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Citation for published version:
Carbery, A 2009, 'Large sets with limited tube occupancy', *Journal of the London Mathematical Society*, vol. 79, no. 2, pp. 529-543. https://doi.org/10.1112/jlms/jdn086

Digital Object Identifier (DOI):
10.1112/jlms/jdn086

Published In:
Journal of the London Mathematical Society

J. London Math. Soc. (2) 79 (2009) 529–543
doi:10.1112/jlms/jdn086

Large sets with limited tube occupancy

Anthony Carbery

Abstract

We study subsets E of euclidean space with the property that for every tube, the amount of mass of E contained in that tube is small, and address via the probabilistic method the question of how large such sets may be. We also study discrete analogues of this question, and relate it to problems in harmonic analysis concerning the extension operator for the Fourier transform.

1. *Introduction*

A bounded subset E of $\mathbb{R}^d$ is a Kakeya-type set (or, more accurately, a Besicovitch–Kakeya–Furstenberg-type set) if each of a large set of tubes (say one in each direction, or one passing through each point of a hyperplane) contains a relatively large amount of E. The natural question for such sets is how small they can be, and this question has received a great deal of attention over the last forty years.

In this paper we are concerned, in contrast, with 'anti-Kakeya'-type sets, that is, subsets E of $\mathbb{R}^d$ such that, for *every* tube, the amount of mass of E contained in the tube is small. The question now is how large such sets may be. In other words, given a bounded subset of $\mathbb{R}^d$, how much mass can one put in it without there being too much mass in any one tube?

This question naturally arises in X-ray tomography, but we are interested in its connections with harmonic analysis and PDEs.

In the late 1970s Stein (see [17]) proposed that the disc multiplier operator should be controlled by a maximal function involving averages over eccentric rectangles via an L^2-weighted inequality. Parallel to this, it is natural to ask the same question (and indeed in some model cases the questions are equivalent; see a forthcoming paper by Carbery and Wisewell) for the extension operator for the Fourier transform associated to a hypersurface in $\mathbb{R}^d$ of nonvanishing Gaussian curvature such as the unit sphere or the base of a paraboloid. The extension operator for the Fourier transform is the operator

$$g \longmapsto \widehat{gd\sigma}(x) = \int g(\omega)e^{-2\pi i x\cdot\omega}d\sigma(\omega),$$

where $\widehat{\ }$ denotes the Fourier transform, and σ is the measure associated to a smooth density supported on the hypersurface. Thus one is led to consider inequalities of the form

$$\int_{\mathbb{R}^d} |\widehat{gd\sigma}(x)|^2 w(x)dx \leqslant C\int |g(\omega)|^2 \mathcal{M}w(\omega)\ d\sigma(\omega),$$

where the maximal operator $\mathcal{M}$ involves averages over highly eccentric tubes or rectangles.

In the mid 1980s, Mizohata and Takeuchi, in connection with estimates for solutions to the Helmholtz equation, and apparently unaware of the connection with Stein's proposal, suggested that the following should hold.

Received 22 August 2008; published online 16 March 2009.

2000 *Mathematics Subject Classification* 28A78, 42B99, 52C99, 60D05.

Part of this research was undertaken at the University of Athens in spring 2006, where the author was supported by a Leverhulme Study Abroad Fellowship and the EC project 'Pythagoras'.

Conjecture 1 (Mizohata–Takeuchi [16]). We have

$$\int_{\mathbb{R}^d} |\widehat{gd\sigma}(x)|^2 w(x)dx \leqslant C \sup_T w(T) \int |g|^2 \, d\sigma,$$

where the sup is taken over all 1-tubes T.

Here and in what follows, an r-tube T is an r-neighbourhood of a (doubly infinite) straight line in $\mathbb{R}^d$. Because of the nature of the Fourier analysis (basically the uncertainty principle), it suffices to consider weights w that are essentially constant on unit scale in this conjecture, so that for such weights the term $\sup_T w(T)$ is equivalent to the sup norm of the X-ray transform of w.

In [2, 7–9] the conjecture was resolved in the affirmative for the case that the weight w is radial, but it remains open in the general case. In the radial case explicit spectral representations for the operator $g \mapsto \widehat{gd\sigma}$ in terms of spherical harmonics and Bessel functions can be exploited.

The papers [7–9] concerned analogues of Riemann's localisation theorem for Fourier transforms in higher dimensions. For $f \in L^2(\mathbb{R}^d)$ let

$$S_R f(x) = \int_{|\xi| \leqslant R} \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

If f is identically zero on the unit ball $\mathbb{B}$ of $\mathbb{R}^d$, then in what senses can we expect pointwise convergence of $S_R f(x)$ to zero on $\mathbb{B}$? The following results were obtained.

Proposition 1. (i) *If $E \subseteq \mathbb{B}$ supports a positive measure μ with*

$$\sup_{r\text{-tubes } T} \frac{\mu(T)}{r^{d-1}} \leqslant C$$

uniformly in r, then, conditional on Conjecture 1 holding, $S_R f(x) \to 0$ almost everywhere $d\mu$.

(ii) *If $d - 1/2 < \beta \leqslant d$, if $0 < \mathcal{H}_\beta(E) < \infty$, and if E is radial, then*

$$\sup_{r\text{-tubes } T} \frac{\mathcal{H}_\beta(T \cap E)}{r^{d-1}} \leqslant C$$

uniformly in r (and so $S_R f(x)$ converges to 0 almost everywhere with respect to $\mathcal{H}_\beta|_E$ if $f \equiv 0$ on $\mathbb{B}$).

(iii) *There is no $E \subseteq \mathbb{B}$ with $0 < \mathcal{H}_{d-1}(E) < \infty$ such that*

$$\sup_{r\text{-tubes } T} \frac{\mathcal{H}_{d-1}(T \cap E)}{r^{d-1}} \leqslant C$$

uniformly in r (and, moreover, if $\mathcal{H}_{d-1}(E)$ is σ-finite, then there is an $f \in L^2(\mathbb{R}^d)$, identically zero on $\mathbb{B}$, such that $S_R f(x)$ diverges on E).

