof. We have $(I/I^2)^{-1} \otimes_X M = \mathcal{O}_X(-4)$. By Serre duality

$$\mathrm{Ext}^i_X(T, T(-4)) = \mathrm{Ext}^{3-i}_X(T, T(1))^*$$

Since T is arithmetically Cohen-Macaulay, we have

$$H^i(X, T(j)) = 0$$

for all j and $i = 1, 2$. Hence by Serre duality

$$\mathrm{Ext}^i_X(T, \mathcal{O}_X(j)) = 0$$

for all j and $i = 1, 2$.

So the only possible issue is the value of

$$\mathrm{Ext}^i_X(C, C(1))$$

for $i = 1, 2$. From (a shift by 1 of) the exact sequence (12.2) we get

$$\mathrm{Ext}^1_X(C, C(1)) = \mathrm{Ext}^2_X(C, C) = 0$$

since C is exceptional. Similarly

$$\mathrm{Ext}^2_X(C, C(1)) \hookrightarrow \mathrm{Ext}^3_X(C, C) = 0. \qquad \square$$

Consider first the functor (see (11.3) and Corollary 11.4)

$$L : D^b(\mathrm{coh}(X)) \to D^b_{w\,\mathrm{coh}(X)}(\mathcal{X}_\eta^{\mathrm{dg}})$$

Put $\mathcal{T} = wT \in \mathrm{Mod}(\mathcal{X})$ and $\tilde{\mathcal{T}} = L(T) \in D^b_{w\,\mathrm{coh}(X)}(\mathcal{X}_\eta^{\mathrm{dg}})$. Since Γ acts on T it also acts on $\tilde{\mathcal{T}}$ and hence $\tilde{\mathcal{T}} \in D(\mathcal{X}_\eta^{\mathrm{dg}})_\Gamma$. According to Lemma 8.3.1 and Corollary 8.3.2, there is a well defined obstruction

$$o_3(\tilde{\mathcal{T}}) \in \mathrm{HH}^3(\Gamma, \mathrm{Ext}^{-1}_{\mathcal{X}_\eta^{\mathrm{dg}}}(\tilde{\mathcal{T}}, \tilde{\mathcal{T}}))$$

which vanishes if $\tilde{\mathcal{T}}$ is in the essential image of

$$D(\mathcal{X}_\eta^{\mathrm{dg}} \otimes_k \Gamma) \to D(\mathcal{X}_\eta^{\mathrm{dg}})_\Gamma$$

Lemma 12.4. *One has*

$$o_3(\tilde{\mathcal{T}}) \neq 0$$

Proof. If $o_3(\tilde{\mathcal{T}})$ were to vanish, then the higher obstructions $o_{3+i}(\tilde{\mathcal{T}})$ (Lemma 8.3.1), which lie in $\mathrm{HH}^{3+i}(\Gamma, \mathrm{Ext}^{-1-i}_{\mathcal{X}_\eta^{\mathrm{dg}}}(\tilde{\mathcal{T}}, \tilde{\mathcal{T}}))$, would also vanish, since $\mathrm{gl}\dim\Gamma = 3$ and hence the Hochschild dimension of Γ is 3 as well. So $\tilde{\mathcal{T}}$ may be viewed as an object in $D(\mathcal{X}_\eta^{\mathrm{dg}} \otimes_k \Gamma) \cong D_\infty(\mathcal{X}_\eta \otimes_k \Gamma)$. By (11.4) we get a distinguished triangle in $D(\mathcal{X}_\eta^{\mathrm{dg}})$

$$\mathcal{T} \xrightarrow{\alpha} \tilde{\mathcal{T}} \xrightarrow{\beta} \Sigma^{-4}\mathcal{M}^{-1} \otimes_{\mathcal{X}} \mathcal{T} \to \tag{12.4}$$

and hence

$$H^*(\tilde{\mathcal{T}}) = \mathcal{T} \oplus \Sigma^{-4}(\mathcal{M}^{-1} \otimes_{\mathcal{X}} \mathcal{T}),$$

and moreover, by construction, this isomorphism is compatible with the $H^*(\mathcal{X}_\eta^{\mathrm{dg}}) = H^*(\mathcal{X}_\eta)$ and Γ-actions. In the terminology of §7.4, $\tilde{\mathcal{T}}$ is a colift of $\mathcal{T} \in D(\mathcal{X} \otimes_k \Gamma)$ to $D_\infty((\mathcal{X}_\eta \otimes_k \Gamma)_{\eta\cup 1})$ (see §7.2).

The obstruction against the existence of such a colift is the image of $\eta \cup 1$ under the characteristic map

$$\mathrm{HH}^6(\mathcal{X} \otimes_k \Gamma, \mathcal{M} \otimes_k \Gamma) \xrightarrow{c_{\mathcal{T}}} \mathrm{Ext}^6_{\mathcal{X}\otimes_k\Gamma}(\mathcal{T}, \mathcal{M} \otimes_{\mathcal{X}} \mathcal{T})$$

(see Lemma 7.4.1, Lemma 7.3.1 and §7.2). Let $c_{\mathcal{T},\Gamma}$ be the composition

$$\mathrm{HH}^6(\mathcal{X}, \mathcal{M}) \xrightarrow{\eta\mapsto\eta\cup 1} \mathrm{HH}^6(\mathcal{X} \otimes_k \Gamma, \mathcal{M} \otimes_k \Gamma) \xrightarrow{c_{\mathcal{T}}} \mathrm{Ext}^6_{\mathcal{X}\otimes_k\Gamma}(\mathcal{T}, \mathcal{M} \otimes_{\mathcal{X}} \mathcal{T})$$

By the Γ-equivariant version of (9.14) we have a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}^6(X, M) \arrow[r, "c_{T,\Gamma}"] \arrow[d, "W"', "\cong"] & \mathrm{Ext}^6_{\mathrm{Qch}(X)_\Gamma}(T, M \overset{L}{\otimes}_X T) \arrow[d, "\cong"', "w"] \\
\mathrm{HH}^6(\mathcal{X}, \mathcal{M}) \arrow[r, "c_{\mathcal{T},\Gamma}"'] & \mathrm{Ext}^6_{\mathcal{X}\otimes_k\Gamma}(\mathcal{T}, \mathcal{M} \overset{L}{\otimes}_{\mathcal{X}} \mathcal{T})
\end{tikzcd}$$

of Γ-equivariant characteristic maps. The rightmost map is an isomorphism by (9.12) and (9.13). The leftmost map is an isomorphism by (9.5). By Proposition 9.9.2, the upper horizontal map is also an isomorphism, finishing the proof that the colift does not exist (as $\eta \neq 0$). $\square$

Applying $\mathrm{RHom}_{\mathcal{X}_\eta^{\mathrm{dg}}}(-,\tilde{\mathcal{T}})$ to (12.4), and using (6.10), we get a distinguished triangle of complexes

$$\mathrm{RHom}_{\mathcal{X}}(\Sigma^{-4}\mathcal{M}^{-1}\otimes_{\mathcal{X}}\mathcal{T},\mathcal{T})\to \mathrm{RHom}_{\mathcal{X}_\eta^{\mathrm{dg}}}(\tilde{\mathcal{T}},\tilde{\mathcal{T}})\to \mathrm{RHom}_{\mathcal{X}}(\mathcal{T},\mathcal{T})\to$$

and hence

$$\mathrm{Ext}^3_{\mathcal{X}}(\mathcal{M}^{-1}\otimes_{\mathcal{X}}\mathcal{T},\mathcal{T})\cong \mathrm{Ext}^{-1}_{\mathcal{X}_\eta^{\mathrm{dg}}}(\tilde{\mathcal{T}},\tilde{\mathcal{T}}) \tag{12.5}$$

For use below, we note that this isomorphism sends a morphism $\phi:\mathcal{M}^{-1}\otimes_{\mathcal{X}}\mathcal{T}\to\mathcal{T}$ of degree three to the composition

$$\tilde{\mathcal{T}}\xrightarrow{\beta}\Sigma^{-4}\mathcal{M}^{-1}\otimes_{\mathcal{X}}\mathcal{T}\xrightarrow{\phi}\Sigma^{-1}\mathcal{T}\xrightarrow{\Sigma^{-1}\alpha}\Sigma^{-1}\tilde{\mathcal{T}}. \tag{12.6}$$

Lemma 12.5. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Ext}^3_{\mathcal{X}}(\mathcal{M}^{-1}\otimes_{\mathcal{X}}\mathcal{T},\mathcal{T}) \arrow[r, "\cong"] \arrow[d, "f_*"'] & \mathrm{Ext}^{-1}_{\mathcal{X}_\eta^{\mathrm{dg}}}(\tilde{\mathcal{T}},\tilde{\mathcal{T}}) \arrow[d, "\tilde{f}_*\circ\psi_{\mathcal{X}_\eta,*}"] \\
\mathrm{Ext}^3_{\mathcal{Y}}(f_*(\mathcal{M}^{-1}\otimes_{\mathcal{X}}\mathcal{T}),f_*(\mathcal{T})) \arrow[r] & \mathrm{Ext}^{-1}_{\mathcal{Y}}(\tilde{f}_*(\tilde{\mathcal{T}}),\tilde{f}_*(\tilde{\mathcal{T}}))
\end{tikzcd} \tag{12.7}$$

where $\tilde{f}$ is as in (11.1)*, the upper map is as in* (12.5) *and the lower map is defined in a similar way as* (12.6)*.*

Proof. This is a tautology starting from (12.6). □

Corollary 12.6. *The map $\tilde{f}_*\circ\psi_{\mathcal{X}_\eta,*}$ in* (12.7) *is an isomorphism.*

Proof. Since $\dim Y=4$ and T, $M^{-1}\otimes T$ are coherent sheaves on X, we have $\mathrm{Ext}^1_{\mathcal{Y}}(f_*(\Sigma^{-4}\mathcal{M}^{-1}\otimes_{\mathcal{X}}\mathcal{T}),f_*(\mathcal{T}))=\mathrm{Ext}^1_Y(f_*(\Sigma^{-4}M^{-1}\otimes_X T),f_*(T))=0$ (the first equality follows from (9.4) and Lemmas 9.4.1, 9.7.1) and hence

$$\tilde{f}_*(\tilde{\mathcal{T}})=f_*(\mathcal{T})\oplus f_*(\Sigma^{-4}\mathcal{M}^{-1}\otimes_{\mathcal{X}}\mathcal{T}),$$

from which it follows right away that the lower arrow on (12.7) is an isomorphism. So it suffices to show that the left arrow is an isomorphism. Again using (9.4) and Lemmas 9.4.1, 9.7.1, it is sufficient to prove that

$$\mathrm{Ext}^3_X(M^{-1}\otimes_X T,T)\xrightarrow{f_*}\mathrm{Ext}^3_Y(f_*(M^{-1}\otimes_X T),f_*(T))$$

is an isomorphism. We have a long exact sequence (see (10.13))

$$\begin{aligned}\mathrm{Ext}^1_X(T,(I/I^2)^{-1}\otimes_X M\otimes_X T))\to \mathrm{Ext}^3_X(T,M\otimes_X T)\xrightarrow{f_*}\\ \mathrm{Ext}^3_Y(f_*(T),f_*(M\otimes_X T))\to \mathrm{Ext}^2_X(T,(I/I^2)^{-1}\otimes_X M\otimes_X T)).\end{aligned}$$

Now by Lemma 12.3 we have

$$\mathrm{Ext}^1_X(T,(I/I^2)^{-1}\otimes_X M\otimes_X T)=\mathrm{Ext}^2_X(T,(I/I^2)^{-1}\otimes_X M\otimes_X T)=0,$$

so that we have

$$\mathrm{Ext}^3_X(M^{-1}\otimes_X T,T)\underset{\cong}{\xrightarrow{f_*}}\mathrm{Ext}^3_Y(f_*(M^{-1}\otimes_X T),f_*T)$$

and we are done. □

Proof of Theorem 1.4. We follow the strategy exhibited in the introduction. Since Ψ is a functor, we obviously have

$$\Psi(T) = (\tilde{f}_* \circ \psi_{\mathcal{X}_\eta,*})(\tilde{\mathcal{T}}) \in D(\mathcal{Y})_\Gamma.$$

If Ψ is Fourier-Mukai then $\Psi(T) \in D(\mathcal{Y} \otimes_k \Gamma)$. It now suffices to use Lemma 12.7 below to obtain a contradiction. □

Lemma 12.7. *The obstruction (see Lemma 8.3.1)*

$$o_3(\Psi(T)) \in \mathrm{HH}^3(\Gamma, \mathrm{Ext}^{-1}_{\mathcal{Y}}(\Psi(T), \Psi(T))$$

is not vanishing.

