Entropy, energy, and proximity to criticality in global earthquake populations

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[1] We examine the question of proximity of the global earthquake population to the critical point characterised by the energy $E$ and entropy $S$ based on annual frequency data from the Harvard CMT catalogue. The results are compared with a theoretical model corresponding to a Boltzmann probability density distribution of the form $p(E) \propto E^{-B^{1}e^{E/2}}$. The data are consistent with the model predictions for fluctuations in the characteristic energy $\theta$ at constant $B$ value, of the form $S \sim -B(\ln E)$. This approximation is valid for large $\theta$, relative to the maximum possible event size, confirming that the Earth is perpetually in a near-critical state, reminiscent of self-organized criticality. However, the results also show fluctuations of ±10% in entropy that may be more consistent with the notion of intermittent criticality. A more precise definition of the two paradigms, and a similar analysis of numerical models, are both needed to distinguish between these competing models. INDEX TERMS: 3220 Mathematical Geophysics: Nonlinear dynamics; 7223 Seismology: Seismic hazard assessment and prediction; 7209 Seismology: Earthquake dynamics and mechanics; 7260 Seismology: Theory and modeling

1. Introduction

[2] It has been suggested that the Earth’s brittle crust is maintained in a state of self-organized criticality, implying a system perpetually near global failure [Bak and Tang, 1989]. This implies large stress correlations would be maintained in the system, hence reducing the degree of predictability of individual earthquakes [Main, 1997; Geller et al., 1997]. However, many observations have been interpreted as implying instead a system maintained predominantly in a sub-critical state, with fluctuations during the largest earthquakes representing states of intermittent criticality [Jaume and Sykes, 1999], implying a degree of predictability in the population dynamics [Main, 1999a, Sykes et al., 1999; Sornette, 2000]. These include the spatial and temporal scaling relations of earthquake sequences [Utsu et al., 1995], accelerating cumulative Benioff ‘strain’ [Bufe and Varnes, 1993], finite correlation lengths inferred from the tail in the frequency-moment distribution [Kagan, 1999; Leonard et al., 2001]; and anomalous stress diffusion in the crust during earthquake triggering [Marsan et al., 2000]. However, the methods that have been used to establish intermittent criticality have been criticised on statistical [Gross and Rundle, 1998; Vere-Jones et al., 2001] and physical [Main, 1999b] grounds. Of particular concern is the inherent retrospective selection bias if an optimisation procedure is used to tune the data to the desired form [Mulargia, 2001]. Here we examine this problem in the most direct way possible using the tools of statistical mechanics, using global earthquake catalogue data to determine temporal fluctuations in the seismic energy release $E$ and entropy $S$. The use of global data minimises the potential for selection bias. The form of the entropy-energy relationship is identical to that predicted for a system that is in a near-critical state, but with short-term fluctuations in correlation length and entropy that may be significant. If confirmed, this would imply a finite but perhaps low degree of statistical predictability in the population dynamics.

2. Theory

[3] In classical statistical physics, macroscopic thermodynamic variables can be calculated from a basic knowledge of the microstates available to the system with probability

$$p(E)dE = \frac{g_{E} e^{-E/\theta}}{Z} dE,$$

(1)

where in thermodynamics $\theta = kT$, $k$ is Boltzmann’s constant, $T$ the temperature, $g_{E}$ is the degeneracy of states, $i$ the number of ways a particular energy level can be filled, and the partition function $Z$ is the total number of micro-states — a normalising constant to ensure unit total probability [Mandel, 1988]. The degeneracy of earthquake populations occurs because energy is related to the surface rupture area $A$ by $E \sim A^{B/2}$, so there are fewer ways of fitting a larger earthquake of a given rupture area into the total fault surface area of a fault population [Main and Burton, 1984; Rundle, 1993]. For earthquakes the density distribution of the degeneracy has the power-law form

$$g_{E} = E^{-B/2}/E_{0}^{-B},$$

(2)

where $E_{0}$ is a scaling constant with the units of energy. Equation (2) corresponds to the Gutenberg-Richter law for the equivalent magnitude where typically the exponent $B \approx 2/3$. For a two-dimensional fault model, the correlation length is related directly to $\theta^{1/3}$, and tends to infinity precisely at the critical point [Stauffer and Aharony, 1994; Bruce and Wallace, 1989].