Here and in what follows, $\mathcal{H}_\beta$ denotes β-dimensional Hausdorff measure.

It is, therefore, an interesting question for $d - 1 < \beta \leqslant d - 1/2$ as to whether there exist sets of positive finite β-dimensional Hausdorff measure for which

$$\sup_r \sup_{r\text{-tubes } T} \frac{\mathcal{H}_\beta(E \cap T)}{r^{d-1}} < \infty.$$

More generally, one can ask to determine those pairs $(\beta, \gamma) \in [0, d] \times [0, d]$ for which there exists a set $E \subseteq \mathbb{B}$ of positive finite β-dimensional Hausdorff measure such that

$$\sup_r \sup_{r\text{-tubes } T} \frac{\mathcal{H}_\beta(E \cap T)}{r^{\gamma}} < \infty. \tag{1}$$

This question thus asks whether there exist 'large' sets (in terms of having *positive* β-dimensional Hausdorff measure) such that the (β-dimensional) mass in any tube is limited by (1).

It is not difficult to see that, if either $\gamma > d-1$ or $\beta < \gamma$, then (1) implies that $\mathcal{H}_\beta(E) = 0$; see Section 4.

THEOREM 1. *If $\gamma < d-1$ and $\beta > \gamma$, then there exists $E \subseteq \mathbb{B}$ with $0 < \mathcal{H}_\beta(E) < \infty$ such that (1) holds.*

Returning now to Conjecture 1, recall the celebrated Stein–Tomas restriction theorem (see [**18**]) asserting that

$$\|\widehat{gd\sigma}\|_{2(d+1)/(d-1)} \leqslant C\|g\|_2.$$

By the converse to Hölder's inequality, this has an immediate restatement as a weighted inequality:

$$\int_{\mathbb{R}^d} |\widehat{gd\sigma}(x)|^2 w(x)dx \leqslant C\|w\|_{(d+1)/2} \int_{S^{d-1}} |g|^2 d\sigma. \tag{2}$$

So, when considering Conjecture 1, it only makes sense to test it on weights w that are constant on unit scale and for which

$$\sup_{1-\text{tubes}\, T} w(T) \ll \|w\|_{(d+1)/2}. \tag{3}$$

Thus we are looking for weights w whose mass in any tube is much smaller than its L^p norm for $p = (d+1)/2$.

Finding such weights is easy: if one places approximately $N^{(d-1)/2}$ unit balls points spaced by approximately $N^{1/2}$ on the sphere of radius N, then, by the curvature of the sphere, no 1-tube meets more than two of them. (This example came to light in conversations with Jonathan Bennett, Ana Vargas, and Laura Wisewell. There are also examples of *infinite* sequences of points such that no 1-tube meets more than two of them: take a strictly convex plane curve without asymptotes, place a unit ball centred at x_1 on this curve, place another centred on this curve at x_2 in the first available place such that it does not meet the 1-tube generated by the tangent at x_1, etc.)

However, this does not necessarily represent the most efficient example exhibiting (3). In order to consider the problem more quantitatively, it is convenient to introduce a scale N, and to consider finding weights w, constant on unit scale and supported in a ball or cube of size N, such that (3) holds. For such weights it is easy to see that, for $p \geqslant 1$, we have

$$\|w\|_p \leqslant C_{d,p} N^{(d-1)/p} \sup_{1\text{-tubes}\, T} w(T), \tag{4}$$

and it is natural to ask if the factor $N^{(d-1)/p}$ appearing here is sharp. For $p = 1$ and $p = \infty$ this is obvious.

A refinement of this situation is as follows. For $2 \leqslant k \leqslant d^{1/2}N$ let $\mathcal{A}^d(N,k)$ be the maximal number of 1-separated points that we can choose in $Q_N^d := \{1, 2, \ldots, N\}^d$, so that no more than k of them lie in any 1-tube. Then the best constant in (4) will be at least

$$\max_{1 \leqslant k \leqslant d^{1/2}N} \frac{\mathcal{A}^d(N,k)^{1/p}}{k}.$$

So our problem can now be loosely recast as finding good lower bounds on $\mathcal{A}^d(N,k)$, especially for k small. (We shall not care about multiplicative factors of dimensional or other absolute constants: the aim is to find the behaviour of $\mathcal{A}^d(N,k)$ for d fixed, $N \gg 1$, and k in various subranges of $\{2, 3, \ldots, d^{1/2}N\}$. With this in mind, it is clear (by changing N to within controlled

factors) that, in the definition of $\mathcal{A}^d(N,k)$, we may replace the condition that the points be 1-separated with insisting that they be distinct lattice points in $\{1,2,\dots,N\}^d$, without altering the essential behaviour of $\mathcal{A}^d(N,k)$.)

The only obvious upper bound for $\mathcal{A}^d(N,k)$ is $\mathrm{O}(kN^{d-1})$. The example mentioned above, where approximately $N^{(d-1)/2}$ points on a sphere of radius N are spaced approximately $N^{1/2}$ apart, shows that $\mathcal{A}^d(N,2) \geqslant C_d N^{(d-1)/2}$. This example can be modified, for example, by adding more concentric spheres and then packing more points into each sphere, to give concrete examples showing that $\mathcal{A}^d(N,k) \geqslant C_d k N^{d-1}$ when $k \geqslant N^{1/2}$ (where we again arrive at what are essentially radial examples!) and that $\mathcal{A}^d(N,k) \geqslant C_d k^d N^{(d-1)/2}$ when $k \leqslant N^{1/2}$. (See Section 5 for details.) This tells us only that the best constant in (4) will be at least $N^{(d-1)/p} N^{-1/2p'}$ when $1 \leqslant p \leqslant d$, and at least $N^{(d-1)/2p}$ when $p \geqslant d$. Nevertheless, we have the following theorem.