Proof. By the naturality of obstructions (see Lemma 8.3.1), we have

$$o_3(\Psi(T)) = (\tilde{f}_* \circ \psi_{\mathcal{X}_\eta,*})(o_3(\tilde{\mathcal{T}})).$$

By Corollary 12.6 we know that $\tilde{f}_* \circ \psi_{\mathcal{X}_\eta,*}$ is an isomorphism and by Lemma 12.4 we have $o_3(\tilde{\mathcal{T}}) \neq 0$. □

Done!

Appendix A. The virtual kernel cohomology

Theorem-Definition A.1. [44, Thm 1.1, proof] *(see also* [16]*) Let X, Y be smooth projective k-varieties and let $F : D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(Y))$ be an exact functor. Choose an ample line bundle $\mathcal{O}_X(1)$ on X, and let $\Gamma_*(X)$ be the corresponding homogeneous coordinate ring.*

Define

$$\widetilde{\mathcal{H}}^i = \bigoplus_j H^i(F(\mathcal{O}_X(j))).$$

Then $\widetilde{\mathcal{H}}^i$ is a noetherian $\Gamma_*(X) \otimes_k \mathcal{O}_Y$-module. Let $\mathcal{H}^i$ be the corresponding sheaf of $\mathcal{O}_{X\times_k Y}$-modules.

If F is a Fourier-Mukai functor with kernel $\mathcal{K} \in D^b(\mathrm{coh}(X \times_k Y))$, then $H^i(\mathcal{K}) \cong \mathcal{H}^i$.

We will refer to $(\tilde{H}^i)_i$ as the *virtual kernel cohomology* of F.

Proposition A.2. *Let $F = \Psi$ where Ψ is as in* (11.6)*. Then*

$$\mathcal{H}^* = i_*\mathcal{O}_X \oplus \Sigma^{-n+2} i_* M^{-1}. \tag{A.1}$$

where $i : X \to X \times_k Y$ is given by $i(x) = (x, f(x))$.

Proof. According to Lemma 11.3 and Lemma 9.4.1 we find

$$H^*(L(\mathcal{O}_X(j))) = w(\mathcal{O}_X(j) \oplus \Sigma^{-n+2}(M^{-1} \otimes_X \mathcal{O}_X(j)))$$

and since the last 3 morphisms in (11.6) are essentially the pushforward by f on the level of cohomology:

$$\tilde{\mathcal{H}}^* = f_*\mathcal{O}_X(j) \oplus \Sigma^{-n+2} f_*(M^{-1} \otimes_X \mathcal{O}_X(j)) \tag{A.2}$$

From (A.2) we easily deduce (A.1). □

Appendix B. (Non-)existence of topological lifts

B.1. **Spectral categories.** Let Sp be the symmetric monoidal category of symmetric spectra [48]. It is useful to think of Sp as an "absolute" analogue of the category of unbounded complexes of abelian groups. A *spectral category/functor* is a category/functor enriched in Sp. These are absolute analogues of DG-categories and functors.

Given a spectral category $\mathcal{A}$, one can form a category $\pi_0\mathcal{A}$ by keeping the same set of objects and putting $(\pi_0\mathcal{A})(x,y) = \pi_0(\mathcal{A}(x,y))$. A spectral functor $F:\mathcal{A}\to\mathcal{B}$ is a *quasi-equivalence* if $\pi_0 F:\pi_0\mathcal{A}\to\pi_0\mathcal{B}$ is an equivalence and moreover F induces isomorphisms $\pi_*(\mathcal{A}(x,y))\to\pi_*(\mathcal{B}(Fx,Fy))$. Let $\mathrm{Cat}_{\mathrm{Sp}}$ be the category of spectral categories/functors and let $\mathrm{Ho}(\mathrm{Cat}_{\mathrm{Sp}})$ be the corresponding homotopy category obtained by inverting quasi-equivalences.

If R is a commutative ring then the Eilenberg Maclane spectrum HR is a commutative monoid in Sp [48, Example 5.25]. Let $\mathrm{Cat}_{\mathrm{Sp}}(HR)$ be the category of HR-linear spectral categories and let $\mathrm{Ho}(\mathrm{Cat}_{\mathrm{Sp}}(HR))$ be the corresponding homotopy category. We use similar notations for DG-categories: $\mathrm{dgcat}(R)$ and $\mathrm{Ho}(\mathrm{dgcat}(R))$. Let $U:\mathrm{Ho}(\mathrm{Cat}_{\mathrm{Sp}}(HR))\to\mathrm{Ho}(\mathrm{Cat}_{\mathrm{Sp}})$ be the forgetful functor.

Lemma B.1.1. *The forgetful functor $U:\mathrm{Ho}(\mathrm{Cat}_{\mathrm{Sp}}(H\mathbb{Q}))\to\mathrm{Ho}(\mathrm{Cat}_{\mathrm{Sp}})$ is fully faithful.*

Proof. The functor U has a left adjoint L given by smashing Hom-spaces with the Eilenberg-MacLane spectrum $H\mathbb{Q}$. Because $H\mathbb{Q}$ is a localization of the sphere spectrum [47, Theorem 7.11], LU is naturally isomorphic to the identity and hence U is fully faithful. □

In [51] (following [49]) Tabuada constructs an equivalence[6] $\mathbb{H}:\mathrm{Ho}(\mathrm{dgcat}(R))\to\mathrm{Ho}(\mathrm{Cat}_{\mathrm{Sp}}(HR))$. The construction of $\mathbb{H}$ is quite involved but it is not hard to verify that for a R-linear DG-category $\mathfrak{c}$ one has an R-linear equivalence between $\pi_0\mathbb{H}(\mathfrak{c})$ and $H^0(\mathfrak{c})$.

If $\mathfrak{c}$, $\mathfrak{d}$ are R-linear DG-categories and $\Psi:H^0(\mathfrak{c})\to H^0(\mathfrak{d})$ is an R-linear functor then a *spectral lift* of Ψ is a morphism $\widetilde{\Psi}:U\mathbb{H}(\mathfrak{c})\to U\mathbb{H}(\mathfrak{d})$ in $\mathrm{Ho}(\mathrm{Cat}_{\mathrm{Sp}})$ such that $\pi_0\widetilde{\Psi}=\Psi$.

The following lemma shows that if $R=\mathbb{Q}$ there is no difference between a spectral lift and a $\mathbb{Q}$-linear DG-lift.

Lemma B.1.2. *If $R=\mathbb{Q}$ then a spectral lift of $\Psi:H^0(\mathfrak{c})\to H^0(\mathfrak{d})$ exists if and only if there exists a morphisms $\overline{\Psi}:\mathfrak{c}\to\mathfrak{d}$ in $\mathrm{Ho}(\mathrm{dgcat}(\mathbb{Q}))$ such that $H^0(\overline{\Psi})=\Psi$.*

Proof. Let $\widetilde{\Psi}:U\mathbb{H}(\mathfrak{c})\to U\mathbb{H}(\mathfrak{d})$ in $\mathrm{Cat}_{\mathrm{Sp}}$ be a spectral lift of Ψ. Since U is fully faithful by Lemma B.1.1 and $\mathbb{H}$ is an equivalence we may put $\overline{\Psi}=(U\mathbb{H})^{-1}\widetilde{\Psi}$. □

B.2. **Our functor.** For a noetherian scheme X we let $D^b_{\mathrm{dg}}(\mathrm{coh}(X))$ be the standard DG-enhancement of $D^b(\mathrm{coh}(X))$ using injective resolutions.

Proposition B.2.1. *If $k=\mathbb{Q}$ then the functor Ψ defined in Theorem 1.4 does not have a spectral lift.*

---

[6] The result is stated for $R=\mathbb{Z}$ but it works for any R, see the comment in [49, §2.2]

Proof. If Ψ has a spectral lift then by Lemma B.1.2 it has a $\mathbb{Q}$-linear DG-lift. But then $\Psi_{\mathbb{Q}}$ has to be a Fourier-Mukai functor by [52, Thm 8.15] which contradicts Theorem 1.4 $\square$

Remark B.2.2. We conjecture that Proposition B.2.1 holds for any field.

Remark B.2.3. Spectral categories form a rigid model for the category of ∞-categories. See [6, Thm. 4.23] for a precise statement. From Proposition B.2.1 one may deduce that the functor Ψ does not lift to an exact ∞-functor in the sense of [34, §1.1.4].

B.3. **Vologodsky's functor.** Let us first remind the reader how Vologodsky's construction [54] works. Let Y be a smooth projective scheme over $\mathbb{Z}_p$ and let $X/\mathbb{F}_p$ be its special fiber. Let $i : X \to Y$ be the corresponding embedding and put $\Phi = Li^* \circ i_*$. For a carefully chosen Y, Vologodsky shows that Φ is not a Fourier-Mukai functor over $\mathbb{F}_p$.

Observation B.3.1. *The functor Φ has a $\mathbb{Z}$-linear DG-lift and hence a $H\mathbb{Z}$-linear spectral lift.*

Proof. Let $i^! : \mathrm{coh}(Y) \to \mathrm{coh}(X)$ be the right adjoint to $i_* : \mathrm{coh}(X) \to \mathrm{coh}(Y)$. It is easy to see that $Li^* \cong \Sigma \circ Ri^!$. Hence it is sufficient to construct a $\mathbb{Z}$-linear DG-lift for $\Phi' := Ri^! \circ i_*$. As Φ' is a composition of two right derived functors it suffices to invoke Lemma B.3.3 below. $\square$

We now recall some standard facts. If $\mathfrak{a}$, $\mathfrak{b}$ are DG-categories then a *co-quasi-functor* $M : \mathfrak{b} \to \mathfrak{a}$ is a $\mathfrak{a}-\mathfrak{b}$-bimodule (i.e. a DG-functor $\mathfrak{b}^\circ \otimes \mathfrak{a} \to C(\mathbf{Ab})$) such that for for every $B \in \mathfrak{b}$ there is an $M(B) \in \mathfrak{a}$ as well as a morphism of DG-functors $\mathfrak{a}(M(B), -) \to M(B, -)$ (by enriched Yoneda this is the same as an element of $\xi_B \in Z^0 M(B, A))$ such that for all $A' \in \mathfrak{a}$ the induced map $\mathfrak{a}(M(B), A') \to M(B, A')$ is a quasi-isomorphism. A co-quasi-functor M induces an honest functor $H^0(\mathfrak{b}) \to H^0(\mathfrak{a})$ by sending $B \mapsto M(B)$ and a corresponding construction for morphisms. We denote this functor by $H^0(M)$.