[4] The expectation value of the system energy is the first moment

$$\langle E \rangle = \int \frac{E_{\text{max}}}{E_{\text{min}}} E p(E) dE,$$

(3)

and the entropy is defined by

$$S = -\int E_{\text{max}} E_{\text{min}} \ln p(E)p(E) dE.$$  

(4)

The maximum energy $E_{\text{max}}$ here is an imposed physical upper bound, determined either by the size of the Earth in the most extreme case, or more likely by characteristic lengths in the structure of plate boundaries. The Gutenberg-Richter law is recovered as $\theta \rightarrow \infty$, preserving a finite energy through finite $E_{\text{max}}$. From (1)–(4), it can be shown, for a system in a near-critical
state, using conventional logarithmic increments $\delta \log E$ to define the relevant frequencies, that

\[ S = S_0 + B \langle \ln E \rangle, \quad (5) \]

where $S_0$ is a constant (see Appendix). In a sub-critical state with finite positive $\theta$ we would expect the local slope to be greater than $B_0$, and in a supercritical state (negative $\theta$) we would expect a local slope less than $B_0$. Corollaries of (5) for finite, positive $\theta$ are $\langle E \rangle \sim \theta^{1-B}$, and $\partial S/\partial \langle E \rangle = \varepsilon$, with $\varepsilon$ a small positive number (see Appendix). Thus, for a system in a near-critical state, with significant fluctuations in $\theta$ at constant $B$-value, we would expect (i) a strong positive correlation between $S$ and $\langle \ln E \rangle$ of slope $B$, (ii) a strong positive correlation between $\langle E \rangle$ and $\theta$, and (iii) a weak positive correlation between $S$ and $\langle E \rangle$.

3. Method

[5] In order to avoid problems of catalogue heterogeneity, we used seismic moment data from the Harvard Centroid Moment Tensor catalogue for the time period 1977–2000 inclusive, i.e. the era of widely-available digital seismic data. For this period the catalogue was found to be complete above moment magnitude 5.8. The bandwidth of seismic moment in the catalogues used corresponded to 4 orders of magnitude. If the smallest moment corresponded to an elemental source area $A_{\text{min}}$, then the total number of elemental source areas required to describe the system is $N = A_{\text{max}}/A_{\text{min}} = 10^{2.75} \approx 500$. We require $N$ to be large for statistical stability in any thermodynamic treatment, although here $N$ is much less that, say, Avogadro’s number. The scalar moment for both catalogues was then converted to seismic energy using standard relationships [Kanamori, 1977]. Data for non-overlapping annual time windows was used to construct incremental frequency-energy data using $\log E$ equivalent to 0.25 magnitude units. This was the highest resolution that could be achieved while avoiding empty bins in the determination of the frequency-energy relation, a requirement for a stable estimate of $S$. We calculated $\langle E \rangle$ from the individual summed energies, and $S$ using a discrete version of (4) after appropriate normalisation to ensure unit total probability. The method is non-parametric - no curve fits were used to smooth the data.

4. Results

[6] The temporal evolution of $S$, $\langle E \rangle$, $\langle \ln E \rangle$, and $E_{\text{max}}$ is shown on Figure 1. The annual maximum energy $E_{\text{max}}$ is a proxy for $\theta$ [Bruce and Wallace, 1989]. Note the relatively large fluctuations in $\langle E \rangle$ (between 2 and $14 \times 10^{14}$ J), associated with small fluctuations in $S$ (between 1 and 1.2 in normalised units). There is a strong positive correlation between fluctuations in $\langle \ln E \rangle$ and $S$, and between $\langle E \rangle$ and $E_{\text{max}}$, but only a weak correlation if any between $\langle E \rangle$ and $S$. The primary data then qualitatively confirm criteria (i)–(iii).