THEOREM 2. *For $2 \leqslant k \leqslant N^{1/2}$ there is a collection of at least $C_d k N^{d-1} N^{-(d-1)/k}$ lattice points (counted according to multiplicities) in $\{1,2,\dots,N\}^d$, so that no 1-tube contains more than k of them.*

Clearly, $C_d k N^{d-1}$ is best possible for no 1-tube to contain more than k points, and observe that, when $k \geqslant \log N$, the term $N^{-(d-1)/k}$ essentially disappears. When $k=2$, we recover the same numerology as given by the example of approximately $N^{(d-1)/2}$ points placed at roughly equal spacings on a sphere of radius approximately N. Thus we do not resolve the situation when $2 \leqslant k \leqslant \log N$; indeed, for small values of k ($k = 2,3,4$, for example) it seems quite difficult to understand $\mathcal{A}^d(N,k)$. Nevertheless, we believe Theorem 2 to be new, even in the case $d=2$ and $k=3$.

COROLLARY 1. *There exists a w that is constant on unit cubes and that takes integer values such that*

$$\sup_{1\text{-tubes } T} w(T) \leqslant C_d \log N,$$

while

$$\|w\|_1 \geqslant C_d \log N \; N^{d-1}.$$

Since for w taking nonzero values that are at least one we have $\|w\|_p \geqslant \|w\|_1^{1/p}$, we obtain the following corollary.

COROLLARY 2. *With w as in Corollary 1, if $p > 1$ then we have*

$$\|w\|_p \geqslant C_{d,p} \frac{N^{(d-1)/p}}{(\log N)^{1/p'}} \sup_{1\text{-tubes } T} w(T).$$

So the constant in (4) is sharp up to logarithmic factors, and such weights as given by Corollary 2 should, in principle, be good candidates on which to test Conjecture 1.

When $d=2$ and $k=2$, what we consider is closely related to a problem of Motzkin [**10**] that asks for the maximum, over all configurations of n points in $[0,1]^2$, of the minimal width of strips such that there are no more than two points in any strip. In turn, Motzkin's problem is closely related to the Heilbronn triangle problem that asks for the maximum over all configurations of n points in Q^2 of the minimal area of triangles with vertices in the configuration. The proof of Theorem 2 is closely related to work of Komlós, Pintz, and Szemerédi [**12**] on lower bounds

for Heilbronn's problem. In fact, there is a logarithmic improvement of the case $k = 2$ and $d = 2$ of Theorem 2 implied by the work of those authors, and our argument bears a close resemblance to a simplified version of that analysis (see [1]). Nontrivial upper bounds that have been established for the Heilbronn problem (see, for example, [11] and the references therein) do not translate directly into upper bounds on $\mathcal{A}^2(N, 2)$ or for Motzkin's problem (in particular, do we have $\mathcal{A}^2(N, 2) = \mathrm{o}(N)$?).

The argument for Theorem 2 is probabilistic. Corollary 1 can also be obtained by a simpler large deviation/Bernoulli trials argument; Michael Christ has also made a similar observation (M. Christ, personal communication). Since the examples are generated probabilistically rather than deterministically, their potential as counterexamples to Conjecture 1 is perhaps limited; for example, it is not hard to show that, if we write $g \in L^2(S^{d-1})$ in its wave packet representation and then put random ± 1s on the coefficients, then Conjecture 1 holds for all weights w almost surely. (In fact, it holds with the smaller constant $\mathrm{O}(\sup_N w(T(N, N^2))/N^{d-1})$ with the sup taken over all finite tubes $T(N, N^2)$ of $d-1$ short sides N and one long side N^2. This unpublished observation is due to Jonathan Bennett and the present author.)

In Section 2 we prove Theorem 2. In Section 3 we give a similar argument to that of Theorem 2 to establish a lower bound for a quantity arising in a generalisation of the Heilbronn problem. In Section 4 we prove Theorem 1 by building Cantor sets based upon the examples furnished by Theorem 2 or Corollary 1. Since future work will require concrete examples on which to test Conjecture 1, we have included the details of such in Section 5, though logically they are subsumed by Theorem 2.

2. *The proof of Theorem* 2

In this section all tubes are 1-tubes and unspecified constants may depend on the dimension d.

Proof of Theorem 2. First choose $k \geqslant 3$ points in $\Omega := \{1, 2, \ldots, N\}^d$ independently and uniformly at random. Then, for each such point p, we have

$$\mathbb{P}\{p \text{ is in a } \textit{given} \text{ tube } T\} \leqslant CN^{-(d-1)}.$$

Thus,

$$\mathbb{P}\{\textit{each} \text{ such } p \text{ is in a } \textit{given} \text{ tube } T\} \leqslant C^k N^{-k(d-1)}.$$

Since there are about $N^{2(d-1)}$ different tubes T, then

$$\mathbb{P}\{\textit{each} \text{ such } p \text{ is in } \textit{some } T\} \leqslant C^k N^{(2-k)(d-1)}.$$

Now let $M \geqslant k \geqslant 3$ and pick a set of M points in Ω independently and uniformly at random. Therefore, for each k-element subset $\{p_1, \ldots, p_k\}$ of this set,

$$\mathbb{E}\left(\chi_{\{p_1,\ldots,p_k \text{ all lie in some } T\}}\right) \leqslant C^k N^{(2-k)(d-1)}.$$

There are $\binom{M}{k}$ choices σ of k distinct points $i_1, \ldots, i_k$ from $\{1, 2, \ldots, M\}$. Therefore,

$$\sum_{\sigma} \mathbb{E}\left(\chi_{\{p_{i_1},\ldots,p_{i_k} \text{ all lie in some } T\}}\right) \leqslant C^k \binom{M}{k} N^{(2-k)(d-1)},$$

that is,

$$\begin{aligned}&\mathbb{E}\left(\sum_{\sigma}\chi_{\{p_{i_1},\dots,p_{i_k}\text{ all lie in some }T\}}\right)\\&\quad=\mathbb{E}(\#k\text{-element subsets }\textit{all}\text{ of whose members lie in some }T)\\&\quad\leqslant C^k\binom{M}{k}N^{(2-k)(d-1)}.\end{aligned}$$