Lemma B.3.2. *If M is a co-quasi-functor then $H^0(M)$ has a DG-lift.*

Proof. Let $\mathfrak{c} = \mathfrak{a} \coprod_M \mathfrak{b}$ be the gluing of $\mathfrak{a}$ and $\mathfrak{b}$ along M as in [40, §3.2]. Let $\mathfrak{c}'$ be the full DG-subcategory of $\mathfrak{c}$ spanned by the objects $(M(B), B, \xi_B)$ for $B \in \mathfrak{b}$. From the fact that M is a co-quasi-functor one deduces that the projection functor $\mathrm{pr}_{\mathfrak{b}} : \mathfrak{c}' \to \mathfrak{b}$ is a quasi-isomorphism. Let $\overline{M} : \mathfrak{b} \to \mathfrak{a}$ be the morphism in $\mathrm{Ho}(\mathrm{dgcat}(\mathbb{Z}))$ given by the composition $\mathfrak{b} \xrightarrow{\mathrm{pr}_{\mathfrak{b}}^{-1}} \mathfrak{c}' \xrightarrow{\mathrm{pr}_{\mathfrak{a}}} \mathfrak{a}$. It is easy to see that $H^0(\overline{M}) \cong H^0(M)$. $\square$

Lemma B.3.3. *Let F be a left exact functor between Grothendieck categories $\mathcal{C} \to \mathcal{D}$. Equip $D^+(\mathcal{C})$, $D^+(\mathcal{D})$ with their standard enhancements $D^+_{\mathrm{dg}}(\mathcal{C})$, $D^+_{\mathrm{dg}}(\mathcal{D})$ given by injective resolutions. Then the right derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{D})$ of F has a DG-lift.*

Proof. For each $C \in \mathrm{Ob}(D^+(\mathcal{C}))$, $D \in \mathrm{Ob}(D^+(\mathcal{D}))$ fix injective resolutions I_C, I_D. Then we may define a co-quasi-fuctor $RF^{\mathrm{dg}} : D^+_{\mathrm{dg}}(\mathcal{C}) \to D^+_{\mathrm{dg}}(\mathcal{D})$ by putting $RF_{\mathrm{dg}}(C, D) = \underline{\mathrm{Hom}}_{C(\mathcal{D})}(FI_C, I_D)$. It is easy to see that $H^0(RF_{\mathrm{dg}}) = RF$. It now suffices to invoke Lemma B.3.2. $\square$

Appendix C. Proof of Theorem 1.2

Concerning (1), again the most general proof follows from Appendix E as follows. Let $B_1 = \tau_{\leq 0}B$ and let B_2 be the pullback of the diagram

$$\begin{tikzcd} & R \arrow[d] \\ B_1 \arrow[r] & H^0(B_1). \end{tikzcd}$$

Then $H^0(B_2) = R$ and $H^i(B_2) = 0$ for $i = -1, \ldots, -m$ and $i > 0$. Let $\operatorname{Proj} R$ the category of projective R-modules. We can then define a functor $L : R \to D(B_2)$ sending R to B_2, and then extend to

$$L : \operatorname{Proj} R \to D(B_2). \tag{C.1}$$

Moreover since B is concentrated in nonpositive degrees we have a functor H given by $- \overset{L}{\otimes}_{B_2} H^0(B_2) : D(B_2) \to D(H^0(B_2)) = D(R)$ which sends $L(R)$ to R and hence $L(P)$ to P for $P \in \operatorname{Proj} R$. In the same way as in Proposition 6.3.1, by considering the good couple $\mathcal{A} = \{\Sigma^n P | P \in L(\operatorname{Proj} R), n \leq m\}$ and $\mathcal{B} = \{\Sigma^n P | P \in L(\operatorname{Proj} R), n \geq 0\}$ we obtain a functor $L : D^b(R) \to D(B_2)$ extending (C.1). The result follows by composing with $- \overset{L}{\otimes}_{B_2} B$.

We now concentrate on (2). So we have an exact functor

$$L : D^b(R) \to D(B)$$

which sends R to B in a way that is compatible with the right R-action on both sides in $D(B)$. Assume now that L is of the form $U \overset{L}{\otimes}_R -$ for U an object in $D(B \otimes_k R^\circ)$. We assume that U represented by a cofibrant object in $\underline{\operatorname{Mod}}(B \otimes_k R^\circ)$ also denoted by U. Since by construction the isomorphism $U = L(R) \cong B$ is compatible with the right R-action on both sides in $D(B)$, it follows that B as an object in $D(B)_{R^\circ}$ lifts to an object in $D(B \otimes_k R^\circ)$. In other words, the obstructions $o_i(B)$ exhibited in §8.3 vanish, which by the proof of Lemma 8.3.1 is the same as saying that there is a B-linear right A_∞-R-action on B lifting the right R-action on $H^*(B)$, i.e. there is a A_∞-morphism $R^\circ \to \operatorname{End}_B(B) = B^\circ$, finishing the proof.

Appendix D. Proof of Proposition 1.3

D.1. **The unital DG-hull of a strictly unital A_∞-category.** In this section we temporarily drop our blanket convention that A_∞-notions are automatically strictly unital.

If $\mathfrak{a}$ is a strictly unital A_∞-category then there exists a universal strictly unital A_∞ morphism $\psi_\mathfrak{a} : \mathfrak{a} \to U^u(\mathfrak{a})$ to a DG-algebra [26, p127].

Concretely, $U^u(\mathfrak{a})$ is a suitable quotient of the non-unital DG-hull $\Omega\mathbb{B}\mathfrak{a}$ with identity morphisms adjoined. Taking the quotient is necessary to make the adjoined identity morphisms compatible with the ones in $\mathfrak{a}$. From this explicit description one shows easily that $\psi_\mathfrak{a}$ is a quasi-isomorphism and in particular one has equivalences of categories

$$D_\infty(\mathfrak{a}) \cong D_\infty(U^u(\mathfrak{a})) \cong D(U^u(\mathfrak{a}))$$

where on the right we have the usual derived category of a DG-algebra. The second equivalence is [26, Cor. 4.1.3.11].

Since $\Omega\mathbb{B}\mathfrak{a} = T^c(\Sigma^{-1}T(\Sigma\mathfrak{a}))$ (tensor (co)categories without (co)unit) we find that $U^u(\mathfrak{a})$ is concentrated in degrees ≤ 0.

D.2. **The proof.** By Theorem 1.2 we have to compute the obstruction against lifting the natural map $R \to H^*(R_\eta^{\mathrm{dg}})$. By construction, there is a A_∞-quasi-isomorphism $R_\eta \to R_\eta^{\mathrm{dg}}$. Hence, by Corollary 8.2.2, it is sufficient to compute the obstructions for lifting the natural map $f : R \to H^0(R_\eta)$. By Remark 8.2.5, the first possible non-vanishing obstruction is $o_n(f) = \eta$, and it is indeed non-vanishing since we have assumed $\eta \neq 0$. So lifting is not possible, finishing the proof.

Appendix E. (by Amnon Neeman)

E.1. Some basic facts about t–structures.

Lemma E.1. *Let $\mathcal{T}$ be a triangulated category with a t–structure, suppose we are given in $\mathcal{T}$ a morphism of triangles*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "0"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "0"] & X[1] \arrow[d, "0"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$

and assume $X \in \mathcal{T}^{\leq 0}$ and $Z' \in \mathcal{T}^{\geq 0}$. Then there exists $\theta : Z \longrightarrow X'$ with $g = u'\theta v$.

Proof. Because $gu = 0$ the map g must factor as hv for some $h : Z \longrightarrow Y'$. But then $0 = v'g = v'hv$, and $v'h$ must factor as kw for some $k : X[1] \longrightarrow Z'$. Since $X[1] \in \mathcal{T}^{\leq -1}$ and $Z' \in \mathcal{T}^{\geq 0}$ we conclude that the map k must vanish, hence $v'h = 0$. Therefore h must factor as $u'\theta$ for some $\theta : Z \longrightarrow X'$, and $g = hv = u'\theta v$. ☐

Lemma E.2. *As in Lemma E.1 let $\mathcal{T}$ be a triangulated category with a t–structure. Assume we are given two triangles*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$
$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$$

with $Y \in \mathcal{T}^{\leq 0}$ and $Z' \in \mathcal{T}^{\geq 0}$. If $\theta : Z \longrightarrow X'$ is a map such that $u'\theta v = 0$ then there exists a morphism $\sigma : X[1] \longrightarrow X'$ with $\theta = \sigma w$.

Proof. We are given that $u'\theta v = 0$, hence θv must factor as $\theta v = w'[-1]\rho$ for some $\rho : Y \longrightarrow Z'[-1]$. But $Y \in \mathcal{T}^{\leq 0}$ and $Z'[-1] \in \mathcal{T}^{\geq 1}$, hence ρ must vanish. Therefore so does $\theta v = w'[-1]\rho$, and we conclude that θ factors as $\theta = \sigma w$ for some $\sigma : X[1] \longrightarrow X'$. ☐

E.2. Main results.

Reminder E.3. We adopt the notation first introduced in Beĭlinson, Bernstein and Deligne [4, 1.3.9]. If $\mathcal{T}$ is a triangulated category and $\mathcal{X}, \mathcal{Z}$ are full subcategories, then the full subcategory $\mathcal{X} * \mathcal{Z}$ has for objects all the $y \in \mathcal{T}$ for which there exists a triangle $x \longrightarrow y \longrightarrow z$ with $x \in \mathcal{X}$ and $z \in \mathcal{Z}$.

Definition E.4. *Let $H : \mathcal{R} \longrightarrow \mathcal{T}$ be a triangulated functor between triagulated categories. The pair of full subcategories $(\mathcal{A} \subset \mathcal{R}, \mathcal{B} \subset \mathcal{R})$ is called a* good couple with respect to H *if*

(i) $\mathcal{A}[-1] \subset \mathcal{A}$ *and* $\mathcal{B}[1] \subset \mathcal{B}$.

(ii) *The map* $\mathcal{R}(a,b) \longrightarrow \mathcal{T}(Ha,Hb)$ *is an isomorphism if* $a \in \mathcal{A}$ *and* $b \in \mathcal{B}$*, and is surjective if* $a \in \mathcal{A}$ *and* $b \in \mathcal{B}[-1]$.

The good couple $(\mathcal{A},\mathcal{B})$ *is called* excellent *if, in addition to (i) and (ii) above, we have*

(iii) $\mathcal{A} * \mathcal{A} \subset \mathcal{A}$ *and* $\mathcal{B} * \mathcal{B} \subset \mathcal{B}$.

Remark E.5. We note the easy facts

(i) If $(\mathcal{A},\mathcal{B})$ is a good couple for H, and $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{B}' \subset \mathcal{B}$ are full subcategories satisfying $\mathcal{A}'[-1] \subset \mathcal{A}'$ and $\mathcal{B}'[1] \subset \mathcal{B}'$, then $(\mathcal{A}',\mathcal{B}')$ is also a good couple for H. In this situation we will say that the good couple $(\mathcal{A}',\mathcal{B}')$ is contained in the good couple $(\mathcal{A},\mathcal{B})$.

(ii) If $(\mathcal{A},\mathcal{B})$ is a good couple for H, then the restriction of H to $\mathcal{A} \cap \mathcal{B} \subset \mathcal{R}$ is fully faithful.

Lemma E.6. *If* $(\mathcal{A},\mathcal{B})$ *is a good couple with respect to* H *then so are the couples* $(\mathcal{A} * \mathcal{A},\mathcal{B})$ *and* $(\mathcal{A},\mathcal{B} * \mathcal{B})$.

Proof. It is enough to prove that $(\mathcal{A},\mathcal{B} * \mathcal{B})$ is a good couple, the statement about $(\mathcal{A} * \mathcal{A},\mathcal{B})$ is obtained by applying this first case to the functor $H^{\mathrm{op}} : \mathcal{R}^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}}$ and the good couple $(\mathcal{B}^{\mathrm{op}} \subset \mathcal{R}^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}} \subset \mathcal{R}^{\mathrm{op}})$.

The fact that $\mathcal{A}[-1] \subset \mathcal{A}$ is given, while $(\mathcal{B} * \mathcal{B})[1] = \mathcal{B}[1] * \mathcal{B}[1] \subset \mathcal{B} * \mathcal{B}$ is obvious. Suppose $\widetilde{b} \in (\mathcal{B} * \mathcal{B})[-1]$; then there exists a triangle in $b \longrightarrow \widetilde{b} \longrightarrow b' \longrightarrow$ with $b, b' \in \mathcal{B}[-1]$. Let $a \in \mathcal{A}$ be an object, then we have a commutative diagram where the rows are exact

$$\begin{tikzcd}
\mathcal{R}(a,b) \arrow[r] \arrow[d, "\beta"] & \mathcal{R}(a,\widetilde{b}) \arrow[r] \arrow[d, "\gamma"] & \mathcal{R}(a,b') \arrow[r] \arrow[d, "\delta"] & \mathcal{R}(a,b[1]) \arrow[d, "\varepsilon"] \\
\mathcal{T}(Ha,Hb) \arrow[r] & \mathcal{T}(Ha,H\widetilde{b}) \arrow[r] & \mathcal{T}(Ha,Hb') \arrow[r] & \mathcal{T}(Ha,Hb[1])
\end{tikzcd}$$

Since $b[1]$ belongs to $\mathcal{B}$ the map ε is an isomorphism, while β and δ are surjective. Hence γ is surjective.