[7] We test equation (5) quantitatively against the data in Figure 2. The fit is very good ($r^2 = 0.941$) with $B = 0.627 \pm 0.033$, confirming criterion (i) quantitatively. This value of $B$ is indistinguishable from that obtained from a maximum likelihood analysis of CMT frequency-moment data [$B = 0.636 \pm 0.013$, Kagan, 1999; $B = 0.625 \pm 0.011$, Leonard et al., 2001]. The assumption of relatively constant value of $B$ is validated, verifying that temporal fluctuations are dominated changes in the correlation length via $\theta$, similar to that found for spatial fluctuations [Kagan, 1999]. In ideal self-organized criticality, with only small fluctuations about the critical state allowed [Bak and Tang, 1989], the data would plot as a small cluster of points of slope $B$, around a central value of $S$ and $\langle \ln E \rangle$. The fluctuations in the macroscopic thermodynamic variable $S$ (on the order of $\pm 10\%$) may be more consistent with intermittent criticality.
[8] Earthquake generation is neither ‘adiabatic’ (constant $S$) nor ‘isothermal’ (constant $E$), but it is more consistent with the former. The regression coefficient between $\langle E \rangle$ and $E_{\text{max}}$ is $r^2 = 0.805$, quantitatively verifying criterion (ii). Similarly the regression between $S$ and $\langle E \rangle$ gives $r^2 = 0.046$, consistent with criterion (iii), but not sufficient to discriminate between models of self-organized criticality and intermittent criticality, because the slope is indistinguishable from zero within the errors of the regression. We conclude that globally the brittle Earth is in a near-critical state, where system fluctuations are dominated by regression. We find exactly that $\text{criticality}$ is in a near-critical state, with large fluctuations in mean energy occurring a for small changes in entropy. The results are consistent with temporally fluctuating correlation length at constant $B$-value. The inferred $B$-value from the entropy-energy relationship of $0.627 \pm 0.033$ agrees well with independent maximum likelihood determinations from the same catalogue. The fluctuations in entropy are on the order of $\pm 10\%$. Whether or not this is large enough to discriminate between competing models of self-organized criticality and intermittent criticality depends on a more precise definition of the two than available at present, specifically the magnitude of entropy fluctuations that are allowed in a self-organized critical point system. The proximity to the critical point indicates that the probability of the system may be finite, but low.

5. Discussion

[9] So far we have considered only the radiated seismic energy, and not the internal potential energy of the system. Although our knowledge of strain has increased remarkably recently due to the advent of global satellite data, our empirical knowledge of the energetic micro-states in the Earth is hampered by difficulties in measuring the local stress, and also by systematic spatial variations in seismic efficiency. However, numerical modelling has shown a strong positive correlation between the energetic micro-states and radiated seismic energy distribution result from variations in energetic micro-states. The results confirm that global seismicity is in a near-critical state, with the slope being indistinguishable from zero within the errors of the regression. We conclude that globally the brittle Earth is in a near-critical state, where system fluctuations are dominated by regression. We find exactly that $\text{global seismicity}$ is in a near-critical state, with large fluctuations in mean energy occurring a for small changes in entropy. The results are consistent with temporally fluctuating correlation length at constant $B$-value. The inferred $B$-value from the entropy-energy relationship of $0.627 \pm 0.033$ agrees well with independent maximum likelihood determinations from the same catalogue. The fluctuations in entropy are on the order of $\pm 10\%$. Whether or not this is large enough to discriminate between competing models of self-organized criticality and intermittent criticality depends on a more precise definition of the two than available at present, specifically the magnitude of entropy fluctuations that are allowed in a self-organized critical point system. The proximity to the critical point indicates that the predictability of the system may be finite, but low.

6. Conclusion

[11] We have examined the relationship between the macroscopic state variables of energy and entropy for global earthquakes as a function of changes in the energy distribution of energetic micro-states. The results confirm that $\text{global seismicity}$ is in a near-critical state, with large fluctuations in mean energy occurring a for small changes in entropy. The results are consistent with temporally fluctuating correlation length at constant $B$-value. The inferred $B$-value from the entropy-energy relationship of $0.627 \pm 0.033$ agrees well with independent maximum likelihood determinations from the same catalogue. The fluctuations in entropy are on the order of $\pm 10\%$. Whether or not this is large enough to discriminate between competing models of self-organized criticality and intermittent criticality depends on a more precise definition of the two than available at present, specifically the magnitude of entropy fluctuations that are allowed in a self-organized critical point system. The proximity to the critical point indicates that the predictability of the system may be finite, but low.