Therefore, there exists a point in the sample space, corresponding to a set $S\subseteq\Omega$ of cardinality M if we allow for possibly repeated membership, such that the number of k-element subsets (again allowing for possibly repeated membership) of S, all of whose members lie in some T, is at most

$$C^k\binom{M}{k}N^{(2-k)(d-1)}.$$

Attach artificial labels to the repeated members of S to make them all distinct. Call the resulting set $\mathbf{S}$. Then $\mathbf{S}$ contains exactly M distinct points, and the number of k-element subsets of $\mathbf{S}$, all of whose members lie in some T, is at most

$$C^k\binom{M}{k}N^{(2-k)(d-1)}.$$

Call a k-element subset of $\mathbf{S}$ *bad* if all of its members lie in some tube. Then the number of bad k-element subsets of $\mathbf{S}$ is at most

$$C^k\binom{M}{k}N^{(2-k)(d-1)}.$$

For each bad subset of $\mathbf{S}$, remove one point of it from $\mathbf{S}$, resulting in a subset $\mathbf{S}'\subseteq\mathbf{S}$ with

$$\#\mathbf{S}'\geqslant\#\mathbf{S}-C^k\binom{M}{k}N^{(2-k)(d-1)}=M-C^k\binom{M}{k}N^{(2-k)(d-1)}$$

such that *no* k-element subset of $\mathbf{S}'$ lies in any tube, that is, so that no tube contains more than $(k-1)$ members of $\mathbf{S}'$.

Given k and N we want to maximise

$$M-C^k\binom{M}{k}N^{(2-k)(d-1)}$$

over $M\geqslant k$. We claim that we can make this as large as $M/2$ provided that M is no larger than $C'kN^{(d-1)(k-2)/(k-1)}$. Choosing M to be about this value, we then see that $\mathbf{S}'$ is a set of cardinality

$$C'kN^{(d-1)(k-2)/(k-1)},$$

and no tube contains more than $k-1$ points of $\mathbf{S}'$.

To see that, for some C' and $M\leqslant C'kN^{(d-1)(k-2)/(k-1)}$, we have

$$C^k\binom{M}{k}N^{(2-k)(d-1)}\leqslant M/2,$$

is a routine exercise based on Stirling's formula. Indeed, $k\,!$ is bounded below by an absolute constant multiple of $k^{k+1/2}e^{-k}$, so that

$$\binom{M}{k}=\frac{M(M-1)\dots(M-k+1)}{k\,!}\leqslant C_0\frac{e^kM^k}{k^{k+1/2}}.$$

Hence

$$C^k\binom{M}{k}N^{(2-k)(d-1)}\leqslant C_0(Ce)^k\frac{M^k}{k^{k+1/2}}N^{(2-k)(d-1)},$$

which will be at most $M/2$ provided that

$$M \leqslant \left(\frac{k^{k+1/2}N^{(k-2)(d-1)}}{2C_0(Ce)^k}\right)^{1/(k-1)}.$$

But for a suitable choice of C' we have

$$\left(\frac{k^{k+1/2}N^{(k-2)(d-1)}}{2C_0(Ce)^k}\right)^{1/(k-1)} \geqslant C'kN^{(d-1)(k-2)/(k-1)},$$

and so the proof is complete. □

Remark 1. The naive approach here is via a large deviation/Bernoulli trials argument. Picking M points at random as above, $\mathbb{P}\{\text{some } j \text{ points are in a given tube}\} \leqslant C^j\binom{M}{j}N^{-j(d-1)}$. Therefore,

$$\mathbb{P}\{\text{at least } k \text{ points are in a given tube}\} \leqslant \sum_{j=k}^{M}\binom{M}{j}C^jN^{-j(d-1)}.$$

Therefore,

$$\mathbb{P}\{\text{at least } k \text{ points are in some tube}\} \leqslant N^{2(d-1)}\sum_{j=k}^{M}\binom{M}{j}C^jN^{-j(d-1)}.$$

Now for large values of k ($k \geqslant \log N$) and $M \sim kN^{d-1}$ we can bound this by $1/2$, and so we can deduce that there is a set of approximately kN^{d-1} points (again counted according to multiplicities) with no more than k in any tube. This suffices for Corollary 1. For smaller values of k the estimate on the probability is useless, but instead we have

$$\begin{aligned}&\mathbb{P}\{\text{exactly } k \text{ points in some } T\}\\ &\quad\leqslant \mathbb{E}(\#k\text{-element subsets all of whose members lie in some } T)\\ &\quad\leqslant \binom{M}{k}C^kN^{-(k-2)(d-1)},\end{aligned}$$

which suffices for the argument to continue as in the proof of Theorem 2.

The argument for Theorem 2 can be made to apply in the case of 1-neighbourhoods of m-planes. We illustrate this in the case of hyperplanes. Note that the trivial upper bound on the number of points in $\{1, 2, \ldots, N\}^d$ such that there are no more than k in any slab of width 1 (that is, a 1-neighbourhood of a hyperplane) is C_dkN

Proposition 2. *For $k \geqslant d+1$ there is a configuration of at least $C_dkNN^{-(d-1)/k}$ lattice points in $\{1, 2, \ldots, N\}^d$ with no more than k in any slab of width 1.*

This will also follow from the result of the next section.