Now suppose $\widetilde{b} \in \mathcal{B} * \mathcal{B}$. Then there exists a triangle $b \longrightarrow \widetilde{b} \longrightarrow b' \longrightarrow$ with $b, b' \in \mathcal{B}$. Let $a \in \mathcal{A}$ be an object, then we have a commutative diagram where the rows are exact

$$\begin{tikzcd}
\mathcal{R}(a,b'[-1]) \arrow[r] \arrow[d, "\alpha"] & \mathcal{R}(a,b) \arrow[r] \arrow[d, "\beta"] & \mathcal{R}(a,\widetilde{b}) \arrow[r] \arrow[d, "\gamma"] & \mathcal{R}(a,b') \arrow[r] \arrow[d, "\delta"] & \mathcal{R}(a,b[1]) \arrow[d, "\varepsilon"] \\
\mathcal{T}(Ha,Hb'[-1]) \arrow[r] & \mathcal{T}(Ha,Hb) \arrow[r] & \mathcal{T}(Ha,H\widetilde{b}) \arrow[r] & \mathcal{T}(Ha,Hb') \arrow[r] & \mathcal{T}(Ha,Hb[1])
\end{tikzcd}$$

We know that $b, b' \in \mathcal{B}$, and as $\mathcal{B}[1] \subset \mathcal{B}$ it follows that also $b[1] \in \mathcal{B}$. Therefore β, δ and ε are isomorphisms. Since $b[-1] \in \mathcal{B}[-1]$ the map α is surjective. The fine 5-lemma now tells us that γ is an isomorphism. $\square$

The following is now immediate

Corollary E.7. *Every good couple* $(\mathcal{A},\mathcal{B})$ *is contained in an excellent couple. In fact: the smallest excellent couple containing* $(\mathcal{A},\mathcal{B})$ *is the pair* $(\mathcal{A}^*,\mathcal{B}^*)$*, where* $\mathcal{X}^*$

is defined to be the union

$$\mathcal{X}^* = \bigcup_{n=1}^{\infty} \underbrace{\mathcal{X} * \mathcal{X} * \cdots * \mathcal{X}}_{n \text{ times}}$$

Lemma E.8. *Suppose $(\mathcal{A}, \mathcal{B})$ is an excellent couple for H. Then the category $\mathcal{C} = H(\mathcal{A} \cap \mathcal{B})$, the essential image of $\mathcal{A} \cap \mathcal{B}$ under H, satisfies $\mathcal{C} * \mathcal{C} \subset \mathcal{C}$.*

Proof. Let c be an object in $\mathcal{C} * \mathcal{C}$. Then there exists in $\mathcal{T}$ a triangle $H(b[-1]) \longrightarrow c \longrightarrow H(a) \xrightarrow{g} H(b)$ with $a, b[-1]$ both objects in $\mathcal{A} \cap \mathcal{B}$. In particular a belongs to $\mathcal{A}$ and $b[-1]$ belongs to $\mathcal{B}$, but as $\mathcal{B}[1] \subset \mathcal{B}$ we have that $b \in \mathcal{B}$. Therefore the map $\mathcal{R}(a, b) \longrightarrow \mathcal{T}(Ha, Hb)$ is an isomorphism, and hence there exists a (unique) morphism $f : a \longrightarrow b$ in $\mathcal{R}$ with $H(f) = g$. Form in $\mathcal{R}$ the triangle $b[-1] \longrightarrow \widetilde{c} \longrightarrow a \xrightarrow{f} b$. Then $\widetilde{c}$ belongs to $(\mathcal{A} \cap \mathcal{B}) * (\mathcal{A} \cap \mathcal{B}) \subset \mathcal{A} \cap \mathcal{B}$, and the functor H takes the triangle above to $Hb[-1] \longrightarrow H\widetilde{c} \longrightarrow Ha \xrightarrow{g} Hb$. Hence $c \cong H\widetilde{c}$ with $\widetilde{c} \in \mathcal{A} \cap \mathcal{B}$. □

Corollary E.9. *Suppose we are given a good couple $(\mathcal{A}, \mathcal{B})$ and let $\mathcal{C}$ be the essential image of $\mathcal{A} \cap \mathcal{B}$ under the functor H. Assume we are also given a subcategory $\mathcal{D} \subset \mathcal{T}$, and suppose further that every object in $\mathcal{D}$ lies in $\mathcal{C}^*$, where the notation $\mathcal{C}^*$ is as in Corollary E.7.*

Then there is an excellent couple $(\mathcal{A}', \mathcal{B}')$, containing $(\mathcal{A}, \mathcal{B})$, and such that $\mathcal{D}$ lies in the essential image under H of $\mathcal{A}' \cap \mathcal{B}'$.

Proof. From Corollary E.7 it follows that the good couple $(\mathcal{A}, \mathcal{B})$ may be included in an excellent couple $(\mathcal{A}', \mathcal{B}')$. Let $\mathcal{C}' \subset \mathcal{T}$ be the essential image of $\mathcal{A}' \cap \mathcal{B}'$ under H; then $\mathcal{C}$ is clearly contained in $\mathcal{C}'$, Lemma E.8 informs us that $\mathcal{C}' * \mathcal{C}' \subset \mathcal{C}'$, and hence $\mathcal{C}'$ contains $\mathcal{C}^*$ which contains $\mathcal{D}$. □

Now for the main result.

Theorem E.10. *Let $H : \mathcal{R} \longrightarrow \mathcal{T}$ be a triangulated functor. Assume the category $\mathcal{R}$ satisfies the axioms of the article* [36]. *Suppose further that $\mathcal{T}$ has a non-degenerate t–structure with heart $\mathcal{T}^\heartsuit$, let $\mathcal{H} : \mathcal{T} \longrightarrow \mathcal{T}^\heartsuit$ be the standard homological functor from $\mathcal{T}$ to the heart, and let $\mathcal{A} \subset \mathcal{T}^\heartsuit$ be a full, abelian subcategory closed under extensions. Assume $(\mathcal{A}', \mathcal{B}')$ is an excellent couple such that $\mathcal{A}$ is contained in the essential image of $\mathcal{A}' \cap \mathcal{B}'$.*

Then there exists a triangulated functor $G : \mathcal{T}^b_{\mathcal{A}} \longrightarrow \mathcal{R}$, where $\mathcal{T}^b_{\mathcal{A}} \subset \mathcal{T}$ is the full subcategory defined by

$$\mathcal{T}^b_{\mathcal{A}} \quad = \quad \left\{ t \in \mathcal{T} \;\middle|\; \begin{array}{c} \mathcal{H}^i(t) = 0 \textit{ for all but finitely many } i \in \mathbb{Z} \\ \mathcal{H}^i(t) \in \mathcal{A} \textit{ for every } i \in \mathbb{Z} \end{array} \right\}$$

and such that the composite $HG : \mathcal{T}^b_{\mathcal{A}} \longrightarrow \mathcal{T}$ is naturally isomorphic to the inclusion.

More precisely: if we let $\mathcal{T}^{[m,n]}_{\mathcal{A}} = \mathcal{T}^{\leq n} \cap \mathcal{T}^{\geq m} \cap \mathcal{T}^b_{\mathcal{A}}$, our construction will be such that $G(\mathcal{T}^{[m,n]}_{\mathcal{A}}) \subset \mathcal{A}'[-m] \cap \mathcal{B}'[-n]$.

Proof. On the category $\mathcal{A} = \mathcal{T}^{[0,0]}_{\mathcal{A}}$ we have little choice: we are looking for an additive functor $G : \mathcal{A} \longrightarrow \mathcal{A}' \cap \mathcal{B}'$ so that the composite $\mathcal{A} \xrightarrow{G} \mathcal{A}' \cap \mathcal{B}' \hookrightarrow \mathcal{R} \xrightarrow{H} \mathcal{T}$ is isomorphic to the inclusion. But H is fully faithful on $\mathcal{A}' \cap \mathcal{B}'$ by Remark E.5(ii), and the essential image $H(\mathcal{A}' \cap \mathcal{B}')$ contains $\mathcal{A}$ by hypothesis. To define G on $\mathcal{A}$ we just choose a quasi-inverse, and let $\varphi : I \longrightarrow HG$ be the natural isomorphism of HG with the inclusion functor. Since we want G and φ to be compatible with the shift this determines G and φ on $\mathcal{T}_{\mathcal{A}}^{[m,m]} = \mathcal{A}[-m]$ for every integer m.

The strategy will be to prove, by induction on $n-m$, that the additive functor G and the natural isomorphism φ may be extended to $G : \mathcal{T}_{\mathcal{A}}^{[m,n]} \longrightarrow \mathcal{R}$, compatibly with the shift. We have proved the case $n - m = 0$, and it remains to extend from $[m, n-1]$ to $[m, n]$. By shifting we may assume $n = 0$, that is for $m \leq -1$ we extend from $[m, -1]$ to $[m, 0]$. It will be handy in the induction to note the following little fact.

(i) Suppose the additive functor $G : \mathcal{T}_{\mathcal{A}}^{[m,n]} \longrightarrow \mathcal{A}'[-m] \cap \mathcal{B}'[-n]$ has been defined, as has the natural isomorphism $\varphi : I \longrightarrow HG$. Then for X, Y objects of $\mathcal{T}_{\mathcal{A}}^{[m,n]}$ we have that any morphism $\beta : HG(X) \longrightarrow HG(Y)$ is equal to $HG(b)$ for some morphism $b : X \longrightarrow Y$. If, for some integer i with $m \leq i \leq n$, we have that X belongs to $\mathcal{T}_{\mathcal{A}}^{[i,n]}$ and Y belongs to $\mathcal{T}_{\mathcal{A}}^{[m,i]}$, then any morphism $\gamma : G(X) \longrightarrow G(Y)$ is equal to $G(g)$ for some $g : X \longrightarrow Y$.

Proof of (i). Because $\varphi : I \longrightarrow HG$ is a natural transformation we have, for any morphism $f : X \longrightarrow Y$ in $\mathcal{T}_{\mathcal{A}}^{[m,n]}$, the commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_X"'] & Y \arrow[d, "\varphi_Y"] \\ HG(X) \arrow[r, "HG(f)"] & HG(Y) \end{tikzcd}$$

Applying this to the morphism $f = \varphi_X : X \longrightarrow HG(X)$ we deduce the commutativity of

$$\begin{tikzcd} X \arrow[r, "\varphi_X"] \arrow[d, "\varphi_X"'] & HG(X) \arrow[d, "\varphi_{HG(X)}"] \\ HG(X) \arrow[r, "HG(\varphi_X)"] & HGHG(X) \end{tikzcd}$$

In other words the two composites

$$\begin{tikzcd} X \arrow[r, "\varphi_X"] & HG(X) \arrow[r, shift left, "\varphi_{HG(X)}"] \arrow[r, shift right, "HG(\varphi_X)"'] & HGHG(X) \end{tikzcd}$$

are equal. Since $\varphi_X : X \longrightarrow HG(X)$ is an isomorphism we deduce that $\varphi_{HG(X)} = HG(\varphi_X)$ are equal maps $HG(X) \longrightarrow HGHG(X)$.

In view of the above the commutative square

$$\begin{tikzcd} HG(X) \arrow[r, "\beta"] \arrow[d, "\varphi_{HG(X)}"'] & HG(Y) \arrow[d, "\varphi_{HG(Y)}"] \\ HGHG(X) \arrow[r, "HG(\beta)"] & HGHG(Y) \end{tikzcd}$$

may be rewritten as

$$\begin{tikzcd} HG(X) \arrow[r, "\beta"] \arrow[d, "HG(\varphi_X)"'] & HG(Y) \arrow[d, "HG(\varphi_Y)"] \\ HGHG(X) \arrow[r, "HG(\beta)"] & HGHG(Y) \end{tikzcd}$$

in other words if $b = \varphi_Y^{-1}\beta\varphi_X$ then $\beta = HG(b)$.