Appendix

[12] From (1) and (2) of the main text we have the incremental probability

$$p(E)dE = \alpha E^{-B-1} \exp(-E/\theta)dE,$$

(A1)

where $\alpha = E_0^{-B} \gamma^{-1}$ is a scaling constant. If we use logarithmic increments of energy $p(E)\ln E$

$$\frac{d\ln E}{dE} = \frac{1}{E} \quad \text{or} \quad dE = Ed\ln E,$$

(A2)

then the partition function is

$$Z = \int_{E_{\text{min}}}^{E_{\text{max}}} \frac{E}{E_0}^{-B} \exp \left( -\frac{E}{\theta} \right) dE,$$

(A3)

the energy is

$$\langle E \rangle = Z^{-1} \int_{E_{\text{min}}}^{E_{\text{max}}} E \left( \frac{E}{E_0} \right)^{-B} \exp \left( -\frac{E}{\theta} \right) dE,$$

(A4)

and the entropy is

$$S = -Z^{-1} \int_{E_{\text{min}}}^{E_{\text{max}}} \exp \left( -\frac{E}{\theta} \right) \ln \left( \frac{E/E_0}{} \right)^{-B} \exp(-E/\theta) \frac{dE}{Z}.$$ 

(A5)

[13] If we expand the logarithmic term in the kernel of (A5), we find exactly that

$$S = \ln Z - B\ln E_0 + B\langle E \rangle/\theta + \langle E \rangle/\theta.$$ 

(A6)

For a system of finite mean energy near the critical point $0 \rightarrow \infty$, $\langle E \rangle/\theta$ tends to zero, and can be neglected. Near the critical point, we find

$$\frac{d\ln Z}{d\theta} = \frac{1}{Z} \frac{\partial Z}{\partial \theta} = \frac{\langle E \rangle}{\theta^2}.$$ 

(A7)
Again as $0 \to \infty$ we find $\partial \ln Z / \partial 0 = 0$, and hence $Z \sim \text{constant}$. Thus from (A6) a cross-plot of $S$ and $\langle \ln E \rangle$ should approximate to a straight line with slope $B$ and intercept $S_0 \approx \ln Z - B \ln E_0$ [equation (5) and criterion (i) of the main text].

From (A2) and (A4) we have after a change of variables to $x = E/E_{\text{max}}$, with $E_{\text{max}} \gg E_{\text{min}}$,

$$
\langle E \rangle \approx \theta^{1 - B} E_{\text{min}}^B \int_0^{E_{\text{max}}} x^{-B} e^{-x} \, dx. \quad (A8)
$$

The integral is the incomplete gamma function. For $B = 2/3$, it is near but less than the value of the complete gamma function of $1.786$ [Kagan, 1993]. Thus $\langle E \rangle \approx \theta^{1 - B} E_{\text{min}}^B$, implying a positive correlation between $\langle E \rangle$ and $0$ [criterion ii].

From (A6) it follows that

$$
\frac{\partial S}{\partial \langle E \rangle} = \frac{\partial \ln Z}{\partial \langle E \rangle} + B \frac{\partial \langle \ln E \rangle}{\partial \langle E \rangle} + \frac{1}{\theta}. \quad (A9)
$$

For $Z \sim \text{constant}$, it can be shown from the definitions of $\langle E \rangle$ and $\langle \ln E \rangle$ that

$$
\frac{\partial \langle \ln E \rangle}{\partial \theta} = \frac{\partial \langle \ln E \rangle}{\partial 0} \left/ \frac{\partial \langle E \rangle}{\partial 0} \right. \approx \langle \ln E \rangle / \langle E \rangle. \quad (A10)
$$

For the values reported here $1/\theta \approx 10^{-17} \, \text{J}^{-1}$, and $\langle \ln E \rangle / \langle E \rangle \approx 31.85/(8 \times 10^{14}) = 4 \times 10^{-14} \ln (1/J)$. Hence, $\partial S / \partial \langle E \rangle \approx \varepsilon$ where $\varepsilon \ll 1$ but $\varepsilon > 1/0$ [criterion iii].

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