3. *A Heilbronn-type problem in higher dimensions*

A higher-dimensional analogue of the Heilbronn problem is also amenable to the method for proving Theorem 2. (Indeed, as mentioned above, the method originated in the study of the Heilbronn problem [12].) Barequet [3] considered configurations of $n \geqslant d+1$ points in $Q^d = [0, 1]^d$ and wanted to maximise (over all possible configurations) the minimal volume Δ_n^d of a

simplex with vertices in the configuration. There is the trivial upper bound $O(1/n)$, obtained by decomposing Q^d into parallel slabs rather than tubes, but this has been improved by Brass [6] in odd dimensions, for example, $\Delta_n^3 \leqslant C_d/n^{6/5}$. For lower bounds, Barequet considered $\{(t, t^2, \ldots, t^d) \mod p : 0 \leqslant t \leqslant p-1\}$, which has no $d+1$ points lying in a hyperplane, and so any simplex with vertices in the set has volume bounded below. Thus $\Delta_n^d \geqslant C_d/n^d$. This was improved by Lefmann [13] to $\Delta_n^d \geqslant C_d(\log n)/n^d$ following the argument of Komlós, Pintz, and Szemerédi [12]. Lefmann and Schmitt [14] had an algorithmic approach giving the same behaviour at least when $d = 3$.

We now consider the volume of the convex hull of k points in a configuration of n points in Q^d, where $n \geqslant k \geqslant d+2$. Again the slab argument gives an upper bound of $C_d k/n$ for the minimal volume $\Delta_{n,k}^d$ of the convex hull of k such points. Below we establish that this upper bound is sharp for sufficiently large k, but first we need a simple lemma (whose proof we include for the convenience of the reader).

LEMMA 1. *Pick $k \geqslant d+1$ points independently and uniformly at random in Q^d and let K be their convex hull. Then*

$$\mathbb{P}\{|K| \leqslant V\} \leqslant C_d^k V^{k-d}.$$

Proof. We first do a calculation. Let $\mathcal{R}$ denote the region of $(\mathbb{R}^d)^k$ consisting of points $(x_1, x_2, \ldots, x_k) \in (\mathbb{R}^d)^k$ that satisfy the following constraints:

- x_1 and x_2 are such that $|x_1 - x_2| \geqslant |x_i - x_j|$ for all choices of pairs x_i and x_j;
- x_3 is such that the area of the triangle with vertices x_1, x_2, and z is maximised when $z = x_3$;
- x_4 is such that the 3-dimensional volume of the simplex with vertices x_1, x_2, x_3, and z is maximised when $z = x_4$;

and so on until

- x_{d+1} is such that the d-dimensional volume of the simplex with vertices $x_1, x_2, x_3, \ldots, x_d$ and z is maximised when $z = x_{d+1}$.

For $(x_1, \ldots, x_k) \in \mathcal{R}$ let α_j denote the j-dimensional volume of the simplex with vertices $x_1, \ldots, x_{j+1}$. Then, for $j \geqslant 3$, the vertex x_j lies in the intersection of the two balls with radii α_1 centred at x_1 and x_2, and it also lies in a $2\alpha_2/\alpha_1$-neighbourhood of the line containing x_1 and x_2. For $j \geqslant 4$, the vertex x_j additionally lies in a $3\alpha_3/\alpha_2$-neighbourhood of the plane containing x_1, x_2, and x_3. Similarly for $j \geqslant 5$, etc. Thus, for $m \geqslant d$, the vertex x_{m+1} lies in the intersection of the two balls with radii α_1 centred at x_1 and x_2, a $2\alpha_2/\alpha_1$-neighbourhood of the line containing x_1 and x_2, a $3\alpha_3/\alpha_2$-neighbourhood of the 2-plane containing x_1, x_2, and x_3, and so on, up to a $d\alpha_d/\alpha_{d-1}$-neighbourhood of the $(d-1)$-plane containing $x_1, \ldots, x_d$. In particular, each x_m with $m > d$ lives in a rectangular box of sides

$$\alpha_1 \times \frac{2\,.\,2\alpha_2}{\alpha_1} \times \ldots \times \frac{2\,.\,d\alpha_d}{\alpha_{d-1}},$$

which has volume

$$d!\,2^{d-1}\alpha_d.$$

(Note that the sequence $m\alpha_m/\alpha_{m-1}$ is monotonic nonincreasing.)

For those $(x_1, \ldots, x_k) \in \mathcal{R} \cap (Q^d)^k$ for which $|K(x_1, \ldots, x_k)| \leqslant V$, we also have

$$\alpha_d \leqslant V.$$

By symmetry, we can cover $(\mathbb{R}^d)^k$ by $\binom{k}{2} \times (k-2) \times \ldots \times (k-d)$ disjoint versions of $\mathcal{R}$ with the special variables that have been singled out permuted around.

Thus,

$$\begin{aligned}
&\mathbb{P}\{|K| \leqslant V\} \\
&\quad = \int_{Q^d} \cdots \int_{Q^d} \chi_{\{x_1,\ldots,x_k : |K(x_1,\ldots,x_k)| \leqslant V\}} dx_1 \ldots dx_k \\
&\quad \leqslant \frac{k(k-1)\ldots(k-d)}{2} \int_{\mathcal{R}\cap(Q^d)^k} \chi_{\{x_1,\ldots,x_k : |K(x_1,\ldots,x_k)| \leqslant V\}} dx_1 \ldots dx_k.
\end{aligned} \tag{5}$$

With $x_1, \ldots, x_d$ fixed, we therefore see that the integral in (5) in each of the variables $x_{d+1}, \ldots, x_k$ is at most

$$2^{d-1} d!\, V.$$

Therefore,

$$\mathbb{P}\{|K| \leqslant V\} \leqslant \frac{k(k-1)\ldots(k-d)}{2} \left(2^{d-1} d!\, V\right)^{k-d} \leqslant C_d^k V^{k-d},$$

as required. □

The case $d = 2$ of the following theorem is already known (with a different proof); see [5] and the references therein.