Now suppose we are given a map $\gamma : G(X) \longrightarrow G(Y)$. Applying the previous assertion to $H(\gamma) : HG(X) \longrightarrow HG(Y)$ we learn that there is a map $g : X \longrightarrow Y$ with $H(\gamma) = HG(g)$. Hence H takes the map $\gamma - G(g)$ to zero. But $\gamma - G(g)$ is a morphism $G(X) \longrightarrow G(Y)$, and as $X \in \mathcal{T}_{\mathcal{A}}^{[i,n]}$ we have $G(X) \in \mathcal{A}'[-i]$ while $Y \in \mathcal{T}_{\mathcal{A}}^{[m,i]}$ implies that $G(Y) \in \mathcal{B}'[-i]$. The fact that H annihilates $\gamma - G(g)$ therefore means $\gamma - G(g) = 0$. □

The preparatory result being proved, it's time to extend G and φ from $\mathcal{T}_{\mathcal{A}}^{[m,-1]}$ to $\mathcal{T}_{\mathcal{A}}^{[m,0]}$. Let us begin with objects: assume Y is an object in $\mathcal{T}_{\mathcal{A}}^{[m,0]}$. The t–structure gives us a triangle $Y^{\leq -1} \xrightarrow{u} Y \xrightarrow{v} Y^{\geq 0} \xrightarrow{w} Y^{\leq -1}[1]$ in $\mathcal{T}_{\mathcal{A}}^{b}$, with $Y^{\leq -1} \in \mathcal{T}_{\mathcal{A}}^{[m,-1]}$ and $Y^{\geq 0} \in \mathcal{T}_{\mathcal{A}}^{[0,0]}$. By induction we have already defined $G(Y^{\leq -1}) \in \mathcal{A}'[-m] \cap \mathcal{B}'[1]$ and $G(Y^{\geq 0}) \in \mathcal{A}' \cap \mathcal{B}'$. The triangle gives a map $w : Y^{\geq 0} \longrightarrow Y^{\leq -1}[1]$, and induction gives isomorphisms $\varphi_{Y^{\geq 0}} : Y^{\geq 0} \cong HG(Y^{\geq 0})$ and $\varphi_{Y^{\leq -1}} : Y^{\leq -1} \cong HG(Y^{\leq -1})$. This permits us to form the commutative square

$$\begin{tikzcd} Y^{\geq 0} \arrow[r, "w"] \arrow[d, "\varphi_{Y^{\geq 0}}"'] & Y^{\leq -1}[1] \arrow[d, "\varphi_{Y^{\leq -1}}[1]"] \\ HG(Y^{\geq 0}) \arrow[r, "\widehat{w}"] & HG(Y^{\leq -1})[1] \end{tikzcd}$$

In other words we define $\widehat{w}$ to be the composite making the square commute. On the other hand $G(Y^{\geq 0}) \in \mathcal{A}'$ and $G(Y^{\leq -1})[1] \in \mathcal{B}'[2] \subset \mathcal{B}'$, and this implies that the map

$$\mathcal{R}\Big(G(Y^{\geq 0}), G(Y^{\leq -1})[1]\Big) \longrightarrow \mathcal{T}\Big(HG(Y^{\geq 0}), HG(Y^{\leq -1})[1]\Big)$$

is an isomorphism. There is a unique $\widetilde{w} : G(Y^{\geq 0}) \longrightarrow G(Y^{\leq -1})[1]$ with $H(\widetilde{w}) = \widehat{w}$.

Let $\mathcal{S}$ be the category of triangles in $\mathcal{R}$ in sense of [36, Axiom 3.4]. By [36, Axiom 3.4(GTR4) and (GTR6)] we may choose an object $S \in \mathcal{S}$ so that $F(S)$ is a candidate triangle where the third morphism is $\widetilde{w} : G(Y^{\geq 0}) \longrightarrow G(Y^{\leq -1})[1]$. Choose and fix such an $S = S(Y)$ for every object $Y \in \mathcal{T}^{[m,0]}$, and declare $F(S(Y))$ to be $G(Y^{\leq -1}) \xrightarrow{\widetilde{u}} G(Y) \xrightarrow{\widetilde{v}} G(Y^{\geq 0}) \xrightarrow{\widetilde{w}} G(Y^{\leq -1})[1]$. In other words we define $G(Y)$ to be the third edge of a triangle on $\widetilde{w}$; but for the sake of future definitions we keep track, in the enriched category of triangles, of the entire triangle $S(Y)$ defining $G(Y)$. Our first observation is that, since $G(Y^{\leq -1}) \in \mathcal{A}'[-m] \cap \mathcal{B}'[1] \subset \mathcal{A}'[-m] \cap \mathcal{B}'$ and $G(Y^{\geq 0}) \in \mathcal{A}' \cap \mathcal{B}' \subset \mathcal{A}'[-m] \cap \mathcal{B}'$, we have that $G(Y)$ belongs to $(\mathcal{A}'[-m] \cap \mathcal{B}') * (\mathcal{A}'[-m] \cap \mathcal{B}') \subset \mathcal{A}'[-m] \cap \mathcal{B}'$.

We are also given, in the category $\mathcal{T}^b_{\mathcal{A}}$, a commutative diagram where the rows are triangles

$$\begin{tikzcd} Y^{\le -1} \arrow[r, "u"] & Y \arrow[r, "v"] & Y^{\ge 0} \arrow[r, "w"] \arrow[d, "\varphi_{Y^{\ge 0}}"] & Y^{\le -1}[1] \arrow[d, "\varphi_{Y^{\le -1}}{[1]}"] \\ HG(Y^{\le -1}) \arrow[r, "H(\widetilde{u})"] & HG(Y) \arrow[r, "H(\widetilde{v})"] & HG(Y^{\ge 0}) \arrow[r, "H(\widetilde{w})"] & HG(Y^{\le -1})[1] \end{tikzcd}$$

which may be extended, in the category $\mathcal{T}^b_{\mathcal{A}}$, to a morphism of triangles

$$\begin{tikzcd} Y^{\le -1} \arrow[r, "u"] \arrow[d, "\varphi_{Y^{\le -1}}"] & Y \arrow[r, "v"] \arrow[d, "\varphi_Y"] & Y^{\ge 0} \arrow[r, "w"] \arrow[d, "\varphi_{Y^{\ge 0}}"] & Y^{\le -1}[1] \arrow[d, "\varphi_{Y^{\le -1}}{[1]}"] \\ HG(Y^{\le -1}) \arrow[r, "H(\widetilde{u})"] & HG(Y) \arrow[r, "H(\widetilde{v})"] & HG(Y^{\ge 0}) \arrow[r, "H(\widetilde{w})"] & HG(Y^{\le -1})[1] \end{tikzcd}$$

Fix such a φ_Y. Since $\varphi_{Y^{\le -1}}$ and $\varphi_{Y^{\ge 0}}$ are both isomorphisms so is φ_Y. For every object $Y \in \mathcal{T}^{[m,0]}$ we have defined the object $G(Y)$ and the isomorphism $\varphi_Y : Y \longrightarrow HG(Y)$. It remains to define the functor G on morphisms. As we will see below we are done making choices, the rest of the construction will be forced on us.

One note: If $Y \in \mathcal{T}^{[m,-1]}_{\mathcal{A}} \subset \mathcal{T}^{[m,0]}_{\mathcal{A}}$ then our choice of triangle $Y^{\le -1} \xrightarrow{u} Y \xrightarrow{v} Y^{\ge 0} \xrightarrow{w} Y^{\le -1}[1]$ in $\mathcal{T}^b_{\mathcal{A}}$ will be $Y \xrightarrow{\mathrm{id}} Y \longrightarrow 0 \longrightarrow Y[1]$, and $S(Y)$ will be the unique object in $\mathcal{S}$ with $F(S(Y))$ being $G(Y) \xrightarrow{\mathrm{id}} G(Y) \longrightarrow 0 \longrightarrow G(Y)[1]$. If $Y \in \mathcal{A} = \mathcal{T}^{[0,0]}_{\mathcal{A}} \subset \mathcal{T}^{[m,0]}_{\mathcal{A}}$ then our choice of triangle $Y^{\le -1} \xrightarrow{u} Y \xrightarrow{v} Y^{\ge 0} \xrightarrow{w} Y^{\le -1}[1]$ in $\mathcal{T}^b_{\mathcal{A}}$ will be $0 \longrightarrow Y \xrightarrow{\mathrm{id}} Y \longrightarrow 0$, and $S(Y)$ is the unique object in $\mathcal{S}$ with $F(S(Y))$ being $0 \longrightarrow G(Y) \xrightarrow{\mathrm{id}} G(Y) \longrightarrow 0$. For the sake of compatibility with earlier constructions, we also make sure that on the category $\mathcal{T}^{[m+1,0]}_{\mathcal{A}}$ our choices are the same as they were when we were dealing with extending from intervals of length $-m-2$ to intervals of length $-m-1$.

Suppose next that we are given in $\mathcal{T}^{[m,0]}_{\mathcal{A}}$ a morphism $g : Y \longrightarrow Z$. The construction above gives, in the category $\mathcal{S}$, the enriched triangles $S(Y)$ and $S(Z)$, with $F(S(Y))$ and $F(S(Z))$ being

$$G(Y^{\le -1}) \xrightarrow{\widetilde{u}} G(Y) \xrightarrow{\widetilde{v}} G(Y^{\ge 0}) \xrightarrow{\widetilde{w}} G(Y^{\le -1})[1]$$
$$G(Z^{\le -1}) \xrightarrow{\widetilde{u}'} G(Z) \xrightarrow{\widetilde{v}'} G(Z^{\ge 0}) \xrightarrow{\widetilde{w}'} G(Z^{\le -1})[1]$$

Induction gives us the vertical maps in the square below

$$\begin{tikzcd} G(Y^{\ge 0}) \arrow[r, "\widetilde{w}"] \arrow[d, "G(g^{\ge 0})"'] \arrow[dr, phantom, "\spadesuit"] & G(Y^{\le -1})[1] \arrow[d, "G(g^{\le -1})[1]"] \\ G(Z^{\ge 0}) \arrow[r, "\widetilde{w}'"] & G(Z^{\le -1})[1] \end{tikzcd}$$

and we would like to show that the square $\spadesuit$ commutes. But $G(Y^{\ge 0})$ belongs to $\mathcal{A}'$ while $G(Z^{\le -1})[1])$ belongs to $\mathcal{B}'[2] \subset \mathcal{B}'$, hence it suffices to show that the two composites become equal after applying the functor H. But the squares

$$\begin{tikzcd} Y^{\geq 0} \arrow[r, "\varphi_{Y^{\geq 0}}"] \arrow[d, "g^{\geq 0}"] & HG(Y^{\geq 0}) \arrow[d, "HG(g^{\geq 0})"] \\ Z^{\geq 0} \arrow[r, "\varphi_{Z^{\geq 0}}"] & HG(Z^{\geq 0}) \end{tikzcd} \qquad \begin{tikzcd} Y^{\leq -1}[1] \arrow[r, "\varphi_{Y^{\leq -1}}[1]"] \arrow[d, "g^{\leq -1}[1]"] & HG(Y^{\leq -1})[1] \arrow[d, "HG(g^{\leq -1})[1]"] \\ Z^{\leq -1}[1] \arrow[r, "\varphi_{Z^{\leq -1}}[1]"] & HG(Z^{\leq -1})[1] \end{tikzcd}$$

commute by induction, more precisely by the naturality of φ on objects of length $< -m$. And the squares

$$\begin{tikzcd} Y^{\geq 0} \arrow[r, "w"] \arrow[d, "\varphi_{Y^{\geq 0}}"] & Y^{\leq -1}[1] \arrow[d, "\varphi_{Y^{\leq -1}}[1]"] \\ HG(Y^{\geq 0}) \arrow[r, "H(\widetilde{w})"] & HG(Y^{\leq -1})[1] \end{tikzcd} \qquad \begin{tikzcd} Z^{\geq 0} \arrow[r, "w'"] \arrow[d, "\varphi_{Z^{\geq 0}}"] & Z^{\leq -1}[1] \arrow[d, "\varphi_{Z^{\leq -1}}[1]"] \\ HG(Z^{\geq 0}) \arrow[r, "H(\widetilde{w}')"] & HG(Z^{\leq -1})[1] \end{tikzcd}$$

commute by the construction of the maps $\widetilde{w} : G(Y^{\geq 0}) \longrightarrow G(Y^{\leq -1})[1]$ and $\widetilde{w}' : G(Z^{\geq 0}) \longrightarrow G(Z^{\leq -1})[1]$ above. We deduce that $H(\spadesuit)$ is isomorphic to the obviously commutative square

$$\begin{tikzcd} Y^{\geq 0} \arrow[r, "w"] \arrow[d, "g^{\geq 0}"'] & Y^{\leq -1}[1] \arrow[d, "g^{\leq -1}[1]"] \\ Z^{\geq 0} \arrow[r, "w'"] & Z^{\leq -1}[1] \end{tikzcd}$$

Hence the square $\spadesuit$ does commute. Next we will prove

(ii) There is a unique morphism $\widetilde{k} : S(Y) \longrightarrow S(Z)$, in the category $\mathcal{S}$, so that $F(\widetilde{k})$ is a morphism of candidate triangles in $\mathcal{R}$

$$\begin{tikzcd} G(Y^{\leq -1}) \arrow[r, "\widetilde{u}"] \arrow[d, "G(g^{\leq -1})"] & G(Y) \arrow[r, "\widetilde{v}"] \arrow[d, "k"] & G(Y^{\geq 0}) \arrow[r, "\widetilde{w}"] \arrow[d, "G(g^{\geq 0})"'] \arrow[dr, phantom, "\spadesuit"] & G(Y^{\leq -1})[1] \arrow[d, "G(g^{\leq -1})[1]"] \\ G(Z^{\leq -1}) \arrow[r, "\widetilde{u}'"] & G(Z) \arrow[r, "\widetilde{v}'"] & G(Z^{\geq 0}) \arrow[r, "\widetilde{w}'"] & G(Z^{\leq -1})[1] \end{tikzcd}$$

and the square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\varphi_Y"'] & Z \arrow[d, "\varphi_Z"] \\ HG(Y) \arrow[r, "H(k)"] & HG(Z) \end{tikzcd}$$

commutes in $\mathcal{T}^b_{\mathcal{A}}$.