THEOREM 3. *Let $n \geqslant k \geqslant d+2$. Then there is a configuration of n points in Q^d such that the volume of the convex hull of any k of these points is at least $C_d(k/n)^{(k-1)/(k-d)}$; that is,*

$$\Delta_{n,k}^d \geqslant C_d(k/n)^{(k-1)/(k-d)}.$$

Proof. Let $M \geqslant k$ and pick a set of M points in Q^d independently and uniformly at random in Q^d. By Lemma 1, for each k-element subset $\{p_1, \ldots, p_k\}$ of this set,

$$\mathbb{E}\left(\chi_{\{p_1,\ldots,p_k \text{ all lie in some convex body } B \text{ of volume } V\}}\right) \leqslant C^k V^{k-d}.$$

There are $\binom{M}{k}$ choices σ of k points $i_1, \ldots, i_k$ from $\{1, 2, \ldots, M\}$. Therefore,

$$\sum_\sigma \mathbb{E}\left(\chi_{\{p_{i_1},\ldots,p_{i_k} \text{ all lie in some } B\}}\right) \leqslant C^k \binom{M}{k} V^{k-d},$$

that is,

$$\begin{aligned}
&\mathbb{E}\left(\sum_\sigma \chi_{\{p_{i_1},\ldots,p_{i_k} \text{ all lie in some } B\}}\right) \\
&\quad = \mathbb{E}(\#\text{k-element subsets } \textit{all} \text{ of whose members lie in some} B) \\
&\quad \leqslant C^k \binom{M}{k} V^{k-d}.
\end{aligned}$$

Therefore, there exists a set S, where $\#S = M$ and $S \subseteq Q^d$, such that the number of k-element subsets of S, all of whose members lie in some B, is at most

$$C^k \binom{M}{k} V^{k-d}.$$

Call a k-element subset of S *bad* if all of its members lie in some convex body of volume V. Then the number of bad k-element subsets of S is at most

$$C^k \binom{M}{k} V^{k-d}.$$

For each bad subset of S remove one point of S, resulting in a subset $S' \subseteq S$ with

$$\#S' \geqslant \#S - C^k \binom{M}{k} V^{k-d} = M - C^k \binom{M}{k} V^{k-d}$$

such that *no* k-element subset of S' lies in any convex body of volume V, that is, so that no convex body of volume Vcontains more than $(k-1)$ members of S'.

Given k and V, we want to maximise

$$M - C^k \binom{M}{k} V^{k-d}$$

over $M \geqslant k$. As before, we can make this as large as $M/2$ provided that

$$M \leqslant C' k V^{-(k-d)/(k-1)}.$$

Choosing M to be about this value, we see that S' is a set of cardinality

$$n := C' k N^{(k-d)/(k-1)}$$

and no convex body of volume V contains more than $k-1$ points of S', provided that $V^{(k-d)/(k-1)} \geqslant C'k/n$, that is, $V \leqslant C''(k/n)^{(k-1)/(k-d)}$. Thus the convex hull of any k points has volume greater than $C''(k/n)^{(k-1)/(k-d)}$. □

4. *Proof of Theorem* 1

Let T be an r-tube. We refer to r as the *width* of T and denote it by $w(T)$. We begin with the easy assertion made in the introduction.

Proposition 3. *If either $\gamma > d-1$ or if $\beta < \gamma$ and if $E \subseteq Q^d$ satisfies*

$$\mathcal{H}_\beta(E \cap T) \leqslant C w(T)^\gamma \tag{6}$$

for all tubes T, then $\mathcal{H}_\beta(E) = 0$.

Proof. Suppose first that $\gamma > d-1$ and that $E \subseteq Q^d$ satisfies $\mathcal{H}_\beta(E \cap T) \leqslant C w(T)^\gamma$ for all T. Fix a width w. Then we can cover E by $\mathrm{O}(w^{-(d-1)})$ disjoint parallel tubes T of width w, each of which satisfies $\mathcal{H}_\beta(E \cap T) \leqslant C w^\gamma$. Summing, we have $\mathcal{H}_\beta(E) \leqslant C w^{\gamma-(d-1)}$. Now let $w \to 0$.

Now suppose that $\beta < \gamma$ and that $E \subseteq Q^d$ satisfies $\mathcal{H}_\beta(E \cap T) \leqslant C w(T)^\gamma$ for all T. We may assume, by taking a tube of width 1, that $\mathcal{H}_\beta(E) < \infty$. Then any projection E' of E onto a coordinate hyperplane satisfies $\mathcal{H}_\beta(E') < \infty$, and, in particular, for $\gamma > \beta$, we have $\mathcal{H}_\gamma(E') = 0$. Let $\epsilon > 0$. Cover E' by $(d-1)$-dimensional balls B_i with diameters r_i such that $\sum_i r_i^\gamma < \epsilon$. Then E is covered by tubes T_i whose widths are r_i. Therefore, $\mathcal{H}_\beta(E) \leqslant \sum_i \mathcal{H}_\beta(E \cap T_i) \leqslant C r_i^\gamma \leqslant C\epsilon$. Now let $\epsilon \to 0$. □

Before proving Theorem 1 we must deal with an annoying technicality that arises because the weights w of Theorem 2 and Corollary 1 are not necessarily exactly characteristic functions of sets. The following is a consequence of Theorem 2.

Corollary 3. *For each $\sigma > d-1$ and N sufficiently large, there is an s with $d-1 < s \leqslant \sigma$ and a set of at least $C_{d,\sigma} N^s / \log N$ unit cubes in $[0,N]^d$ with no 1-tube meeting more than N^{s-d+1} of them.*

Proof. In Theorem 2 take $k = N^{\sigma-d+1} \geqslant 2$. Then there is a w with $\int w \geqslant C_{d,\sigma} N^{\sigma}$ and $\int_T w \leqslant N^{\sigma-d+1}$ for all 1-tubes T.

Let $E_j = \{x : 2^j \leqslant w(x) < 2^{j+1}\}$ for $1 \leqslant 2^j \leqslant N^{\sigma-d+1}$. Then we have

$$\sum_j 2^j |E_j \cap T| \leqslant N^{\sigma-d+1}$$

for all 1-tubes T, and also

$$\sum_j 2^j |E_j| \geqslant C_{d,\sigma} N^{\sigma}.$$

Thus there exists a j with

$$2^j |E_j| \geqslant C_{d,\sigma} \frac{N^{\sigma}}{\log N},$$

and also

$$2^j |E_j \cap T| \leqslant N^{\sigma-d+1}$$

for all 1-tubes T. Thus, for some j, we have

$$|E_j| \geqslant C_{d,\sigma} \frac{N^{\sigma}}{2^j \log N}$$

and

$$|E_j \cap T| \leqslant \frac{N^{\sigma-d+1}}{2^j}$$

for all 1-tubes T. Letting s be defined by $N^{\sigma}/2^j = N^s$, we have

$$|E_j| \geqslant C_{d,\sigma} \frac{N^s}{\log N}$$

and

$$|E_j \cap T| \leqslant N^{s-d+1}$$

for all 1-tubes T. □

In all likelihood it is the case $2^j = 1$ that actually occurs in the argument; it would be too optimistic to expect large values to occur.

Theorem 1 now follows from the next result together with the trivial observation that, if (6) holds for a certain β and γ, then it also holds for the same β and all γ' with $\gamma' \leqslant \gamma$.

Proposition 4. *For each $\epsilon > 0$ sufficiently small there exists a $0 < \delta \leqslant \epsilon$ such that, if $\gamma < d-1$ and $\beta = \gamma + \delta$, then there exists a set $E \subseteq Q^d$ of positive finite β-dimensional Hausdorff measure such that $\mathcal{H}_\beta(E \cap T) \leqslant C_{\beta,\gamma,\epsilon,d} w(T)^{\gamma}$ for all tubes T.*

Proof. Let $\sigma = d - 1 + \epsilon > d - 1$ and let $C_{d,\sigma}$ be the constant implied in Corollary 3. Choose N such that $N^{\gamma-d+1} \log N = C_{d,\sigma}$ and so that $N^{\epsilon} \geqslant 2$. We can do this because $\gamma < d-1$ and because we can adjust $C_{d,\sigma}$ to be smaller if necessary (depending on ϵ) to accommodate the condition $N^{\epsilon} \geqslant 2$. Set $a = C_{d,\sigma}/\log N$, and note that

$$d - 1 - \gamma = \frac{\log(1/a)}{\log N}.$$

By Corollary 3, there is an $s \in (d-1, \sigma]$ such that there is a set of (exactly) $C_{d,\sigma} N^s / \log N$ unit cubes in $[0, N]^d$ with no 1-tube meeting more than N^{s-d+1} of them. Define $\delta := s - (d-1)$.

Rescale this set so that it consists of small cubes of side N^{-1} contained inside the unit cube Q^d. We now build a self-similar Cantor set whose first stage is this set. Thus the first stage $\mathcal{E}_1$ consists of aN^s cubes of side N^{-1} such that no tube of width N^{-1} meets more than N^{s-d+1} of these cubes.

For the second stage we put a $1/N$-scaled copy of our basic set inside each cube from the first stage, and so the second stage $\mathcal{E}_2$ consists of a^2N^{2s} cubes of side N^{-2} with the property that any tube of width N^{-2} meets only at most N^{s-d+1} cubes of the second stage within each cube of the first stage. Since the expansion of any N^{-2}-tube by a factor of N meets only at most N^{s-d+1} cubes from the *first* stage, the same holds for any N^{-2}-tube itself. Thus any N^{-2}-tube meets at most $N^{2(s-d+1)}$ cubes of the second stage altogether.

We continue in this manner. Thus, at the kth stage we have a family $\mathcal{E}_k$ of a^kN^{ks} cubes of side N^{-k} with the property that any tube of width N^{-k} meets only at most N^{s-d+1} cubes of the kth stage within each cube of the $(k-1)$th stage, and thus (by induction) at most $N^{k(s-d+1)}$ cubes of $\mathcal{E}_k$ altogether.

We define

$$E = \bigcap_{k=1}^{\infty} \bigcup_{Q\in\mathcal{E}_k} Q.$$

Note that the Minkowski dimension of E is

$$s - \frac{\log(1/a)}{\log N} = s - (d-1-\gamma) = \gamma + \delta = \beta.$$

A standard argument (see [**15**, p. 63]) gives that the Hausdorff dimension of E is also β and that E has positive finite β-dimensional Hausdorff measure equal to H, say. Note by self-similarity that, if $Q \in \mathcal{E}_k$, then $\mathcal{H}_\beta(Q) = a^{-k}N^{-ks}H$.

We next show that $\mathcal{H}_\beta(E\cap T) \leqslant Cw(T)^\gamma$ for all tubes T. Indeed, we shall show that, if $w(T) = N^{-k}$, then we have $\mathcal{H}_\beta(E\cap T) \leqslant CN^{-\gamma k}$. (The general case follows from this one at the expense of a power of N.) Since the number of cubes of $\mathcal{E}_k$ that a given tube of width N^{-k} can meet is at most $N^{(s-d+1)k}$ and the total number of cubes in $\mathcal{E}_k$ is a^kN^{ks}, then

$$\frac{\mathcal{H}_\beta(E\cap T)}{N^{k\gamma}} \leqslant \frac{N^{(s-d+1)k}H}{a^kN^{ks}N^{k\gamma}} = H\left(\frac{N^{\gamma-d+1}}{a}\right)^k = H$$

by the choice of N. This concludes the proof. □

Notice that the argument gives no result on the line $\beta = \gamma$; moreover, even if we had the best possible starting situation of a set of c_dkN^{d-1} points with at most k in any 1-tube for all $k \geqslant 2$, we would not be able to obtain an example in the case $\beta > \gamma = d-1$ with this argument.