Proof of (ii). We begin with the proof of existence. From [36, Axiom 3.4(GTR5) and (GTR6)] we may extend the commutative square $\spadesuit$ to a morphism of triangles. That is there exists in the category $\mathcal{S}$ a morphism $\widetilde{h} : S(Y) \longrightarrow SZ)$ so that $F(\widetilde{h})$

is a map

$$\begin{tikzcd}
G(Y^{\leq -1}) \arrow[r, "\widetilde{u}"] \arrow[d, "G(g^{\leq -1})"] & G(Y) \arrow[r, "\widetilde{v}"] \arrow[d, "h"] & G(Y^{\geq 0}) \arrow[r, "\widetilde{w}"] \arrow[d, "G(g^{\geq 0})"] & G(Y^{\leq -1})[1] \arrow[d, "G(g^{\leq -1})[1]"] \\
G(Z^{\leq -1}) \arrow[r, "\widetilde{u}'"] & G(Z) \arrow[r, "\widetilde{v}'"] & G(Z^{\geq 0}) \arrow[r, "\widetilde{w}'"] & G(Z^{\leq -1})[1]
\end{tikzcd}$$

Applying the functor H we obtain in $\mathcal{T}^b_{\mathcal{A}}$ the morphism of triangles

$$\begin{tikzcd}
HG(Y^{\leq -1}) \arrow[r, "H(\widetilde{u})"] \arrow[d, "HG(g^{\leq -1})"] & HG(Y) \arrow[r, "H(\widetilde{v})"] \arrow[d, "H(h)"] & HG(Y^{\geq 0}) \arrow[r, "H(\widetilde{w})"] \arrow[d, "HG(g^{\geq 0})"] & HG(Y^{\leq -1})[1] \arrow[d, "HG(g^{\leq -1})[1]"] \\
HG(Z^{\leq -1}) \arrow[r, "H(\widetilde{u}')"] & HG(Z) \arrow[r, "H(\widetilde{v}')"] & HG(Z^{\geq 0}) \arrow[r, "H(\widetilde{w}')"] & HG(Z^{\leq -1})[1]
\end{tikzcd}$$

There is no reason for this morphism of triangles to agree with the composite

$$\begin{tikzcd}
HG(Y^{\leq -1}) \arrow[r, "H(\widetilde{u})"] & HG(Y) \arrow[r, "H(\widetilde{v})"] & HG(Y^{\geq 0}) \arrow[r, "H(\widetilde{w})"] & HG(Y^{\leq -1})[1] \\
Y^{\leq -1} \arrow[r, "u"] \arrow[u, "\varphi_{Y^{\leq -1}}"] \arrow[d, "g^{\leq -1}"] & Y \arrow[r, "v"] \arrow[u, "\varphi_Y"] \arrow[d, "g"] & Y^{\geq 0} \arrow[r, "w"] \arrow[u, "\varphi_{Y^{\geq 0}}"] \arrow[d, "g^{\geq 0}"] & Y^{\leq -1}[1][1] \arrow[u, "\varphi_{Y^{\leq -1}}[1]"] \arrow[d, "g^{\leq -1}"] \\
Z^{\leq -1} \arrow[r, "u'"] \arrow[d, "\varphi_{Z^{\leq -1}}"] & Z \arrow[r, "v'"] \arrow[d, "\varphi_Z"] & Z^{\geq 0} \arrow[r, "w'"] \arrow[d, "\varphi_{Z^{\geq 0}}"] & Z^{\leq -1}[1] \arrow[d, "\varphi_{Z^{\leq -1}}"] \\
HG(Z^{\leq -1}) \arrow[r, "H(\widetilde{u}')"] & HG(Z) \arrow[r, "H(\widetilde{v}')"] & HG(Z^{\geq 0}) \arrow[r, "H(\widetilde{w}')"] & HG(Z^{\leq -1})[1]
\end{tikzcd}$$

but the difference is a morphism of triangles

$$\begin{tikzcd}
HG(Y^{\leq -1}) \arrow[r, "H(\widetilde{u})"] \arrow[d, "0"] & HG(Y) \arrow[r, "H(\widetilde{v})"] \arrow[d, "H(h)-\varphi_Z g \varphi_Y^{-1}"] & HG(Y^{\geq 0}) \arrow[r, "H(\widetilde{w})"] \arrow[d, "0"] & HG(Y^{\leq -1})[1] \arrow[d, "0"] \\
HG(Z^{\leq -1}) \arrow[r, "H(\widetilde{u}')"] & HG(Z) \arrow[r, "H(\widetilde{v}')"] & HG(Z^{\geq 0}) \arrow[r, "H(\widetilde{w}')"] & HG(Z^{\leq -1})[1]
\end{tikzcd}$$

Since $HG(Y^{\leq -1}) \cong Y^{\leq -1}$ belongs to $\mathcal{T}^{\leq -1}$ and $HG(Z^{\geq 0}) \cong Z^{\geq 0}$ belongs to $\mathcal{T}^{\geq 0}$ Lemma E.1 applies, and tells us that there exists a morphism $\theta : HG(Y^{\geq 0}) \longrightarrow HG(Z^{\leq -1})$ with $H(h) - \varphi_Z g \varphi_Y^{-1} = H(\widetilde{u}')\theta H(\widetilde{v})$. But now $G(Y^{\geq 0})$ belongs to $\mathcal{A}'$ and $G(Z^{\leq -1})$ belongs to $\mathcal{B}'[1] \subset \mathcal{B}'$, and hence the map $\theta : HG(Y^{\geq 0}) \longrightarrow HG(Z^{\leq -1})$ can be expressed (uniquely) as $H(\rho)$ for a morphism $\rho : G(Y^{\leq 0}) \longrightarrow G(Z^{\leq -1})$. By [36, Axiom 3.4(GTR2) and (GTR6)] the morphisms of triangles $\widetilde{h} : S(Y) \longrightarrow S(Z)$ in $\mathcal{S}$, whose images under the functor F are of the form

$$\begin{tikzcd}
G(Y^{\leq -1}) \arrow[r, "\widetilde{u}"] \arrow[d, "G(g^{\leq -1})"] & G(Y) \arrow[r, "\widetilde{v}"] \arrow[d, "h"] & G(Y^{\geq 0}) \arrow[r, "\widetilde{w}"] \arrow[d, "G(g^{\geq 0})"] & G(Y^{\leq -1})[1] \arrow[d, "G(g^{\leq -1})[1]"] \\
G(Z^{\leq -1}) \arrow[r, "\widetilde{u}'"] & G(Z) \arrow[r, "\widetilde{v}'"] & G(Z^{\geq 0}) \arrow[r, "\widetilde{w}'"] & G(Z^{\leq -1})[1]
\end{tikzcd}$$

are acted on transitively by the group $\widetilde{u}' \circ \mathrm{Hom}\big(G(Y^{\geq 0}), G(Z^{\leq -1})\big) \circ \widetilde{v}$. Hence we may choose a morphism of triangles $\widetilde{k} : S(Y) \longrightarrow S(Z)$ whose image under F is

$$\begin{tikzcd}
G(Y^{\leq -1}) \arrow[r, "\widetilde{u}"] \arrow[d, "G(g^{\leq -1})"] & G(Y) \arrow[r, "\widetilde{v}"] \arrow[d, "k=h-\widetilde{u}'\rho\widetilde{v}"] & G(Y^{\geq 0}) \arrow[r, "\widetilde{w}"] \arrow[d, "G(g^{\geq 0})"] & G(Y^{\leq -1})[1] \arrow[d, "G(g^{\leq -1})[1]"] \\
G(Z^{\leq -1}) \arrow[r, "\widetilde{u}'"] & G(Z) \arrow[r, "\widetilde{v}'"] & G(Z^{\geq 0}) \arrow[r, "\widetilde{w}'"] & G(Z^{\leq -1})[1]
\end{tikzcd}$$

But $H(k) = H(h - \widetilde{u}'\rho\widetilde{v}) = H(h) - H(\widetilde{u}')\theta H(\widetilde{v}) = \varphi_Z g \varphi_Y^{-1}$. Hence the diagram

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "\varphi_Y"'] & Z \arrow[d, "\varphi_Z"] \\
HG(Y) \arrow[r, "H(k)"] & HG(Z)
\end{tikzcd}$$

commutes, proving the existence.

Next we need to show the uniqueness. Suppose $\widetilde{h}, \widetilde{h}'$ are two morphisms $S(Y) \longrightarrow S(Z)$ as in (i): then F takes $\widetilde{h} - \widetilde{h}'$ to the morphism of triangles

$$\begin{tikzcd}
G(Y^{\leq -1}) \arrow[r, "\widetilde{u}"] \arrow[d, "0"] & G(Y) \arrow[r, "\widetilde{v}"] \arrow[d, "h-h'"] & G(Y^{\geq 0}) \arrow[r, "\widetilde{w}"] \arrow[d, "0"] & G(Y^{\leq -1})[1] \arrow[d, "0"] \\
G(Z^{\leq -1}) \arrow[r, "\widetilde{u}'"] & G(Z) \arrow[r, "\widetilde{v}'"] & G(Z^{\geq 0}) \arrow[r, "\widetilde{w}'"] & G(Z^{\leq -1})[1]
\end{tikzcd}$$

and [36, Axioms 3.4 (GRT2) and (GTR6)] guarantee that there exists a $\rho : G(Y^{\geq 0}) \longrightarrow G(Z^{\leq -1})$ with $h - h' = \widetilde{u}'\rho\widetilde{v}$. Since $H(h)$ and $H(h')$ are both equal to $\varphi_Z g \varphi_Y^{-1}$ we have $0 = H(h - h') = H(\widetilde{u}')H(\rho)H(\widetilde{v})$. But now the triangles in $\mathcal{T}$

$$HG(Y^{\leq -1}) \xrightarrow{H(\widetilde{u})} HG(Y) \xrightarrow{H(\widetilde{v})} HG(Y^{\geq 0}) \xrightarrow{H(\widetilde{w})} HG(Y^{\leq -1})[1]$$
$$HG(Z^{\leq -1}) \xrightarrow{H(\widetilde{u}')} HG(Z) \xrightarrow{H(\widetilde{v}')} HG(Z^{\geq 0}) \xrightarrow{H(\widetilde{w}')} HG(Z^{\leq -1})[1]$$

are such that $HG(Y) \cong Y$ lies in $\mathcal{T}^{\leq 0}$ and $HG(Z^{\geq 0}) \cong Z^{\geq 0}$ belongs to $\mathcal{T}^{\geq 0}$. From Lemma E.2 we learn that $H(\rho)$ must factor as $\sigma H(\widetilde{w})$, for some $\sigma : HG(Y^{\leq -1})[1] \longrightarrow HG(Z^{\leq -1})$. Let $\gamma : Z^{\leq -2} \longrightarrow Z^{\leq -1}$ be the canonical t–structure map. It is a morphism in $\mathcal{T}_{\mathcal{A}}^{[m,-1]}$, on which G and the isomorphism $\varphi : I \longrightarrow HG$ are already defined—hence γ is isomorphic to $HG(\gamma) : HG(Z^{\leq -2}) \longrightarrow HG(Z^{\leq -1})$. In particular $HG(\gamma)$ identifies as the map from the t–structure truncation. Therefore $\sigma : HG(Y^{\leq -1})[1] \longrightarrow HG(Z^{\leq -1})$, which is a map from $HG(Y^{\leq -1})[1] \cong Y^{\leq -1}[1] \in \mathcal{T}^{\leq -2}$ to the object $HG(Z^{\leq -1})$, must factor (uniquely) as

$$HG(Y^{\leq -1})[1] \xrightarrow{\ \beta\ } HG(Z^{\leq -2}) \xrightarrow{HG(\gamma)} HG(Z^{\leq -1})$$

The objects $Y^{\leq -1}[1]$ and $Z^{\leq -2}$ both belong to $\mathcal{T}_{\mathcal{A}}^{[m-1,-2]}$, on which G and the isomorphism $I \longrightarrow HG$ are already defined. By (i) above there exists a morphism $b : Y^{\leq -1}[1] \longrightarrow Z^{\leq -2}$ with $\beta = HG(b)$. Define $\theta : G(Y^{\leq -1}[1]) \longrightarrow G(Z^{\leq -1})$ to be the composite

$$G(Y^{\leq -1})[1] \xrightarrow{G(b)} G(Z^{\leq -2}) \xrightarrow{G(\gamma)} G(Z^{\leq -1})$$

then $H(\theta) = \sigma$. Hence $\rho - \theta\widetilde{w}$ satisfies the identity $H(\rho - \theta\widetilde{w}) = H(\rho) - H(\theta)H(\widetilde{w}) = H(\rho) - \sigma H(\widetilde{w}) = 0$.