For values of (β,γ) far from $(d-1,d-1)$ there are explicit examples establishing the conclusion of Theorem 1. Thus the unit sphere demonstrates it for $\beta = d-1$ and $\gamma = d-3/2$. We already know (see Proposition 1) that all radial sets of dimension $\beta > d-1/2$ are examples when $\gamma = d-1$. Similarly, suitable radial Cantor sets furnish examples when $d-1 < \beta < d-1/2$ and $\gamma \geqslant \beta - 1/2$. We can improve on this by building Cantor sets based upon the constructions of Section 5 below to obtain examples for $\beta > \gamma$ and (β,γ) strictly under the line joining $((d-1)/2,(d-1)/2)$ to $(d-1/2,d-1)$. We leave the details to the interested reader.

It would be interesting to know if there are examples of *rectifiable* sets demonstrating the conclusion of Theorem 1 when $\beta = d-1$ and $d-3/2 < \gamma < d-1$.

5. *Concentric spheres*

For $1 \leqslant k \leqslant N^{1/2}$ consider a collection $\mathcal{C}_k$ of $C_d k^d N^{(d-1)/2}$ points obtained by placing points roughly equally spaced by approximately $N^{1/2}/k$ on each of k concentric spheres in $\mathbb{R}^d$ with equally-spaced radii in $[N/2, N]$. (Note that a *single* such sphere contains approximately $C_d k^{d-1} N^{(d-1)/2}$ points and no 1-tube contains more than $\mathrm{O}(k)$ of its points.)

Proposition 5. *No* 1*-tube meets more than* $\mathrm{O}(k)$ *members of* $\mathcal{C}_k$.

Proof. A 1-tube T will typically meet points from two types of sphere. The first type consists of those spheres that contribute multiple points to T; the second type consists of those that contribute at most 1 point. The overall contribution of those of the second type is clearly $\mathrm{O}(k)$, and so it suffices to deal with those of the first type.

For a sphere of radius λ to contribute multiple points to T it must be that the cap where the sphere of radius λ meets T is nonempty and has diameter ℓ_λ, which is at least the spacing of the points on this sphere, that is, at least $N^{1/2}/k$. Then the number of points so contributed from this sphere will be $\mathrm{O}(\ell_\lambda k/N^{1/2})$ (because the cap will be elliptical with $(d-2)$ short sides of length approximately 1). Therefore, the total number of points contributed by the spheres contributing multiply is

$$\frac{k}{N^{1/2}} \sum_{\lambda : \ell_\lambda \geqslant N^{1/2}/k} \ell_\lambda. \tag{7}$$

Suppose that T has distance ρ from the origin, with $\rho \leqslant \lambda$ to ensure that T actually meets the annulus of radius λ. If $\lambda \geqslant \rho + 1$, then ℓ_λ is about $N^{1/2}/(\lambda-\rho)^{1/2}$, so that multiple contributions only occur when $\lambda - \rho \leqslant k^2$. If $\rho \leqslant \lambda \leqslant \rho + 1$, then ℓ_λ is about $N^{1/2}$, which is good. Therefore, (7) is effectively at most

$$\frac{k+k}{N^{1/2}} \sum_{\lambda : \lambda - \rho \leqslant k^2} N^{1/2} (\lambda - \rho)^{-1/2} = k \left(1 + \sum_{\lambda : \lambda - \rho \leqslant k^2} (\lambda - \rho)^{-1/2} \right).$$

Now the values of λ are equally spaced in $[N/2, N]$ with spacing N/k, and so

$$\begin{aligned} \sum_{\lambda : \lambda - \rho \leqslant k^2} (\lambda - \rho)^{-1/2} &= \sum_{j : jN/k - \rho \leqslant k^2} \left(\frac{jN}{k - \rho} \right)^{-1/2} \\ &= \left(\frac{k}{N} \right)^{1/2} \sum_{j : j - \rho k/N \leqslant k^3/N} \left(\frac{j - \rho k}{N} \right)^{-1/2} \\ &\leqslant \left(\frac{k}{N} \right)^{1/2} \left(\frac{k^3}{N} \right)^{1/2} = \frac{k^2}{N} \leqslant 1. \end{aligned}$$

Therefore, (7) is dominated by k, as required. □

Remark 2. A weak point of the argument is the simple estimate for the tubes contributing at most one point. Clearly, there is scope for choosing rotations of the spheres to make it very unlikely that a given tube would meet many points on these spheres. This probabilistic approach leads ultimately to the considerations of Section 2, where in choosing points at random (rather

than choosing random rotations of the fixed configurations on spheres that we have built here) leads to a situation where the details are somewhat cleaner.

So, for the set $\mathcal{C}_k$, Conjecture 1 predicts that, for $1 \leqslant k \leqslant N^{1/2}$, we have

$$\sum_{x_\alpha \in \mathcal{C}_k} |\widehat{gd\sigma}(x_\alpha)|^2 \leqslant C_d k \int_{S^{d-1}} |g|^2 d\sigma, \tag{8}$$

while the Stein–Tomas restriction theorem (2) gives

$$\sum_{x_\alpha \in \mathcal{C}_k} |\widehat{gd\sigma}(x_\alpha)|^2 \leqslant C_d k^{2d/(d+1)} N^{(d-1)/(d+1)} \int_{S^{d-1}} |g|^2 d\sigma;$$

clearly, the former is a much stronger inequality for all $1 \leqslant k \leqslant N^{1/2}$. As a first indication that some of the inequalities (8) may have a chance of being true, we mention a result from [**4**] (Corollary 3) that implies that, when $d = 2$ and we place $N^{2/3}$ points x_α at roughly equal spacings of approximately $N^{1/3}$ on a circle of radius N, then we have

$$\sum_{x_\alpha} |\widehat{gd\sigma}(x_\alpha)|^2 \leqslant C_d \log N \int_{S^1} |g|^2 d\sigma.$$

Thus, when $d = 2$, we have at least some handle on the extreme cases $k = 1$ and $k = N^{1/2}$ of (8). We hope to return to these matters elsewhere.

Acknowledgements. The author would like to thank Apostolos Giannopoulos for some valuable conversations during his visit to Athens. He would also like to thank Michael Christ for discussions on some aspects of this material.

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