But $\rho - \theta\widetilde{w}$ is a morphism $G(Y^{\geq 0}) \longrightarrow G(Z^{\leq -1})$, with $G(Y^{\geq 0}) \in \mathcal{A}'$ and $G(Z^{\leq -1}) \in \mathcal{B}'[1] \subset \mathcal{B}'$. The fact that $H(\rho - \theta\widetilde{w}) = 0$ implies that $\rho - \theta\widetilde{w} = 0$. It follows that $h - h' = \widetilde{u}'\rho\widetilde{v} = \widetilde{u}'\theta\widetilde{w}\widetilde{v} = 0$, where the vanishing is because the composite $\widetilde{w}\widetilde{v}$ vanishes. $\square$

This completes the proof of (ii), and allows us to make the key definition

(iii) If $g : Y \longrightarrow Z$ is a morphism in $\mathcal{T}_{\mathcal{A}}^{[m,0]}$, let $\widetilde{g} : S(Y) \longrightarrow S(Z)$ be the unique morphism satisfying the conditions of (ii). We define $G(g) : G(Y) \longrightarrow G(Z)$ by letting $F(\widetilde{g})$ be the morphism

$$\begin{tikzcd}
G(Y^{\leq -1}) \arrow[r, "\widetilde{u}"] \arrow[d, "G(g^{\leq -1})"] & G(Y) \arrow[r, "\widetilde{v}"] \arrow[d, "G(g)"] & G(Y^{\geq 0}) \arrow[r, "\widetilde{w}"] \arrow[d, "G(g^{\geq 0})"] & G(Y^{\leq -1})[1] \arrow[d, "G(g^{\leq -1})[1]"] \\
G(Z^{\leq -1}) \arrow[r, "\widetilde{u}'"] & G(Z) \arrow[r, "\widetilde{v}'"] & G(Z^{\geq 0}) \arrow[r, "\widetilde{w}'"] & G(Z^{\leq -1})[1]
\end{tikzcd}$$

The reader will note (ii) guarantees the commutativity of the square

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "\varphi_Y"'] & Z \arrow[d, "\varphi_Z"] \\
HG(Y) \arrow[r, "HG(g)"] & HG(Z)
\end{tikzcd}$$

which is precisely what we need to prove the naturality of φ.

The fact that $G(\mathrm{id}) = \mathrm{id}$, $G(g + h) = G(g) + G(h)$ and $G(hg) = G(h)G(g)$ are all immediate from the uniqueness proved in (ii). We have extended the additive functor G and the natural transformation $\varphi : I \longrightarrow HG$ from $\mathcal{T}_{\mathcal{A}}^{[m,-1]}$ to $\mathcal{T}_{\mathcal{A}}^{[m,0]}$. It remains to show that the functor G, which by induction has been extended to all of $\mathcal{T}_{\mathcal{A}}^{b}$, is a triangulated functor. We need to prove that G takes triangles to triangles. The next little fact will help.

(iv) Suppose $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is a triangle in $\mathcal{T}_{\mathcal{A}}^{b}$. If we can exhibit in $\mathcal{R}$ some triangle of the form $G(X) \xrightarrow{G(u)} G(Y) \xrightarrow{G(v')} G(Z') \xrightarrow{G(w')} G(X)[1]$, then the sequence $G(X) \xrightarrow{G(u)} G(Y) \xrightarrow{G(v)} G(Z) \xrightarrow{G(w)} G(X)[1]$ is also a triangle in $\mathcal{R}$.

Proof of (iv). Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\varphi_X"] & Y \arrow[r, "v'"] \arrow[d, "\varphi_Y"] & Z' \arrow[r, "w'"] \arrow[d, "\varphi_Z"] & X[1] \arrow[d, "\varphi_X[1]"] \\
HG(X) \arrow[r, "HG(u)"] & HG(Y) \arrow[r, "HG(v')"] & HG(Z') \arrow[r, "HG(w')"] & HG(X)[1]
\end{tikzcd}$$

which commutes by the naturality of φ. The bottom row is obtained by applying the triangulated functor H to the triangle $G(X) \xrightarrow{G(u)} G(Y) \xrightarrow{G(v')} G(Z') \xrightarrow{G(w')} G(X)[1]$ in the category $\mathcal{R}$. Hence the bottom row is a triangle. Since φ is an isomorphism the top row is isomorphic to the bottom row, hence a triangle in $\mathcal{T}_{\mathcal{A}}^{b}$. But in the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v'"] \arrow[d, equal] & Z' \arrow[r, "w'"] & X[1] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1] \end{tikzcd}$$

the rows are triangles, hence we may extend to an isomorphism

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v'"] \arrow[d, equal] & Z' \arrow[r, "w'"] \arrow[d, "\psi"] & X[1] \arrow[d, equal] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1] \end{tikzcd}$$

Applying the functor G we have in $\mathcal{R}$ an isomorphism of the top and bottom rows

$$\begin{tikzcd} G(X) \arrow[r, "G(u)"] \arrow[d, equal] & G(Y) \arrow[r, "G(v')"] \arrow[d, equal] & G(Z') \arrow[r, "G(w)'"] \arrow[d, "G(\psi)"] & G(X)[1] \arrow[d, equal] \\ G(X) \arrow[r, "G(u)"] & G(Y) \arrow[r, "G(v)"] & G(Z) \arrow[r, "G(w)"] & G(X)[1] \end{tikzcd}$$

Since the top row is a triangle so is the bottom. $\square$

The preliminaries are now out of the way, we have to prove that G takes triangles to triangles. Let us begin with two easy special cases.

(v) Let Y be an object in $\mathcal{T}^{\leq 0} \cap \mathcal{T}^b_{\mathcal{A}}$ and consider the triangle $Y^{\leq -1} \xrightarrow{u} Y \xrightarrow{v} Y^{\geq 0} \xrightarrow{w} Y^{\leq -1}[1]$. Then G takes it to a triangle.

Proof of (v). Choose an $m \leq 0$ with $Y \in \mathcal{T}^{[m,0]}_{\mathcal{A}}$; then $Y^{\leq -1}$ and $Y^{\geq 0}$ also belong to $\mathcal{T}^{[m,0]}_{\mathcal{A}}$, hence we can work inside $\mathcal{T}^{[m,0]}_{\mathcal{A}}$ to compute what G does. Recalling the construction that allowed us to extend from $\mathcal{T}^{[m,-1]}_{\mathcal{A}}$ to $\mathcal{T}^{[m,0]}_{\mathcal{A}}$, the triangles $S(Y^{\leq -1})$, $S(Y)$ and $S(Y^{\geq 0})$ are such that $F(S(Y^{\leq -1}))$, $F(S(Y))$ and $F(S(Y^{\geq 0}))$ are the rows in the diagram below

$$\begin{tikzcd} G(Y^{\leq -1}) \arrow[r, "\mathrm{id}"] & G(Y^{\leq -1}) \arrow[r] & 0 \arrow[r] \arrow[d] & G(Y^{\leq -1})[1] \arrow[d, "\mathrm{id}"] \\ G(Y^{\leq -1}) \arrow[r, "\widetilde{u}"] & G(Y) \arrow[r, "\widetilde{v}"] & G(Y^{\geq 0}) \arrow[r, "\widetilde{w}"] \arrow[d, "\mathrm{id}"] & G(Y^{\leq -1})[1] \arrow[d] \\ 0 \arrow[r] & G(Y^{\geq 0}) \arrow[r, "\mathrm{id}"] & G(Y^{\geq 0}) \arrow[r] & 0 \end{tikzcd}$$

and there is only one way to extend to morphisms of triangles. Thus the complicated definition of (iii) specializes to

$$\begin{tikzcd}
G(Y^{\le -1}) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & G(Y^{\le -1}) \arrow[r] \arrow[d, "G(u)=\widetilde{u}"] & 0 \arrow[r] \arrow[d] & G(Y^{\le -1})[1] \arrow[d, "\mathrm{id}"] \\
G(Y^{\le -1}) \arrow[r, "\widetilde{u}"] \arrow[d] & G(Y) \arrow[r, "\widetilde{v}"] \arrow[d, "G(v)=\widetilde{v}"] & G(Y^{\ge 0}) \arrow[r, "\widetilde{w}"] \arrow[d, "\mathrm{id}"] & G(Y^{\le -1})[1] \arrow[d] \\
0 \arrow[r] & G(Y^{\ge 0}) \arrow[r, "\mathrm{id}"] & G(Y^{\ge 0}) \arrow[r] & 0
\end{tikzcd}$$

The morphisms $\widetilde{w}$ and $G(w)$ have the property that $H(\widetilde{w}) = \varphi_{Y^{\le -1}}[1] w \varphi^{-1}_{Y^{\ge 0}} = HG(w)$. Thus $\widetilde{w}, G(w)$ are morphisms from $G(Y^{\ge 0}) \in \mathcal{A}'$ to $G(Y^{\le -1})[1] \in \mathcal{B}'$, whose images under H agree—we conclude that $G(w) = \widetilde{w}$. Therefore G takes the triangle $Y^{\le -1} \xrightarrow{u} Y \xrightarrow{v} Y^{\ge 0} \xrightarrow{w} Y^{\le -1}[1]$ to $G(Y^{\le -1}) \xrightarrow{\widetilde{u}} G(Y) \xrightarrow{\widetilde{v}} G(Y^{\ge 0}) \xrightarrow{\widetilde{w}} G(Y^{\le -1})[1]$, which is a triangle by construction. $\square$

(vi) Suppose we are given in $\mathcal{T}^b_{\mathcal{A}}$ a triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ with $A, B, C \in \mathcal{A} = \mathcal{T}^{[0,0]}_{\mathcal{A}}$. Then G takes it to a triangle.

Proof of (vi). Complete $G(w) : G(C) \longrightarrow G(A)[1]$ to a triangle $G(A) \xrightarrow{u'} Y \xrightarrow{v'} G(C) \xrightarrow{G(w)} G(A)[1]$. Then Y belongs to $(\mathcal{A}' \cap \mathcal{B}') * (\mathcal{A}' \cap \mathcal{B}') \subset \mathcal{A}' \cap \mathcal{B}'$. Now consider the diagram

$$\begin{tikzcd}
HG(A) \arrow[r, "HG(u)"] & HG(B) \arrow[r, "HG(v)"] & HG(C) \arrow[r, "HG(w)"] \arrow[d, equal] & HG(A)[1] \arrow[d, equal] \\
HG(A) \arrow[r, "H(u')"] & H(Y) \arrow[r, "H(v')"] & HG(C) \arrow[r, "HG(w)"] & HG(A)[1]
\end{tikzcd}$$

The top row is isomorphic to $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$, hence a triangle in $\mathcal{T}^b_{\mathcal{A}}$. And the bottom row is obtained by applying the functor H to the triangle $G(A) \xrightarrow{u'} Y \xrightarrow{v'} G(C) \xrightarrow{G(w)} G(A)[1]$. Hence both rows are triangles and we may complete to a morphism of triangles

$$\begin{tikzcd}
HG(A) \arrow[r, "HG(u)"] \arrow[d, equal] & HG(B) \arrow[r, "HG(v)"] \arrow[d, "\psi"] & HG(C) \arrow[r, "HG(w)"] \arrow[d, equal] & HG(A)[1] \arrow[d, equal] \\
HG(A) \arrow[r, "H(u')"] & H(Y) \arrow[r, "H(v')"] & HG(C) \arrow[r, "HG(w)"] & HG(A)[1]
\end{tikzcd}$$

with ψ an isomorphism. Since the functor H is fully faithful on $\mathcal{A}' \cap \mathcal{B}'$ we learn first that the isomorphism $\psi : HG(B) \longrightarrow H(Y)$ must be $H(\rho)$ for some isomorphism $\rho : G(B) \longrightarrow Y$. But the diagram

$$\begin{tikzcd}
G(A) \arrow[r, "G(u)"] \arrow[d, equal] & G(B) \arrow[r, "G(v)"] \arrow[d, "\rho"] & G(C) \arrow[d, equal] \\
G(A) \arrow[r, "u'"] & Y \arrow[r, "v'"] & G(C)
\end{tikzcd}$$

is a diagram in $\mathcal{A}' \cap \mathcal{B}'$ whose image under H commutes, hence the diagram commutes. We conclude that $G(A) \xrightarrow{G(u)} G(B) \xrightarrow{G(v)} G(C) \xrightarrow{G(w)} G(A)[1]$ is isomorphic in $\mathcal{R}$ to the triangle $G(A) \xrightarrow{u'} Y \xrightarrow{v'} G(C) \xrightarrow{G(w)} G(A)[1]$, hence is a triangle. $\square$

(vii) Suppose we are given in $\mathcal{T}^b_{\mathcal{A}}$ a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ with $X, Y \in \mathcal{T}^{\leq 0} \cap \mathcal{T}^b_{\mathcal{A}}$ and $Z \in \mathcal{A} = \mathcal{T}^{[0,0]}_{\mathcal{A}}$. Then G takes it to a triangle.

Proof of (vii). In the category $\mathcal{T}^b_{\mathcal{A}}$ factor the map v canonically as $Y \xrightarrow{v'} Y^{\geq 0} \xrightarrow{\widetilde{v}} Z$. Complete to an octahedron

$$\begin{tikzcd}
 & X^{\leq -1} \arrow[r, "u^{\leq -1}"] \arrow[d, "u''"] & Y^{\leq -1} \arrow[d, "u'"] & \\
Z[-1] \arrow[r, "-w[-1]"] \arrow[d, equal] & X \arrow[r, "u"] \arrow[d, "v''"] & Y \arrow[r, "v"] \arrow[d, "v'"] & Z \arrow[d, equal] \\
Z[-1] \arrow[r, "-\widetilde{w}[-1]"] & X^{\geq 0} \arrow[r, "u^{\geq 0}"] \arrow[d, "w''"] & Y^{\geq 0} \arrow[r, "\widetilde{v}"] \arrow[d, "w'"] & Z \\
 & X^{\leq -1}[1] \arrow[r, "u^{\leq -1}[1]"] & Y^{\leq -1}[1] &
\end{tikzcd}$$

In the category $\mathcal{R}$ complete to an octahedron the composable morphisms $G(X^{\geq 0}) \xrightarrow{G(u^{\geq 0})} G(Y^{\geq 0}) \xrightarrow{G(w')} G(Y^{\leq -1}[1])$. We obtain a diagram where the rows and columns are triangles

$$\begin{tikzcd}
 & G(X^{\leq -1}) \arrow[r, "G(u^{\leq -1})"] \arrow[d, "G(u'')"] & G(Y^{\leq -1}) \arrow[d, "G(u')"] & \\
G(Z)[-1] \arrow[r, "\alpha"] \arrow[d, equal] & G(X) \arrow[r, "\beta"] \arrow[d, "G(v'')"] & G(Y) \arrow[r, "G(v)"] \arrow[d, "G(v')"] & G(Z) \arrow[d, equal] \\
G(Z)[-1] \arrow[r, "-G(\widetilde{w})[-1]"] & G(X^{\geq 0}) \arrow[r, "G(u^{\geq 0})"] \arrow[d, "G(w'')"] & G(Y^{\geq 0}) \arrow[r, "G(\widetilde{v})"] \arrow[d, "G(w')"] & G(Z) \\
 & G(X^{\leq -1}[1]) \arrow[r, "G(u^{\leq -1}[1])"] & G(Y^{\leq -1}[1]) &
\end{tikzcd}$$

The third row is the triangle in $\mathcal{R}$ obtained by applying (vi) to the triangle $X^{\geq 0} \xrightarrow{u^{\geq 0}} Y^{\geq 0} \xrightarrow{\widetilde{v}} Z \xrightarrow{\widetilde{w}} X^{\geq 0}[1]$. The second and third column are by applying (v) to the triangles $X^{\leq -1} \xrightarrow{u''} X \xrightarrow{v''} X^{\geq 0} \xrightarrow{w''} X^{\leq -1}[1]$ and $Y^{\leq -1} \xrightarrow{u'} Y \xrightarrow{v'} Y^{\geq 0} \xrightarrow{w'} Y^{\leq -1}[1]$. The map α is a morphism $G(Z[-1]) \longrightarrow G(X)$ with $Z[-1] \in \mathcal{T}^{\geq 0} \cap \mathcal{T}^b_{\mathcal{A}}$ and $X \in \mathcal{T}^{\leq 0} \cap \mathcal{T}^b_{\mathcal{A}}$, and (i) tells us that $\alpha = G(a)$ with $a : Z[-1] \longrightarrow X$ a morphism in $\mathcal{T}^b_{\mathcal{A}}$. Now the fact that G is a functor gives us a commutative square

$$\begin{tikzcd}
G(X^{\le -1}) \arrow[r, "G(u^{\le -1})"] \arrow[d, "G(u'')"] & G(Y^{\le -1}) \arrow[d, "G(u')"] \\
G(X) \arrow[r, "G(u)"] & G(Y)
\end{tikzcd}$$

and hence $[\beta - G(u)]G(u'') = 0$. It follows that $\beta - G(u)$ factors as $\gamma G(v'')$ for some $\gamma : G(X^{\ge 0}) \longrightarrow G(Y)$. Since $X^{\ge 0}$ belongs to $\mathcal{T}^{\ge 0} \cap \mathcal{T}^b_{\mathcal{A}}$ and Y belongs to $\mathcal{T}^{\le 0} \cap \mathcal{T}^b_{\mathcal{A}}$, part (i) tells us that there must be a map $g : X^{\ge 0} \longrightarrow Y$ with $G(g) = \gamma$. Thus $\beta = G(u) + \gamma G(v'') = G(u) + G(g)G(v'') = G(u + gv'')$. Putting $b = u + gv''$ we have that $G(Z)[-1] \xrightarrow{G(a)} G(X) \xrightarrow{G(b)} G(Y) \xrightarrow{G(v)} G(Z)$ is a triangle in $\mathcal{R}$, and by (iv) so is $G(Z)[-1] \xrightarrow{G(-w[1]))} G(X) \xrightarrow{G(u)} G(Y) \xrightarrow{G(v)} G(Z)$. $\square$

It remains to conclude the proof of Theorem E.10, we must show that G takes any triangle in $\mathcal{T}^b_{\mathcal{A}}$ to a triangle in $\mathcal{R}$. The proof will be by induction on the length of the cohomology sequence. We have a homological functor $\mathcal{H} : \mathcal{T}^b_{\mathcal{A}} \longrightarrow \mathcal{A}$, from $\mathcal{T}^b_{\mathcal{A}}$ to the heart of its t–structure. Every triangle in $\mathcal{T}^b_{\mathcal{A}}$ maps under this functor to a long exact sequence in $\mathcal{A}$ which vanishes outside a bounded interval. We let ℓ be the smallest integer so that the long exact sequence has nonzero terms all contained in an interval of length ℓ.

If $\ell \le 3$ then a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ of length ℓ may be rotated to have $X, Y, Z \in \mathcal{A}$, and (vi) tells us that G takes it to a triangle. It remains to prove the induction step: we must show that if all triangles of length $< \ell$ map under G to triangles, then so do triangles of length ℓ. Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be a triangle of length ℓ. By rotating we may assume that X, Y, Z all belong to $\mathcal{T}^{\le 0} \cap \mathcal{T}^b_{\mathcal{A}}$ and $Z^{\ge 0} \ne 0$. In the category $\mathcal{T}^b_{\mathcal{A}}$ complete the composable maps $Y \longrightarrow Z \longrightarrow Z^{\ge 0}$ to an octahedron

$$\begin{tikzcd}
 & Z^{\ge 0}[-1] \arrow[r, equal] \arrow[d, "w'"] & Z^{\ge 0}[-1] \arrow[d, "w''"] & \\
X \arrow[r, "\widetilde{u}"] \arrow[d, equal] & \widehat{Y} \arrow[r, "\widetilde{v}"] \arrow[d, "u'"] & Z^{\le -1} \arrow[r, "\widetilde{w}"] \arrow[d, "u''"] & X[1] \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[d, "v'"] & Z \arrow[r, "w"] \arrow[d, "v''"] & X[1] \\
 & Z^{\ge 0} \arrow[r, equal] & Z^{\ge 0} &
\end{tikzcd}$$

Now complete the composable maps $G(Z^{\geq 0}[-1]) \xrightarrow{G(w')} G(\widehat{Y}) \xrightarrow{G(\widetilde{v})} G(Z^{\leq -1})$ to an octahedron in $\mathcal{R}$

$$\begin{tikzcd}
 & G(Z^{\geq 0}[-1]) \arrow[r, equal] \arrow[d, "G(w')"] & G(Z^{\geq 0}[-1]) \arrow[d, "G(w'')"] & \\
G(X) \arrow[r, "G(\widetilde{u})"] \arrow[d, equal] & G(\widehat{Y}) \arrow[r, "G(\widetilde{v})"] \arrow[d, "G(u')"] & G(Z^{\leq -1}) \arrow[r, "G(\widetilde{w})"] \arrow[d, "G(u'')"] & G(X[1]) \arrow[d, equal] \\
G(X) \arrow[r, "G(u)"] & G(Y) \arrow[r, "\beta"] \arrow[d, "G(v')"] & G(Z) \arrow[r, "\gamma"] \arrow[d, "G(v'')"] & G(X[1]) \\
 & G(Z^{\geq 0}) \arrow[r, equal] & G(Z^{\geq 0}) &
\end{tikzcd}$$

The fact that G takes $Z^{\geq 0}[-1] \longrightarrow Z^{\leq -1} \longrightarrow Z \longrightarrow Z^{\geq 0}$ to a triangle is by (v). The fact that G takes $Z^{\geq 0}[-1] \longrightarrow \widehat{Y} \longrightarrow Y \longrightarrow Z^{\geq 0}$ to a triangle is by (vii). And the fact that G takes $X \longrightarrow \widehat{Y} \longrightarrow Z^{\leq -1} \longrightarrow X[1]$ to a triangle is by induction on ℓ.

Applying the functor G to the octahedron in $\mathcal{T}^b_{\mathcal{A}}$ gives a commutative diagram, in particular the two squares in

$$\begin{tikzcd}
G(\widehat{Y}) \arrow[r, "G(\widetilde{v})"] \arrow[d, "G(u')"] & G(Z^{\leq -1}) \arrow[r, "G(\widetilde{w})"] \arrow[d, "G(u'')"] & G(X[1]) \arrow[d, equal] \\
G(Y) \arrow[r, "G(v)"] & G(Z) \arrow[r, "G(w)"] & G(X[1])
\end{tikzcd}$$

both commute. Hence $[\beta - G(v)]G(u') = 0$ and $[\gamma - G(w)]G(u'') = 0$. It follows that there exist maps $\beta' : G(Z^{\geq 0}) \longrightarrow G(Z)$ and $\gamma' : G(Z^{\geq 0}) \longrightarrow G(X[1])$ with $\beta - G(v) = \beta' G(v')$ and $\gamma - G(w) = \gamma' G(v'')$. But $Z^{\geq 0}$ belongs to $\mathcal{T}^{\geq 0} \cap \mathcal{T}^b_{\mathcal{A}}$ and $Z, X[1]$ are both in $\mathcal{T}^{\leq 0} \cap \mathcal{T}^b_{\mathcal{A}}$, and (i) tells us that there exist morphisms $b : Z^{\geq 0} \longrightarrow Z$ and $g : Z^{\geq 0} \longrightarrow X[1]$ with $\beta' = G(b)$ and $\gamma' = G(g)$. Therefore $\beta = G(v + bv')$ and $\gamma = G(w + gv'')$. Thus we have produced a triangle

$$G(X) \xrightarrow{G(u)} G(Y) \xrightarrow{G(v+bv')} G(Z) \xrightarrow{G(w+gv'')} G(X[1])$$

and (iv) tells us that G takes $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ to a triangle. □

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